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BOUNDARY VALUE PROBLEM OF HARMONIC MAPS INTO $\mathbb{CP}^n$ AND $\mathbb{QP}^n$

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We construct equivariant maps from $\mathbb{B}^{m+1}$ into $\mathbb{CP}^n$ and $\mathbb{QP}^n$, and prove the existence of such harmonic maps for equivariant boundary data which are not a priori required to have small range.

1. Introduction.

The boundary value problem for harmonic maps is solvable when the boundary image lies in a geodesic convex neighbourhood of the target manifold (see. [**Ha**], [**H-K-W**]). In general, a size restriction of the boundary map is necessary and the result in [**H-K-W**] is optimal. But, one still expects the solvability for the boundary value problem with large image range when the boundary condition is 'sufficiently nice'. In [**J-K**] and [**E-L1**] the authors consider the rotationally symmetric harmonic maps from $\mathbb{B}^m$ into $\mathbb{S}^n$ whose boundary values lie just outside of a geodesic convex neighbourhood. Recently, many works have been written on maps from $\mathbb{B}^3$ into $\mathbb{S}^2$ (see [**Ha**], [**H-K-L1**], [**H-K-L2**], [**H-L-P**] and [**Z**]).

It is natural to investigate the problem with large boundary data when the target manifold with variable sectional curvature. The first candidate is complex projective space with the Fubini-Study metric. In author's previous work [**X2**] a reduction method for large and equivariant boundary data has been exhibited, and the simplist case has been solved by the heat flow method. In the present paper we study more general situations by the variational method. Besides the concrete results, the research on this problem might give some implications on the obstruction of the solvability of certain boundary value problems for a system of elliptic PDE.

Let $\mathbb{B}^{m+1}$ be the (m+1)-dimensional unit ball. Under an $\mathbb{S}^{m-1}$ action the base region $D \in \mathbb{R}^2$ is given by

$$D = \left\{(r,z) \in \mathbb{R}^2; \quad r^2+z^2<1, \quad r>0\right\}.$$

Then $\tilde{r} = (r,z) : \mathbb{B}^{m+1} \to D$ is an isoparametric map of rank 2. We denote $\partial_1 D = \partial D \setminus \{r=0\}$. On the other hand the distance function from a fixed point in $\mathbb{CP}^n$ (res. $\mathbb{QP}^n$) is an isoparametric function $\phi \quad (0 \leq \phi \leq$

$\frac{\pi}{2}$). Let $f_1 : \mathbb{S}^{m-1} \to \mathbb{CP}^{n-1}\,(\text{res}.\ \mathbb{QP}^{n-1})$, $f_2 : \mathbb{S}^{m-1} \to \mathbb{S}^1\,(\text{res}.\ \mathbb{S}^3)$ be harmonic maps of constant energy densities $\frac{\lambda_1}{2}$ and $\frac{\lambda_2}{2}$, respectively. By using the geometric properties of the geodesic sphere in $\mathbb{CP}^n\,(\text{res}.\ \mathbb{QP}^n)$ we can construct equivariant maps from $\mathbb{B}^{m+1}$ into $\mathbb{CP}^n\,(\text{res}.\ \mathbb{QP}^n)$ from f_1 and f_2 as described in Section 3.

We first study the case m=2 which has interest in its own right and obtain the following result.

Theorem A. *For any equivariant boundary condition with respect to the isoparametric map $\tilde{r}$ and the isoparametric function ϕ, whose restriction on $\partial_1 D$ is a regular function $\phi_0 \quad (0 \le \phi_0 \le \frac{\pi}{2})$ which is of order $O(r^{\sqrt{\lambda_1+\lambda_2}})$ as $r \to 0$, there exists a solution to the boundary value problem of equivariant harmonic maps from $\mathbb{B}^3$ into $\mathbb{CP}^n$ with the above boundary data.*

Remark 1. It is well-known that for complex projective space with the Fubini-Study metric, the sectional curvature lies between 1 and 4, the radius of the geodesic convex ball is $\frac{\pi}{4}$ and its diameter is $\frac{\pi}{2}$. The boundary condition in our theorem overpasses the convex ball and can reach any possible range.

Remark 2. The above result holds true for quaternionic projective space.

We then study the higher dimensional case $m > 2$. The relevant ODE is different from the former case and has no first integral. By employing the stability theory in ODE and some estimates we are able to overcome the difficulty and obtain the following Theorem B. As a by-product, our construction can also supply the rotationally symmetric maps from $\mathbb{B}^m$ into $\mathbb{CP}^n\,(\text{res}.\ \mathbb{QP}^n)$. The reduced harmonicity equation is also Eq. (4.9). Lemma 4.7 below shows that there always exist the rotationally symmetric harmonic maps from $\mathbb{B}^m$ into $\mathbb{CP}^n\,(\text{res}.\ \mathbb{QP}^n)$ for the given large boundary condition $\phi(1) = \psi < \frac{\pi}{2}$ without any dimension limitation. This phenomenon is in sharp contrast with the case when the target manifold is sphere (see [**J-K**] and [**E-L1**]).

Theorem B. *Given an equivariant boundary condition with respect to the isoparametric map $\tilde{r}$ and the isoparametric function ϕ, whose restriction on $\partial_1 D$ is a regular function ϕ_0 $\left(0 \le \phi_0 < \frac{\pi}{2}\right)$, which is of order $O(r^{\sqrt{\lambda_1+\lambda_2}})$ as $r \to 0$, if $\lambda_1 > \lambda_2$ then there exists a solution to the boundary value problem of equivariant harmonic maps from $\mathbb{B}^{m+1}$ into $\mathbb{QP}^n$ with the given boundary condition.*

Remark. Theorem B is also valid for $\mathbb{CP}^n$ as the target manifold. But $\lambda_2 = 0$ in this case.

