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SEMICROSSED PRODUCTS OF THE DISK ALGEBRA: CONTRACTIVE REPRESENTATIONS AND MAXIMAL IDEALS

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Given the disk algebra $\mathcal{A}(\mathbb{D})$ and an automorphism α, there is associated a non-self-adjoint operator algebra $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ called the semicrossed product of $\mathcal{A}(\mathbb{D})$ with α. We consider those algebras where the automorphism arises via composition with parabolic, hyperbolic, and elliptic conformal maps φ of $\mathbb{D}$ onto itself. To characterize the contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$, a noncommutative dilation result is obtained. The result states that given a pair of contractions S, T on some Hilbert space $\mathcal{H}$ which satisfy $TS = S\varphi(T)$, there exist unitaries U, V on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ which dilate S and T respectively and satisfy $VU = U\varphi(V)$. It is then shown that there is a one-to-one correspondence between the contractive (and completely contractive) representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ on a Hilbert space $\mathcal{H}$ and pairs of contractions S and T on $\mathcal{H}$ satisfying $TS = S\varphi(T)$. The characters, maximal ideals, and strong radical of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ are then computed. In the last section, we compare the strong radical to the Jacobson radical.

I. Introduction.

A semicrossed product of the disk algebra is an operator algebra associated to the pair $(\mathcal{A}(\mathbb{D}), \alpha)$, where $\mathcal{A}(\mathbb{D})$ is the disk algebra and α an automorphism of $\mathcal{A}(\mathbb{D})$. Any such α has the form $\alpha(f) = f \circ \varphi$ $(f \in \mathcal{A}(\mathbb{D}))$ for a linear fractional transformation φ. It is well-known there is a one-to-one correspondence between contractions (i.e., bounded linear operators T on some Hilbert space with $\|T\| \leq 1$) and contractive representations of $\mathcal{A}(\mathbb{D})$. Here, analogously, there is a one-to-one correspondence between pairs S, T of contractions satisfying the relation $TS = S\varphi(T)$ and contractive representations of the semicrossed product, denoted $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$, associated to $(\mathcal{A}(\mathbb{D}), \alpha)$. This is meaningful since linear fractional transformations map contractions to contractions (cf. [**Sz-NF**]). The question of whether contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ can be dilated to representations of the C^*-crossed product $\mathbb{Z} \times_\alpha C(\mathbb{T})$ is equivalent to the following: given a pair of contractions S, T on some Hilbert space $\mathcal{H}$ satisfying $TS = S\varphi(T)$ do there exist unitaries U, V on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ which are dilations of S, T respectively, and satisfy $VU = U\varphi(V)$? This question, which is of interest in its own right, has an affirmative solution [Theorem II.4]. Furthermore, it marks the starting point in our study of semicrossed products of the disk algebra, by giving faithful representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$.

Section V deals with the characterization of the maximal ideal space. The character space, or equivalently the space of maximal ideals of codimension one, was easily obtained from the representation theory [Corollary III.11]. It turns out there are no maximal ideals of codimension greater than one – unless the automorphism (i.e., the linear fractional transformation) is elliptic of finite period, say K, in which case the maximal ideals have codimension either 1 or K^2. Finally, these results, together with [**HPW**] are used to compare the strong radical with the Jacobson radical: the two radicals coincide except when α is elliptic and nonperiodic (that is, an irrational rotation). (Theorem VI.1.)

II. Dilating Noncommuting Contractions.

It is well-known that each contraction T on a Hilbert space $\mathcal{H}$ can be dilated to a unitary U on a Hilbert space $\mathcal{K} \subseteq \mathcal{H}$ [**Sz-NF**]. That is, for $n \geq 1, T^n h = P_{\mathcal{H}} U^n h \ \forall \ h \in \mathcal{H}$. This result then yields the fact that the contractive representations of the disk algebra $\mathcal{A}(\mathbb{D})$ on $\mathcal{H}$ are in a one-to-one correspondence with contractions on $\mathcal{H}$ [**DP**], [**Sz-NF**]. Moreover, it shows that each contractive representation is completely contractive [**DP**]. Andô then generalized this result by showing that every commuting pair of contractions S and T on $\mathcal{H}$ have a unitary dilation on some $\mathcal{K} \supseteq \mathcal{H}$ [**A**]. That is, there exist unitaries U and V on $\mathcal{K}$ such that $\forall \ m \geq 1, \ n \geq 1, \ S^m T^n h = P_{\mathcal{H}} U^m V^n h$. Hence the contractive representations of the bidisk algebra $\mathcal{A}(\mathbb{D}^2)$ on $\mathcal{H}$ are in one-to-one correspondence with commuting pairs of contractions on $\mathcal{H}$. Furthermore, each representation of $\mathcal{A}(\mathbb{D}^2)$ is completely contractive [**DP**]. Recently, Sebestyén showed that every anti-commuting pair of contractions have such a dilation [**S**]. In this section we show that when φ is a conformal automorphism of $\mathbb{D}$ and S and T are contractions on $\mathcal{H}$ satisfying $TS = S\varphi(T)$ then a unitary dilation exists. This result is then used to characterize the contractive representations of a semicrossed product. Proofs in this section closely resemble those in [**S**]. Lemma II.1 is directly lifted from [**S**].

Lemma II.1. *Let $\mathcal{K}$ and $\mathcal{K}'$ be Hilbert spaces, $\mathcal{H} \subset \mathcal{K}$ and $\mathcal{H}' \subset \mathcal{K}'$ be subspaces and $X : \mathcal{H} \to \mathcal{K}'$ and $X' : \mathcal{H}' \to \mathcal{K}$ be given bounded linear transformations. Then, there exists an operator $Y : \mathcal{K} \to \mathcal{K}'$ extending X so that Y^* extends X' if and only if $\langle Xh, h' \rangle = \langle h, X'h' \rangle \ \forall \ h \in \mathcal{H}, h' \in \mathcal{H}'$.*

Moreover, $\|Y\| \le \max\{\|X\|, \|X'\|\}$.

Suppose now that S and T are contractions on $\mathcal{H}$ which satisfy $TS = S\varphi(T)$ for some linear fractional transformation φ of $\mathbb{D}$. Note that $\varphi(T)$ is a well-defined contraction by the functional calculus found in [**Sz-NF**]. Let U be the minimal isometric dilation of S acting on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. Then, U^* extends S^*, where $\mathcal{K} = \bigvee_{n=0}^{\infty} \mathcal{K}_n$ and $\mathcal{K}_n = \bigvee_{k=0}^{n} U^k(\mathcal{H})$.

