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NONCOMMUTATIVE END THEORY

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The corona algebra $M(\mathfrak{A})/\mathfrak{A}$ contains essential information on the global structure of $\mathfrak{A}$, as demonstrated for instance by Busby theory. It is an interesting and surprisingly difficult task to determine the ideal structure of $M(\mathfrak{A})/\mathfrak{A}$ by means of the internal structure of $\mathfrak{A}$.

Toward this end, we generalize Freudenthal's classical theory of *ends* of topological spaces to a large class of C^*-algebras. However, mirroring requirements necessary already in the commutative case, we must restrict attention to C^*-algebras $\mathfrak{A}$ which are σ-unital and have connected and locally connected spectra. Furthermore, we must study separately a certain pathological behavior which occurs in neither commutative nor stable C^*-algebras.

We introduce a notion of sequences determining ends in such a C^*-algebra $\mathfrak{A}$ and pass to a set of equivalence classes of such sequences, the *ends* of $\mathfrak{A}$. We show that ends are in a natural 1–1 correspondence with the set of components of $M(\mathfrak{A})/\mathfrak{A}$, hence giving a complete description of the complemented ideals of such corona algebras.

As an application we show that corona algebras of primitive σ-unital C^*-algebras are prime. Furthermore, we employ the methods developed to show that, for a large class of C^*-algebras, the end theory of a tensor product of two nonunital C^*-algebras is always trivial.

0.1. Introduction.

The *corona algebra* $M(\mathfrak{A})/\mathfrak{A}$ ([**34**]) of a non-unital C^*-algebra $\mathfrak{A}$ contains essential information on the global structure "at infinity" of $\mathfrak{A}$. An important instance of this is the bijective correspondence between $*$-homomorphisms from $\mathfrak{B}$ to $M(\mathfrak{A})/\mathfrak{A}$ and equivalence classes of extensions of C^*-algebras

$$0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{X} \longrightarrow \mathfrak{B} \longrightarrow 0$$

noted by Busby ([**13**]). This observation is fundamental in BDF-theory ([**11**]) and its generalizations, which apply to describe the set of extensions by K-theory in certain cases.

The objective of the present paper is to develop and then apply a generalized form of *end theory* to describe the ideal structure of a corona algebra

$M(\mathfrak{A})/\mathfrak{A}$ in terms of the original algebra $\mathfrak{A}$. Questions about the ideal structure of the corona are notoriously difficult. They have been considered, predominantly with methods related to K-theory, in [21], [29], [30], [35], and [42]. Very accurate information has been achieved for C^*-algebras which are either stable or of real rank zero. Our program applies to far more general C^*-algebras, but our results are less accurate in the sense that we only describe ideal structure up to indecomposability and primeness.

Our approach is based on the topological notion of *ends*, due to Hans Freudenthal and developed in [23], [24]. A decreasing sequence of nonempty, open and connected subsets G_k of the topological space X is said to *determine an end* if ∂G_k is compact for every k and if

$$\bigcap_{k=1}^{\infty} \overline{G_k} = \emptyset.$$

An *end* of X is an equivalence class of sequences determining ends under the relation

$$(G_k) \approx (H_k) \Longleftrightarrow \forall k \in \mathbb{N} : G_k \cap H_k \neq \emptyset,$$

which turns out to be transitive. Local connectivity of X is essential to building a theory of ends in X, and end theory may be very elegantly done in the class of topological spaces we will denote by the term *Raum*: Hausdorff spaces that are also connected, locally connected, locally compact, and σ-compact. That name was used by Hopf in [26, 1.1] to denote a very similar class of topological spaces.

An important tool in the developing of end theory, interesting in its own right, is the *Freudenthal compactification* of X which can be constructed from the end theory of X and can be characterized as the maximal compactification φX of X with a totally disconnected remainder $\varphi X\backslash X$. Freudenthal devises an algorithm to define certain sequences determining ends, and then proves, using φX, that all sequences determining ends thus arise. The reader is referred to the original sources [23], [24], [26], or, for an English version slightly more geared to a C^*-algebraic point of view, [20]. A central point in the philosophy behind the present work is the observation in [20] that the end structure of a Raum X gives information on the component structure of the remainder $\gamma X\backslash X$ for any compactification γX, and actually describes completely the component structure of the *corona* $\beta X\backslash X$.

We take the theory of ends to a noncommutative setting using the noncommutative topology associated to Akemann and Pedersen's notion of closed, open and compact projections in the enveloping von Neumann algebra. In [19], notions of connectivity, local connectivity and components based on this were investigated, and these will play a key role in the program.

In defining a noncommutative generalization of the Raum spaces, we encounter a certain pathological behavior which does not occur in the commutative case, and we prove that assuming that this behavior does not occur is both necessary and sufficient for a consistent end theory. Focusing on the class of C^*-algebras thus determined, we generalize Freudenthal's algorithm to devise a number of ends for the C^*-algebra and prove that all ends in fact occur this way. As we do not, at our current stage, have a theory of Freudenthal compactification available to us, we must take a different path than Freudenthal to prove this. Also, we must employ quite elaborate essentially non-central methods to prove that all components of the corona arise from a sequence determining an end.

With the fundamentals of a theory of ends laid down, we go on to consider applications. A result by Zhang concerning primeness of corona algebras is generalized substantially. We also investigate the end theory of tensor products; in the commutative case it follows almost immediately from the definitions that the end theory of a noncompact product Raum $X \times Y$ is determined by the end theory of the noncompact factor (say X) if the other one (say Y) is compact. Also, it follows almost as readily that the end theory of $X \times Y$ collapses into one end if *both* of the factors are noncompact. The reader may consider $X = \mathbb{R}, Y = [0, 1]$ as an example of the first phenomenon, $X = Y = \mathbb{R}$ as an example of the second. Using the methods developed in the present paper, we prove that this behavior carries over to the noncommutative case. Finally, we compute the end theory of the C^*-algebra of the real Heisenberg group.

It is our hope that the beautiful results on the ends of covering spaces of groups have a C^*-algebra counterpart, cf. [**22**]. On an even more ambitious note, the reader is referred to the recent book of Hughes and Ranicki [**27**] for a summary of more than 30 years of work on what might be called "algebraic topology of ends". This present paper and [**19**] lay the noncommutative "point set" groundwork from which an attack on noncommutative versions of the homotopy and homology results in [**27**] can be based.

This revised and shortened version replaces an earlier preprint of the same name.

0.2. Notation.

Let $(\pi_u, \mathfrak{H}_u)$ denote the universal representation of a C^*-algebra $\mathfrak{A}$. We tacitly consider $\mathfrak{A}$ and its multiplier algebra $M(\mathfrak{A})$ as operators on $\mathfrak{H}_u$ and denote the weak closure of $\mathfrak{A}$ here, the enveloping von Neumann algebra, by $\mathfrak{A}^{**}$. The unit of $\mathfrak{A}^{**}$ is denoted by $\mathbf{1}$, and the normal extension of a representation π of $\mathfrak{A}$ to $\mathfrak{A}^{**}$ is denoted by π^{**}. In $\mathfrak{A}^*$ we consider the sets $Q(\mathfrak{A})$, $S(\mathfrak{A})$ and $P(\mathfrak{A})$; the quasi-states, states and pure states, respectively.

When a projection p of $\mathfrak{A}^{**}$ is given, we denote by $F(p)$ the set

$$\{\varphi \in Q(\mathfrak{A}) | \varphi(\mathbf{1}-p)=0\}$$

supported by p, by $P(p)$ the subset $F(p)\cap P(\mathfrak{A})$. In $\mathfrak{A}^{**}$ we denote by z the central cover of the reduced atomic representation, i.e. the cover of all the irreducible representations. The central cover of p in $\mathfrak{A}^{**}$ is denoted by $\mathsf{c}(p)$. $\operatorname{Prim}(\mathfrak{A})$ is the primitive ideal spectrum of $\mathfrak{A}$.

When M is a subset of some C^*-algebra, $C^*(M)$ refers to the smallest C^*-subalgebra containing M. By $\prod$ we refer to the unrestricted sum of C^*-algebras, by $\sum$ to the restricted (Kaplansky) sum.

We shall use the letter β to refer both to the strict topology in a multiplier algebra as well as referring to the Stone-Čech compactification βX of a topological space X. This should cause no confusion. The map $\kappa_{\mathfrak{A}}$ is the canonical map from $M(\mathfrak{A})$ to the *corona algebra* $M(\mathfrak{A})/\mathfrak{A}$.

We shall work repeatedly with projections in the enveloping von Neumann algebra $\mathfrak{A}^{**}$ of a C^*-algebra $\mathfrak{A}$. Recall from [**1**] that $p\in\mathfrak{A}^{**}$ is *closed* when $F(p)$ is closed, and that p is *open* when $\mathbf{1}-p$ is closed. For several equivalent conditions, see [**32**, 3.11.9]. We denote by $\overline{p}$ the least closed projection dominating p, and say that p is *regular* when $\overline{F(p)}=F(\overline{p})$. For basic results about these classes of projections, we shall refer to [**1**] and [**17**], noting that although the C^*-algebras considered in these papers are assumed to be unital, the results needed here hold true in general. Details can be found in [**18**].

When p is open, we denote by $\operatorname{her}(p)$ the hereditary C^*-subalgebra of $\mathfrak{A}$ covered by p, and by $\operatorname{her}_M(p)$ the hereditary C^*-subalgebra of $M(\mathfrak{A})$ covered by p, i.e.

$$\operatorname{her}(p) = p\mathfrak{A}^{**}p\cap\mathfrak{A}$$
$$\operatorname{her}_M(p) = p\mathfrak{A}^{**}p\cap M(\mathfrak{A}).$$

Abbreviating $\mathfrak{B}=\operatorname{her}(p)$ and letting $\mathfrak{L}=\mathfrak{A}^{**}p\cap\mathfrak{A}$ and $\mathfrak{R}=p\mathfrak{A}^{**}\cap\mathfrak{A}$ — the closed left and right ideals covered by p, we have (cf. [**19**, 4.10])

$$\operatorname{her}_M(p)=\{x\in M(\mathfrak{A})|\mathfrak{A}x\subseteq\mathfrak{L}, x\mathfrak{A}\subseteq\mathfrak{R}\}=\overline{\mathfrak{B}}^{\beta}.$$

This hereditary uwaveC^*-subalgebra is also denoted by $M(\mathfrak{A},\mathfrak{B})$ in the literature ([**10**], [**33**]). We will not use this notation here.

When p is an open projection and q a subprojection of p, we say that q is central, closed or open *relative to* p if it has these properties considered as an element of $\operatorname{her}(p)^{**}$. Note that q is open relative to p exactly when it is open in $\mathfrak{A}^{**}$.

0.3. Connectivity.

In [**19**], the second author investigated notions of connectivity and components in the setting of C^*-algebras. A C^*-algebra is *connected* when it can not be decomposed into a nontrivial direct sum of two ideals. A projection p in $\mathfrak{A}^{**}$ is *connected* if whenever it can be decomposed

$$p = px_0 + px_1 \tag{1}$$

where the x_i are open, central projections, the decomposition is trivial, i.e.

$$\{px_0, px_1\} = \{0, p\}.$$

A *component projection* is a maximal connected projection, and component projections are automatically closed and central. The class of connected projections has several closure properties, but fewer than what one might expect from the commutative case.

It is proven in [**19**] that with these definitions, $\mathfrak{A}$ is connected precisely when the two spectra $P(\mathfrak{A})$ and $\mathrm{Prim}(\mathfrak{A})$ are connected, as topological spaces. Furthermore, there are natural 1–1 correspondences between the set of component projections, the set of components of $P(\mathfrak{A})$, and the set of components of $\mathrm{Prim}(\mathfrak{A})$. The common cardinal of the set of components is denoted by $\#_{\mathsf{K}}\mathfrak{A}$.

A C^*-algebra is *locally connected* if and only if every hereditary C^*-subalgebra only has open component projections. As with connectivity, $\mathfrak{A}$ will be locally connected exactly when $P(\mathfrak{A})$ and $\mathrm{Prim}(\mathfrak{A})$ are, as topological spaces.

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1. Preliminaries.

1.1. Compact and bounded projections.

In this section, we shall give a short survey of the classes of *compact* and *bounded* projections. The latter class (as well as classes closely related to it) has been studied extensively under the name of relatively compact projections in [**9**] and [**8**].

The following are the defining properties for a *compact* projection (cf. [4, 2.4] and [3, II.5]):

Lemma 1.1.1. *For a projection $p \in \mathfrak{A}^{**}$, the following conditions are equivalent*

(i) $F(p) \cap S(\mathfrak{A})$ *is weak* closed in* $\mathfrak{A}^*$.

(ii) p *is closed and* $a \in \mathfrak{A}_{sa}$ *exists with* $p \le a \le \mathbf{1}$.

(ii′) p *is closed and* $0 \le a \le \mathbf{1}$ *exists with* $p = ap$.

(iii) p *is closed and* $a \in \mathfrak{A}_{sa}$ *exists with* $p \le a$.

For more equivalent conditions, see [10, 2.47]. Note that when (ii') holds, since a and p commute, we may assume by functional calculus that $\|a\| = 1$. By (i) and the finite intersection property, the compact projections share the following property with their commutative counterparts:

Proposition 1.1.2. *Let q be a compact projection and $(p_i)_I$ a family of closed projections such that for all finite subsets I_0 of I, $(\bigwedge_{I_0} p_i) \wedge q \neq 0$. Then*

$$\left(\bigwedge_{i\in I} p_i\right) \wedge q \neq 0.$$

We now come to a crucial concept in this paper, that of a *bounded* (or precompact) projection. As noted by Brown in [8] there is, in fact, an entire continuum of possible definitions generalizing the commutative case. We choose the strongest one: invoking the closure operation on projections in $\mathfrak{A}^{**}$, we will say that a (usually open) projection p is *bounded* when $\overline{p}$ is compact. However, we shall also occasionally need a weaker form of boundedness. This is what we need to know.

Proposition 1.1.3. *Let $\mathfrak{A}$ be a σ-unital C^*-algebra. When p is an open projection of $\mathfrak{A}^{**}$, consider the conditions*

(i) p *is bounded.*

(ii) *There exists $a \in \mathfrak{A}_{sa}$ with $p \le a \le \mathbf{1}$.*

(iii) *There is no sequence (φ_n) in $P(p)$ with $\varphi_n \to 0$ weak*.*

(iv) $\mathrm{her}(p)$ *is strictly closed.*

Then

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)} \iff \text{(iv)}$$

and all conditions are equivalent when p is central.

Proofs for those of the implications above which are not direct consequences of Lemma 1.1.1 can be found in [**9**, Section 4]. See also [**25**] for a detailed exposition and related results.

We say that an ideal $\mathfrak{I}$ is *bounded* when its (central) open cover (cf. [**32**, 3.11.10]) satisfies (i)–(iv) above. A counterexample to (i)$\Longleftrightarrow$(iii) of Proposition 1.1.3, even for a regular projection, was shown to us by L.G. Brown, [**8**, 4.10]. The following lemma describes how to work with bounded projections. As the restrictions on (ii) and (iii) indicate, one can not rely too heavily on intuition from the commutative case.