2. The Geometry of $\mathbb{CP}^n$ and $\mathbb{QP}^n$.

Let $\pi : \mathbb{S}^{2n+1}\,(\text{res}\,.\ \mathbb{S}^{4n+3}) \to \mathbb{CP}^n\,(\text{res}\,.\ \mathbb{QP}^n)$ be the usual Riemannian submersion with totally geodesic fiber $\mathbb{S}^1\,(\text{res}\,.\ \mathbb{S}^3)$. For any $Z \in \mathbb{S}^{2n+1}$ (res. $\mathbb{S}^{4n+3}$) there exist $X \in \mathbb{S}^{2n-1}\,(\text{res}\,.\ \mathbb{S}^{4n-1})$ and $Y \in \mathbb{S}^1\,(\text{res}\,.\ \mathbb{S}^3)$ such that

$$Z = (X \sin\phi,\, Y\cos\phi), \quad 0 \le \phi \le \frac{\pi}{2}, \tag{2.1}$$

where ϕ is an isoparametric function on $\mathbb{S}^{2n+1}\,(\text{res}\,.\ \mathbb{S}^{4n+3})$ which equivariant with respect to Riemannian submersion π. This induces an isoparametric function on $\mathbb{CP}^n\,(\text{res}\,.\ \mathbb{QP}^n)$. We denote it by the same letter ϕ. The level hypersurfaces of ϕ are given by

$$\begin{aligned} &M_\phi = \mathbb{S}^{2n-1}(\sin\phi) \times \mathbb{S}^1(\cos\phi)/\mathbb{S}^1 \\ &\Big(\text{res}\,.\ M_\phi = \mathbb{S}^{4n-1}(\sin\phi) \times \mathbb{S}^3(\cos\phi)/\mathbb{S}^3\Big), \qquad 0 < \phi < \frac{\pi}{2} \end{aligned} \tag{2.2}$$

with the focal point $A \in \mathbb{CP}^n\,(\text{res}\,.\ \mathbb{QP}^n)$ and the focal variety $\mathbb{CP}^{n-1}$ (res. $\mathbb{QP}^{n-1}$). One can easily see that M_ϕ is the geodesic sphere at the distance ϕ from A.

Every geodesic emanating from the point A lies in certain complex (res. quaternionic) projective line passing through A. It follows that these projective lines are the integral manifolds of the distribution $\{n = \operatorname{grad}\phi, Jn\}$, (res. $\{n = \operatorname{grad}\phi, J_1 n,\ J_2 n,\ J_3 n\}$) where $J\,(\text{res}\,.\ J_1,\ J_2,\ J_3)$ is the complex (res. quaternionic) structure of $\mathbb{CP}^n\,(\text{res}\,.\ \mathbb{QP}^n)$. We know that $\mathbb{CP}^1 = \mathbb{S}^2(\frac{1}{2})$ (res. $\mathbb{QP}^1 = \mathbb{S}^4(\frac{1}{2})$) of constant sectional curvature 4, which is totally geodesic in $\mathbb{CP}^n$ (res. $\mathbb{QP}^n$). The integral curves of $n = \operatorname{grad}\phi$ are geodesics in $\mathbb{CP}^n$ (res. $\mathbb{QP}^n$). Thus $\mathbb{CP}^1$ (res. $\mathbb{QP}^1$) has the metric form in polar coordinates

$$d\phi^2 + \left(\frac{1}{2}\sin 2\phi\right)^2 d\alpha^2, \tag{2.3}$$

where $0 \le \alpha \le 2\pi\,(\text{res}\,.\ d\alpha^2$ is the metric form of $\mathbb{S}^3)$. It follows that $Jn\,(\text{res}\,.\ J_i n)$ lies in the principal direction corresponding to the principal curvature $-2\cot 2\phi$.

For any Z in a level hypersurface M_ϕ we draw a geodesic $\gamma(\phi)$ connecting the points A and Z (γ is unique and perpendicularly intersects M_ϕ, since the cut locus distance is $\frac{\pi}{2}$), then extend it to the focal variety $\mathbb{CP}^{n-1}\,(\text{res}\,.\ \mathbb{QP}^n)$. This yields a unique intersection point $A' \in \mathbb{CP}^{n-1}\,(\text{res}\,.\ \mathbb{QP}^{n-1})$. These two points A and A' uniquely determine a complex (res. quaternionic) projective line $\mathbb{CP}^1 = \mathbb{S}^2(\frac{1}{2})\,(\text{res}\,.\ \mathbb{QP}^1 = \mathbb{S}^4(\frac{1}{2}))$ which perpendicularly intersects the geodesic sphere M_ϕ at $\mathbb{S}^1(\frac{1}{2}\sin 2\phi)\,(\text{res}\,.\ \mathbb{S}^3(\frac{1}{2}\sin 2\phi))$. Choosing a local orthonormal frame field in $\mathbb{CP}^{n-1}\,(\text{res}\,.\ \mathbb{QP}^{n-1})$ near A', then parallel translating it back to the Z along $\gamma(\phi)$, it can be proved that all of those lie in principal directions corresponding to the principal curvature $-cot\,\phi$ (see [**X1**]). Hence, we have:

Proposition 2.1. *The geodesic sphere*

$$M_\phi = \mathbb{S}^{2n-1}(\sin\phi)\times\mathbb{S}^1(\cos\phi)/\mathbb{S}^1$$
$$\Big(\text{res.}\;\; M_\phi = \mathbb{S}^{4n-1}(\sin\phi)\times\mathbb{S}^3(\cos\phi)/\mathbb{S}^3\Big)$$

in $\mathbb{CP}^n$ (*res.* $\mathbb{QP}^n$) *has principal curvatures* $-\cot\phi$ *of multiplicity* $2n-2$ (*res.* $4n-4$) *and* $-2\cot 2\phi$ (*res. of multiplicity* 3).