Lemma II.2. *Let S and T be contractions on $\mathcal{H}$ such that $TS = S\varphi(T)$. If U is the minimal isometric dilation of S acting on $\mathcal{K}$, then there exists T_φ an operator on $\mathcal{K}$ such that T_φ^* extends T^*, $\|T_\varphi\| \le 1$, and $T_\varphi U = U\varphi(T_\varphi)$.*

Proof. The proof is similar to [**S**]. At the n-th step of induction, Lemma II.1 is applied to the maps $U\varphi(T_{n-1})U^*|_{\mathcal{H}_{n-1}} : \mathcal{H}_{n-1} \equiv U(\mathcal{K}_{n-1}) \to \mathcal{K}_n$ and $T_{n-1}^* : \mathcal{K}_{n-1} \to \mathcal{K}_n$ and $\|T_n\| \le 1$ since $\|\varphi(T_{n-1})\| \le 1$. Supposing that $\varphi = \mu\varphi_a$ where $|\mu| = 1$ and $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$, the conditions of Lemma II.1 are satisfied since $\langle x, [\mu\varphi_a(T_n)]^* y\rangle = \langle x, \overline{\mu}\varphi_{\overline{a}}(T_n^*)y\rangle = \langle x, \overline{\mu}\varphi_{\overline{a}}(T_{n-1}^*)y\rangle = \langle x, [\mu\varphi_a(T_{n-1})]^* y\rangle$ for $x \in \mathcal{H}_{n-1}$ and $y \in \mathcal{K}_{n-1}$ [**Sz-NF**, I.4]. $\square$

Lemma II.3. *Let S and T be contractions on $\mathcal{H}$ such that $TS = S\varphi(T)$. If U is the minimal unitary dilation of S acting on a Hilbert space $\mathcal{K}$, then there exists T_φ on $\mathcal{K}$ which is a dilation of T with $\|T_\varphi\| \le 1$ and $T_\varphi U = U\varphi(T_\varphi)$.*

Proof. Let U_+ be the minimal isometric dilation of S acting on $\mathcal{K}_+; \mathcal{H} \subseteq \mathcal{K}_+ \subseteq \mathcal{K}$. By Lemma II.2, there exists a dilation T_{φ_+} of T to $\mathcal{K}_+$ with $\|T_{\varphi_+}\| \le 1$ and $T_{\varphi+}U_+ = U_+\varphi(T_{\varphi+})$. By considering a sequence of polynomials $p_n \to \varphi^{-1}$, it follows from $T_{\varphi+}U_+ = U_+\varphi(T_{\varphi+})$ that $\varphi^{-1}(T_{\varphi+})U_+ = U_+T_{\varphi+}$. Taking adjoints yields $T_{\varphi+}^*U_+^* = U_+^*\varphi^{-1}(T_{\varphi+})^*$. Since U^* is the unique minimal *isometric* dilation of U_+^* it follows by Lemma II.2 and [**Sz-NF**, I.4] that there exists an operator T_φ^* on $\mathcal{K}$ such that T_φ extends $T_{\varphi+}$, $\|T_\varphi^*\| \le 1$, and $T_\varphi^*U^* = U^*\varphi^{-1}(T_\varphi)^*$. Reasoning as above, $T_\varphi U = U\varphi(T_\varphi)$. $\square$

Theorem II.4. *Let S and T be contractions on $\mathcal{H}$ such that $TS = S\varphi(T)$. Then there exists a pair of unitaries U and V such that $VU = U\varphi(V)$ and $S^mT^n = P_{\mathcal{H}}U^mV^n|_{\mathcal{H}}$ for every $m, n \in \mathbb{N}$.*

Proof. As in [**S**], let U_0 be the minimal unitary dilation of S and T_0 a contractive dilation of T with $T_0U_0 = U_0\varphi(T_0)$. Then let V be the minimal unitary dilation of T_0 and proceed to extend U_0 to a unitary such that $VU = U\varphi(V)$. The proof follows [**S**] after it is shown that U defined on $\mathcal{K} = \bigvee_{-\infty}^{\infty} V^n(\mathcal{K}_0)$ by $U(V^nk_0) = \varphi^{-1}(V)^nU_0k_0$ is isometric. However, since V is the minimal unitary dilation of T_0, it follows that $\varphi^{-1}(V)$ is the minimal unitary dilation of $\varphi^{-1}(T_0)$ [**Sz-NF**, I.4.3] and so $\langle U(V^mh_0), U(V^nk_0)\rangle =$

$\langle\varphi^{-1}(V)^m U_0 h_0, \varphi^{-1}(V)^n U_0 k_0\rangle = \langle\varphi^{-1}(T_0)^m U_0 h_0, \varphi^{-1}(T_0)^n U_0 k_0\rangle = \langle U_0 T_0^m h_0, U_0 T_0^n k_0\rangle = \langle V^m h_0, V^n k_0\rangle$. $\square$

III. The Semicrossed Product.

Example III.1. Let $\mathcal{D}$ be an operator algebra and α an automorphism of $\mathcal{D}$. Let $(\rho, \mathcal{K})$ be a contractive representation of $\mathcal{D}$ and let $\mathcal{H} = H^2(\mathcal{K})$, the space of square summable elements in $\bigoplus_0^\infty \mathcal{K}$. Define a contractive representation π of $\mathcal{D}$ on $\mathcal{H}$ by $\pi(f)(\xi_0, \xi_1, \xi_2, \dots) = (\rho(f)\xi_0, \rho(\alpha(f))\xi_1, \rho(\alpha^2(f))\xi_2, \dots)$. If U_+ is the unilateral shift on $\mathcal{H}$, then $U_+\pi(\alpha(f)) = \pi(f)U_+ \forall\ f \in \mathcal{D}$. Write $\pi = \widetilde{\rho}$ for the contractive representation constructed above.

Example III.2. The disk algebra $\mathcal{A}(\mathbb{D})$ can be considered as an operator algebra acting (via multiplication) on the Hilbert space $L^2(\mathbb{T})$. Let $\alpha : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D})$ be an (isometric) automorphism of $\mathcal{A}(\mathbb{D})$ so that $\alpha(f) = f \circ \varphi$ for some conformal mapping φ of $\mathbb{D}$. Let U be the forward unilateral shift on $\oplus_{i=0}^\infty L^2(\mathbb{T})$, $U(\xi_0, \xi_1, \xi_2, \dots) = (0, \xi_0, \xi_1, \dots)$. For $f \in \mathcal{A}(\mathbb{D})$, let D_f be the diagonal operator on $\oplus_{i=0}^\infty L^2(\mathbb{T})$ given by $D_f(\xi_0, \xi_1, \xi_2, \dots) = (f\xi_0, \alpha(f)\xi_1, \alpha^2(f)\xi_2,\ \alpha^3(f)\xi_3, \dots)$. Then $UD_{\alpha(f)} = D_f U\ \forall f \in \mathcal{A}(\mathbb{D})$. We let $\mathfrak{A}_\alpha$ denote the norm closed subalgebra of $B\left(\oplus_{i=0}^\infty L^2(\mathbb{T})\right)$ generated by U and $D_f, f \in \mathcal{A}(\mathbb{D})$.

Note that $UD_{\alpha(f)} = D_f U$ so that $\mathfrak{A}_\alpha$ is commutative if and only if α (and hence φ) is the identity. Further, we remark that every conformal map of $\mathbb{D}$ onto itself has the form $\varphi(z) = \mu\varphi_a(z)$ where $\mu \in \mathbb{C}, |\mu| = 1$, and $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}(a \in \mathbb{D})$. We classify these as hyperbolic, parabolic, or elliptic [**B**]. In the hyperbolic case, $\varphi_a : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ has two distinct fixed points which lie on $\partial\mathbb{D}$. If φ_a is parabolic, it has a unique fixed point lying in $\partial\mathbb{D}$. In the elliptic case, φ_a has one fixed point in $\mathbb{D}$ (and one outside $\mathbb{D}$).

By Lemma 10 of [**HPW**], the study of $\mathfrak{A}_\alpha$ can be reduced to three specific cases. If α is hyperbolic, parabolic, or elliptic, we can assume φ fixes $\{-1, 1\}, \{1\}$, or $\{0\}$ respectively.