Lemma 1.1.4. *Let p, q be projections of $\mathfrak{A}^{**}$.*

(i) *If $p \leq q$ and q is bounded, p is bounded.*

(ii) *If $a \in \mathfrak{A}$ exists with $p \leq a$, and p is regular, p is bounded.*

(iii) *If p, q are bounded and $\|\overline{pq} - \overline{p} \wedge \overline{q}\| < \mathbf{1}$, then $p \vee q$ is bounded.*

Proof. The first assertion is obvious. To prove (ii), let $G = F(\overline{p}) \cap S(\mathfrak{A})$. We will show that G is compact, i.e. that G is closed in $F(\overline{p})$. Suppose not, and let ψ_λ be a net in G converging to ψ with $\|\psi\| = 1 - \delta < 1$. Since $F(p)$ is dense in $F(\overline{p})$ by regularity we can assume that the net lies in $F(p)$. After normalizing if necessary we may assume that $\psi_\lambda(p) = 1$ for each λ. By assumption, $p \leq a$ for some $a \in \mathfrak{A}$. Pick n such that $\|a\|^{1/n} < 1 + \delta$ and note that also $\|a^{1/n}\| < 1 + \delta$. By [**32**, 1.3.9], $a^{1/n}$ also dominates p, so without loss of generality we may assume that in fact $\|a\| < 1 + \delta$. We have $\psi_\lambda(a) \geq 1$ for all λ, hence $\psi(a) \geq 1$. But

$$\psi(a) \leq \|\psi\|\|a\| < (1 - \delta)(1 + \delta) = 1 - \delta^2 < 1,$$

For (iii), note that by [**1**, II.7], $\overline{p} \vee \overline{q}$ is closed. It clearly dominates $\overline{p \vee q}$, which is then compact by Lemma 1.1.1(iii), since

$$\overline{p \vee q} \leq \overline{p} \vee \overline{q} \leq \overline{p} + \overline{q} \leq a + b,$$

where $0 \leq a, b \leq \mathbf{1}$ in $\mathfrak{A}$ exist by Lemma 1.1.1(ii). □

Proposition 1.1.5.

(i) *When p is a projection in $\mathfrak{A}^{**}$, and $\mathbf{1} - p$ is compact, then*

$$\kappa_{\mathfrak{A}}(\mathrm{her}_M(p)) = M(\mathfrak{A})/\mathfrak{A}.$$

(ii) *When $\mathfrak{A}$ is a σ-unital C^*-algebra and $\mathfrak{I}$ a bounded ideal thereof, then*

$$M(\mathfrak{A}/\mathfrak{I})\big/\mathfrak{A}/\mathfrak{I} \simeq M(\mathfrak{A})/\mathfrak{A}.$$

Proof. For (i), take $a \in \mathfrak{A}$, $0 \leq a \leq \mathbf{1}$ with $a(\mathbf{1}-p) = \mathbf{1}-p$. It follows that $\mathbf{1}-a \leq p$, so that for every $x \in M(\mathfrak{A})$, $(\mathbf{1}-a)x(\mathbf{1}-a) \in \mathrm{her}_M(p)$. Clearly $\kappa_{\mathfrak{A}}(x) = \kappa_{\mathfrak{A}}((\mathbf{1}-a)x(\mathbf{1}-a))$.

For (ii), we apply the results in [**33**]. Letting ρ denote the canonical epimorphism $\rho : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$, we get from [**33**, Theorem 10] that ρ^{**} restricts to a surjection $M(\mathfrak{A}) \to M(\mathfrak{A}/\mathfrak{I})$. The induced map

$$\tilde{\rho} : M(\mathfrak{A})/\mathfrak{A} \to M(\mathfrak{A}/\mathfrak{I})\Big/\mathfrak{A}/\mathfrak{I}$$

is also onto, and by [**33**, Theorem 23], we need only show that $\overline{\mathfrak{I}}^{\beta} = \mathfrak{I}$ to see that it is 1–1. This follows directly from Proposition 1.1.3. □

1.2. Semicentral boundaries.

An important technical notion in this paper is that of a projection with compact *semicentral boundary* $\overline{\mathsf{c}(p)} - p$. The following lemma explains its basic importance. Recall the notions of connectivity, local connectivity, connected projections and component projections from Section 0.3.

Lemma 1.2.1. *Let p be an open connected projection of a C^*-algebra $\mathfrak{A}^{**}$. The following conditions are equivalent:*

(i) *p has compact semicentral boundary $\overline{\mathsf{c}(p)} - p$.*

(ii) *p is central and clopen relative to $\mathbf{1}-r$ for some compact projection r.*

Proof. First note that the conditions on p and r in (ii) can be stated as

$$\mathbf{1} - r - p \text{ is an open projection} \tag{2}$$

$$\mathsf{c}(p)(\mathbf{1}-r) = p. \tag{3}$$

Assume that (i) holds, and let $r = \overline{\mathsf{c}(p)} - p$. We get (2) by $\mathbf{1}-r-p = \mathbf{1}-\overline{\mathsf{c}(p)}$ and (3) by

$$\mathsf{c}(p)(\mathbf{1}-r) = \mathsf{c}(p) - \mathsf{c}(p)\overline{\mathsf{c}(p)} + \mathsf{c}(p)p = p.$$

In the other direction, note that when such an r exists, we get $[\mathbf{1}-r-p]\mathsf{c}(p) = 0$ by (3) and conclude that $[\mathbf{1}-r-p]\overline{\mathsf{c}(p)} = 0$ by (2). That leads in turn to $\overline{\mathsf{c}(p)} - p = \overline{\mathsf{c}(p)}r \leq r$, and $\overline{\mathsf{c}(p)} - p$ will be compact. □

The lemma above is particularly useful when $\mathfrak{A}$ is locally connected. In this case, any component projection of $\mathbf{1}-r$, where r is compact, will satisfy the conditions in (ii).

The following lemma enables us to circumvent the complications caused by the fact that there is more than one notion generalizing bounded sets to a noncommutative setting. With the observation below in hand, we may work with projections whose central covers are bounded, and employ the fact that all notions of boundedness coincide for central projections.

Lemma 1.2.2. *Let r be a compact projection of a C^*-algebra $\mathfrak{A}$. For p a projection which is central and clopen relative to $\mathbf{1} - r$,*

$$p \text{ is bounded} \Longleftrightarrow \mathsf{c}(p) \text{ is bounded.}$$

Proof. By Lemma 1.2.1, $\overline{\mathsf{c}(p)} - p$ is compact. Writing

$$\overline{\mathsf{c}(p)} = (\overline{\mathsf{c}(p)} - p) + p$$

we see that when p is bounded, $\overline{\mathsf{c}(p)}$ is the sum of two bounded projections. Clearly $\overline{p}$ commutes with $\overline{\mathsf{c}(p)} - p$, and Lemma 1.1.4(iii) applies. □

In the rest of this paragraph we will need to work in both $\mathfrak{A}^{**}$ and $M(\mathfrak{A})^{**}$ at the same time. As $\mathfrak{A}$ is an ideal of $M(\mathfrak{A})$, there is an open central projection of $M(\mathfrak{A})^{**}$ covering $\mathfrak{A}^{**}$. We will denote this projection by $\mathbf{1}_{\mathfrak{A}}$, while denoting the unit of $M(\mathfrak{A})$ by $\mathbf{1}_{M(\mathfrak{A})}$. Note now that $M(\mathfrak{A})$ embeds into $M(\mathfrak{A})^{**}$ via $\mathfrak{A}^{**}$. This embedding is *not* compatible with the natural embedding of $M(\mathfrak{A})$ into $M(\mathfrak{A})^{**}$ — the latter preserves the unit, the former does not — and we will denote the embedding of $\mathfrak{A}^{**}$ into $M(\mathfrak{A})^{**}\mathbf{1}_{\mathfrak{A}}$ explicitly by ι_M. When we work in $\mathfrak{A}^{**}$, we will refer to the common unit for $\mathfrak{A}$ and $M(\mathfrak{A})$ here by $\mathbf{1}$. Hence, $\iota_M(\mathbf{1}) = \mathbf{1}_{\mathfrak{A}}$.

Recall that if $\mathfrak{I}$ is a norm closed two sided ideal of the C^*-algebra $\mathfrak{B}$ with covering projection x in $\mathfrak{B}^{**}$ and $\kappa : \mathfrak{B} \to \mathfrak{B}/\mathfrak{I}$ is the quotient map, then κ^{**} is an isometry from $(\mathbf{1} - x)\mathfrak{B}^{**}$ onto $(\mathfrak{B}/\mathfrak{I})^{**}$. This is a consequence of [**36**, Theorem 4.9(b)]. Applying this to $\mathfrak{B} = M(\mathfrak{A})$ and $\mathfrak{I} = \mathfrak{A}$, we get an isomorphism of $(M(\mathfrak{A})/\mathfrak{A})^{**}$ with $M(\mathfrak{A})^{**}(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$. As this does not in the same way as ι_M lend itself to confusion, we shall identify these two von Neumann algebras in what follows.

For a projection p in $\mathfrak{A}^{**}$, $\overline{p}^M$ denotes the closure of $\iota_M(p)$ in $M(\mathfrak{A})^{**}$. If p is open, $\Theta(p)$ denotes the cover of $\mathrm{her}_M(p)$ in $M(\mathfrak{A})^{**}$.

Lemma 1.2.3. *When p is closed and q is open in $\mathfrak{A}^{**}$,*

(i) *$\iota_M(q)$ is open in $\mathfrak{A}^{**}$.*

(ii) *$\overline{p}^M \mathbf{1}_{\mathfrak{A}} = \iota_M(p)$.*

(iii) *$\Theta(\mathbf{1} - p) = \mathbf{1}_{M(\mathfrak{A})} - \overline{p}^M$.*

Proof. When (a_λ) is a net in $\mathfrak{A}$ increasing to q in $\mathfrak{A}^{**}$, considering a_λ as an increasing net in $M(\mathfrak{A})$ we get that $\iota_M(q)$ is open also, proving (i). Obviously,

$\overline{p}^M \mathbf{1}_{\mathfrak{A}} \geq \iota_M(p)\mathbf{1}_{\mathfrak{A}} = \iota_M(p)$. For the other inequality, note that $\iota_M(\mathbf{1}-p)$ is open by (i). As $\mathbf{1}_{\mathfrak{A}}$ is central and open, we get by [1, II.7] that also $\mathbf{1}_{\mathfrak{A}}\iota_M(\mathbf{1}-p)$ is open, hence that

$$\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}\iota_M(\mathbf{1}-p) = \mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}} + \mathbf{1}_{\mathfrak{A}}\iota_M(p) = (\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \vee \iota_M(p)$$

is closed. Then $\overline{p}^M \leq (\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \vee \iota_M(p)$, and $\overline{p}^M \mathbf{1}_{\mathfrak{A}} \leq \iota_M(p)$, proving (ii). The last claim can be found in [10, 3.F]; one notes that

$$\mathrm{her}(\mathbf{1}_{M(\mathfrak{A})} - \overline{p}^M) = \{x \in M(\mathfrak{A}) | xp = px = 0\} = \mathrm{her}_M(\mathbf{1}-p).$$

□

Lemma 1.2.4. *Let p be an open projection of $\mathfrak{A}^{**}$. If $\overline{\mathsf{c}(p)} - p$ is compact, then*

$$\overline{p}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = \overline{\mathsf{c}(p)}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}).$$

Proof. To prove the nontrivial inequality, we prove that the set of states taking the value 1 at the rightmost projection also take the value 1 at the leftmost projection. Note that $\iota_M(\mathsf{c}(p))$ is central, hence regular in $M(\mathfrak{A})^{**}$, and fix $\varphi \in F(\overline{\mathsf{c}(p)}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})) \cap S(\mathfrak{A})$. By regularity, we can take $\varphi_\lambda \in F(\iota_M(\mathsf{c}(p)))$ with $\varphi_\lambda \to \varphi$ *weak** in $M(\mathfrak{A})^*$. Choose $a \in (\mathfrak{A}_+)_1$ dominating $\overline{\mathsf{c}(p)} - p$, and note that

$$0 \leq \varphi_\lambda(\iota_M(\overline{\mathsf{c}(p)} - p)) \leq \varphi_\lambda(a) \to \varphi(a) \leq \varphi(\mathbf{1}_{\mathfrak{A}}) = 0.$$

As $\varphi_\lambda(\iota_M(\overline{\mathsf{c}(p)})) \geq \varphi_\lambda(\iota_M(\mathsf{c}(p))) = 1$, $\varphi_\lambda(\iota_M(p)) \to 1$. Choose a net b_μ in $M(\mathfrak{A})_1^+$ decreasing to $\overline{p}^M$, and note that for every μ,

$$1 \geq \varphi_\lambda(b_\mu) \geq \varphi_\lambda(\overline{p}^M) \geq \varphi_\lambda(\iota_M(p)) \to 1,$$

so that $\varphi(b_\mu) = 1$. As φ is normal, $\varphi(\overline{p}^M) = 1$ and $\varphi \in F(\overline{p}^M) \cap S(\mathfrak{A})$. □

Lemma 1.2.5. *Suppose r is a compact projection in $\mathfrak{A}^{**}$ and $\mathbf{1} - r$ decomposes into a finite number of relatively central open projections $c_1, \dots, c_N$. Set $d_n = \overline{c_n}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$. Then*

$$d_n = \overline{\mathsf{c}(c_n)}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = \Theta(c_n)(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}),$$

and $d_1, \dots, d_N$ are orthogonal, central projections of $M(\mathfrak{A})/\mathfrak{A}$ adding up to one.

Proof. The first equality follows by combining Lemmas 1.2.1 and 1.2.4. For the second, note that the $\Theta(c_i)$ are orthogonal and that by [**10**, 3.46a],

$$\sum_{i=1}^{N} \Theta(c_i) = \Theta(\mathbf{1} - r).$$

Since obviously $\Theta(c_i)\iota_M(c_j) = 0$ when $i \neq j$, we have that $\Theta(c_i)\overline{c_j}^M = 0$. By Proposition 1.1.5(i), $\Theta(\mathbf{1} - r)(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}$, and the $\Theta(c_i)(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$ must add up to the unit of $M(\mathfrak{A})/\mathfrak{A}$. We then conclude that

$$\overline{c_j}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \le \Theta(c_j)(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}),$$

in particular, the $\overline{c_i}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$ are orthogonal. Note that $\sum_1^N \mathsf{c}(c_i)$ has compact complement, so that by Lemma 1.2.3(iii) and Proposition 1.1.3(iv)

$$\begin{aligned}
\overline{\textstyle\sum_1^N \mathsf{c}(c_i)}^M &= \mathbf{1}_{M(\mathfrak{A})} - \Theta(\mathbf{1}_{\mathfrak{A}} - \overline{\textstyle\sum_1^N \mathsf{c}(c_i)}) \\
&= \mathbf{1}_{M(\mathfrak{A})} - (\mathbf{1}_{\mathfrak{A}} - \overline{\textstyle\sum_1^N \mathsf{c}(c_i)}) \\
&\ge \mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}},
\end{aligned}$$

whence by an application of the first equality

$$\begin{aligned}
\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}} &= \overline{\textstyle\sum_{i=1}^N \mathsf{c}(c_i)}^M (\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \\
&\le \left(\bigvee_{i=1}^{N} \overline{\mathsf{c}(c_i)}^M \right) (\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \\
&= \sum_{i=1}^{N} \left[\overline{\mathsf{c}(c_i)}^M (\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \right] \\
&= \sum_{i=1}^{N} \left[\overline{c_i}^M (\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \right] \\
&\le \mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}.
\end{aligned}$$

In the first inequality, we used that $\bigvee_{i=1}^N \overline{\mathsf{c}(c_i)}^M$ is closed by [**1**, II.7].

Since every d_k is closed in $(M(\mathfrak{A})/\mathfrak{A})^{**}$, every d_k is open. They are hence elements of $M(\mathfrak{A})/\mathfrak{A}$ by [**32**, 3.12.9]. □

2. Some classes of C^*-algebras.

2.1. The Raum algebras.

The definition of a *Raum* algebra given below is a direct translation to the noncommutative setting of the definition of the class of topological spaces considered by Freudenthal, and it appears at first to be the natural class of C^*-algebras in which to do end theory. However, a certain property which is automatic in the commutative case may fail to hold in this setting, and, since this property is shown to be essential for a fully satisfactory end theory, we must pass to the subclass of *Raum*$^+$ algebras for some of our results. In this section we give definitions and examples, and show that a Raum algebra which fails to be a Raum$^+$ algebra has a pathologically rich component structure (in the sense of [**19**, Section 3]) in its corona algebra. After we have developed a theory of ends, we will be able to prove that if the component structure of the corona of a given Raum $\mathfrak{A}$ is not of this pathological form, then $\mathfrak{A}$ is in fact a Raum$^+$ algebra.

Definition 2.1.1. A C^*-algebra $\mathfrak{A}$ is called a *Raum* algebra if it is connected, locally connected and σ-unital.