3. Construction of Equivariant Maps into $\mathbb{CP}^n$ and $\mathbb{QP}^n$.

Let (M,g) and (N,h) be Riemannian manifolds with metric tensors g and h, respectively. Harmonic maps are described as critical points of the following energy functional

$$E(f) = \frac{1}{2}\int_M e(f) * 1, \tag{3.1}$$

where $e(f)$ stands for the energy density. The Euler-Lagrange equation of the energy functional is

$$\tau(f) = 0, \tag{3.2}$$

where $\tau(f)$ is the tension field. In local coordinates

$$\begin{aligned} e(f) &= g^{ij}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}h_{\beta\gamma},\\ \tau(f) &= \left(\Delta_M f^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right)\frac{\partial}{\partial y^\alpha}, \end{aligned} \tag{3.3}$$

where $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel's symbols of the target manifold N. Here and in the sequal we use the summation convention. For more detail knowledge of harmonic maps please consult [**E-L2**].

Let $\pi_1 : M \to \overline{M}$, $\pi_2 : N \to \overline{N}$ be Riemannian submersions. If $f : M \to N$ is a fiber-preserving map, namely for the points $x_1, x_2 \in M$, $\pi_2(f(x_1)) = \pi_2(f(x_2))$ provided $\pi_1(x_1) = \pi_1(x_2)$, then f is called an equivariant map with respect to Riemannian submersions π_1 and π_2. Due to the structure of the Riemannian submersion there are vertical vector fields which are tangent to fiber submanifolds and horizontal vector fields which are orthogonal complements of the vertical vector fields. A map f is called horizontal if it maps any horizontal vector field to a horizontal one.

Now we are going to define a concrete equivariant map from $\mathbb{B}^{m+k}$ into $\mathbb{CP}^n$.

Let $\mathbb{B}^{m+k}$ be a unit $(m+k)$–dimensional open ball in $\mathbb{R}^{m+k}$. Consider the generalized cylindrical coordinates in $\mathbb{B}^{m+k}$. For any $Z \in \mathbb{B}^{m+k}$ there exist $X \in \mathbb{S}^{m-1}$ and $(r, z_1, \dots, z_k) \in D$ such that

$$Z = (rX, z_1, \dots, z_k),$$

where

$$D = \left\{ (r, z_1, \dots, z_k) \in \mathbb{R}^{k+1};\ r^2 + \sum_{\ell=1}^{k} z_\ell^2 < 1, \quad r > 0 \right\}.$$

The Euclidean metric in $\mathbb{B}^{m+k}$ is then given by

$$ds^2 = dr^2 + r^2 ds_1^2 + \sum_{\ell=1}^{k} dz_\ell^2,$$

where ds_1^2 is the standard metric in $\mathbb{S}^{m-1}$. It is easily seen that

$$|dr|^2 = |dz|^2 = 1\ .$$

By a direct computation

$$\Delta r = \frac{m-1}{r},$$
$$\Delta z_\ell = 0.$$

Hence $\tilde{r} = (r, z_1, \dots, z_k) : \mathbb{B}^{m+k} \to D$ is an isoparametric map of rank $k+1$ with fiber submanifolds $\mathbb{S}^{m-1}(r)$. In D we define a usual flat metric $dr^2 + \sum_{\ell=1}^{k} dz_\ell^2$ such that $\tilde{r} : \mathbb{B}^{m+k} \setminus \{r = 0\} \to D$ is a Riemannian submersion.

On the other hand, on the target manifold $\mathbb{CP}^n$ (res. $\mathbb{QP}^n$), as described in the last section, there is an isoparametric function ϕ with focal point A and the focal variety $\mathbb{CP}^{n-1}$ (res. $\mathbb{QP}^{n-1}$).

Let $f_1 : \mathbb{S}^{m-1} \to \mathbb{CP}^{n-1}$ (res. $\mathbb{QP}^{n-1}$) be a harmonic map with the constant energy density $\frac{\lambda_1}{2}$, $f_2 : \mathbb{S}^{m-1} \to \mathbb{S}^1$ (res. $\mathbb{S}^3$) be a harmonic map with the constant energy density $\frac{\lambda_2}{2}$ (if $m > 2$, $\lambda_2 = 0$ in the complex case). Now we define a map $f : \mathbb{B}^{m+k} \to \mathbb{CP}^n$ (res. $\mathbb{QP}^n$) as follows. For any $Z = (rX, z) \in \mathbb{B}^{m+k} \setminus \{r = 0\}$ we join A and $f_1(X) \in \mathbb{CP}^{n-1}$ (res. $\mathbb{QP}^{n-1}$) by the unique complex (res. quaternionic) projective line which intersects a level hypersurface $M_{\phi(r,z)}$ at a circle $\mathbb{S}^1(\frac{1}{2}\sin 2\phi)$ (res. a sphere $\mathbb{S}^3(\frac{1}{2}\sin 2\phi)$). By then using f_2 we have a point $f(Z) \in M_{\phi(r,z)} \in \mathbb{CP}^n$, where the smooth function $\phi(r, z)$ on D will be determined later by the harmonicity equation. It is easily seen that f is an equivariant map with respect to Riemannian submersions in both domain and target manifolds. It induces a harmonic map between fiber submanifolds. It is also a horizontal map.

Thus, we can use the following reduction theorem (see [**X1**], pp. 273-275) to derive the harmonicity equation.

Theorem 3.1. *Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be Riemannian submersions, H_1 the mean curvature vector of the fiber submanifold F_1 in E_1, and B_2 the second fundamental form of the fiber submanifold F_2 in E_2. Let $f : E_1 \to E_2$ be a horizantal equivariant map, $\bar{f}$ its induced map from M_1 into M_2 with tension field $\tau(\bar{f})$. $f^{\perp}$ denotes the restriction of the fiber F_1. Then f is a harmonic map if and only if $f^{\perp}$ is harmonic and the following equation is satisfied*

$$\tau^*(\bar{f}) + B_2(f_*e_t, f_*e_t) - f_*H_1 = 0,$$

where $\{e_t\}$ $(t = m_1 + 1, \cdots, n_1)$ is a local orthonormal frame field of fiber F_1 and $\tau^*(\bar{f})$ denotes the horizontal lift of $\tau(\bar{f})$.