Example III.3. For $f \in \mathcal{A}(\mathbb{D})$ and α as in Example III.2, the composition operator $C_{\varphi^{-1}}$ and the Toeplitz operator T_f on $H^2(\mathbb{D})$ satisfy $C_{\varphi^{-1}}T_{\alpha(f)} = T_f C_{\varphi^{-1}}$. We define $\mathcal{B}_\alpha$ to be the norm closed subalgebra of $B(H^2(\mathbb{D}))$ generated by $C_{\varphi^{-1}}$ and $T_f(f \in \mathcal{A}(\mathbb{D}))$ where $\alpha(f)(z) = f(\varphi(z))$.

Definition III.4. Let α be an automorphism of $\mathcal{D}, \rho$ a contractive representation of $\mathcal{D}$ on $\mathcal{H}$, and V a contraction (isometry) on $\mathcal{H}$. We say that (ρ, V) is a contractive (isometric) covariant representation of $(\mathcal{D}, \alpha)$ if $V\rho(\alpha(f)) = \rho(f)V\ \forall\ f \in \mathcal{D}$.

Remark III.5. Contractive (isometric) covariant representations of $(\mathcal{D}, \alpha)$ exist as exhibited by the examples above. However, unlike covariant representations of C^*-algebras, there are isometric covariant representations (ρ, V) satisfying $\rho(\alpha(f))V = V\rho(f)$ but not $\rho(\alpha(f))V^* = V^*\rho(f)$. (Examples III.1 and III.3 satisfy both conditions.)

Example III.6. Let α be an elliptic automorphism of $\mathcal{A}(\mathbb{D})$. Let ρ be a contractive representation of $\mathcal{A}(\mathbb{D})$ on $\ell^2(\mathbb{C})$ given by $\rho(z)(\xi_0, \xi_1, \xi_2, \dots) = (0, \mu\xi_0, \mu^2\xi_1, \mu^3\xi_2, \dots)$ and V the unilateral shift. Then $\rho(\alpha(f))V = V\rho(f)$ but $\rho(\alpha(f))V^* \neq V^*\rho(f)$.

Consider an automorphism α of $C(\mathbb{T})$ given by composition with the restricted Möbius transformation $\varphi|_{\mathbb{T}}$. If δ_n denotes the Kronecker delta on $\mathbb{Z}$, the algebra $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ consists of all formal sums $\sum_{-\infty}^{\infty} \delta_n \otimes f_n$ with $f_n \in C(\mathbb{T}), \sum_{-\infty}^{\infty} \|f_n\| < \infty$. An adjoint and multiplication can be defined (on simple tensors) by $(\delta_n \otimes f)^* = \delta_{-n} \otimes \alpha^{-n}(\overline{f})$ and $(\delta_n \otimes f)(\delta_m \otimes g) = \delta_{n+m} \otimes f\alpha^n(g)$.

A multiplication could also be defined by letting $\mathbb{Z}$ act on the left side by $(\delta_n \otimes f)(\delta_m \otimes g) = \delta_{n+m} \otimes \alpha^m(f)g$. If the Banach space $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ is provided with this alternate multiplication, and the adjoint is left unchanged, we obtain a new Banach algebra denoted $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$. The Banach algebras $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ and $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ are isomorphic [**P1**].

Define the Banach algebra $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ to be the subalgebra of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ consisting of elements of the form $F = \sum_{n \geq 0} \delta_n \otimes f_n$ with $f_n \in \mathcal{A}(\mathbb{D})$ and $\|F\|_1 = \sum_{n\geq 0} \|f_n\| < \infty$. Endow $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ with a multiplication $(\delta_n \otimes f)(\delta_m \otimes g) = \delta_{n+m} \otimes \alpha^m(f)g$ so that it is a Banach algebra without adjoint.

If (ρ, V) is a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$ on $\mathcal{H}$, then $\pi : \ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha) \to B(\mathcal{H})$ defined by $\pi\left(\sum_{n\geq 0} \delta_n \otimes f_n\right) = \sum_{n \geq 0} V^n \rho(f_n)$ is a contractive representation. Denote this representation by $\pi = V \times \rho$.

Proposition III.7. *The correspondence $(\rho, V) \leftrightarrow V \times \rho$ is a bijection between the collection of contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$ and contractive representations of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$.*

Proof. By the preceding remarks, we need only show that π, a contractive representation of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ on $\mathcal{H}$, gives rise to a contractive covariant pair (ρ, V) of $(\mathcal{A}(\mathbb{D}), \alpha)$ and that $\pi = V \times \rho$. Define a contraction V on $\mathcal{H}$ by $V = \pi(\delta_1 \otimes 1)$. Define a (contractive) representation ρ of $\mathcal{A}(\mathbb{D})$ by $\rho(f) = \pi(\delta_0 \otimes f)$. Then (ρ, V) is a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$ since $\rho(f)V = \pi(\delta_0 \otimes f \cdot \delta_1 \otimes 1) = \pi(\delta_1 \otimes \alpha(f)) = \pi(\delta_1 \otimes 1 \cdot \delta_0 \otimes \alpha(f)) = V\rho(\alpha(f))$. To complete the proof, note that $\pi = V \times \rho$ on a dense subset of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ and hence everywhere. $\square$

Recall that the crossed product $\mathbb{Z}\times_\alpha C(\mathbb{T})$ is the completion of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ under the norm $\|F\| = \sup\{\|\pi(F)\| : \pi$ is a contractive representation of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)\}$ for $F \in \ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ [**MM**].

Lemma III.8. *The Banach algebra $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ admits a faithful contractive representation.*

Proof. Let ρ be the (faithful) representation of $\mathcal{A}(\mathbb{D})$ on $L^2(\mathbb{T})$ given by multiplication. As in Example III.1, $(\widetilde{\rho}, U_+)$ is a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$. Thus, $U_+ \times \widetilde{\rho}$ is a contractive representation of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ on $\bigoplus_0^\infty L^2(\mathbb{T})$. Suppose $(U_+ \times \widetilde{\rho})(\sum_{n\geq 0} \delta_n \otimes f_n) = 0$. Then $\sum_{n\geq 0} U^n D_{f_n} = 0$ in $\mathfrak{A}_\alpha$ and hence $\left(\sum_{n\geq 0} U^n D_{f_n}\right)(\xi_0, 0, 0, \dots) = (0, 0, 0, \dots)$ $\forall\ \xi_0 \in L^2(\mathbb{T})$. It follows that $f_k \cdot \xi_0 = 0\ \forall \xi_0 \in L^2(\mathbb{T})$. By the faithfulness of ρ, $f_k \equiv 0\ \forall\ k \geq 0$. Thus, $U_+ \times \widetilde{\rho}$ is faithful. □

With that motivation, we define an operator enveloping norm on $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$.

Definition III.9. For $F \in \ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$, set $\|F\| = \sup\{\|(V \times \rho)(F)\| : (\rho, V)$ is a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)\}$. Define the semicrossed product of $\mathcal{A}(\mathbb{D})$ with α, denoted $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$, to be the completion of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ with respect to this norm.

Since contractive representations ρ of $\mathcal{A}(\mathbb{D})$ on $\mathcal{H}$ correspond bijectively with contractions on $\mathcal{H}$ [**Sz-NF**], it follows from Proposition III.7 that there is a bijection between the contractive representations of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ and the contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. Since $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ is dense in $\mathbb{Z}^+\times_\alpha\mathcal{A}(\mathbb{D})$, the contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$ give rise to all contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$.