In the definition below, and the rest of the paper, we apply the following convention. The reader is asked to recall from [**19**, 5.8] that any open projection in a locally connected C^*-algebra can always be expressed as the sum of its (necessarily open) component projections.

Convention 2.1.2. *In a locally connected C^*-algebra $\mathfrak{A}$, suppose that an open projection p has components $(c_i)_I$. We shall partition I into two subsets I_U and I_B so that c_i with $i \in I_B$ is bounded and c_i with $i \in I_U$ is unbounded.*

Definition 2.1.3. If $\mathfrak{A}$ is a Raum algebra and p is an open projection in $\mathfrak{A}^{**}$, then p is said to be *behaved* if, when p is written out in components as in 2.1.2,

(i) I_U is finite.

(ii) $\sum_{I_B} c_i$ is bounded.

A *Raum*$^+$ algebra is a Raum algebra for which $\mathbf{1} - r$ is behaved for every compact r.

As we shall see, the notions of Raum and Raum$^+$ coincide in the commutative case, and we will say that a locally compact Hausdorff space X is a *Raum* when $C_0(X)$ is a Raum algebra, i.e. when X is connected, locally connected and σ-compact.

The definition of a Raum$^+$ algebra is not easy to work with directly. We find below certain simpler criteria with which it is equivalent.

Proposition 2.1.4. *Let p and q be compact projections in a locally connected C^*-algebra. If $\|qp - p\| < 1$, and if $\mathbf{1} - p$ and $\mathbf{1} - q$ are written out in components*

$$\mathbf{1} - p = \sum_{i \in I} c_i \qquad \mathbf{1} - q = \sum_{j \in J} d_j,$$

(*cf.* [**19**, 5.8]) *there is a unique map* $\Gamma : J \to I$ *with the property that* $\mathsf{c}(d_j) \leq \mathsf{c}(c_{\Gamma(j)})$. *If* $\mathbf{1}-q$ *is behaved, so is* $\mathbf{1}-p$, *and in the sense of* 2.1.2, $\Gamma(J_U) = I_U$. *Furthermore, if* $q \geq p$, Γ *has the property that* $d_j \leq c_{\Gamma(j)}$.

Proof. We first prove

$$\mathsf{c}(\mathbf{1}-p) \geq \mathsf{c}(\mathbf{1}-q). \tag{4}$$

To see this, we only need to prove that if π is an irreducible representation and $\pi^{**}(p) = \mathbf{1}$ then $\pi^{**}(q) = \mathbf{1}$, for then (4) will hold under z, which suffices by [**1**, II.17] as the central covers are open by [**19**, 0.1]. Let x be the cover of π. When $\pi^{**}(p) = \mathbf{1}$, $px = x$, whence $\|x - qx\| = \|(qp-p)x\| < 1$. Since $x - qx$ is a projection, $qx = x$.

When d is a fixed component of $\mathbf{1}-q$, $\mathsf{c}(d)$ is a component of $\mathsf{c}(\mathbf{1}-q)$ by [**19**, 5.8]. By (4) and the fact that $\mathsf{c}(d)$ is connected under $\mathsf{c}(\mathbf{1}-p)$, $\mathsf{c}(d)$ is dominated by a unique component of $\mathsf{c}(\mathbf{1}-p)$. This has the form $\mathsf{c}(c)$ for a unique component c of $\mathbf{1}-p$ by [**19**, 5.8], and the map Γ is defined according to this procedure. When d_j is unbounded, so are the larger projections $\mathsf{c}(d_j)$ and $\mathsf{c}(c_{\Gamma(j)})$, and hence $c_{\Gamma(j)}$ by Lemma 1.2.2. We hence have $\Gamma(J_U) \subseteq I_U$.

Assume now that $\mathbf{1}-q$ is behaved. Since $\mathsf{c}(c_i)\mathsf{c}(d_j) = 0$ when $i \neq \Gamma(j)$, we have

$$\sum_{i \in I \setminus \Gamma(J_U)} \mathsf{c}(c_i) \leq p \vee \left(\sum_{j \in J_B} \mathsf{c}(d_j) \right) = p \vee \mathsf{c}\left(\sum_{j \in J_B} d_j \right).$$

By Lemma 1.2.2, we may conclude from behavedness of $\mathbf{1}-q$ that the central cover in the last expression is bounded, and hence so is the supremum by Lemma 1.1.4(iii). As then the sum of components of $\mathsf{c}(\mathbf{1}-p)$ with indices in $I \setminus \Gamma(J_U)$ is bounded, it can contain no unbounded components. We learn from this that $I_U \subseteq \Gamma(J_U)$; in particular, the set is finite. Also, as $I_B \subseteq I \setminus \Gamma(J_U)$ by the above, the second requirement for $\mathbf{1}-p$ being behaved is met by 1.2.2 again.

For the last claim, note that for any fixed j, d_j is dominated by a unique component c of $\mathbf{1}-p$. As $d_j \leq \mathsf{c}(c_{\Gamma(j)})$, c must be the same as $c_{\Gamma(j)}$. ☐

The following concept is crucial in the paper.

Lemma 2.1.5. *Any* σ*-unital* C^**-algebra possesses an increasing sequence* $(r_k)_1^\infty$ *of compact projections of* $\mathfrak{A}^{**}$ *for which* $\bigwedge_{k=1}^\infty \overline{\mathbf{1}-r_k} = 0$ *holds.*

Definition 2.1.6. A sequence $\mathbf{r} = (r_k)$ with these properties is called a *compact nest* for $\mathfrak{A}$. We adopt the convention that $r_0 = 0$.

Proof of Lemma 2.1.5. Fix a strictly positive element h and let $r_k = 1_{[1-\alpha_k,1]}(h)$ where (α_k) is some sequence increasing to 1 in $[0,1]$. This is a compact nest since

$$\bigwedge_{k=1}^{\infty} \overline{\mathbf{1}-r_k} \leq \bigwedge_{k=1}^{\infty} 1_{[0,1-\alpha_k]}(h) = 1_{\{0\}}(h) = 0.$$

□

Lemma 2.1.7. *A compact nest (r_n) is an approximate unit in the sense that for any $a \in \mathfrak{A}$,*

$$\|a - ar_n\| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. Suppose, to the contrary, that $b \in \mathfrak{A}$ has the property that

$$\|b - br_n\| \geq \delta > 0 \qquad (5)$$

for infinitely many n. We may assume that $0 \leq b \leq \mathbf{1}$ and that (5) holds for all n. By the C^*-equality, $\|(\mathbf{1}-r_n)b^2(\mathbf{1}-r_n)\| \geq \delta^2$, and we may hence choose a sequence $\varphi_n \in F(\mathbf{1}-r_n)$ with $\varphi_n(b) \geq \varphi(b^2) \geq \delta^2$. By compactness of $Q(\mathfrak{A})$, a subnet of (φ_n) converges to ψ here. We get that $\psi(b) \geq \delta^2$. On the other hand, as $(\mathbf{1}-r_n)$ is decreasing, $\psi \in \overline{F(\mathbf{1}-r_n)} \subseteq F(\overline{\mathbf{1}-r_n})$ for each n and this yields the contradiction $\psi = 0$ by the definition of compact nests. □

Proposition 2.1.8. *If a Raum algebra $\mathfrak{A}$ possesses a compact nest $\mathbf{r}$ with $\mathbf{1}-r_n$ behaved for every n, $\mathfrak{A}$ is a Raum$^+$ algebra.*

Proof. Let a compact projection p be given. By Lemma 1.1.1(ii), p is dominated by $a \in \mathfrak{A}$, and by Lemma 2.1.7, we can choose n such that $\|a - ar_n\| < 1$. Applying the C^*-equality, we get that $\|pr_n - p\| < 1$. Apply Proposition 2.1.4. □

Lemma 2.1.9. *When r is a compact projection in a Raum algebra $\mathfrak{A}$,*

$$\mathbf{1}-r \text{ is behaved} \Longleftrightarrow \mathsf{c}(\mathbf{1}-r) \text{ is behaved}.$$

In particular, a Raum algebra $\mathfrak{A}$ is a Raum$^+$ algebra precisely when $\mathbf{1}-r$ is behaved for every central compact projection r.

Proof. Let $(c_i)_I$ be the set of component projections of $\mathbf{1}-r$. By [19, 5.8], $(\mathsf{c}(c_i))_I$ is the set of component projections of $\mathsf{c}(\mathbf{1}-r)$. Lemma 1.2.2 applies to every component projection as well as any sum of component projections to show that $\mathbf{1}-r$ and $\mathsf{c}(\mathbf{1}-r)$ are behaved simultaneously. □

Here is a consequence of that result.

Proposition 2.1.10. *Assume that a finite number of the irreducible representations* $\pi_1, \dots, \pi_N$ *of* $\mathfrak{A}$ *add up to a representation with bounded kernel. If* $\mathfrak{A}$ *is a Raum algebra, it is a Raum*$^+$ *algebra.*

Proof. We prove that *every* open central projection p is behaved, which will suffice by Lemma 2.1.9. Denoting the components of p by $c_i, i \in I$, we partition I according to 2.1.2 and let y_j denote the central covers of the π_j. Setting $y = \sum_1^N y_j$, we note that $\mathbf{1} - \overline{y}$ is bounded. As y_j is minimal central, if $py_j \neq 0$ we can write $py_j = c_i y_j$ for a unique i according to [**19**, 2.3(iii)]. Let

$$I' = \{i \in I | c_i y_j = py_j \text{ for some } j \in \{1, \dots, N\}\}$$

and note that I' has exactly N elements by the uniqueness mentioned above. In particular, the set is finite.

By definition, $\sum_{I \setminus I'} c_i y = 0$. Noting that $\sum_{I \setminus I'} c_i$ is open by [**1**, II.5], we get $\mathbf{1} - (\sum_{I \setminus I'} c_i) \geq \overline{y}$. Apply Lemma 1.1.4(i) to $\sum_{I \setminus I'} c_i \leq \mathbf{1} - \overline{y}$ to see that $\sum_{I \setminus I'} c_i$ is bounded. We conclude that $I \setminus I' \subseteq I_B$, and by the above, both I_U and $I_B \cap I'$ are finite. The first behavedness condition is then clearly met, and the second is met as

$$\sum_{i \in I_B} c_i \leq \sum_{i \in I_B \cap I} c_i + \sum_{i \in I \setminus I'} c_i$$

is bounded as a consequence of Lemma 1.1.4(iii). □

The reason why Raum$^+$ algebras are necessary in our setting is that a Raum algebra $\mathfrak{A}$ is a Raum$^+$ algebra exactly when $\#_{\mathsf{K}}(M(\mathfrak{A})/\mathfrak{A}) < 2^{\mathfrak{c}}$. Here the letter $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$ (note that we are not assuming the continuum hypothesis). There is hence, by sheer cardinality reasoning, no hope of describing the component structure of the corona of a Raum which is not a Raum$^+$ algebra by a tree with only finitely many branches at each vertex, as we shall do in the Raum$^+$ case in the next section. We must, however, postpone the proof of the equivalence of these two conditions to after our end theory has been fully developed, and only give one implication here.

Proposition 2.1.11. *If* $\mathfrak{A}$ *is a Raum algebra, and* $\#_{\mathsf{K}}(M(\mathfrak{A})/\mathfrak{A}) < 2^{\mathfrak{c}}$, $\mathfrak{A}$ *is a Raum*$^+$ *algebra.*

Proof. We employ Lemma 2.1.9 above. Let r be a central and compact projection, and write $\mathbf{1} - r$ out in components. Denote $\mathrm{her}_M(r)$ by $\mathfrak{B}$.

Assume first that $c = \sum_{I_B} c_i$ is unbounded. By the second half of Proposition 1.1.3, since the components are central in $\mathfrak{A}^{**}$, we can choose a sequence $\varphi_n \in P(c)$ with $\varphi_n \to 0$ *weak**. As $\varphi_n(c_i)$ is either 0 or 1 for every n and i, and since $\varphi_n(c_{i_0})$ can only be nonzero for finitely many n by Proposition 1.1.3 again, we can find a sequence (i_n) from I_B such that $\varphi_n(c_{i_n}) = 1$ for every n. Put $q_n = c_{i_n}$ and note that by Proposition 1.1.3 no sum $\sum_{n\in N} q_n$ is bounded when N is infinite because in that case, $(\varphi_n)_N$ could be considered as a sequence tending to zero. If I_U is infinite, we make such a sequence by setting $q_n = c_{i_n}$ where i_n is any sequence of different elements in I_U.

Let N be a subset of $\mathbb{N}$ and consider the ideal $\mathfrak{I}_N$ of $\mathfrak{B}$ covered by $\sum_N q_n$. As its central cover is a relatively clopen projection, $\mathfrak{I}_N$ is a complemented ideal in $\mathrm{her}(p)$, and by construction, it is unbounded, and so $\overline{\mathfrak{I}_N}^\beta \not\subseteq \mathfrak{A}$. By [**19**, 4.11], we have that $\overline{\mathfrak{I}}^\beta$ is complemented in $\mathrm{her}_M(p)$. Letting $\mathfrak{I}'_N = \kappa_{\mathfrak{A}}(\overline{\mathfrak{I}}^\beta)$ we also get that $\mathfrak{I}'_N$ is a complemented ideal in $M(\mathfrak{A})/\mathfrak{A}$ as a consequence of Proposition 1.1.5(i). Note that $\mathfrak{I}'_N \neq 0$ if N is not finite, and that $\mathfrak{I}_{N\cap N'} = \mathfrak{I}_N\mathfrak{I}_{N'}$. Denote by x_N the clopen central projections in $M(\mathfrak{A})/\mathfrak{A}$ corresponding to $\mathfrak{I}_N$, and let ω denote a free ultrafilter of $\mathbb{N}$. As no element of ω is finite, $(x_N)_{N\in\omega}$ has the finite intersection property, and since $M(\mathfrak{A})/\mathfrak{A}$ is unital, $\bigwedge_{N\in\omega} x_N$ is not zero by Lemma 1.1.2. As every component of $M(\mathfrak{A})/\mathfrak{A}$ is either completely contained in x_N or completely disjoint from it, the infimum above dominates at least one component, and invoking the axiom of choice, we pick out one and call it c_ω. When ω and ω' are different free ultrafilters, there exist $N \in \omega$ and $N' \in \omega'$ with $N \cap N' = \emptyset$, whence $c_\omega c_{\omega'} \leq x_N x_{N'} = 0$ and the two components are different. As $\mathrm{card}(\beta\mathbb{N}\backslash\mathbb{N}) = 2^{\mathfrak{c}}$ by [**39**, 3.2,3.9], we have produced a contradiction. □

2.2. Examples of Raum algebras.

Example 2.2.1. $C_0(X)$, where X is a noncompact Raum.

Example 2.2.2. A primitive, nonunital, σ-unital C^*-algebra.

Example 2.2.3. $\mathfrak{B} = C^*(\mathfrak{A}\oplus C(X), p\oplus\mathbf{1})$, where $\mathfrak{A}$ is a primitive, nonunital, σ-unital Raum algebra, X a compact Raum ($X \neq \{\mathrm{pt}\}$) and p a projection in $M(\mathfrak{A})\backslash\mathfrak{A}$.

The gluing technique applied in the above example is the simplest way to demonstrate certain phenomena. It appears in the following example as well. In this, we are working in the set of bounded block diagonal operators on an infinite sum of separable Hilbert spaces $\mathfrak{H}$. The elements here will interchangeably be called $\mathbf{a}$ or (a_k), where a_k is the restriction of $\mathbf{a}$ to the k'th copy of $\mathbb{B}$ or $\mathbb{K}$.

Example 2.2.4. $\mathfrak{A}_{\mathbf{q}} = C^*(\sum_1^\infty \mathbb{K}, (q_n)_1^\infty)$; every q_n a projection of $\mathbb{B}\backslash\mathbb{K}$, infinitely many $q_n \neq \mathbf{1}$, infinitely many $\mathbf{1} - q_n \in \mathbb{K}$.