Let H_1 be the mean curvature vector of the fiber submanifold $\mathbb{S}^{m-1}$ in $\mathbb{B}^{m+k}$. Then by a computation we have

$$H_1 = -\frac{m-1}{r}\frac{\partial}{\partial r},$$
$$f_*H_1 = -\frac{m-1}{r}\frac{\partial \phi}{\partial r}\frac{\partial}{\partial \phi}.$$

Let B_2 be the second fundamental form of the fiber submanifold M_ϕ in $\mathbb{CP}^n$ (res. $\mathbb{QP}^n$). Let $\{\frac{1}{r}e_t\}$ be a local orthonormal frame field of $\mathbb{S}^{m-1}(\frac{1}{r})$. By a direct computation

$$B_2\left(f_*\frac{1}{r}e_t, f_*\frac{1}{r}e_t\right) = -\frac{\lambda_1 \sin\phi\cos\phi}{r^2} - \frac{\lambda_2 \sin 2\phi \cos 2\phi}{2r^2}.$$

The defined map f induces a map $\bar{f}$ between base manifolds. Its tension field is

$$\tau(\bar{f}) = \frac{\partial^2 \phi}{\partial r^2} + \sum_{\ell=1}^{k}\frac{\partial^2\phi}{\partial z_\ell^2}$$

with horizontal lift

$$\tau^*(\bar{f}) = \left(\frac{\partial^2 \phi}{\partial r^2} + \sum_{\ell=1}^{k}\frac{\partial^2\phi}{\partial z_\ell^2}\right)\frac{\partial}{\partial \phi}.$$

In what follows we only consider the case $k = 1$. The equation reduces to

$$(3.4)\quad \frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{m-1}{r}\frac{\partial\phi}{\partial r} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi\cos 2\phi = 0,$$
$$(r,z) \in D, \quad 0 < \phi < \frac{\pi}{2}.$$

If $\lim_{r\to 0}\phi(r,z) = 0$, then f can be continuously extended to whole $\mathbb{B}^{m+1}$. Furthermore, $f(\mathbb{B}^{m+1})$ does not lie in a complex (res. quaternionic) projective line $\mathbb{CP}^1$ (res. $\mathbb{QP}^1$) for $\lambda_1 \neq 0$, since any complex (res. quaternionic) projective line starting from A intersects the focal variety $\mathbb{CP}^{n-1}$ (res. $\mathbb{QP}^{n-1}$) at only one point. We are interested in the general case when both λ_1 and λ_2 do not vanish.

If the boundary data are also equivariant with respect to isoparametric map $\tilde{r}$ and the isoparametric function ϕ, then the boundary condition is also reduced to the boundary $\partial_1 D$. Furthermore, suppose that the function $\phi_0 = \phi|_{\partial_1 D}$ satisfies the following conditions:

$$(3.5)\qquad 1)\ \lim_{r\to 0}\phi_0 = 0;$$
$$2)\ \max\,\phi_0 \leq \frac{\pi}{2}.$$

Any solution to Equation (3.4) with boundary conditions (3.5) supplies us a continuous map f from $\mathbb{B}^{m+1}$ into $\mathbb{CP}^n$ (res. $\mathbb{QP}^n$), which is smooth harmonic on $\mathbb{B}^{m+1}\setminus\{r = 0\}$. One can prove that the map is weakly harmonic on whole $\mathbb{B}^{m+1}$ by a cut-off function technique. Thus, by main regularity theorem for harmonic maps (see [**Hi**] or [**E-L2**, p. 397]), f is a smooth harmonic map.

4. Proof of the main theorems.

It is not difficult to obtain the energy functional of the above defined map (up to a constant factor) as follows

$$(4.1)\qquad E = \int_D \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{\lambda_1}{r^2}\sin^2\phi + \frac{\lambda_2}{4r^2}\sin^2 2\phi\right] r^{m-1}\,dr dz,$$

which is defined on the Hilbert space

$$X = \Big\{ f \in L_1^2(\mathbb{B}^{m+1}, \mathbb{CP}^n\ (\text{res.}\ \ \mathbb{QP}^n));$$
$$f\ \text{ is equivariant with respect to the Riemannian submersions}\Big\}.$$

We will solve Equation (3.4) by finding a smooth critical point of the functional (4.1).

The following Lemma 4.1 and Lemma 4.2 are similar to that in [**Z**]. For convenience to readers we record their proofs here.

Lemma 4.1. *Any critical point ϕ of E is a local minimum in the sense that*

$$E(\phi) \leq E(\phi + \eta)$$

provided $\eta \in C_0^\infty(D)$ and

$$\begin{aligned} d(\partial D, \operatorname{supp}(\eta)) &\geq C\rho\,(\operatorname{supp}(\eta)), & \text{when} \quad m = 2, \\ [d(\partial D, \operatorname{supp}(\eta))]^{m-1} &\geq C[\rho\,(\operatorname{supp}(\eta))]^2, & \text{when} \quad m > 2, \end{aligned}$$

where $d(\cdot,\cdot)$ and $\rho(\cdot)$ denote the distance and the diameter, and C is a constant.