Theorem III.10. *The contractive representations of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ are in a one-to-one corespondence with pairs of contractions S and T satisfying $TS = S\varphi(T)$.*

Corollary III.11. *The character space of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is $\mathcal{M} = \{(z_0, \xi_0) \in \mathbb{C}^2 : |z_0| \leq 1, |\xi_0| \leq 1$ and either $\xi_0 = 0$ or $\varphi(z_0) = z_0\}$.*

Proof. Any character γ (a contractive representation of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ on $\mathbb{C}$) is determined by a pair $(z_0, \xi_0) \in \mathbb{C}^2$ satisfying $|z_0| \leq 1, |\xi_0| \leq 1$, and $z_0\xi_0 = \xi_0\varphi(z_0)$. However, $z_0\xi_0 = \xi_0\varphi(z_0)$ if and only if $\xi_0 = 0$ or z_0 is a fixed point of φ. □

Proposition III.12. $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ *is isomorphic to a non-self-adjoint sub-*

algebra of $\mathbb{Z} \times_\alpha C(\mathbb{T})$.

Proof. Since $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ can be considered to be a subalgebra of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \gamma)$, there exists an embedding $\imath$ of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ into $\mathbb{Z} \times_\alpha C(\mathbb{T})$. If, for $F \in \ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha), \|F\| = \|\imath(F)\|$, then $\imath$ can be extended to an isometric isomorphism $\widehat{\imath} : \mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}) \to \mathbb{Z} \times_\alpha C(\mathbb{T})$ so that

$$\begin{tikzcd}
\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha) \arrow[r, "\imath"] \arrow[d, "\subseteq"'] & \mathbb{Z} \times_\alpha C(\mathbb{T}) \arrow[d, "id"] \\
\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}) \arrow[r, "\widehat{\imath}"'] & \mathbb{Z} \times_\alpha C(\mathbb{T})
\end{tikzcd}$$

commutes. Since every covariant representation (π, V) of $(C(\mathbb{T}), \alpha)$ restricts to a contractive covariant representation of $(\mathcal{A}(\mathbb{D}), \alpha)$, it follows that $\|F\| \geq \|\imath(F)\|$. We show $\|\imath(F)\| \geq \|F\|$ to complete the proof. If π is any contractive representation of $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ on $\mathcal{H}$, it is determined by two contractions $S = \pi(\delta_1 \otimes 1)$ and $T = \pi(\delta_0 \otimes z)$ which satisfy $TS = S\varphi(T)$. By Theorem II.4 , there exist unitaries U and V on $\mathcal{K} \supseteq \mathcal{H}$ such that $VU = U\varphi(V)$ and $S^m T^n = P_{\mathcal{H}} U^m V^n|_{\mathcal{H}}\ \forall\ m, n \in \mathbb{N}$. Then π can be extended to a contractive Banach $*$-representation $\widetilde{\pi}$ of $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)^{op}$ on $\mathcal{K}$ by defining $\widetilde{\pi}\left(\sum_{n=-\infty}^{\infty} \delta_n \otimes f_n\right) = \sum_{n=-\infty}^{\infty} U^n f_n(V)$. Hence $\|F\| \leq \|\imath(F)\|$. □

Recall that the semicrossed product norm (Definition III.9) was defined by taking a supremum over the collection of contractive covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. By Theorem II.4, we could equally well have defined this norm by taking a supremum over the (smaller) collection of isometric covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. In fact, by Proposition IV.1, we could also have defined this norm by taking a supremum over the pure isometric covariant representations of $(\mathcal{A}(\mathbb{D}), \alpha)$. Moreover, as Corollary III.14 shows, this norm makes every representation of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ completely contractive.

The proof of the following proposition, which is used only in the subsequent corollary, is left to the reader.

Proposition III.13. *The C^*-envelope of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$, $C^*(\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}))$, is isometrically isomorphic to $\mathbb{Z} \times_\alpha C(\mathbb{T})$.*

Corollary III.14. *Every contractive representation of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is completely contractive.*

Proof. By a fundamental theorem of Arveson [**Ar**], a contractive representation ρ of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ on $\mathcal{H}$ is completely contractive if and only if there exists a triple $(\mathcal{K}, \widetilde{\rho}, X)$ where $\widetilde{\rho}$ is a $*$-representation of $C^*(\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})) = \mathbb{Z} \times_\alpha$

$C(\mathbb{T})$ on $\mathcal{K}$ and an isometry $X : \mathcal{H} \to \mathcal{K}$ such that $\rho(F) = X^*\widetilde{\rho}(F)X$ $\forall F \in \mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$. However, each contractive representation ρ of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ on $\mathcal{H}$ is completely determined by a pair of contractions S and T satisfying $TS = S\varphi(T)$. Let U and V be the pair of unitaries on $\mathcal{K}$ generated by Theorem II.4 and take $X : \mathcal{H} \to \mathcal{K}$ to be the inclusion map. Define then $\widetilde{\rho}$ on a dense subset of $\mathbb{Z} \times_\alpha C(\mathbb{T})$ by $\widetilde{\rho}\left(\sum_{-\infty}^{\infty} \delta_n \otimes f_n\right) = \sum_{-\infty}^{\infty} U^n f_n(V)$. □

IV. A Concrete Representation.

Proposition IV.1. *$\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is completely isometrically isomorphic to $\mathfrak{A}_\alpha$.*

Proof. Let $\pi : \mathbb{Z} \times_\alpha C(\mathbb{T}) \to B\left(\bigoplus_{-\infty}^{\infty} L^2(\mathbb{T})\right)$ be defined on the dense subset $\ell^1(\mathbb{Z}, C(\mathbb{T}), \alpha)$ by $\sum_{-\infty}^{\infty} \delta_n \otimes f_n \mapsto \sum_{-\infty}^{\infty} U^n M_{f_n}$ where U is the bilateral shift and $M_f(\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots) = (\ldots, \alpha^{-1}(f)\xi_{-1}, f\xi_0, \alpha(f)\xi_1, \ldots)$. By [**Pe**, 7.7.5], π is an isometry since $\pi(f) \cdot \xi = f \cdot \xi$ ($\xi \in L^2(\mathbb{T})$) is faithful. In fact, π is completely isometric as it is a $*$-homomorphism. Hence $\widetilde{\pi} \equiv \pi|_{\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})}$ is a complete isometry. Note that $\bigoplus_0^{\infty} L^2(\mathbb{T})$ is invariant under $\widetilde{\pi}(\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}))$ so that $\widetilde{\widetilde{\pi}} : \mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}) \to B\left(\bigoplus_0^{\infty} L^2(\mathbb{T})\right)$ defined on a dense subset by $\widetilde{\widetilde{\pi}}\left(\sum_{n\geq 0} \delta_n \otimes f_n\right) = \widetilde{\pi}\left(\sum_{n\geq 0} \delta_n \otimes f_n\right)|_{\bigoplus_0^{\infty} L^2(\mathbb{T})}$ is clearly completely contractive onto its range $\mathfrak{A}_\alpha$. It is completely isometric if the induced map $\widehat{\pi}_k : (\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})) \otimes M_k(\mathbb{C}) \to \mathfrak{A}_\alpha \otimes M_k(\mathbb{C})$ is isometric for all $k \geq 1$. Let $F = (F_{ij}) \in (\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})) \otimes M_k(\mathbb{C})$. We show that

$$\|\widehat{\pi}(F)\| = \|(\widetilde{\pi}(F_{ij}))\| = \left\| (\widetilde{\pi}(F_{ij}))|_{\bigoplus\limits_1^k \left(\bigoplus\limits_0^{\infty} L^2(\mathbb{T})\right)} \right\|.$$