Proposition 2.2.5. *The examples above satisfy the conditions is the previous section according to the following table.*

Example	2.2.1	2.2.2	2.2.3	2.2.4
Raum	+	+	+	+
Raum$^+$	+	+	+	

Sketch of proof Example 2.2.1. That a commutative Raum algebra has the Raum$^+$ property is a key step in Freudenthals algorithmic aproach to ends, see [**23**]. For an exposition with more emphasis on this step, see [**20**].
Example 2.2.2. See [**19**, 1.11].
Example 2.2.3. If π is a faithful irreducible representation of $\mathfrak{A}$, $\pi' = \pi \oplus 0$ is an irreducible representation of $\mathfrak{B}$ with bounded kernel, so that $\mathfrak{A}$ is a Raum$^+$ will follow from Lemma 2.1.10 when we have established that $\mathfrak{B}$ is in fact a Raum algebra. To see that the algebra is connected, note that with ρ the 1-dimensional representation with kernel $\mathfrak{A} \oplus C(X)$, $\ker\rho$ lies in every nonempty closed subset of $\mathrm{Prim}(\mathfrak{B})$. Hence $\mathrm{Prim}(\mathfrak{B})$ must be connected. To check local connectivity we must see that every ideal has only open components, cf. [**19**, 5.6ii]. We have seen that the trivial ideal $\mathfrak{B}$ has his property. Every proper ideal $\mathfrak{J}$ decomposes as $\mathfrak{J}\cap\mathfrak{A} \oplus \mathfrak{J}\cap C(X)$, and we may prove the property in each summand separately. Now assuming that $\mathfrak{J}$ is contained in either $\mathfrak{A}$ or $C(X)$, the result follows since these two C^*-algebras are locally connected by assumption.
Example 2.2.4. The algebra is nonunital since $\mathbf{1} - \mathbf{q} \notin \sum_1^\infty \mathbb{K}$. The prime ideals of $\mathfrak{A}_{\mathbf{q}}$ are

$$\mathfrak{J}_\infty = \sum_1^\infty \mathbb{K} \qquad \mathfrak{J}_k = \underbrace{\mathbb{K} \oplus \cdots \oplus \mathbb{K}}_{k-1} \oplus 0 \oplus \sum_{k+1}^\infty \mathbb{K},$$

To prove that that $\mathfrak{A}_{\mathbf{q}}$ is a Raum algebra, we only need to prove by [**19**, 1.7] and [**19**, 5.6] that $\mathrm{Prim}(\mathfrak{A}_{\mathbf{q}}) = \{\mathfrak{J}_\infty, \mathfrak{J}_1, \mathfrak{J}_2, \dots\}$ is connected and locally connected. This is clear as the topology of $\mathrm{Prim}(\mathfrak{A}_{\mathbf{q}})$ is given by $\overline{E} = E \cup_{\{} \mathfrak{J}_\infty\}$. locally connected since every set $\{\mathfrak{J}_k\}$ is open. To see why $\mathfrak{A}_{\mathbf{q}}$ is not a Raum$^+$ algebra, decompose $\mathfrak{J}_\infty = \sum_1^\infty \mathbb{K}$. Every summand corresponding to an n for which $\mathbf{1} - q_n \in \mathbb{K}$ is bounded, but an infinite sum of $\mathbb{K}$'s will never be bounded. This can be seen using (ii) of Proposition 1.1.3. □

3. Sequences determining ends.

The central notion in Freudenthal's work is that of a sequence determining an end. In the first subsection, we give similar noncommutative definitions for Raum algebras. We then demonstrate how sequences determining ends may be found in an algorithmic fashion in the Raum$^+$ case.

3.1. The definition.

Definition 3.1.1. We say that a decreasing sequence (p_k) of nonzero projections of $\mathfrak{A}^{**}$ *determines an end* of $\mathfrak{A}$ if for all $k \in \mathbb{N}$

(i) p_k is open.

(ii) p_k is connected.

(iii) p_k has compact semicentral boundary $\overline{\mathsf{c}(p_k)} - p_k$, and

(iv) $\bigwedge_{k=1}^{\infty} \overline{p_k} = 0$.

We say that (p_k) determines an end *weakly* if instead of (iii), (p_k) satisfies

(iii′) p_k has compact central boundary $\overline{\mathsf{c}(p_k)} - \mathsf{c}(p_k)$.

Example 3.1.2. 1°: Consider the Raum$^+$ algebra $C_0(\mathbb{R})$. Defining projections

$$p_n^- = 1_{(-\infty,-n)} \qquad p_n^+ = 1_{(n,\infty)}$$

we get two sequences determining ends (p_n^-) and (p_n^+).

2°: Consider the Raum$^+$ algebra $\mathbb{K}(\mathfrak{H})$, and choose an orthonormal basis (ξ_n) for $\mathfrak{H}$. When

$$q_n = [\mathrm{span}\{\xi_i | i \geq n\}] \qquad q_n' = [\mathrm{span}\{\xi_{2i} | 2i \geq n\}],$$

(q_n) is a sequence determining an end and (q_n') is a sequence weakly determining an end of $\mathfrak{A}$.

To prove the claims in 2°, one must note that $\mathsf{c}(q_n) = \mathsf{c}(q_n') = \mathbf{1}$ for every n.

We will impose an equivalence relation on the set of sequences weakly determining ends.

Proposition 3.1.3. *Let (p_k) and (q_k) be sequences weakly determining ends of $\mathfrak{A}$. The following conditions are equivalent.*

(i) *For all $l \in \mathbb{N}$, $\mathsf{c}(p_l)\mathsf{c}(q_l) \neq 0$.*

(ii) *For all $l \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with $p_k \leq \mathsf{c}(q_l)$.*

Proof. If (i) holds true and l is given, then since $\overline{\mathsf{c}(q_l)} - \mathsf{c}(q_l)$ is central and compact, every $\overline{p_k}(\overline{\mathsf{c}(q_l)} - \mathsf{c}(q_l))$ is a compact projection. As

$$\bigwedge_{k=1}^{\infty} \left(\overline{p_k}\left(\overline{\mathsf{c}(q_l)} - \mathsf{c}(q_l)\right)\right) = \left(\bigwedge_{k=1}^{\infty} \overline{p_k}\right)\left(\overline{\mathsf{c}(q_l)} - \mathsf{c}(q_l)\right) = 0,$$

we get $\overline{p_k}(\overline{\mathsf{c}(q_l)} - \mathsf{c}(q_l)) = 0$ for some $k \in \mathbb{N}$ by Proposition 1.1.2. *A fortiori*, $p_k(\overline{\mathsf{c}(q_l)} - \mathsf{c}(q_l)) = 0$. With $m = \max(k, l)$,

$$0 < \mathsf{c}(p_m)\mathsf{c}(q_m) \leq \mathsf{c}(p_k)\mathsf{c}(q_l),$$

so $p_k\mathsf{c}(q_l) \neq 0$ by [**32**, 2.6.7].

Note now that if $p_k(\overline{\mathsf{c}(q_l)} - \mathsf{c}(q_l)) = 0$, we can write $p_k\overline{\mathsf{c}(q_l)} = p_k\mathsf{c}(q_l)$, and p_k is separated (cf. [**19**, 2.1]) by $\{\mathsf{c}(q_l), 1 - \overline{\mathsf{c}(q_l)}\}$. We conclude that $p_k \leq \mathsf{c}(q_l)$. Suppose (ii) holds and let l be given. By assumption, there is a $k \in \mathbb{N}$ with $p_k \leq \mathsf{c}(q_l)$. We may assume that $k \geq l$. Consequently,

$$\mathsf{c}(p_l)\mathsf{c}(q_l) \geq \mathsf{c}(p_k)\mathsf{c}(q_l) \geq p_k\mathsf{c}(q_l) = p_k > 0.$$

□

First note that by (i) above, "≈" is reflexive and symmetric, by (ii) it is transitive, and that by (i) and the fact that sequences determining ends are decreasing, a sequence determining an end is equivalent to any of its subsequences.

Definition 3.1.4. If (p_k) and (q_k) satisfy (i)–(ii) above, we say that the sequences are *equivalent* and write $(p_k) \approx (q_k)$. Denoting equivalence classes by "[·]", we define

$$\mathsf{E}(\mathfrak{A}) = \{[(p_k)] | (p_k) \text{ weakly determines an end of } \mathfrak{A}\}.$$

and denote the cardinal of $\mathsf{E}(\mathfrak{A})$ by $\#_{\mathsf{E}}\mathfrak{A}$.

We shall refer to an equivalence class of sequences weakly determining ends simply as an *end*.

The sequences determining ends (p_n^-) and (p_n^+) in $C_0(\mathbb{R})$ given in Example 3.1.2(1°) are *not* equivalent. The sequences weakly determining ends (q_n) and (q_n') in $\mathbb{K}$ given in Example 3.1.2(2°) are. In fact, as will become evident shortly, $\mathsf{E}(C_0(\mathbb{R})) = \{[(p_n^-)], [(p_n^+)]\}$ and $\mathsf{E}(\mathbb{K}) = \{[(q_n)]\}$.

Remark 3.1.5. Several remarks are due on the definitions given above. First, the reader should notice that although the conditions (ii) and (iii) in the definition of sequences determining ends, as well as the definition of equivalence of ends, are essentially concerning the structure of the set of open central projections of $\mathfrak{A}$ — i.e. the ideal structure of $\mathfrak{A}$ — we could not have phrased our definition without invoking noncentral projections. To guarantee a fair supply of vanishing sequences of closed projections, we need to allow noncentral ones, cf. the simple case.

To understand the importance of (iii) and (iii') in Definition 3.1.1, the reader is advised to first consider the commutative Raum case — that an unbounded set has compact boundary means, in a sense, that it is connected at infinity. One may consider subsets of the real line as in Example 3.1.2(1°). Generalizing that notion to a noncommutative setting we have the option of considering $\overline{p}-p$, $\overline{\mathsf{c}(p)}-p$ and $\overline{\mathsf{c}(p)}-\mathsf{c}(p)$. As the first choice is not compatible with our definition of connectivity, this does not lead to an equivalence relation, but both the other ones do. The difference between the two notions of sequences determining ends is rather subtle, and we choose to focus on the condition (iii) for purposes of reaching corona components only. This will be further explained in Remark 4.1.11 below.

3.2. An algorithmic approach to ends.

Recall from Lemma 2.1.5 that every Raum algebra possesses a compact nest $\mathbf{r}$. We shall prove that in a general Raum algebra, every equivalence class of sequences determining ends has a representative (s_n) which comes from $\mathbf{r}$ in the strong sense that every s_n is a component projection of $\mathbf{1}-r_n$. This fact may be employed to give an algorithmic approach to finding all sequences determining ends of a Raum$^+$ algebra; essentially the same as the one Freudenthal devises in [**23**], see also [**20**]. The algorithm is based on arranging the set of unbounded components of the complements of the elements in the compact nest into a tree of connected projections. A truly algorithmic approach is not possible in a general Raum algebra, as there is no way of deciding at a finite stage whether or not a given projection is part of a full sequence determining an end.

To be more specific, consider a fixed compact nest $\mathbf{r}$ of the Raum$^+$ algebra $\mathfrak{A}$, let $p = r_n$ and $q = r_{n+1}$ and note that Proposition 2.1.4 applies in its full force since $q \geq p$. By this result, since $\mathbf{1} - q$ is behaved by the Raum$^+$ condition, every unbounded component of $\mathbf{1} - r_n$ dominates at least one unbounded component of $\mathbf{1} - r_{n+1}$. It is also clear from the definition of behaved projections that the set of unbounded components of $\mathbf{1} - r_n$ can not be empty, since $\mathbf{1} - r_n$ is itself unbounded. With this in hand we may construct a *family tree* of components. The elements of the nth level or *generation* is the set of unbounded component projections of $\mathbf{1} - r_n$, and given unbounded component projections c of $\mathbf{1} - r_n$ and d of $\mathbf{1} - r_{n+1}$, we say that d is a *descendant* of c, if c dominates d.

In this language, then, we have inferred from Proposition 2.1.4 that every element of the nth generation has at least one descendant and that it has a forbear when $n > 0$. We hence get a family tree with every branch of infinite length in this fashion. Notationally, we arrange the family tree by means of multiindices of integers. Every element of the nth generation is denoted by $s_{i_1\cdots i_n}$, and $s_{j_1\cdots j_{n+1}}$ is a descendant of $s_{i_1\cdots i_n}$ precisely when $i_k = j_k$ for $k \leq n$. We call an integer sequence (i_k) such that $s_{i_1\cdots i_n}$ is in the nth generation for every n a *branch*. In this setting, we denote the set of branches by $T_{\mathbf{r}}$, and call a sequence $(s_{i_1\cdots i_n})$ given by a branch (i_k) an $\mathbf{r}$-*sequence*.

Proposition 3.2.1. *Let $\mathfrak{A}$ be a Raum$^+$ algebra with a compact nest $\mathbf{r}$, and apply the notation above.*

(i) *$(s_{i_1\cdots i_n})$ is a sequence determining an end for every $(i_n) \in T_{\mathbf{r}}$.*

(ii) *$(s_{i_1\cdots i_n}) \approx (s_{j_1\dots j_n})$ if and only if $(i_n) = (j_n)$ in $T_{\mathbf{r}}$.*

Proof. By the definitions of the $s_{i_1\cdots i_n}$ and $T_{\mathbf{r}}$ above, $(s_{i_1\cdots i_n})$ is a decreasing sequence of open, nonempty and connected projections. By Lemma 1.2.1, $\overline{\mathsf{c}(s_{i_1\cdots i_n}) - s_{i_1\cdots i_n}}$ is compact for every k-tuple $i_1, \dots, i_n$. Finally, we note that $\overline{s_{i_1\cdots i_n}} \leq \overline{\mathbf{1} - 1_{[1/n,1]}(h)} \to 0$, proving (i).

When $(i_n) \neq (j_n)$ in $T_{\mathbf{r}}$, $s_{i_1\cdots i_n}$ and $s_{j_1\dots j_n}$ are different unbounded component projections of $\mathbf{1} - 1_{[1/n,1]}(h)$ for some $n \in \mathbb{N}$. As thus, by [**19**, 5.8], $\mathsf{c}(s_{i_1\cdots i_n})\mathsf{c}(s_{j_1\cdots j_n}) = 0$, the two sequences determining ends are inequivalent. ☐

Proposition 3.2.2. *Let (p_n) be a sequence weakly determining an end of a Raum algebra $\mathfrak{A}$, and let $\mathbf{r}$ be a compact nest of $\mathfrak{A}$. There exists a unique $\mathbf{r}$-sequence (s_n) with $(p_n) \approx (s_n)$.*

Proof. Let $r_n = 1_{[1/n,1]}(h)$. As $\overline{p_n} \searrow 0$, we have for every m_0 by Proposition 1.1.2 that

$$\bigwedge_{n=1}^{\infty} [\mathbf{1} - \mathsf{c}(\mathbf{1} - r_{m_0})]\overline{p_n} = 0,$$

whence by compactness, $\overline{p_n} \leq \mathsf{c}(\mathbf{1} - r_{m_0})$ eventually. We may assume that in fact $\overline{p_n} \leq \mathsf{c}(\mathbf{1} - r_n)$ for every n by replacing (p_n) with a subsequence. By the correspondence between components of $\mathbf{1} - r_n$ and $\mathsf{c}(\mathbf{1} - r_n)$ established in [**19**, 5.8], we get a unique component projection s_n of $\mathbf{1} - r_n$ with $\mathsf{c}(s_n)$ dominating $\mathsf{c}(p_n)$. By Lemma 1.2.1, $\overline{\mathsf{c}(s_n) - s_n}$ is compact, and clearly $\bigwedge_1^\infty \overline{s_n} = 0 \leq \bigwedge_1^\infty \overline{r_n} = 0$. We will prove that s_n is a decreasing sequence, after which it will follow by Proposition 3.1.3 that $(s_n) \approx (p_n)$. Assume that for some n, $s_n \not\geq s_{n+1}$. Since s_{n+1} is connected in her$(\mathbf{1} - r_n)$, and s_n is a component projection here, we must have $s_n d = 0$, where d is the component projection of $\mathrm{her}(\mathbf{1}-r_n)$ dominating s_{n+1}. By [19, 5.8], also $\mathsf{c}(s_n)\mathsf{c}(d)=0$, contradicting that $\mathsf{c}(s_n)\mathsf{c}(d)\geq \mathsf{c}(p_n)\mathsf{c}(p_{n+1})=\mathsf{c}(p_{n+1})>0$. □

Corollary 3.2.3.