Proof. Suppose that $\phi \in X$ is a critical point of E and η is a map in $C_0^\infty(D)$. Using integral by parts, Equation (3.4) and Taylor's theorem, we have

$$\begin{aligned} &2(E(\phi) - E(\phi + \eta)) \\ &= \int_D \Big\{ r^{m-1}\left(|\nabla\phi|^2 - |\nabla(\phi+\eta)|^2\right) + \lambda_1(\sin^2\phi - \sin^2(\phi+\eta))r^{m-3} \\ &\qquad\qquad + \frac{\lambda_2(\sin^2 2\phi - \sin^2 2(\phi+\eta))r^{m-3}}{4} \Big\} drdz \\ &= \int_D \Big\{ - r^{m-1}|\nabla\eta|^2 + \lambda_1(\sin^2\phi - \sin^2(\phi+\eta) + \eta\sin 2\phi)r^{m-3} \\ &\qquad\qquad + \frac{\lambda_2(\sin^2 2\phi - \sin^2 2(\phi+\eta) + 4\eta\sin 2\phi\cos 2\phi)\, r^{m-3}}{4} \Big\} drdz \\ &\leq - \int_D r^{m-1}|\nabla\eta|^2 drdz + \int_D (\lambda_1 + \lambda_2)\eta^2 r^{m-3} drdz. \end{aligned}$$

When $m = 2$,

$$\begin{aligned} &2(E(\phi) - E(\phi+\eta)) \\ &\leq -d(\partial D, \operatorname{supp}(\eta)) \int_D |\nabla\eta|^2 drdz + \frac{\int_D (\lambda_1+\lambda_2)\eta^2 drdz}{d(\partial D, \operatorname{supp}(\eta))}. \end{aligned}$$

When $m > 2$,

$$\begin{aligned} &2(E(\phi) - E(\phi+\eta)) \\ &\leq -[\,d(\partial D, \operatorname{supp}(\eta))]^{m-1} \int_D |\nabla\eta|^2 drdz + \int_D (\lambda_1+\lambda_2)\eta^2 drdz. \end{aligned}$$

Noting that the first eigenvalue of the Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^n$ is bounded from below by $\{\rho(\Omega)\}^{-2}$, we obtain the required inequality, provided the condition on $\mathrm{supp}(\eta)$ is satisfied. $\square$

Lemma 4.2. *Suppose ϕ_1 and ϕ_2 are critical points of E and $0 \leq \phi_1 \leq \phi_2$ on D. Let ψ_i be the boundary values of ϕ_i, $i = 1, 2$. Suppose ψ is a function on $\partial_1 D$ lying between ψ_1 and ψ_2, i.e. $\psi_1 \leq \psi \leq \psi_2$. Then*

$$c \equiv \inf\{\, E(\phi);\ \phi_1 \leq \phi \leq \phi_2,\ \phi|_{\partial_1 D} = \psi\}$$

is a critical value of E and is achieved by a critical point ϕ, i.e. there exists $\phi \in X$ such that

$$\frac{d}{dt}E(\phi + t\eta)|_{t=0} = 0, \quad \text{for all } \ \eta \in C_0^\infty(D). \tag{4.2}$$

Moreover, $\phi_1 \leq \phi \leq \phi_2$ on D and $\phi|_{\partial_1 D} = \psi$.

Proof. Consider a minimizing sequence $\{\phi_i\}$ which has a weak limit ϕ in L^2_1 sense. It is easy to see that $E(\phi) = c$; in fact, ϕ is a strong limit of $\{\phi_i\}$ in L^2 sense. Therefore, ϕ_i converges to ϕ pointwisely almost everywhere, $\phi_1 \leq \phi \leq \phi_2$ and $\phi|_{\partial_1 D} = \psi$. In order to show that ϕ is a critical point of E, we need to prove that (4.2) holds for all $\eta \in C_0^\infty$. But it is enough to show that (4.2) holds in the case $m = 2$ for those $\eta \in C_0^\infty$ satisfying (for $m > 2$ there are minor modifications)

$$d(\partial D, \mathrm{supp}(\eta)) \geq C\,\rho(\mathrm{supp}(\eta)).$$

Let η be such a C_0^∞ function and consider

$$\phi_t = \phi + t\eta, \quad \text{for all } \ t \in \mathbb{R}.$$

We define

$$S_t = \{(r, z) \in D;\ \phi + t\eta > \phi_2\},$$
$$M_t = \{(r, z) \in D;\ \phi + t\eta < \phi_1\}.$$

Since $\phi_1 \leq \phi \leq \phi_2$, S_t and M_t are subsets of $\mathrm{supp}(\eta)$ and they satisfy the following conditions

$$d(\partial D, S_t) \geq C\,\rho(S_t),$$
$$d(\partial D, M_t) \geq C\,\rho(M_t).$$

Define

$$\phi^t = \begin{cases} \phi_2 & \text{on } S_t, \\ \phi + t\eta & \text{on } D \setminus (S_t \cup M_t), \\ \phi_1 & \text{on } M_t; \end{cases}$$

$$\phi_1^t = \begin{cases} \phi + t\eta & \text{on } M_t, \\ \phi_1 & \text{on } D \setminus M_t; \end{cases}$$

$$\phi_2^t = \begin{cases} \phi + t\eta & \text{on } S_t, \\ \phi_2 & \text{on } D \setminus S_t. \end{cases}$$

By the definition of ϕ, we know that $E(\phi) \leq E(\phi^t)$. By using Lemma 4.1 we have

$$E(\phi^t) - E(\phi + t\eta) = \{E(\phi_1) - E(\phi_1^t)\} + \{E(\phi_2) - E(\phi_2^t)\} \leq 0.$$

Combining these two inequalities, we get

$$E(\phi) \leq E(\phi + t\eta).$$

The lemma is proved. □

We now consider the case $m = 2$. The following Lemma 4.3 and Lemma 4.5 are refined versions of that in [**X2**]. Lemma 4.4 is cited from [**X2**] which is waiting for appearence.