Define

$$\ell^2_N(L^2(\mathbb{T})) = \left\{ \xi = (\xi_k)_{k=-\infty}^{\infty} \in \bigoplus_{-\infty}^{\infty} L^2(\mathbb{T}) : \xi_k = 0 \text{ for } k < -N \right\}.$$

Then each $\ell^2_N(L^2(\mathbb{T}))$ is invariant under $\widetilde{\pi}(\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D}))$ and $\bigcup_{N\geq 0} \ell^2_N(L^2(\mathbb{T}))$ is dense in $\bigoplus_{-\infty}^{\infty} L^2(\mathbb{T})$. Hence, $\bigcup_{N\geq 0} \bigoplus_1^k \ell^2_N(L^2(\mathbb{T}))$ is dense in $\bigoplus_1^k (\bigoplus_{-\infty}^{\infty} L^2(\mathbb{T}))$. It follows that

$$\|(\widetilde{\pi}(F_{ij}))\| = \sup_{N\geq 0} \sup_{\substack{\xi \in \bigoplus\limits_1^k \ell^2_N(L^2(\mathbb{T})) \\ \|\xi\|=1}} \|(\widetilde{\pi}(F_{ij}))\xi\|$$

$$= \sup_{N\geq 0} \sup_{\substack{\xi \in \bigoplus_1^k \ell^2_N(L^2(\mathbb{T})) \\ \|\xi\|=1}} \|(\widetilde{\pi}(F_{ij}))\xi_U\|$$

where $\xi_U = (U^N\xi_1, U^N\xi_2, ..., U^N\xi_k)$, U is the bilateral shift, and $\xi_i \in \ell^2_N L^2(\mathbb{T})$. Thus,

$$\begin{aligned}\|(\widetilde{\pi}(F_{ij}))\| &= \sup_{\substack{\eta \in \bigoplus_1^k \ell^2_0(L^2(\mathbb{T})) \\ \|\eta\|=1}} \|(\widetilde{\pi}(F_{ij}))\eta\| \\ &= \left\| (\widetilde{\pi}(F_{ij}))|_{\bigoplus_1^k \left(\bigoplus_0^\infty L^2(\mathbb{T})\right)} \right\|.\end{aligned}$$

□

Let us now reconsider the algebra $\mathcal{B}_\alpha$ defined in Example III.3. In what follows we discuss the isomorphism question regarding $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ and $\mathcal{B}_\alpha$ in the case where α is elliptic and irrational. Recall that a change of variables [**HPW**, Lemma 10] reduces the analysis to the case where $\varphi(z) = \mu z$ where μ is not a root of unity.

Proposition IV.2. *If α is elliptic and nonperiodic, then $\mathfrak{A}_\alpha$ is completely isometrically isomorphic to $\mathcal{B}_\alpha$.*

Proof. Let $\mathcal{C}$ denote the irrational rotation algebra, i.e. the C^*-algebra generated by *any* two unitaries S and T satisfying $TS = \mu ST$ [**Rf2**], [**Br**]. In particular, $\mathcal{C}$ can be realized as $\mathbb{Z} \times_\mu C(\mathbb{T})$ or as the C^*-algebra of operators on $B(L^2(\mathbb{T}))$ generated by the composition operator $C_{\varphi^{-1}}$ and multiplication operators $M_f (f \in C(\mathbb{T}))$. Since $\ell^1(\mathbb{Z}^+, \mathcal{A}(\mathbb{D}), \alpha)$ can be isometrically embedded in $\ell^1(\mathbb{Z}, C(\mathbb{T}), \mu)$, it follows that $\rho : \mathfrak{A}_\alpha \to \mathcal{C}$ defined on a dense subset by $\rho\left(\sum_{n=0}^N U^n D_{f_n}\right) = \sum_{n=0}^N C^n_{\varphi^{-1}} M_{f_n}$ is an isometric representation on $L^2(\mathbb{T})$. Let $\rho_{H^2(\mathbb{D})} : \mathfrak{A}_\alpha \to B(H^2(\mathbb{D}))$ be given by $\rho_{H^2}(F) = \rho(F)|_{H^2(\mathbb{D})}$. Then ρ_{H^2} is a contractive representation of $\mathfrak{A}_\alpha$ onto $\mathcal{B}_\alpha$. To show that ρ_{H^2} is isometric, we show $\|\rho(F)\| = \|\rho(F)|_{H^2(\mathbb{D})}\| \ \forall F \in \mathfrak{A}_\alpha$. This follows as in Proposition IV.1.

By Propositions IV.1 and III.13, $C^*(\mathfrak{A}_\alpha) \cong \mathbb{Z} \times_\alpha C(\mathbb{T})$. Let π be a C^*-representation of $C^*(\mathfrak{A}_\alpha)$ on $L^2(\mathbb{T})$ defined by $\pi(U) = C_{\varphi^{-1}}$ and $\pi(D_f) = M_f$ where U is the bilateral shift on $\oplus_{-\infty}^\infty L^2(\mathbb{T})$ and $D_f(..., \xi_{-1}, \xi_0, \xi_1, ...) = (..., \alpha^{-1}(f)\xi_{-1}, f\xi_0, \alpha(f)\xi_1, ...)$. Let $X : H^2(\mathbb{D}) \to L^2(\mathbb{T})$ be inclusion. Then $\rho_{H^2}(F) = X^*\pi(F)X \ \forall\ F \in \mathfrak{A}_\alpha$, and ρ_{H^2} is completely contractive.

By the above comments and Proposition III.13, $C^*(\mathfrak{B}_\alpha) \cong \mathbb{Z} \times_\alpha C(\mathbb{T})$. Let π' be a C^*-representation of $C^*(\mathfrak{B}_\alpha)$ on $\oplus_{-\infty}^{\infty} L^2(\mathbb{T})$ defined by $\pi'(C_{\varphi^{-1}}) = U$ and $\pi'(M_f) = D_f$ where U and D_f are as above. Let $X : \oplus_0^\infty L^2(\mathbb{T}) \to \oplus_{-\infty}^{\infty} L^2(\mathbb{T})$ be inclusion. Then $\rho_{H^2}^{-1}(F) = X^* \pi'(F) X \ \forall\ F \in \mathfrak{B}_\alpha$ and $\rho_{H^2}^{-1}$ is also completely contractive. □

When α is elliptic and periodic we can construct a contractive, but not faithful, representation of $\mathfrak{A}_\alpha$ onto $\mathcal{B}_\alpha$.

Proposition IV.3. *$\pi : \mathfrak{A}_\alpha \to \mathcal{B}_\alpha$ determined by $U \mapsto C_{\varphi^{-1}}$ and $D_f \mapsto T_f$ is a contractive, surjective homomorphism.*

Proof. The result follows by Proposition IV.1 and the fact that a contractive representation of $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is completely determined by two contractions S and T satisfying $TS = S\varphi(T)$. □

Remark IV.4. Proposition IV.3 shows that algebraically $\mathfrak{A}_\alpha / \ker \pi \cong \mathcal{B}_\alpha$. However, $\ker \pi \neq (0)$. For example, if $f \in \mathcal{A}(\mathbb{D})$ then $0 = C_{\varphi^{-1}} T_f + C_{\varphi^{-1}}^{K+1} T_{-f} = \pi(UD_f + U^{K+1} D_{-f})$. This algebraic isomorphism explains the disparity in the character spaces of $\mathfrak{A}_\alpha$ and $\mathcal{B}_\alpha$ [**H**] when α is periodic.