(i) *When $\mathfrak{A}$ is a Raum algebra,*

$$\mathsf{E}(\mathfrak{A})=\{[(p_k)]|(p_k) \textit{ determines an end of } \mathfrak{A}\}.$$

(ii) *When $\mathfrak{A}$ is a Raum algebra and $\mathbf{r}$ a compact nest,*

$$\mathsf{E}(\mathfrak{A})=\{[s_n]|(s_n) \textit{ is a } \mathbf{r}\textit{-sequence}\}.$$

Corollary 3.2.4. *If $\mathfrak{A}$ is a Raum$^+$ algebra, $1\leq \#_{\mathsf{E}}\mathfrak{A}\leq \mathfrak{c}$.*

Proof. In every $\mathrm{her}(\mathbf{1}-r_n)$ there is a finite, positive number of unbounded component projections. The tree defined above will hence have at least one, and at most $\mathbb{N}^{\mathbb{N}}=\mathfrak{c}$ branches. □

Example 3.2.5. A Raum algebra with no sequences determining ends.

Construction. Consider Example 2.2.4 with *every* q_n of finite corank. We will argue on $\mathsf{z}\mathfrak{H}_u$, and start out by fixing some notation and making a few observations here. Let x_k be the central cover of $\mathfrak{I}_k$ for $k\in\mathbb{N}\cup\{\infty\}$. Note that $x_\infty+\sum_{k=1}^{\infty}x_k=\mathsf{z}$ and that x_∞ is a minimal projection, hence compact by [1, II.4]. Assume that (p_n) is a sequence determining an end of $\mathfrak{A}_{\mathbf{q}}$, and note that since $\bigwedge_1^\infty \overline{p_n}=0$ and x_∞ is compact, $\overline{p_n}x_\infty$ will be zero eventually as a consequence of Proposition 1.1.2. By discarding a finite number of elements from the sequence we get an equivalent sequence determining an end, and so we may assume that in fact $\overline{p}_1x_\infty=0$.

For every $N\subseteq\mathbb{N}$, there exists an open projection y_N in $\mathfrak{A}_{\mathbf{q}}^{**}$ with $y_n\mathsf{z}=\sum_{k\in N}x_k$, as can be seen by producing a sequence of elements in $\mathfrak{A}_{\mathbf{q}}$ increasing to it. Now assume that p_1x_k and p_1x_l are both nonzero and fix some set $N\subseteq\mathbb{N}$ with $k\in N$ but $l\notin N$. As then $p_1\mathsf{z}=p_1y_N\mathsf{z}+p_1y_{\mathbb{N}\setminus N}\mathsf{z}$ with nonzero summands on the right, we conclude from [1, II.17] that $p_1=p_1y_N+p_1y_{\mathbb{N}\setminus N}$, whence p_1 is not connected, a contradiction. We thus get that $p_1\mathsf{z}$, and hence $p_n\mathsf{z}$ for every $n\in\mathbb{N}$, is dominated by x_k for some k. We may assume, replacing $\mathbf{q}$ if necessary, that in fact $q_k=\mathbf{1}$. Note that $\mathbf{1}-\overline{p_1}$ is open and dominates x_∞, and if $\mathbf{a}_\mu=\mathbf{b}_\mu+\lambda_\mu\mathbf{q}$ is a net of positive operators of $\mathfrak{A}_{\mathbf{q}}$ increasing to $\mathbf{1}-\overline{p_1}$, then $\lambda_\mu\geq\frac{1}{2}$ eventually. From that stage on,

$$\pi_k(\mathbf{a}_\mu)=b_{\mu k}+\lambda_\mu\mathbf{1}\geq b_{\mu k}+\tfrac{1}{2}\mathbf{1},$$

and every projection dominating the latter operator must have finite corank. This means that $\overline{p_1}x_k$ has finite rank, whence so does $p_1x_k=p_1$. As no p_n is zero, $p_n=p_{n_0}$ for all $n\geq n_0$ for some n, and $\bigwedge_1^\infty\overline{p_n}=\overline{p_{n_0}}\neq 0$, a contradiction.

4. Decomposing the corona.

In this section, we shall demonstrate how knowledge of the structure of the sequences determining ends of $\mathfrak{A}$ leads to detailed knowledge about the component structure of the corona algebra $M(\mathfrak{A})/\mathfrak{A}$. In the commutative case, a correspondence between the set of sequences determining ends and the set of corona components may be established by the map sending a representative for an sequence determining an end (G_k) to the set

$$\bigcap_{k=1}^{\infty} \overline{G_k}^{\beta}$$

where the closure is taken in the Stone-Čech compactification βX. This turns out to be a component in $\beta X \backslash X$, and the map a 1–1 correspondence between the set of *equivalence classes* of sequences determining ends and the set of corona components. For details, see [20]. In our setting, it is easy to see that the map sending a sequence determining an end (p_k) to

$$\bigwedge_{k=1}^{\infty} \overline{p_k}^{M}$$

is well-defined and injective as a map from $\mathsf{E}(\mathfrak{A})$ to the set of closed projections of $(M(\mathfrak{A})/\mathfrak{A})^{**}$, but establishing that the infimum is even connected turns out to be surprisingly technical. When that hurdle has been passed, showing that the analogy with the commutative case is complete in the Raum$^+$ case is relatively easy.

4.1. Ends and corona components.

We first establish the easier of the claims made above.

Lemma 4.1.1. *Let (p_n) and (q_n) be sequences weakly determining ends in a Raum algebra $\mathfrak{A}$. Then $(p_n) \approx (q_n)$ if and only if*

$$\bigwedge_{n=1}^{\infty} \overline{\mathsf{c}(p_n)}^{M} = \bigwedge_{n=1}^{\infty} \overline{\mathsf{c}(q_n)}^{M}.$$

In fact, the infima are orthogonal when $(p_n) \not\approx (q_n)$.

Proof. If (p_n) and (q_n) are equivalent, note that for any given m_0, there is an n_0 such that

$$\overline{\mathsf{c}(p_{m_0})}^{M} \geq \overline{\mathsf{c}(q_{n_0})}^{M} \geq \bigwedge_{1}^{\infty} \overline{\mathsf{c}(q_n)}^{M},$$

and the result follows by symmetry. In the other direction, assume that $(p_n) \not\approx (q_n)$ and choose a compact nest $\mathbf{r}$. By the first half of the proof, we may take sequences determining ends $(p'_n) \approx (p_n)$ and $(q'_n) \approx (q_n)$ and prove instead that

$$\bigwedge_{n=1}^{\infty} \overline{\mathsf{c}(p'_n)}^M \neq \bigwedge_{n=1}^{\infty} \overline{\mathsf{c}(q'_n)}^M.$$

By Proposition 3.2.2, we may assume that p_n and q_n are components of $\mathbf{1}-r_n$. As the sequences determining ends are not equivalent, p_n and q_n are different components of $\mathbf{1}-r_n$ eventually by Proposition 3.2.1(ii), and so also $\overline{\mathsf{c}(p_n)}^M(\mathbf{1}_{M(\mathfrak{A})}-\mathbf{1}_{\mathfrak{A}})$ and $\overline{\mathsf{c}(q_n)}^M(\mathbf{1}_{M(\mathfrak{A})}-\mathbf{1}_{\mathfrak{A}})$ are orthogonal according to Lemma 1.2.5. This proves the last claim of the lemma and also yields the other implication in the main assertion as neither infimum is zero by compactness in $M(\mathfrak{A})$, cf. Proposition 1.1.2. □

Proposition 4.1.2. *Let (p_n) and (q_n) be sequences determining ends in a Raum algebra $\mathfrak{A}$. Then*

(i) $\bigwedge_1^\infty \overline{p_n}^M$ *is dominated by* $\mathbf{1}_{M(\mathfrak{A})}-\mathbf{1}_{\mathfrak{A}}$.

(ii) $\bigwedge_1^\infty \overline{p_n}^M = \bigwedge_1^\infty \overline{\mathsf{c}(p_n)}^M(\mathbf{1}_{M(\mathfrak{A})}-\mathbf{1}_{\mathfrak{A}})$

(iii) $(p_n) \approx (q_n)$ *if and only if* $\bigwedge_1^\infty \overline{p_n}^M = \bigwedge_1^\infty \overline{q_n}^M$.

Proof. The first claim follows from the definition of sequences determining ends by

$$\left(\bigwedge_{n=1}^{\infty} \overline{p_n}^M\right)\mathbf{1}_{\mathfrak{A}} = \bigwedge_{n=1}^{\infty} \overline{p_n}^M \mathbf{1}_{\mathfrak{A}} = \bigwedge_{n=1}^{\infty} \overline{\iota_M(p_n)} = \iota_M\left(\bigwedge_{n=1}^{\infty} \overline{p_n}\right) = 0,$$

cf. Lemma 1.2.3(ii).

As Lemma 1.2.4 applies to every p_n, we get that

$$\bigwedge_1^{\infty}\left(\overline{p_n}^M[\mathbf{1}_{M(\mathfrak{A})}-\mathbf{1}_{\mathfrak{A}}]\right) = \bigwedge_1^{\infty}\left(\overline{\mathsf{c}(p_n)}^M[\mathbf{1}_{M(\mathfrak{A})}-\mathbf{1}_{\mathfrak{A}}]\right)$$

from which (ii) follows in combination with (i).

Finally, if $(p_n) \approx (q_n)$, the infima in (iii) agree as a consequence of Lemma 4.1.1 and (ii) above. If $(p_n) \not\approx (q_n)$, we get that the infima are orthogonal by the last claim of Lemma 4.1.1. They are nonzero by Proposition 1.1.2. □

We now turn to the preliminaries of our main technical result. For an example showing that the e of the lemma below may not always be chosen to be central, see [**19**, 3.2].

Lemma 4.1.3. *Let $\mathfrak{B}$ be a unital C^*-algebra and let $x_n \in \mathfrak{B}$ be a decreasing sequence of central projections. If*

$$\bigwedge_{n=1}^{\infty} x_n = y_0 + y_1,$$

where the y_i are nontrivial closed central projections in $\mathfrak{B}^{**}$, there exists a projection $e \in \mathfrak{B}$ with $ey_0 = 0$, $ey_1 = y_1$.

Proof. There exists $a \in \mathfrak{B}$ with $ay_0 = 0$ and $ay_1 = y_1$ by [2, II.1], see also [6, 2.7]. As $x_n \searrow y_0 + y_1$, we have

$$b_n = (a - a^2)x_n(a - a^2) \searrow (a - a^2)(y_0 + y_1)(a - a^2) = 0,$$

whence by Dini's lemma applied to $\widehat{b_n} \in \mathrm{Aff}(S(\mathfrak{B}))$, cf. [32, 3.10], we can choose n_0 with $\|(a - a^2)x_{n_0}\| \leq \frac{1}{4}$. We have $\mathrm{sp}(ax_{n_0}) \subseteq [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, and can hence define $e = 1_{[\frac{1}{2},1]}(ax_{n_0}) \in \mathfrak{B}$. Since x_{n_0}, y_0 and y_1 are central, $ey_i = ay_i$. □

Recall from [4, p. 257] that a positive functional φ is said to be *definite* on $a \in \mathfrak{A}^{**}$ if $|\varphi(a)|^2 = \|\varphi\|\varphi(a^*a)$ and that for φ definite on a, $\varphi(ab)\|\varphi\| = \varphi(a)\varphi(b)$ for every $b \in \mathfrak{A}^{**}$.

Convention 4.1.4. Consider a Raum algebra $\mathfrak{A}$. Given a strictly positive element h of norm one in $\mathfrak{A}$ and projections e_0, e_1 in $M(\mathfrak{A})/\mathfrak{A}$ adding up to the unit, we shall fix the following notation:

(i) $a_i \in M(\mathfrak{A})$ are orthogonal elements, $0 \leq a_i \leq \mathbf{1}$ with $\kappa_{\mathfrak{A}}(a_i) = e_i$, $i \in \{0, 1\}$.

(ii) $b_i = a_i(\mathbf{1} - h)a_i$, $i \in \{0, 1\}$.

(iii) $q^i[\alpha] = 1_{(\alpha,1]}(b_i)$, $i \in \{0, 1\}$, $\alpha \in (0, 1)$.

(iv) $r[\alpha] = \mathbf{1} - q^0[\alpha] - q^1[\alpha]$, $\alpha \in (0, 1)$.

The a_i in (i) exist according to [5, 2.6]. Note that the $q^i[\alpha_n]$ are open, *relative to* $\mathfrak{A}$, according to [3, III.7]. We can say more:

Lemma 4.1.5. *With notation as in 4.1.4, when $\alpha_n \nearrow 1$, we have for $i \in \{0, 1\}$:*

(i) *If $\varphi_n \in P(\mathfrak{A})$ satisfies $\varphi_n(q^i[\alpha_n]) = 1$ for every n, then $\varphi_n \xrightarrow[n\to\infty]{} 0$ weak* in $\mathfrak{A}^*$.*

(ii) $\|q^i[\alpha_n]h\| \xrightarrow[n\to\infty]{} 0$.

(iii) *$(r[\alpha_n])_1^\infty$ is a compact nest.*

(iv) $\overline{q^i[\alpha]}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = e_i$.

Proof. Fix $i \in \{0, 1\}$. For (i), note that

$$1 \geq \varphi_n(b_i) \geq \alpha_n \varphi_n(q^i[\alpha_n]) = \alpha_n \underset{n\to\infty}{\longrightarrow} 1$$

so that $\varphi_n(b_i) \to 1$. Assume that $\varphi_n(d) \not\to 0$ for some $d \in \mathfrak{A}$. As $M(\mathfrak{A})$ is unital, $S(M(\mathfrak{A}))$ is compact, and we can find a subnet φ_{n_λ} converging to ψ in $S(M(\mathfrak{A}))$ with $\psi(d) \neq 0$. As ψ does not vanish on $\mathfrak{A}$, $\psi(h) > 0$, but as $\varphi_n(b_i) \to 1$, we get $\psi(a_i) \geq \psi(a_i^2) \geq \psi(b_i) = 1$. Thus ψ is definite on a_i, and we get the contradiction

$$1 = \psi(b_i) = \psi(a_i)\psi(\mathbf{1} - h)\psi(a_i) = 1 - \psi(h).$$

To prove (ii), as the $q^i[\alpha_n]$ are open in $\mathfrak{A}^{**}$ according to [**3**, III.7], we may apply the theory of projections tending to infinity developed in [**4**]. The claim then follows from (i) by [**4**, 2.2].

For (iii), note that $r[\alpha]$ is a closed projection in $\mathfrak{A}^{**}$ by orthogonality of the $q^i[\alpha]$. Write $c = \mathbf{1} - b_0 - b_1$ and note that $c \in \mathfrak{A}$ as $\kappa_{\mathfrak{A}}(c) = \mathbf{1} - e_0 - e_1 = 0$. We then have

$$r[\alpha] = \mathbf{1} - 1_{(\alpha,1]}(b_0 + b_1) = \mathbf{1} - 1_{(\alpha,1]}(\mathbf{1} - c) = 1_{(1-\alpha,1]}(c) \leq (1-\alpha)^{-1} c,$$

thus proving by Lemma 1.1.1(iii) that $r[\alpha]$ is a compact projection. Then, as also

$$\overline{\mathbf{1} - r[\alpha_n]} = \overline{q^0[\alpha_n] + q^1[\alpha_n]} \leq 1_{[\alpha_n,1]}(b_0) + 1_{[\alpha_n,1]}(b_1) \leq q^0[\alpha_{n-1}] + q^1[\alpha_{n-1}],$$

$(r[\alpha_n])_1^\infty$ is a compact nest.