The key point is the existence of the upper barrier functions. Let us consider the z-independent solutions of (3.4), which are solutions to the following ODE

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi\cos 2\phi = 0, \qquad 0 < r < 1, \quad 0 < \phi < \frac{\pi}{2}. \tag{4.3}$$

To solve Equation (4.3) with the condition $\lim_{r\to 0}\phi(r) = 0$ we make the change of variable $r = e^x$, $-\infty < x \leq 0$. Then (4.3) becomes

$$\frac{d^2\phi}{dx^2} - \frac{\lambda_1}{2}\sin 2\phi - \frac{\lambda_2}{2}\sin 2\phi\cos 2\phi = 0, \tag{4.4}$$

$$-\infty < x < 0, \quad 0 < \phi < \frac{\pi}{2}, \quad \lim_{x\to-\infty}\phi(x) = 0. \tag{4.5}$$

Multiplying (4.4) by $\frac{d\phi}{dx}$ and integrating, we obtain

$$\left(\frac{d\phi}{dx}\right)^2 - \lambda_1 \sin^2\phi - \frac{\lambda_2}{4}\sin^2 2\phi = c, \tag{4.6}$$

where c is a constant. Due to the conditions (4.5) the constant c has to be zero and (4.6) becomes

$$\frac{d\phi}{dx} = \pm\sqrt{\lambda_1 + \lambda_2\cos^2\phi}\,\sin\phi.$$

Noting (4.5), the minus sign of the right hand side of the above equation is impossible. Therefore, (4.6) reduces to

$$\frac{d\phi}{dx} = \sqrt{\lambda_1 + \lambda_2\cos^2\phi}\,\sin\phi. \tag{4.7}$$

For any initial condition $\phi(x_0) = \tau$, $-\infty < x_0 < \infty$, there is a unique solution $\phi(x)$ to (4.7), which can be extended to whole line $(-\infty, \infty)$ since the right hand side of (4.7) is bounded. Notice that the constant solutions of (4.7) are $\phi = k\pi$, $(k = 0, \pm 1, \cdots, \pm\infty)$. Let us consider the solutions ϕ of (4.7) with initial condition $0 < \tau < \pi$. By uniqueness the solution curve on (x, ϕ) plane lies within two lines $\phi \equiv 0$ and $\phi \equiv \pi$. Thus $\lim\limits_{x\to-\infty}\phi(x)$ exists, which implies that there exists a sequence of points $\{x_k\} \to -\infty$ such that $\frac{d\phi}{dx}(x_k) \to 0$. Considering Equation (4.7) on those points gives $\lim\limits_{x\to-\infty}\phi(x) = 0$. Similarly, $\lim\limits_{x\to\infty}\phi(x) = \pi$. In summarizing, we have:

Lemma 4.3. *For any $\tau < \pi$ there exists a unique solution $\phi_\tau(x)$ of (4.7) satisfying the boundary conditions $\phi_\tau(0) = \tau$ and $\lim\limits_{x\to-\infty}\phi(x) = 0$. Furthermore,*

$$\phi_{\tau_1}(x) < \phi_{\tau_2}(x),$$

where $\tau_1 < \tau_2$.

Lemma 4.4. *Let $\phi(r)$ be a solution to (4.3) satisfying the boundary conditions*

$$\lim_{r\to 0}\phi(r) = 0 \quad \textit{and} \quad \phi(r_0) = 2\arctan\left(cr_0^{\sqrt{\lambda_1}}\right) \le \frac{\pi}{2}.$$

Then we have the estimates

$$2\arctan\left(cr^{\sqrt{\lambda_1+\lambda_2}}\right) \le \phi(r) \le 2\arctan\left(cr^{\sqrt{\lambda_1}}\right), \tag{4.8}$$

where $0 < r \le r_0 \le 1$ and c is a positive constant.

Proof. Let

$$L_1(\psi) = \frac{d\psi}{dx} - \sqrt{\lambda_1}\sin\psi$$

for $-\infty < x \leq \ln r_0$. It can be verified that $\psi = 2\arctan(c\exp\sqrt{\lambda_1}x)$ is a solution to the equation

$$L_1(\psi) = 0.$$

For a solution $\phi(x)$ to (4.7) with

$$\phi(\ln r_0) = 2\arctan(cr_0^{\sqrt{\lambda_1}}) \leq \frac{\pi}{2}$$

$$\begin{aligned} L_1(\phi) &= \frac{d\phi}{dx} - \sqrt{\lambda_1}\sin\phi \\ &= \frac{d\phi}{dx} - \sqrt{\lambda_1 + \lambda_2\cos^2\phi}\sin\phi + \sqrt{\lambda_1 + \lambda_2\cos^2\phi}\sin\phi - \sqrt{\lambda_1}\sin\phi \\ &= \sqrt{\lambda_1 + \lambda_2\cos^2\phi}\sin\phi - \sqrt{\lambda_1}\sin\phi \geq 0. \end{aligned}$$

Notice that

$$\phi(\ln r_0) = \psi(\ln r_0), \quad \lim_{x\to-\infty}\phi(x) = \lim_{x\to-\infty}\psi(x) = 0.$$

If there exists a point $x \in (-\infty, \ln r_0)$ such that $(\phi - \psi)(x) > 0$, then there is a positive maximum point $x_0 \in (-\infty, \ln r_0)$ of $\phi - \psi$. We have

$$0 \leq L_1(\phi)(x_0) - L_1(\psi)(x_0) = \sqrt{\lambda_1}(\sin\psi(x_0) - \sin\phi(x_0)),$$

but the right hand side of the above expression is negative. The contradiction implies $\phi \leq \psi$ on $(-\infty, \ln r_0]$. Reversing back to the original variable gives

$$\phi(r) \leq 2\arctan\left(cr^{\sqrt{\lambda_1}}\right).$$

Let

$$L_2(\psi) = \frac{d\psi}{dx} - \sqrt{\lambda_1 + \lambda_2}\sin\psi.$$

Then any solution of (4.7) is a supersolution of $L_2(\psi) = 0$. By the similar argument as the above we will obtain another inequality of (4.8) □

Lemma 4.5. *There exists a solution* $\bar\phi$ *to Equation* (4.3) *satisfying the condition* $\lim_{r\to0}\bar\phi(r) = 0$ *such that* $\bar\phi|_{\partial_1 D} \geq \psi$ *where* ψ *satisfies conditions:*

(1) ψ *is a regular function on* $\partial_1 D$ *and is of order* $O(r^{\sqrt{\lambda_1+\lambda_2}})$ *as* $r \to 0$;
(2) $\max\psi < \pi$.