V. The Maximal Ideal Space.

In this section we show that the maximal ideal space of $\mathfrak{A}_\alpha$ is the same as the character space except in the case where α is elliptic and periodic. We use an ergodic argument for the nonperiodic elliptic case and a spectral argument for the hyperbolic and parabolic cases. We then characterize the maximal ideal space in the case where α is periodic.

Recall from Corollary III.11 that the maps $\gamma_{z_0}^{(\xi_0)} : \mathfrak{A}_\alpha \to \mathbb{C}$ (where $|z_0| \leq 1$, $|\xi_0| \leq 1$, and $\varphi(z_0) = z_0$ if $\xi_0 \neq 0$) defined on a dense subset by $\gamma_{z_0}^{(\xi_0)}(\sum_{i=0}^n U^i D_{f_i}) = \sum_{i=0}^n f_i(z_0)\xi_0^i$ are the characters of $\mathfrak{A}_\alpha$. We remark that the multiplicative linear functionals of $\mathfrak{A}_\alpha$ could also be calculated by using a technique similar to that found in [**H**] and [**HH**].

Remark V.I. When α is elliptic and β is either parabolic or hyperbolic, it is known that $\mathfrak{A}_\alpha$ is not isomorphic to $\mathfrak{A}_\beta$. This follows since the radical in the elliptic case is $\{0\}$ whereas in the other cases the radical is the nontrivial set of quasinilpotents (Theorems 11 and 12 of [**HPW**]). Knowing the characters of $\mathfrak{A}_\alpha$ allows us to conclude that $\mathfrak{A}_\alpha \ncong \mathfrak{A}_\beta$ when α is parabolic and β is hyperbolic; for if such an isomorphism Γ existed, it would induce a homeomorphism τ of the character spaces defined by $\tau(\gamma)(F) = \gamma(\Gamma(F))$.

To each $F \in \mathfrak{A}_\alpha$ we may associate a unique Fourier series, $F \sim \sum_{n=0}^{\infty} U^n D_{f_n}$. We denote by $\pi_n(F)$ the nth Fourier coefficient of F. Some useful properties of these Fourier coefficients are listed in the following lemma from [**HPW**].

Lemma V.2. *For $n = 0, 1, 2, \ldots$, there is a linear mapping $\pi_n : \mathfrak{A}_\alpha \to \mathcal{A}(\mathbb{D})$ satisfying*

(i) $\|\pi_n(F)\| \le \|F\|,\ F \in \mathfrak{A}_\alpha$.

(ii) $\pi_0(FG) = \pi_0(F)\pi_0(G)$ *for* $F, G \in \mathfrak{A}_\alpha$.

(iii) $\pi_n\left(\sum_{k=0}^{N} U^k D_{f_k}\right) = \begin{Bmatrix} f_n & 0 \le n \le N \\ 0 & n > N \end{Bmatrix}$.

(iv) $\pi_n(F) = 0\ \forall\ n \ge 0 \Rightarrow F \equiv 0$.

Consider the case where α is either parabolic or hyperbolic. From Theorem 12 of [**HPW**], the Jacobson radical is Rad $(\mathfrak{A}_\alpha) = \{F \in \mathfrak{A}_\alpha : \pi_0(F) = 0$ and $\pi_n(F)(z_0) = 0$ for $\varphi(z_0) = z_0\}$. That is, the radical is precisely the set of quasinilpotent elements. We show by way of contradiction that every maximal ideal $\mathcal{M}$ in $\mathfrak{A}_\alpha$ contains the commutator ideal, denoted $\mathcal{C}$, and hence is of codimension one.

Lemma V.3. *If $\mathcal{B}$ is a (unital) Banach algebra and $\mathcal{M}$ is a maximal ideal in $\mathcal{B}$ not containing the commutator ideal $\mathcal{C}$, then $\mathcal{B} = \mathcal{M} + \mathcal{C}$.*

Proof. By the maximality of $\mathcal{M}$, we can find $b_0 = m_0 + c_0 \in (\mathcal{M} + \mathcal{C}) \cap \{b \in \mathcal{B} : \|b - 1\| < \frac{1}{2}\}$ where $m_0 \in \mathcal{M}$ and $c_0 \in \mathcal{C}$. Since b_0 is invertible, $1 = b_0^{-1} m_0 + b_0^{-1} c_0 \in \mathcal{M} + \mathcal{C}$ and hence $\mathcal{B} = \mathcal{M} + \mathcal{C}$. □

Proposition V.4. *Let α be parabolic or hyperbolic. The maximal ideals of $\mathfrak{A}_\alpha$ are precisely the kernels of its characters.*

Proof. We show that any maximal ideal $\mathcal{M}$ contains the commutator ideal. Suppose it does not. By the above lemma, $\exists\ F \in \mathcal{M}$ and $C \in \mathcal{C}$ such that $D_1 = F + C$. Since $\gamma_z^{(0)}(C) = 0$ it follows that $\gamma_z^{(0)}(F) = 1\ \forall\ z \in \overline{\mathbb{D}}$. Write $F = D_1 + G$ so that $\pi_0(G) \equiv 0$ and $\pi_n(F) \equiv \pi_n(G)$ for $n \ge 1$. Let z_0 be a fixed point of φ. Since $\gamma_{z_0}^{(\xi)}(C) = 0$ it follows that $\gamma_{z_0}^{(\xi)}(F) = \gamma_{z_0}^{(\xi)}(D_1) = 1\ \forall\ \xi \in \overline{\mathbb{D}}$. Hence, $\gamma_{z_0}^{(\xi)}(F - D_1) = \sum_{n=1}^{\infty} \pi_n(F)(z_0)\xi^n = 0\ \forall\ \xi \in \overline{\mathbb{D}}$. Thus, $\pi_n(F)(z_0) = 0$ for $n \ge 1$, and so $\pi_0(G) \equiv 0$ and $\pi_n(G)(z_0) = 0$ for $n \ge 1$. By the preceeding remarks, $G \in$ Rad $(\mathfrak{A}_\alpha)$. Hence $\mathrm{sp}(G) = \{0\}$ so that $\mathrm{sp}(F) = \mathrm{sp}(D_1 + G) = \{1\}$ by the spectral mapping theorem. But then $F \in \mathcal{M}$ is invertible, contradicting the maximality of $\mathcal{M}$. □

We now consider the case where α is elliptic. Recall that we are assuming w.l.o.g. that $\alpha(f) = f \circ \varphi$ where $\varphi(z) = \mu z$ for some $|\mu| = 1$. The structure of $\mathfrak{A}_\alpha$ is closely tied to whether μ is a root of unity or not.

By the special structure of α and the definition of $\mathfrak{A}_\alpha$, it is an easy calculation to show that $\tilde{\alpha} : \mathfrak{A}_\alpha \to \mathfrak{A}_\alpha$ defined on a dense subset by $\tilde{\alpha}\left(\sum_{i=0}^n U^i D_{f_i}\right) = \sum_{i=0}^n U^i D_{\alpha(f_i)}$ is an isometric automorphism. If μ is a Kth root of unity, define $\#{:}\mathfrak{A}_\alpha \to \mathfrak{A}_\alpha$ by $F \mapsto \frac{1}{K}\sum_{k=0}^{K-1} \tilde{\alpha}^k(F)$. Note that $\#$ is $\tilde{\alpha}$-invariant. Defining $\mathfrak{A}_0$ to be the closed subalgebra of $\mathfrak{A}_\alpha$ generated by $\{U, D_f : f$ is α-invariant$\}$, it is easy to verify that $\mathfrak{A}_0$ is maximal abelian and $\#$ is a linear projection onto $\mathfrak{A}_0$. As in [**P2**, V.8] we can define a map $\#$ with similar properties when α is nonperiodic by $\#\left(\sum_{i=0}^n U^i D_{f_i}\right) \equiv \sum_{i=0}^n U^i D_{\int\limits_{\mathbb{T}} f_i dm(z)} = \sum_{i=0}^n U^i D_{f_i(0)}$.