To prove (iv), first note that for any continuous function $f : [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$, $f(b_i)(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = e_i$. Then $\overline{q^i[\alpha]}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \geq e_i$, as we can choose a continuous function f of the form described above for which $f \leq 1_{(\alpha,1]}$. Since, in $\mathfrak{A}^{**}$, $\overline{q^i[\alpha]} \leq 1_{[\alpha,1]}(b_i)$, we have that $\overline{q^0[\alpha]}$ and $\overline{q^1[\alpha]}$ are orthogonal. By [**10**, 3.33], also $\overline{q^0[\alpha]}^M$ and $\overline{q^1[\alpha]}^M$ are orthogonal, and equality in (iv) follows. ☐

In the commutative case, a sequence determining an end corresponds to a sequence (G_k) of open sets. The sequence $\overline{G_k}^\beta$ will be a decreasing sequence of connected, closed sets in the compact Hausdorff space βX and so will have connected intersection. In general, the infimum of a decreasing sequence of connected, closed projections may fail to be connected in a unital C^*-algebra, as demonstrated in [**19**, 2.4]. We must hence take a different and far more laborious path to prove the following key result.

Proposition 4.1.6. *Let (p_n) be a sequence determining an end of a Raum algebra $\mathfrak{A}$. Then $\bigwedge_1^\infty \overline{p_n}^M$ is connected.*

Proof. Denote the infimum by f and assume that f is not connected. This means that we can write f nontrivially as $fx_0 + fx_1$ for some open central projections x_0, x_1. By the fact that f is closed combined with [**1**, II.5], the elements $d_i = fx_i$ are closed, as they may be written $d_i = f(\mathbf{1}_{M(\mathfrak{A})} - x_i)$. Furthermore, they are central as f is central by Proposition 4.1.2(ii).

Note that by Proposition 4.1.2(ii), f is the infimum of a decreasing sequence of closed projections of $(M(\mathfrak{A})/\mathfrak{A})^{**}$, so as $M(\mathfrak{A})/\mathfrak{A}$ is a unital C^*-algebra, we may apply Lemma 4.1.3. This result leaves us with a projection e_0 in $M(\mathfrak{A})/\mathfrak{A}$ with

$$e_0 d_0 = 0, \quad e_0 d_1 = d_1. \tag{6}$$

We let $e_1 = \mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}} - e_0$, fix a strictly positive element h in $\mathfrak{A}$ of norm one, and define $a_i, b_i, q^i[\alpha]$ and $r[\alpha]$ according to 4.1.4. We also put $b = b_0 + b_1$, choose a sequence $\alpha_n \nearrow 1$, and abbreviate $q^i_n = q^i[\alpha_n]$, $r_n = r[\alpha_n]$. Note that (r_n) is a compact nest by Lemma 4.1.5(iii). We shall denote this by $\mathbf{r}$.

We will prove, successively, the following claims:

1° Passing to a subsequence of $((q^0_n, q^1_n, r_n))_1^\infty$, we can find an $\mathbf{r}$-sequence (s_n) with $(s_n) \approx (p_n)$ and

- (1.i) $\|(\mathbf{1} - r_n)h\| \leq 2^{-n}$.
- (1.ii) $s_n q^i_n \neq s_n$, $i \in \{0, 1\}$.

2° Passing to a subsequence of $((q^0_n, q^1_n, r_n, s_n))_1^\infty$, there exists $v_n, w_n \in \mathrm{her}(\mathbf{1} - r_n), \varphi^i_n \in P(s_n)$, $i \in \{0, 1\}$, such that for every $n, k \in \mathbb{N}$:

- (2.i) $v_{n-1}(\mathbf{1} - r_n) = (\mathbf{1} - r_n)v_{n-1} = 0$
- (2.ii) $\varphi^0_n(q^0_n) = \varphi^1_n(q^1_n) = 1$.
- (2.iii) $w_n^* \varphi^0_n w_n = \varphi^1_n$.
- (2.iv) $\|w_{n-1} - v_{n-1}\| < 2^{-n-1}$.
- (2.v) $\varphi^0_n(r_{n+k}) \leq 2^{-n-k}$.

3° There exists $\psi_i \in S(M(\mathfrak{A})/\mathfrak{A})$, $i \in \{0, 1\}$, and $v \in M(\mathfrak{A})/\mathfrak{A}$ with

- (3.i) $v\psi_0 v^* = \psi_1$
- (3.ii) $\psi_0(d_0) = \psi_1(d_1) = 1$.

This gives the desired contradiction. For as the d_i are orthogonal, $\psi_i(d_{1-i}) = 0$, whence by centrality of d_1

$$1 = \psi_1(d_1) = \psi_0(vd_1v^*) = \psi_0(d_1vv^*) \leq \psi_0(d_1)^{\frac{1}{2}} \psi_0((vv^*)^2)^{\frac{1}{2}} = 0.$$

Now to the proof of the claims 1°-3° above.

1°: By Lemma 4.1.5(ii), (1.i) can be met after passing to a subsequence which we shall also call r_n. Combining Lemma 3.2.2 and the fact that (r_n) is a compact nest by Lemma 4.1.5(iii), we get a representative (s_n) of $[(p_n)]$ which is an $\mathbf{r}$-sequence. Assume now that for infinitely many n,

$$s_n \leq q^1_n. \tag{7}$$

Passing to a subsequence again, we may assume that (7) holds for all $n \in \mathbb{N}$, whence

$$\overline{s_n}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) \leq \overline{q_n^1}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = e_1$$

according to Lemma 4.1.5(iv). This contradicts the fact that by Lemma 4.1.2(iii),

$$\bigwedge_1^\infty \overline{s_n}^M = \bigwedge_1^\infty \overline{s_n}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = d_0 + d_1 \not\leq e.$$

By symmetry, the case that $s_n \leq q_n^0$ infinitely often is also ruled out. Hence, (1. ii) may be arranged.

2°: Suppose that projections s, q^0 and q^1 satisfy the condition (1. ii), i.e. that $sq^i \neq q^i$ for both $i \in \{0,1\}$, and suppose that s is a component projection of $q^0 + q^1$. This setting occurs for every n in the sequences chosen above. Since s commutes with the q^i, the sq^i are open projections. However, according to the definition of connectivity, they are not central relative to s. Hence there exists an irreducible representation $(\pi, \mathfrak{H})$ of $\mathrm{her}(s)$ such that $\pi(q^i s)$ is not central. We can hence find unit vectors $\xi^j \in \mathfrak{H}, j \in \{0,1\}$ with

$$\pi^{**}(q^i)\xi^j = \delta_{ij}\xi^j \tag{8}$$

for all $i, j \in \{0,1\}$, and by Kadison's transitivity theorem, we can also find $w \in (\mathrm{her}(s))_1$ with

$$\pi(w)\xi^j = \xi^{1-j}. \tag{9}$$

We will construct the subsequence (n_m) explicitly. Let $w_0 = v_0 = 0$, $n_1 = 1$, and take an irreducible representation $(\pi_1, \mathfrak{H}_1)$, unit vectors $\xi_1^i \in \mathfrak{H}_1$ and an element $w_1 \in \mathrm{her}(s_1)$ satisfying (8) and (9) as above. Let φ_1^i be the extension of $\langle \pi_1(\cdot)\xi_1^i, \xi_1^i \rangle$ to $\mathfrak{A}$. This is a pure state by [**32**, 4.1.5]. We then have all the conditions in 2° satisfied for $n = 1$ (and $k = 0$).

Now suppose sequences $(n_m)_1^M, (w_m)_0^M, (v_m)_0^{M-1}, (\varphi_m^i)_1^M$ have been chosen, satisfying all conditions in 2° when $n, n+k \leq M$. Applying Lemma 4.1.5(ii), we may choose $\beta > \alpha_{n_M}$ such that $\|(\mathbf{1} - r[\beta])w_M\|, \|w_M(\mathbf{1} - r[\beta])\| \leq 2^{-M-1}$, and N satisfying

$$\alpha_N > \beta \tag{10}$$

$$\|(\mathbf{1} - r_N)h\| \leq 2^{-M-1} \tag{11}$$

$$\varphi_m^i(\mathbf{1} - r_N) \leq 2^{-m-M-1}, m \in \{1,\ldots,M\}, i \in \{0,1\} \tag{12}$$

Taking a continuous function $f_M : [0,1] \to [0,1]$ which is 1 on $[0,\beta]$ and 0 on $[\alpha_N, 1]$, we set

$$v_M = f_M(b)w_M f_M(b),$$

and note that

$$\begin{aligned}\|w_M - v_M\| &\le \|(\mathbf{1} - f_M(b))w_n\| + \|f_M(b)\|\|w_n(\mathbf{1} - f_M(b))\| \\ &\le \|(\mathbf{1} - r[\beta])w_M\| + \|w_M(\mathbf{1} - r[\beta])\| \le 2^{-M}\end{aligned}$$

by the C^*-identity. As $f(b)$ commutes with r_M, we get that $v_M \in \operatorname{her}(\mathbf{1}-r_M)$ because w_M is. Furthermore, as $f_M 1_{(\alpha_N,1]} = 0$, v_M satisfies (2. i). When we let $n_{M+1} = N$ and choose $(\pi_i, \mathfrak{H}_i)$, $\xi_n^i \in \mathfrak{H}_n$ and w_{M+1} satisfying (8) and (9), all conditions are satisfied for all n, k with $n, n+k \le M+1$ when we define φ_n^i as the extensions of $\langle \pi_n(\cdot)\xi_n^i, \xi_n^i\rangle$.

3°: Note that when $m > n$,

$$v_m v_n = v_m(\mathbf{1} - r_m)v_n = v_m(\mathbf{1} - r_m)(\mathbf{1} - r_{n+1})v_n = 0$$

by (2. i). Similar reasoning proves that whenever $m \ne n$, $v_n v_m = 0$ and $v_n v_m^* = 0$. The first fact shows that $v' = \sum_{n=1}^\infty v_n$ defines an element of $\mathfrak{A}^{**}$ which has norm 1 since $v_n \in \operatorname{her}(\mathbf{1} - r_n)_1$. We have

$$v'h = \sum_{n=1}^\infty v_n h \in \mathfrak{A}$$

as $\|v_n h\| \le \|(\mathbf{1} - r_n)h\| \le 2^{-n}$ by (2. ii), and from a similar calculation with h on the right, we may conclude that $v' \in M(\mathfrak{A})$. Now

$$(\mathbf{1} - r_n)v' = \sum_{m=1}^\infty (\mathbf{1} - r_n)v_m = (\mathbf{1} - r_n)\sum_{m=n+1}^\infty v_m$$

and with $V_n = \sum_{m=n+1}^\infty v_m$ we have

$$\begin{aligned}\varphi_n^i(V_n^* V_n) &= \varphi_n^i\left(\sum_{m=n+1}^\infty v_m^* v_m\right) \\ &\le \sum_{m=n+1}^\infty |\varphi_n^i(\mathbf{1} - r_m)| \\ &\le \sum_{j=1}^\infty 2^{-n-j} = 2^{-n}\end{aligned}$$

according to (2. v). The same estimate holds with the factors in the opposite order. Combining these facts with (2. iii), we get

$$\begin{aligned}\|{v'}^*\varphi_n^0 v' - \varphi_n^1\| &= \|({v'}^*(\mathbf{1} - r_n))\varphi_n^0((\mathbf{1} - r_n)v') - \varphi_n^1\| \\ &= \|(v_n^* + V_n^*)\varphi_n^0(v_n + V_n) - \varphi_n^1\| \\ &\le \varphi_n^0({V_n}^* V_n)^{\frac12} + \varphi_n^0(V_n V_n^*)^{\frac12} + \varphi_n^0({V_n}^* V_n) + \|v_n^*\varphi_n^0 v_n - \varphi_n^1\| \\ &\le 2^{-\frac{n}{2}+1} + 3\cdot 2^{-n} \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Consider the φ_n^i as states on $M(\mathfrak{A})$. As $S(M(\mathfrak{A})) \times S(M(\mathfrak{A}))$ is compact, we can find two subnets $(\varphi_{n_\lambda}^i)_\Lambda$ converging to $\psi_i' \in S(M(\mathfrak{A}))$. Since $\|v'^*\varphi_n^0 v' - \varphi_n^1\| \to 0$, we have $v'^*\psi_0' v' = \psi_1'$ as

$$\|\psi_0'(v'bv'^*) - \psi_1'(b)\| \leq \|\psi_0'(v'bv'^*) - \varphi_{n_\lambda}(v'bv'^*)\| + \|b\|\|v'^*\varphi_{n_\lambda}^0 v' - \varphi_{n_\lambda}^1\| \\ +\|\varphi_{n_\lambda}(b) - \psi_1'(b)\| \underset{n\to\infty}{\longrightarrow} 0$$

for every $b \in M(\mathfrak{A})$.

Note that $\varphi_n^i \to 0$ *weak** on $\mathfrak{A}$ by Lemma 4.1.5(i) and (2. ii), so that $\varphi_{n_\lambda}^i(x) \to 0$ for all $x \in \mathfrak{A}$, and the ψ_i' vanish on $\mathfrak{A}$. As $\kappa_{\mathfrak{A}}^* : (M(\mathfrak{A})/\mathfrak{A})^* \to M(\mathfrak{A})^*$ is isometric onto the annihilator of $\mathfrak{A}$ by [**36**, Theorem 4.9(b)] and we just showed that the ψ_i' lie in that annihilator, we can define $\psi_i \in S(M(\mathfrak{A})/\mathfrak{A})$ by $\psi_i = \kappa_{\mathfrak{A}}^{*-1}(\psi_i')$. Then $\psi_i \circ \kappa_{\mathfrak{A}} = \psi_i'$ and with $v = \kappa_{\mathfrak{A}}(v')$, $v^*\psi_0 v = \psi_1$. By (2. iii) above,

$$1 \geq \psi_i(e_i) = \psi_i(\kappa_{\mathfrak{A}}(b_i)) = \psi_i'(b_i) = \lim_{\mathbb{N}} \varphi_n^i(b_i) = 1.$$

Finally, as we have

$$\varphi_n^i(\overline{s_{n_0}}^M) \geq \varphi_n^i(s_{n_0}) \geq \varphi_n^i(s_n) = 1$$

for all $n \geq n_0$, $\psi_i(\overline{s_{n_0}}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})) \geq 1$ for both i and every n_0. Then also, using normality and Proposition 4.1.2(iii), $\psi_i(\bigwedge_1^\infty \overline{s_n}^M) = \psi_i(d_0 + d_1) = 1$, and we get

$$\psi_i(d_i) = \psi_i(e_i(d_0 + d_1)) = \psi_i(e_i) = 1$$

as required. □

We are now able to collect our results into our main theorem.

Theorem 4.1.7. *When $\mathfrak{A}$ is a Raum algebra, the map*

$$[(p_n)] \mapsto \bigwedge_{n=1}^{\infty} \overline{p_n}^M$$

sends equivalence classes of sequences determining ends of $\mathfrak{A}$ to component projections of $M(\mathfrak{A})/\mathfrak{A}$. The map is injective. It is onto when $\mathfrak{A}$ is a Raum$^+$ algebra.

Proof. The map does not depend on the choice of representative for $[(p_n)]$ as a consequence of Proposition 4.1.2(iii). To see that $\bigwedge_{n=1}^\infty \overline{p_n}^M$ is a component projection, note that by Proposition 4.1.6 it is connected and let c be the component projection of $M(\mathfrak{A})/\mathfrak{A}$ dominating it. Since $\overline{p_n}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$ is central and clopen according to Lemma 1.2.5, $c \leq \overline{p_n}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$ whence $\bigwedge_{n=1}^{\infty} \overline{p_n}^M = c$.

For the other claims we fix a compact nest $\mathbf{r}$. As every equivalence class has a unique representative among the $\mathbf{r}$-sequences by Proposition 3.2.2, we need only prove that if $(i_n) \neq (j_n)$ in $T_{\mathbf{r}}$, then

$$\bigwedge_1^{\infty} \overline{s_{i_1 \cdots i_n}}^M \neq \bigwedge_1^{\infty} \overline{s_{j_1 \dots j_n}}^M.$$

To see this, first note that the infima are both nonzero by uniticity of $M(\mathfrak{A})$, cf. Lemma 1.1.2. If n is given with $i_n \neq j_n$, we get that $s_{i_1 \cdots i_n}$ and $s_{j_1 \dots j_n}$ are different components of $\mathbf{1} - r_n$. Choosing projections d, e which are clopen and central relative to $\mathbf{1} - r_n$, have $d + e = \mathbf{1} - r_n$ and dominate $s_{i_1 \cdots i_n}$ and $s_{j_1 \dots j_n}$, respectively, we need only prove that $\overline{d}^M$ and $\overline{e}^M$ are orthogonal. This follows from Lemma 1.2.5.