Proof. By the condition (1) there are constants K and δ such that

$$\psi(r,z) \leq Kr^{\sqrt{\lambda_1+\lambda_2}} \quad \text{when} \quad r \leq \delta \quad \text{and} \quad (r,z) \in \partial_1 D.$$

On the other hand, there is $\delta_1 > 0$, such that for $r \le \delta_1$

$$\phi_c = 2\arctan\left(cr^{\sqrt{\lambda_1+\lambda_2}}\right) \ge \sin\phi_c = \frac{2cr^{\sqrt{\lambda_1+\lambda_2}}}{1+c^2r^{2\sqrt{\lambda_1+\lambda_2}}} \ge Kr^{\sqrt{\lambda_1+\lambda_2}},$$

when $c > \frac{K}{2}$ and $c\delta_1^{\sqrt{\lambda_1+\lambda_2}} \le 1$. Let ϕ_τ is the solution to (4.7) as shown in Lemma 4.3. Define $\phi^\tau(r)$ by

$$\phi^\tau(r) = \phi_\tau(\ln r),$$

which is a solution of (4.3). We can use Lemma 4.4 to conclude that there is τ_0, such that

$$\phi^{\tau_0}(r) \ge \psi \quad \text{on} \quad \partial_1 D \cap \{r < \delta_0\},$$

where $\delta_0 = \min(\delta, \delta_1)$. By Lemma 4.3 the above inequality holds for all $\tau \ge \tau_0$ with the same δ_0.

We see that $\phi^\tau(r)$ converges to π when $\tau \to \pi$ for each $r \in (0,1]$. Hence, there is $\tau_1 \ge \tau_0$ such that

$$\phi^{\tau_1}(\delta_0) \ge \max\psi$$

which gives

$$\phi^{\tau_1} \ge \psi \quad \text{on} \quad \partial D \cap \{\delta_0 \le r \le 1\}.$$

Therefore,

$$\bar{\phi} = \phi^{\tau_1}$$

meets the lemma. □

Now we are in a position to prove Theorem A which is an immediate conclusion of the following result.

Theorem 4.6. *Let*

$$\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi\cos 2\phi = 0$$

define on

$$D = \left\{(r,z) \in \mathbb{R}^2; \quad r^2+z^2 < 1, \quad r > 0\right\}.$$

Let ψ be a regular function on $\partial_1 D$ and be of order $O(r^{\sqrt{\lambda_1+\lambda_2}})$ as $r \to 0$ with $\max\psi < \pi$. Then there exists a smooth solution ϕ to the above equation such that $\phi|_{\partial_1 D} = \psi$ and $\lim_{r\to 0}\phi = 0$.

Proof. We take the zero function as our ϕ_1. By Lemma 4.5 we also have the upper barrier ϕ_2. According to Lemma 4.2 and Lemma 4.3 we obtain a local minimum ϕ of E. By the standard elliptic regularity theorem ϕ is a regular solution of Eq. (3.4) with the boundary condition $\phi|_{\partial_1 D} = \psi$. It is easily seen that $\lim_{r\to 0} \phi = 0$. $\square$

Let us study the case $m \geq 3$. The corresponding ODE to (3.4) is

$$\frac{d^2\phi}{dr^2} + \frac{m-1}{r}\frac{d\phi}{dr} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi \cos 2\phi = 0. \tag{4.9}$$

It is the Euler-Lagrange equation of the following functional

$$\overline{E} = \int_0^1 \left[\left(\frac{d\,\phi}{d\,r}\right)^2 + \frac{\lambda_1}{r^2}\sin^2\phi + \frac{\lambda_2}{4r^2}\sin^2 2\phi \right] r^{m-1}\,dr. \tag{4.10}$$

Obviously, Lemma 4.1 and Lemma 4.2 are also valid for one dimensional case. Select $\phi_1 \equiv 0$ and $\phi_2 \equiv \frac{\pi}{2}$. By using those Lemmas we have a solution ϕ to (4.9) with the given condition $\phi(1) = \phi_1 < \frac{\pi}{2}$. we know that $0 \leq \phi \leq \frac{\pi}{2}$. To analyze the behavior of the solution ϕ we make the change of variable $r = e^x,\ -\infty < x \leq 0$. Then (4.9) becomes

$$\frac{d^2\phi}{dx^2} + (m-2)\frac{d\,\phi}{d\,x} - \frac{\lambda_1}{2}\sin 2\phi - \frac{\lambda_2}{2}\sin 2\phi \cos 2\phi = 0. \tag{4.11}$$

If there exists $x_0 \in (-\infty, 0)$ such that $\phi'(x_0) < 0$, then $\phi'(x) < 0$ and $\phi''(x) > 0$ when $x \leq x_0$. This contradicts $\phi \leq \frac{\pi}{2}$. Therefore, $\phi' \geq 0$ on $(-\infty, 0]$. It turnes out that the existence of $\lim_{x\to-\infty} \phi(x)$ which has to be zero by Eq. (4.11). From Equation (4.11) we see that the solution ϕ can be extended to whole line $(-\infty, \infty)$, since ϕ' is bounded (say, $|\phi'| < \frac{\lambda_1+\lambda_2}{2(m-2)}$).