Proposition V.5. *If α is nonperiodic, then $\mathfrak{A}_0$ is the subalgebra of $\mathfrak{A}_\alpha$ generated by $\{U, D_1\}$. Furthermore, $\#$ is a linear projection onto the maximal abelian subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}_\alpha$.*

Proof. Let $F = \sum\limits_{i=0}^n U^i D_{f_i} \in \mathfrak{A}_0$. Then by α-invariance $f_i\left(\mu^k \cdot \frac{1}{2}\right) = f_i\left(\frac{1}{2}\right)$ $\forall\ k \in \mathbb{N}$, $0 \le i \le n$, so that analyticity, nonperiodicity of α, and the ergodic theorem gives $f_i \equiv f_i(0)$ on $\mathbb{D}$ and hence $\overline{\mathbb{D}}$. Since $\#$ is clearly a linear projection onto $\mathfrak{A}_0$, we need only show $\mathfrak{A}_0$ is a maximal abelian subalgebra of $\mathfrak{A}_\alpha$. By definition, $\mathfrak{A}_0$ is commutative. Suppose that $F \in \mathfrak{A}_\alpha$, $F \sim \sum_{n=0}^\infty U^n D_{f_n}$, commutes with $\mathfrak{A}_0$. Then $FU = UF$ and $\alpha(f_n) = f_n\ \forall\ n \ge 0$. Each f_n is then constant by the nonperiodicity of α. $\square$

The characters of $\mathfrak{A}_0$, which are easy to compute, will be used to characterize the maximal ideals in $\mathfrak{A}_\alpha$. Since $\mathfrak{A}_0$ is a commutative Banach algebra, its maximal ideal space corresponds in a one-to-one fashion with the kernels of its characters. If α is nonperiodic, then $\mathfrak{A}_0 \cong \mathcal{A}(\mathbb{D})$ (given by $U \mapsto z$) and its characters are determined by $U \mapsto \xi \in \overline{\mathbb{D}}$. Denote these by $\gamma^{(\xi)}_{\mathfrak{A}_0}$. If α is periodic with period K, there are more characters. In fact, if we denote by $\gamma^{(\xi)}_{\mathfrak{A}_0, z_0}$ the map determined by $U \mapsto \xi$ and $D_z \mapsto z_0$, the maximal ideal space of $\mathfrak{A}_0$ can be computed as the characters of $\mathfrak{A}_\alpha$ were using the technique found in [**H**] and [**HH**]. If $\psi = \min\{\theta : e^{i\theta} = \mu^k,\ 0 \le k \le K-1,\ 0 < \theta < 2\pi\}$, then $\mathcal{M}_{\mathfrak{A}_0} = \{\gamma^{(\xi)}_{\mathfrak{A}_0, re^{i\theta_0}} : 0 \le r \le 1,\ |\xi| \le 1,\ 0 \le \theta_0 < \psi\}$ is the set of characters on $\mathfrak{A}_0$.

As in [**P2**, V.9], for an ideal $\mathcal{I} \subseteq \mathfrak{A}_0$ define $\widetilde{\mathcal{I}} = \{F \in \mathfrak{A}_\alpha\colon \#(H\tilde{\alpha}^n(F)G) \in \mathcal{I}\ \forall H, G \in \mathfrak{A}_\alpha, n \ge 0\}$. Using $\#$ we can then construct a one-to-one correspondence between the maximal ideals in $\mathfrak{A}_0$ and the maximal $\tilde{\alpha}$-invariant ideals in $\mathfrak{A}_\alpha$.

Proposition V.6. (i) *If $\mathcal{M}_0 \subseteq \mathfrak{A}_0$ is a maximal ideal, then $\widetilde{\mathcal{M}}_0 \subseteq \mathfrak{A}_\alpha$ is a maximal $\tilde{\alpha}$-invariant ideal in $\mathfrak{A}_\alpha$.*

(ii) *If $\mathcal{R}$ is a maximal $\tilde{\alpha}$-invariant ideal, then $\#(\mathcal{R}) \subseteq \mathfrak{A}_0$ is a maximal* *ideal. Furthermore,* $\mathcal{R} = \widetilde{\#(\mathcal{R})}$.

Let $\mathcal{M}$ be a maximal ideal in $\mathfrak{A}_\alpha$ Let $\langle U\rangle$ denote the closed ideal in $\mathfrak{A}_\alpha$ generated by U. Then, $\mathcal{M}\cdot\langle U\rangle \subseteq \tilde{\alpha}(\mathcal{M})$. Furthermore, $\tilde{\alpha}(\mathcal{M})$ maximal implies $\tilde{\alpha}(\mathcal{M})$ is prime so that either $\langle U\rangle \subseteq \tilde{\alpha}(\mathcal{M})$ or $\mathcal{M}\subseteq \tilde{\alpha}(\mathcal{M})$.

Theorem V.7. *If α is nonperiodic, the maximal ideal space of $\mathfrak{A}_\alpha$ is precisely the space of characters.*

Proof. Let $\mathcal{M}$ be maximal in $\mathfrak{A}_\alpha$. If $\mathcal{M}\subseteq\tilde{\alpha}(\mathcal{M})$ then $\mathcal{M}=\tilde{\alpha}(\mathcal{M})$. Thus, $\mathcal{M}$ is $\tilde{\alpha}$-invariant and $\mathcal{M}=\widetilde{\#(\mathcal{M})}$ $=\{F\in\mathfrak{A}_\alpha : \#(H\tilde{\alpha}^n(F)G)\in\ker\gamma\ \forall n\geq 0$ and $H,G\in\mathfrak{A}_\alpha\}$ for some γ a character on $\mathfrak{A}_0$. To show that $\mathcal{M}=\ker\gamma_0^{(\xi)}$ for some $\xi\in\overline{\mathbb{D}}$, we need only show $\mathcal{M}\subseteq\ker\gamma_0^{(\xi)}$. But $F\sim\sum_{n=0}^\infty U^nD_{f_n}$, $F\in\mathcal{M}$ implies $0=\gamma(\#(F))=\gamma\left(\sum_{n=0}^\infty U^nD_{f_n(0)}\right)=\sum_{n=0}^\infty f_n(0)\xi^n=\gamma_0^{(\xi)}(F)$ for some $\xi\in\overline{\mathbb{D}}$.