To see that the map is onto when $\mathfrak{A}$ is a Raum$^+$ algebra, let a component projection f be given. By the Raum$^+$ condition, there are only finitely many branches at each stage, and we may apply Lemma 1.2.5 to the set of components of $\mathbf{1} - r_n$. By the fact that the $\overline{s_{i_1 \cdots i_n}}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$ are elements of $M(\mathfrak{A})/\mathfrak{A}$, f must meet each of them trivially. As they add upp to the unit, there will be a unique $\overline{s_{i_1 \cdots i_n}}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}})$ dominating f. Hence there exists a branch $(i_n) \in T_{\mathbf{r}}$ with

$$f \leq \bigwedge_1^{\infty} \overline{s_{i_1 \cdots i_n}}^M(\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}) = \bigwedge_1^{\infty} \overline{s_{i_1 \cdots i_n}}^M.$$

As the infimum is connected by Proposition 4.1.6, there is actually equality here. □

Corollary 4.1.8. *When $\mathfrak{A}$ is a Raum$^+$ algebra, $\#_{\mathsf{E}}\mathfrak{A} = \#_{\mathsf{K}}(M(\mathfrak{A})/\mathfrak{A})$.*

Corollary 4.1.9. *When $\mathfrak{A}$ is a σ-unital C^*-algebra with a bounded primitive ideal, $M(\mathfrak{A})/\mathfrak{A}$ is connected.*

Proof. Assume first that the ideal is (0), so that in fact $\mathfrak{A}$ is primitive. As every hereditary C^*-subalgebra of $\mathfrak{A}$ is connected by [**19**, 1.11], $\mathfrak{A}$ is obviously a Raum$^+$ algebra with only one sequence determining an end. The general case follows from Lemma 1.1.5(ii). □

Corollary 4.1.10. *Let $\mathfrak{A}$ be a Raum algebra. The following are equivalent:*

(i) *$\mathfrak{A}$ is a Raum$^+$ algebra.*

(ii) *$\#_{\mathsf{K}}(M(\mathfrak{A})/\mathfrak{A}) \leq \mathfrak{c}$.*

(iii) $\#_{\mathsf{K}}(M(\mathfrak{A})/\mathfrak{A}) < 2^{\mathfrak{c}}$.

Proof. If $\mathfrak{A}$ is a Raum$^+$ algebra, the number of sequences determining ends is less than $\mathfrak{c}$ according to Corollary 3.2.4. Then so is the number of corona components according to Corollary 4.1.8 above, proving (i)$\Longrightarrow$(ii). Clearly, (ii) implies (iii), and (iii)$\Longrightarrow$(i) was established in Proposition 2.1.11. □

This result explains why we need not concern ourselves with the continuum hypothesis.

Remark 4.1.11. We are now in the position to explain our previous comments (3.1.5) on the difference between sequences determining ends and sequences *weakly* determining ends. As we saw in Corollary 3.2.3, every equivalence class of sequences weakly determining ends $[(q_n)]$ has at least one representative (p_n) which is in fact a sequence determining an end. By Lemmas 4.1.1 and 1.2.4

$$\bigwedge_{n=1}^{\infty} \overline{q_n}^M \leq \bigwedge_{n=1}^{\infty} \overline{q_n}^M[\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}] = \bigwedge_{n=1}^{\infty} \overline{\mathsf{c}(q_n)}^M[\mathbf{1}_{M(\mathfrak{A})} - \mathbf{1}_{\mathfrak{A}}] = \bigwedge_{n=1}^{\infty} \overline{p_n}^M,$$

where the right hand side is a certain component of $M(\mathfrak{A})/\mathfrak{A}$. The general infimum of a sequence weakly determining an end may be strictly smaller — consider the q_n' in our Example 3.1.2($2°$). We do not know whether the infimum $\bigwedge_1^\infty \overline{q_n}^M$ will be connected for a general sequence weakly determining an end.

5. Applications.

In this section we give applications of our results. In the first section, we show how a small variation of the methods leading to Corollary 4.1.9 gives that the corona algebra of an essentially primitive, σ-unital C^*-algebra is in fact *prime*.

Then we go on to determine the end theory of certain tensor products and of a group C^*-algebra and note the consequences for the structure of their corona algebras. The reader should note that in both these examples, we rely heavily on the fact that we may choose a particularly convenient strictly positive element to use in our determination of the set of ends.

5.1. Prime corona algebras.

We generalize Zhang's result from [**41**, 6.3(i)] that simple, σ-unital C^*-algebras of real rank zero have prime corona. Actually, Zhang's methods give some sort of σ-primeness; the intersection of countably many ideals of $M(\mathfrak{A})/\mathfrak{A}$ will be nonzero for the $\mathfrak{A}$ he considers. The condition on σ-unitality in both results is essential. In fact, examples are known of simple C^*-algebras of real rank zero with $M(\mathfrak{A})/\mathfrak{A}$ nonconnected; indeed, every finite-dimensional C^*-algebra — e.g. $\mathbb{C} \oplus \mathbb{C}$ — can be obtained as the corona of some hereditary C^*-subalgebra of a II_1 factor, as described in [**37**, Corollary 4]. Such a C^*-algebra has real rank zero by [**12**, 1.3,2.8].

As the proof of the following theorem is very similar to that of Proposition 4.1.6, we shall only sketch it here.

Theorem 5.1.1. *When $\mathfrak{A}$ is a σ-unital C^*-algebra with a bounded primitive ideal, $M(\mathfrak{A})/\mathfrak{A}$ is prime.*

Scetch of proof. By Proposition 1.1.5(ii), we may assume that the bounded ideal is in fact trivial so that $\mathfrak{A}$ is primitive. Assume that $M(\mathfrak{A})/\mathfrak{A}$ is not prime and let two nonzero ideals $\mathfrak{I}_0, \mathfrak{I}_1$ with $\mathfrak{I}_0\mathfrak{I}_1 = 0$ be given. Let p_0 and p_1 be the two orthogonal open central projections of $(M(\mathfrak{A})/\mathfrak{A})^{**}$ covering $\mathfrak{I}_0$ and $\mathfrak{I}_1$, respectively. Pick $d_i \in \mathfrak{I}_i$, $0 \leq d_i \leq \mathbf{1}$ with $\|d_0\| = \|d_1\| = 1$. By [**5**, 2.6], we can lift d_0, d_1 to orthogonal elements $a_i \in M(\mathfrak{A})$ with $0 \leq a_i \leq \mathbf{1}$. Set $b_i = a_i(\mathbf{1} - h)a_i$. Let π be a faithful irreducible representation and note that since $\pi^{**}\|_{M(\mathfrak{A})}$ is also faithful (the kernel is an ideal which intersects $\mathfrak{A}$ trivially), we have

$$\|\pi^{**}(b_i)\| \geq \|\kappa_{\mathfrak{A}}(b_i)\| = \|d_i^2\| = 1.$$

As a definiteness argument shows, $\varphi(a_i(\mathbf{1} - h)a_i) < 1$ for all $\varphi \in S(\mathfrak{A})$. In particular, $\pi(b_i)$ does not have an eigenvector on $\mathfrak{H}$ corresponding to the eigenvalue 1, and hence 1 is not an isolated point of $\mathrm{sp}(\pi(b_i))$. Considering b_0 and b_1 simultaneously, we can hence choose $\alpha_n, \beta_n \in (0,1)$ with

$$0 = \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots \to 1$$

and the property

$$(\alpha_n, \beta_n) \cap \mathrm{sp}(\pi(b_i)) \neq \emptyset, \qquad n \in \mathbb{N}, i \in \{0,1\}. \tag{13}$$

Choose positive, piecewise linear functions f_n vanishing outside of $(\beta_{n-1}, \alpha_{n+1})$, bounded by 1 with the constant value 1 on $[\alpha_n, \beta_n]$. Let $c_{in} = f_n(b_i)$ and note that c_{in} is orthogonal to c_{jm} unless $n = m$ and $i = j$. Letting $q_n^i = 1_{(\beta_{n-1},1]}(b_i)$, we have that $q_n^i h$ tends to 0 in norm as in Lemma 4.1.5(ii).

Passing to subsequences simultaneously in the cases $i = 0$ and $i = 1$ we may assume that $\|q_n^i h\| \leq 2^{-n-1}$. Then also

$$\|c_n^i h\| = \|hc_n^i h\|^{\frac{1}{2}} \leq \|hq_n^i h\|^{\frac{1}{2}} = \|q_n^i h\| \leq 2^{-n-1},$$

and when we define $c_n = c_{0n} + c_{1n}$, we get orthogonal elements of norm one with the property that $\|c_n h\| \leq 2^{-n}$.

By (13), we can find unit vectors ξ_n^i in the image of $1_{[\alpha_n,\beta_n]}(\pi(b_i))$. We have that $\langle \xi_n^i, \xi_m^j \rangle = 0$ unless $i = j$ and $n = m$. By Kadison's transitivity theorem, we may find an element $w_n \in \mathfrak{A}$, $\|w_n\| = 1$, such that $\pi(w_n)\xi_n^i = \xi_n^{1-i}$ for $i \in \{0, 1\}$. Note that

$$\begin{aligned} \mathbf{1} \geq \pi(c_n) &\geq \pi^{**}(1_{[\alpha_n,\beta_n]}(b_0) + 1_{[\alpha_n,\beta_n]}(b_1)) \\ &= 1_{[\alpha_n,\beta_n]}(\pi(b_0)) + 1_{[\alpha_n,\beta_n]}(\pi(b_1)) \end{aligned}$$

and $\pi(c_n)$ acts as the unit on ξ_n^i. Let $v_n = c_n w_n c_n$. Let $\varphi_n^i \in P(\mathfrak{A})$ be the pure states given by the ξ_n^i. Then $v_n^* \varphi_n^1 v_n = \varphi_n^0$. As in the proof of 3° in Proposition 4.1.6, we get $v' = \sum_1^\infty v_n \in M(\mathfrak{A})$ and $v'^* \varphi_n^1 v' = \varphi_n^0$. Passing to subnets we have $\varphi_{n_\lambda}^i \to \psi_i'$ which drop to $\psi_i \in S(M(\mathfrak{A})/\mathfrak{A})$ as in 3° of Proposition 4.1.6. We let $v = \kappa_{\mathfrak{A}}(v')$ and conclude that $v^* \psi_1 v = \psi_0$. By construction

$$1 \geq \psi_i(p_i) \geq \psi_i(d_i) \geq \psi_i(d_i^2) = \psi_i(\kappa_{\mathfrak{A}}(b_i)) = \psi_i'(b_i) = \lim_{\mathbb{N}} \varphi_n^i(b_i) = 1,$$

and as p_0 and p_1 are orthogonal projections, we conclude that $\psi_i(p_{1-i}) = 0$. Since p_i is a central projection in $M(\mathfrak{A})/\mathfrak{A}$, we arrive at the contradiction

$$\begin{aligned} 1 = \psi_i(d_i) &= \psi_i(v^* d_{1-i} v) \leq \psi_i(v^* p_{1-i} v) \\ &= \psi_i(p_{1-i} v^* v) \leq \psi_i(p_{1-i})^{1/2} \psi_i((v^* v)^2)^{1/2} \\ &= 0. \end{aligned}$$

□

5.2. Ends and tensor products.

The following lemma allows us to ask natural questions about the end structure of tensor product of Raum algebras.

Lemma 5.2.1. *When $\mathfrak{A}$ and $\mathfrak{B}$ are Raum algebras and $\mathfrak{A}$ is nuclear, then $\mathfrak{A} \otimes \mathfrak{B}$ will be a Raum algebra.*

Proof. As the countable approximate units of $\mathfrak{A}$ and $\mathfrak{B}$ combine to one of the tensor product by [**38**, 4.1], $\mathfrak{A} \otimes \mathfrak{B}$ will be σ-unital. That $\mathfrak{A} \otimes \mathfrak{B}$ is connected and locally connected follows by noting that the nuclearity condition ensures, cf. [**7**], that $\operatorname{Prim} \mathfrak{A} \otimes \mathfrak{B}$ is homeomorphic to $\operatorname{Prim} \mathfrak{A} \times \operatorname{Prim} \mathfrak{B}$, for the product of connected (resp. locally connected) spaces is connected (resp. locally connected). See [**19**, 4.3] for details. □

We do not know whether nuclearity is necessary for the result above.

Consider a C^*-norm β on $\mathfrak{A} \odot \mathfrak{B}$ and the C^*-algebraic tensor product $\mathfrak{A} \otimes_\beta \mathfrak{B}$. We shall mainly work with the minimal tensor product, which will be identified using $\beta = *$. From the universal representation $(\pi_u, \mathfrak{H}_u)$ of the tensor product, according to [**38**, 4.1], we get representations $(\pi_\mathfrak{A}, \mathfrak{H}_u)$ and $(\pi_\mathfrak{B}, \mathfrak{H}_u)$ of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, with the properties that

$$\pi_\mathfrak{A}(a)\pi_\mathfrak{B}(b) = \pi_u(a \otimes b) = \pi_\mathfrak{B}(b)\pi_\mathfrak{A}(a)$$

for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. In particular, $\pi_\mathfrak{A}(\mathfrak{A})'' \subseteq \pi_\mathfrak{B}(\mathfrak{B})'$, so that the images of $\pi_\mathfrak{A}^{**}$ and $\pi_\mathfrak{B}^{**}$ commute. Hence, when $p \in \mathfrak{A}^{**}$ and $q \in \mathfrak{B}^{**}$ are projections, the element $\pi_\mathfrak{A}^{**}(p)\pi_\mathfrak{B}^{**}(q)$ is a projection in $\pi_u(\mathfrak{A} \otimes_\beta \mathfrak{B})'' = (\mathfrak{A} \otimes_\beta \mathfrak{B})^{**}$. We denote this projection by $p \bar{\otimes} q$.

Lemma 5.2.2.

(i) *When $p \in \mathfrak{A}^{**}$ and $q \in \mathfrak{B}^{**}$ are both open, resp. closed, resp. compact, so is $p \bar{\otimes} q$.*

(ii) *There exist compact nests (p_n) of $\mathfrak{A}$ and (q_n) of $\mathfrak{B}$ such that $(p_n \bar{\otimes} q_n)$ is a compact nest of $\mathfrak{A} \otimes_* \mathfrak{B}$.*

Proof. The claims in (i) follow from [**32**, 3.11.9] straightforwardly by appealing to the fact that multiplication is strongly continuous on bounded sets. We will apply (i) to compact nests of the special form $p_n = 1_{[1/n,1]}(h)$ and $q_n = 1_{[1/n,1]}(k)$ derived from strictly positive elements $h \in \mathfrak{A}$ and $k \in \mathfrak{B}$, cf. Lemma 2.1.5. Note that

$$\mathbf{1} - \overline{\mathbf{1} - p_n} \leq 1_{[1-1/n,1]}(h) \underset{n\to\infty}{\longrightarrow} 0,$$

and similarly for q_n, so that

$$\overline{\mathbf{1} - p_n \bar{\otimes} q_n} \leq \mathbf{1} - \left[\mathbf{1} - \overline{\mathbf{1} - p_n}\right] \bar{\otimes} \left[\mathbf{1} - \overline{\mathbf{1} - q_n}\right] \underset{n\to\infty}{\longrightarrow} 0.$$

The $p_n \bar{\otimes} q_n$ are compact by (i), and clearly an increasing sequence. ☐

Lemma 5.2.3. *Let $\mathfrak{A}, \mathfrak{B}$ be connected C^*-algebras. Let p and q be closed, nontrivial projections of $\mathfrak{A}^{**}$ and $\mathfrak{B}^{**}$, respectively. Relative to $\mathfrak{A} \otimes_* \mathfrak{B}$, the set $P(\mathbf{1} - p \bar{\otimes} q)$ is connected.*

Proof. Let $r = \mathbf{1} - p \bar{\otimes} q$ and $\mathfrak{C} = \mathrm{her}(r)$. We shall denote the set of tensor pure states of $\mathfrak{A} \otimes_* \mathfrak{B}$ by $\mathcal{T}$ and consider the subset

$$\mathcal{T}_0 = \{\varphi \otimes_* \psi | \varphi^{**}(p) = 0 \text{ or } \psi^{**}(q) = 0\}.$$

We first prove that $\mathcal{T}_0$ is connected. To see this, first note that the sets $P(\mathfrak{A}) \otimes_* \psi$ and $\varphi \otimes_* P(\mathfrak{B})$ are connected as they are continuous images of the connected sets $P(\mathfrak{A}), P(\mathfrak{B})$, cf. [19, 1.7]. Since p and q are not identities, we may take $\varphi_0 \in P(\mathfrak{A})$ and $\psi_0 \in P(\mathfrak{B})$ with $\varphi_0(p) = \psi_0(q) = 0$, for if, e.g., $p\mathsf{z}_{\mathfrak{A}} = \mathsf{z}_{\mathfrak{A}}$, $p = \mathbf{1}$ by [1, II.16]. Note that

$$\begin{aligned}\mathcal{T}_0 = &\left[\bigcup_{\varphi^{**}(p)=0} [\varphi \otimes_* P(\mathfrak{B}) \cup P(\mathfrak{A}) \otimes_* \psi_0]\right] \cup \\ &\left[\bigcup_{\psi^{**}(q)=0} [P(\mathfrak{A}) \otimes_* \psi \cup \varphi_0 \otimes_* P(\mathfrak{B})]\right].\end{aligned}$$

In each of the five unions taken here, $\varphi_0 \otimes_* \psi_0$ is a common point, and so the union is connected since every subset is.