We use the standard method (see [**Har**], Ch. VIII) to analyze the qualitative properties of ϕ near $\frac{\pi}{2}$. Let $u(x) = \phi(x),\ v(x) = \phi'(x)$. Then Eq. (4.11) becomes a plane autonomous system

$$\begin{cases} u' = v, \\ v' = (2-m)v + \frac{\lambda_1}{2}\sin 2u + \frac{\lambda_2}{2}\sin 2u \cos 2u. \end{cases} \tag{4.12}$$

Its linearized system at the critical point $(\frac{\pi}{2}, 0)$ is

$$\begin{cases} u' = v, \\ v' = (\lambda_2 - \lambda_1)(u - \frac{\pi}{2}) + (2-m)v. \end{cases} \tag{4.13}$$

The characteristic equation of (4.13) is

$$\mu^2 + (m-2)\mu + \lambda_1 - \lambda_2 = 0.$$

When $m > 2+2\sqrt{\lambda_1 - \lambda_2}$, $(\frac{\pi}{2}, 0)$ is a stable improper node. The solution ϕ is monotone increasing to $\frac{\pi}{2}$ as $x \to \infty$. In the case of $m < 2+2\sqrt{\lambda_1 - \lambda_2}$, $(\frac{\pi}{2}, 0)$ is an attracting spriral point which means that the solution ϕ oscillates around $\frac{\pi}{2}$ and approaches $\frac{\pi}{2}$ when $x \to \infty$. It is easily seen that even in the late case ϕ always keeps positive.

In conclusion we summarize as follows.

Lemma 4.7. *Assume $m \geq 3$ and $\lambda_1 > \lambda_2$. There is a positive solution ϕ to (4.11) on $(-\infty, \infty)$ with $\lim_{x\to-\infty} \phi(x) = 0$.*

(1) *If $m < 2+2\sqrt{\lambda_1 - \lambda_2}$, then ϕ is monotone increasing before $\phi = \frac{\pi}{2}$ and then oscillates around $\frac{\pi}{2}$ and approaches $\frac{\pi}{2}$ when $x \to \infty$.*

(2) *If $m > 2+2\sqrt{\lambda_1 - \lambda_2}$, then ϕ is monotone increasing and approaches to $\frac{\pi}{2}$.*

Remark. Eq. (4.11) is autonomous. For any constant c, if $\phi(x)$ is its solution, so is $\phi(x+c)$. Therefore, for any x_0 there always exists a solution ϕ with $\phi(x_0) = \phi_0 < \frac{\pi}{2}$ which have the properties shown in the Lemma 4.7.

Lemma 4.8. *Let ψ and ϕ be solutions to (4.4) and (4.11), respectively. If*

$$\lim_{x\to-\infty} \psi(x) = \lim_{x\to-\infty} \phi = 0 \quad \textit{and} \quad \psi(x_0) = \phi(x_0) \leq \frac{\pi}{8},$$

then $\phi \geq \psi$ on $(-\infty, x_0]$. Furthermore, there exists $r_0 < 1$, such that

$$\phi(\ln r) \geq 2 \arctan\left(c\, r^{\sqrt{\lambda_1+\lambda_2}}\right) \tag{4.14}$$

for $0 \leq r \leq r_0 < 1$ and constant c.

Proof. Let

$$L(\eta) = \eta'' - \frac{\lambda_1}{2} \sin 2\eta - \frac{\lambda_2}{2} \sin 2\eta \cos 2\eta$$

on $(-\infty, x_0]$. Then $L(\psi) = 0$ and $L(\phi) = -(m-2)\phi' \leq 0$. If there is $x \in (-\infty, x_0)$ such that $(\psi - \phi)(x) > 0$, then there exists a positive maximum point $\bar{x} \in (-\infty, x_0)$ of $\psi - \phi$. We have

$$\begin{aligned} 0 \leq L(\psi)(\bar{x}) - L(\phi)(\bar{x}) = (\psi - \phi)''(\bar{x}) &+ \frac{\lambda_1}{2}(\sin 2\phi(\bar{x}) - \sin 2\psi(\bar{x})) \\ &+ \frac{\lambda_2}{4}(\sin 4\phi(\bar{x}) - \sin 4\psi(\bar{x})), \end{aligned}$$

but the right hand side of the above inequality is negative. The contradiction shows $\phi \geq \psi$ on $(-\infty, x_0)$. By using Lemma 4.4 we obtain (4.14) immediately. □

Lemma 4.7 and Lemma 4.8 enable us to have the following conclusion.

Lemma 4.9. *There exists a solution* $\bar{\phi}$ *to Eq.* (4.9) *with* $\lim_{r\to 0}\bar{\phi}(r) = 0$, *such that* $\bar{\phi}|_{\partial_1 D} \geq \psi$, *where* ψ *satisfies condition:*

(1) ψ *is a regular fuction on* $\partial_1 D$ *and is of order* $O\left(r^{\sqrt{\lambda_1+\lambda_2}}\right)$ *as* $r \to 0$;

(2) $\max \psi < \frac{\pi}{2}$.

Then, we also have

Theorem 4.10. *Let*

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{m-1}{r}\frac{\partial \phi}{\partial r} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi \cos 2\phi = 0, \tag{4.15}$$

define on

$$D = \left\{(r,z) \in \mathbb{R}^2; \quad r^2 + z^2 < 1, \quad r > 0\right\}.$$

Let ψ *be a regular function on* $\partial_1 D$ *and be of order* $O\left(r^{\sqrt{\lambda_1+\lambda_2}}\right)$ *as* $r \to 0$ *with* $\max \psi < \frac{\pi}{2}$. *There exists a smooth solution* ϕ *to the above equation such that* $\phi|_{\partial_1 D} = \psi$ *and* $\lim_{r\to 0} \phi = 0$.

We omit the proofs of Lemma 4.9 and Theorem 4.10 which are similar to that of Lemma 4.5 and Theorem 4.6. When $m = 2$ Theorem 4.10 is a special case of Theorem 4.6

Theorem B follows from Theorem 4.10 immediately.

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Note: There were two Equations (4.9) in the paper version. The second has been renumbered as (4.15).