For M a subset of $P(\mathfrak{A} \otimes_* \mathfrak{B})$, we set

$$\operatorname{sat}_{\mathfrak{C}} M = \{u\varphi u^* | \varphi \in M, u \in \mathcal{U}(\tilde{\mathfrak{C}})\}$$

We claim that

$$\begin{aligned}(\operatorname{sat}_{\mathfrak{C}} \mathcal{T}_0)^\circ &= \{a \in (\mathfrak{A} \otimes_* \mathfrak{B})_{sa} | \chi(uau^*) \geq -1 \forall \chi \in \mathcal{T}_0, u \in calU(\tilde{\mathfrak{C}})\} \\ &= \{a \in (\mathfrak{A} \otimes_* \mathfrak{B})_{sa} | (\pi \otimes_* \rho)^{**}(r(a + \mathbf{1})r) \geq 0 \forall \pi \in \hat{\mathfrak{A}}, \rho \in \hat{\mathfrak{B}}\} \\ &= \{a \in (\mathfrak{A} \otimes_* \mathfrak{B})_{sa} | rar \geq -r\} \\ &= F(r)^\circ.\end{aligned}$$

Let us go through this set of equalities. The first and last follow from the definition of polars. For the second, note that if a satisfies the condition involving irreducible representations, it will satisfy the condition involving pure states by the GNS construction. For the other inclusion, let a in the polar and irreducible representations $(\pi, \mathfrak{H})$ and $(\rho, \mathfrak{K})$ be given.

If $(\pi \otimes_* \rho)^{**}(r) = 0$, we are done. If not, since

$$r = \mathbf{1} - (p \otimes \mathbf{1}) \wedge (\mathbf{1} \otimes q) = (\mathbf{1} - p) \otimes \mathbf{1} \vee \mathbf{1} \otimes (\mathbf{1} - q),$$

$(\pi \otimes_* \rho)^{**}$ can not vanish on both of these projections. Assume that $(\pi \otimes_* \rho)^{**}((\mathbf{1} - p) \otimes \mathbf{1}) = \pi^{**}(\mathbf{1} - p) \otimes \mathbf{1} \neq 0$, and choose a unit vector $\xi \in \pi^{**}(\mathbf{1} - p)\mathfrak{H}$. Take any unit vector $\eta \in \mathfrak{K}$ and define a pure state

$$\chi(a) = \langle \pi \otimes_* \rho(a)(\xi \otimes \eta), \xi \otimes \eta \rangle.$$

We have

$$\chi^{**}(r) \geq \chi^{**}((\mathbf{1} - p) \otimes \mathbf{1}) = \langle \pi^{**}(\mathbf{1} - p)\xi, \xi \rangle = 1,$$

so $\chi \in \mathcal{T}_0$. Now let a unit vector ζ under $(\pi \otimes_* \rho)^{**}(r)$ be given. By transitivity, there is a unitary u in $\tilde{\mathfrak{C}}$ with $\pi \otimes_* \rho(u)(\xi \otimes \eta) = \zeta$, whence

$$\langle \pi \otimes_* \rho(a + \mathbf{1})\zeta, \zeta\rangle = \langle \pi \otimes_* \rho(u^*(a + \mathbf{1})u)\xi \otimes \eta, \xi \otimes \eta\rangle = \chi(u(a + \mathbf{1})u^*) \geq 0$$

by our assumption. This proves that relative to $(\pi \otimes_* \rho)^{**}(r)$, $\pi \otimes_* \rho(a+\mathbf{1})$ is positive. The third equality follows by appealing to regularity of the (central!) cover of the sum of all irreducible representations on tensor form, which is faithful as the tensor product is the minimal one, cf. [**31**, Theorem 6.4.19].

We conclude by the double polar theorem that $\overline{F(r)} = \overline{\mathrm{co}}(\mathrm{sat}_{\mathfrak{C}}\, \mathcal{T}_0 \cup \{0\})$, and from [**16**, Appendice B 14] that the extreme points of $\overline{F(r)}$ are contained in $\overline{\mathrm{sat}_{\mathfrak{C}}\, \mathcal{T}_0} \cup \{0\}$. This yields

$$\mathrm{sat}_{\mathfrak{C}}\, \mathcal{T}_0 \subseteq P(r) \subseteq \mathrm{ext}\, \overline{F(r)} \subseteq \overline{\mathrm{sat}_{\mathfrak{C}}\, \mathcal{T}_0}$$

where now closures are relative to $P(\mathfrak{A} \otimes_* \mathfrak{B})$. Since clearly $\mathrm{sat}_{\mathfrak{C}}\, \mathcal{T}_0$ is connected, so is $P(r)$. □

Proposition 5.2.4. *Let $\mathfrak{A}$, $\mathfrak{B}$ be nonunital Raum algebras, one of which is nuclear. Then $\mathfrak{A} \otimes \mathfrak{B}$ is a Raum$^+$ algebra, and $\#_{\mathsf{E}}(\mathfrak{A} \otimes \mathfrak{B}) = 1$.*

Proof. The tensor product is a Raum algebra by Lemma 5.2.1. By Lemma 5.2.2(ii), we can choose compact nests p_n and q_n for $\mathfrak{A}$ and $\mathfrak{B}$, respectively, such that $r_n = p_n \otimes q_n$ is a compact nest for $\mathfrak{A} \otimes_* \mathfrak{B}$. By Proposition 5.2.3 above combined with [**19**, 2.2ii], $\mathbf{1} - r_n$ is connected. Then $\mathbf{1} - r_n$ is behaved, and $\mathfrak{A} \otimes_* \mathfrak{B}$ is a Raum$^+$ algebra by Proposition 2.1.8. This implies in turn that $\#_{\mathsf{E}}(\mathfrak{A} \otimes_* \mathfrak{B}) = 1$ by Proposition 3.2.1(iii). □

Corollary 5.2.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ both be nonunital Raum algebras, one of which is nuclear. The corona algebra $M(\mathfrak{A} \otimes \mathfrak{B})/(\mathfrak{A} \otimes \mathfrak{B})$ is connected.*

Proof. Combine the proposition above with Theorem 4.1.7. □

The situation for tensor products where one factor is unital is quite different, cf. the commutative case. We first need a lemma:

Lemma 5.2.6. *When p is an open projection of the nuclear Raum algebra $\mathfrak{A}$ written out in components $(c_i)_I$, and when $\mathfrak{B}$ is connected and unital, the set of components of* $\mathrm{her}(p \otimes \mathbf{1})$ *in $\mathfrak{A} \otimes \mathfrak{B}$ is exactly $(c_i \otimes \mathbf{1})_I$.*

Proof. By local connectivity ([**19**, 5.8]), the components (c_i) of p are open projections. Hence, the projections $c_i \otimes \mathbf{1}$ are also open, and they add up to $p \otimes \mathbf{1}$ by normality of $\pi_{\mathfrak{A}}^{**}$. This means that they are closed relative to $p \otimes \mathbf{1}$, and clearly they are also relatively central. We hence need only prove that $c_i \otimes \mathbf{1}$ is connected to establish the claims above.

We prove that

$$\operatorname{her}(q\otimes\mathbf{1}) = (\operatorname{her}(q)\odot\mathfrak{B})^{=} \tag{14}$$

for every open projection q. This will prove the claim as $\operatorname{her}(q)$ is nuclear by [**28**, p. 389], so that the closure of $\operatorname{her}(q)\odot\mathfrak{B}$ is the unique tensor product, which is connected by [**19**, 4.6].

The inclusion from right to left in (14) is obvious. In the other direction, choose a net $(a_\lambda)_\Lambda$ from $(\mathfrak{A}_+)_1$ increasing to q and fix a $d \in \operatorname{her}(q\otimes\mathbf{1})$. Given ε, choose $x_i \in \mathfrak{A}$ and $y_i \in \mathfrak{B}$ with $\|r\| < \varepsilon$, where

$$r = d - \sum_{i=1}^{N} x_i \otimes y_i.$$

We then have

$$d = \lim_{\Lambda}(a_\lambda\otimes\mathbf{1})d(a_\lambda\otimes\mathbf{1}) = \lim_{\Lambda}\sum_{i=1}^{N} a_\lambda x_i a_\lambda \otimes y_i - (q\otimes\mathbf{1})r(q\otimes\mathbf{1}),$$

and since then $\operatorname{dist}(d, \operatorname{her}(q)\odot\mathfrak{B})) < \varepsilon$, d is contained in $(\operatorname{her}(q)\odot\mathfrak{B})^{=}$ by completeness. □

Proposition 5.2.7. *Let $\mathfrak{A}$ be a nuclear, nonunital Raum$^+$ algebra and $\mathfrak{B}$ a unital Raum algebra. Then $\mathfrak{A}\otimes\mathfrak{B}$ is a Raum$^+$ algebra, and $\#_{\mathsf{E}}(\mathfrak{A}\otimes\mathfrak{B}) = \#_{\mathsf{E}}\mathfrak{A}$.*

Proof. The tensor product is a Raum algebra by Lemma 5.2.1. Let $\mathbf{r}$ be a compact nest for $\mathfrak{A}$ and note that as the constant sequence $(\mathbf{1})$ is a compact nest in the unital case, $r_n\otimes\mathbf{1}$ is a compact nest as in Lemma 5.2.2. We prove that for any open projection q in $\mathfrak{A}$,

$$q \text{ is bounded} \Longleftrightarrow q\otimes\mathbf{1} \text{ is bounded.}$$

Clearly, if $a \in \mathfrak{A}$ satifies $q \le a \le \mathbf{1}$, we have

$$q\otimes\mathbf{1} \le a\otimes\mathbf{1} \le \mathbf{1}\otimes\mathbf{1},$$

proving the forward implication by Proposition 1.1.3(ii). In the other direction, assume that $c \in (\mathfrak{A}\otimes\mathfrak{B})_1$ dominates $q\otimes\mathbf{1}$. We fix a pure state φ of $\mathfrak{B}$ and an increasing net $a_\lambda \nearrow q$. Recall from e.g. [**40**, 1.5.4(b)] that the slice map L_φ from $\mathfrak{A}\otimes\mathfrak{B}$ to $\mathfrak{A}$ is order preserving and norm decreasing. Hence $L_\varphi(c)$ dominates a_λ for every λ, and $q \le L_\varphi(c) \le \mathbf{1}$.

Employing Lemma 5.2.6, $\mathfrak{A}\otimes\mathfrak{B}$ is a Raum$^+$ algebra according to Proposition 2.1.8 by the fact that $\mathfrak{A}$ is a Raum$^+$ algebra. Also, since tensoring with the unit of $\mathfrak{B}$ preserves order, the two family trees of sequences determining ends will be naturally isomorphic. □

5.3. The real Heisenberg group C^*-algebra.

In this section, we will determine the end structure of $C^*(H_3)$, the group C^*-algebra over the *real Heisenberg group*

$$H_3 = \left\{ \left[\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right] \in M_3(\mathbb{R}) \middle| x, y, z \in \mathbb{R} \right\}.$$

The group is amenable, and so we need not specify whether the C^*-algebra in questions is full or reduced. Note that H_3 is homeomorphic, as a topological space, to $\mathbb{R}^3$, but is a non-commutative topological group.

We shall prove that $C^*(H_3)$ is a Raum$^+$ algebra with exactly one end. This implies, by Corollary 4.1.8, that $M(C^*(H_3))/C^*(H_3)$ is connected. Determining the end structure of $C^*(H_3)$ is very much facilitated by Dixmier's results ([**14**, Proposition 4], [**15**, Proposition 1]) below.

1°: The set of (equivalence classes of) irreducible representations of $C^*(H_3)$ can be parametrized as follows:

$$(\pi_\alpha, \mathfrak{H}) \ , \alpha \in \mathbb{R}\backslash\{0\}$$
$$(\rho_{\beta,\gamma}, \mathbb{C}) \ , (\beta, \gamma) \in \mathbb{R}^2$$

Furthermore, $\pi_\alpha(C^*(H_3)) = \mathbb{K}(\mathfrak{H})$ for every α.

2°: Since, by 1°, $C^*(H_3)$ is a CCR algebra, we may parametrize $\mathrm{Prim}(C^*(H_3))$ by the α, β, γ above. Letting ℓ_0 denote the subset of $\mathrm{Prim}(C^*(H_3))$ corresponding to $\mathbb{R}\backslash\{0\}$ and $\wp$ the set corresponding to $\mathbb{R}^2$, we may describe the topology on $\mathrm{Prim}(C^*(H_3))$ as follows: G is open if and only if

(i) $G \cap \ell_0$ is open.

(ii) $G \cap \wp$ is open.

(iii) If $G \cap \wp \neq \emptyset$, $(G \cap \ell_0) \cup \{0\}$ is open in $\mathbb{R}$.

Lemma 5.3.1. *$C^*(H_3)$ is a Raum algebra.*

Proof. As $C^*(H_3)$ is separable, hence σ-unital, we need only prove that the set $\mathrm{Prim}(C^*(H_3))$ is connected and locally connected. This is straightforward given 2° above. □

Proposition 5.3.2. *$C^*(H_3)$ is a Raum$^+$ algebra with $\#_{\mathsf{E}}(C^*(H_3)) = 1$.*

Proof. Let h be a strictly positive element of $C^*(H_3)$. We shall consider the compact nest given by $r_n = 1_{[1/n,1]}(h)$ and prove that $\mathbf{1} - r_n$ is connected. This will suffice by Propositions 2.1.8 and 3.2.1. First note that as $\pi_\alpha(h)$ is a compact operator which does not have 0 as an eigenvalue, its spectrum is a sequence converging to zero. Hence, $\pi_\alpha^{**}(\mathbf{1}-r_n)$ is never zero. Also note that since

$$\{(\beta,\gamma)\in\mathbb{R}^2|\rho_{\beta,\gamma}(h)\geq\tfrac{1}{n}\}$$

must be a compact set according to [32, 4.4.4], its complement is never empty and $\rho_{\beta_0,\gamma_0}^{**}(\mathbf{1}-r_n)$ is nonzero for some $(\beta_0,\gamma_0)\in\mathbb{R}^2$.

Consequently (cf. [32, 4.1.10]), Prim(her($\mathbf{1}-r_n$)) is homeomorphic to an open subset G of Prim($C^*(H_3)$) with the properties

$$\ell_0\subseteq G \qquad G\cap\wp\neq\emptyset.$$

Such a set must be connected by 2° above. □

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