If $\langle U\rangle\subseteq\tilde{\alpha}(\mathcal{M})$, then by applying $\widetilde{\alpha}^{-1}$ it follows that $\langle U\rangle\subseteq\mathcal{M}$. $\mathcal{M}/\langle U\rangle$ is then a maximal ideal in $\mathfrak{A}_\alpha/\langle U\rangle$. But $\mathfrak{A}_\alpha/\langle U\rangle\cong\mathcal{A}(\mathbb{D})$. Hence, $\mathcal{M}/\langle U\rangle$ corresponds to a maximal ideal in $\mathcal{A}(\mathbb{D})$; namely a kernel of point evaluation. So, $\mathcal{M}/\langle U\rangle=\ker\gamma_z^{(0)}$ for some $z\in\overline{\mathbb{D}}$. □

We now show that if α has period K, there are maximal ideals in $\mathfrak{A}_\alpha$ of codimension 1 and K^2. Define S_K to be the $K\times K$ shift matrix given by $S_{ij}=1$ if $i-j=1 \bmod K$ and 0 otherwise and $T(f,\mu)$ to be the $K\times K$ diagonal matrix given by $T(f,\mu)_{j,j}=f(\mu^{j-1}z_0)$. For $|w_0|\leq 1$ and $|z_0|\leq 1$, define $\rho_{z_0,w_0}:\mathfrak{A}_\alpha\to\mathcal{M}_K(\mathbb{C})$ on a dense subset by $\sum_{\ell=0}^{KL-1}U^\ell D_{f_\ell}\mapsto\sum_{\ell=0}^{KL-1}w_0^\ell S_K^\ell T(f_\ell,\mu)$.

Lemma V.8. *If $|z_0|\leq 1,\ |w_0|\leq 1$, then ρ_{z_0,w_0} is a contractive representation.*

Proof. This follows by Theorem III.10 since ρ_{z_0,w_0} is determined by two contractions S_K and $T(z,\mu)$ satisfying $T(z,\mu)S_K=S_K\varphi(T(z,\mu))$. □

By the simplicity of $\mathcal{M}_K(\mathbb{C}),\ker\rho_{z_0,w_0}$ is a maximal ideal in $\mathfrak{A}_\alpha$ if $z_0\neq 0$ and $w_0\neq 0$.

Lemma V.9. *If $z_0\neq 0$ and $w_0\neq 0$, then ρ_{z_0,w_0} is a contractive representation of $\mathfrak{A}_\alpha$ onto $\mathcal{M}_K(\mathbb{C})$.*

Proof. We need only show that if $z_0\neq 0$ and $w_0\neq 0$, then ρ_{z_0,w_0} is onto. For $0\leq i,j\leq K-1$, define

$$f_{i,j}(z)=\frac{1}{w_0^{K+i-j(\mathrm{mod}\ K)}}\cdot\frac{\prod_{\substack{l=0\\ l\neq K-i}}^{K-1}(\mu^l z-z_0)}{\prod_{l=1}^{K-1}(\mu^l z_0-z_0)}.$$

Then, for $0 \leq k \leq K-1$,

$$\begin{aligned} f_{i,j}(\mu^k z_0) &= \frac{1}{w_0^{K_i-u(\text{mod } K)}} \cdot \frac{\displaystyle\prod_{\substack{l=0 \\ l \neq K-i(\text{mod } K)}}^{K-1} (\mu^{k+l} z_0 - z_0)}{\displaystyle\prod_{l=1}^{K-1} (\mu^l z_0 - z_0)} \\ &= \frac{1}{w_0^{K+i-j(\text{mod } K)}} \begin{Bmatrix} 0 & \text{if} & k \neq i \\ 1 & \text{if} & k = i \end{Bmatrix} \\ &= \frac{1}{w_0^{K+i-j(\text{mod } K)}} \, \delta_{k,i}. \end{aligned}$$

Hence, ρ_{z_0,w_0} is onto $\mathcal{M}_K(\mathbb{C})$ as $E_{ij} = \rho_{z_0,w_0}(U^{K+i-j(\text{mod } K)} D_{f_{i,j}})$. $\square$

Theorem V.10. *If α has period K and $\mathcal{M}$ is a maximal ideal in $\mathfrak{A}_\alpha$, then $\mathcal{M} = \ker \rho_{z_0,w_0}$ for some $z_0, w_0 \in \overline{\mathbb{D}}$.*

Proof. As in the nonperiodic case, $\tilde{\alpha}(\mathcal{M})$ is maximal and hence prime with either $\mathcal{M} \subseteq \tilde{\alpha}(\mathcal{M})$ or $\langle U \rangle \subseteq \tilde{\alpha}(\mathcal{M})$. If $\langle U \rangle \subseteq \tilde{\alpha}(\mathcal{M})$, then since $\langle U \rangle$ is $\tilde{\alpha}$-invariant and α is periodic, $\langle U \rangle \subseteq \tilde{\alpha}^K(\mathcal{M}) = \mathcal{M}$. Thus, $\mathcal{M} = \ker \rho_{z_0,0}$ for some $z_0 \in \overline{\mathbb{D}}$. Suppose then $\mathcal{M} \subseteq \tilde{\alpha}(\mathcal{M})$ so that $\mathcal{M} = \tilde{\alpha}(\mathcal{M})$. By Proposition V.6, $\#(\mathcal{M}) = \ker \gamma$ for some character γ on $\mathfrak{A}_0$ and hence $\mathcal{M} = \widetilde{\#(\mathcal{M})} = \ker \gamma^{(\xi')}_{\mathfrak{A}_0, z'_0}$ for some $\xi' \in \overline{\mathbb{D}}$ and $z'_0 = re^{i\theta}$ where $0 \leq r \leq 1$, $0 \leq \theta < \psi$, and $\psi = \min\{\theta : e^{i\theta} = \mu^k,\ 0 \leq k \leq K-1,\ 0 < \theta < 2\pi\}$. Since $\ker \rho_{z_0,w_0}$ is maximal in $\mathfrak{A}_\alpha$, we need only show that $\ker \rho_{z_0,w_0} \subseteq \mathcal{M}$ for some z_0, w_0. But, $\ker \rho_{z'_0,\xi'}$ is $\tilde{\alpha}$-invariant so that $\rho_{z'_0,\xi'}(\tilde{\alpha}^n(F)) = 0$ $\forall n \geq 0$ and $F \in \ker \rho_{z'_0,\xi'}$. Hence $\rho_{z'_0,\xi'}(\#(H\tilde{\alpha}^n(F)G)) = 0$ $\forall$ $n \geq 0$ and $H, G \in \mathfrak{A}_\alpha$ yielding $\gamma^{(\xi')}_{z'_0}(\#(H\tilde{\alpha}^n(F)G)) = 0$ and $F \in \mathcal{M}$. $\square$

1. VI. The Strong Radical.

Having computed the maximal ideal space of $\mathfrak{A}_\alpha$, we can now compute its strong radical and compare it to its Jacobson radical. For the remainder α will be fixed.

Theorem VI.1. *Let $\mathfrak{A}_J$ and $\mathfrak{A}_S$ denote the Jacobson and strong radicals of $\mathfrak{A}_\alpha$ respectively.*

(i) *If α is parabolic or hyperbolic, $\mathfrak{A}_J = \mathfrak{A}_S$.*

(ii) *If α is elliptic and nonperiodic, $\mathfrak{A}_J \subsetneq \mathfrak{A}_S$.*

(iii) *If α is elliptic and periodic, $\mathfrak{A}_J = \mathfrak{A}_S = (0)$.*

Proof. From [**HPW**], the Jacobson radical is precisely the set of quasinilpotents. If α is parabolic or hyperbolic, then $\mathfrak{A}_J = \{F \in \mathfrak{A} : \pi_0(F) \equiv 0$ and $\pi_n(F)(z_0) = 0 \ \forall\ n \geq 1$ for z_0 fixed by $\alpha\}$. Since the maximal ideals are precisely the kernels of the characters in these cases, (i) follows as

$$\begin{aligned}\mathfrak{A}_S &= \{F \in \mathfrak{A}_\alpha : F \in \ker \gamma_z^{(0)} \ \forall\ z \in \overline{\mathbb{D}} \text{ and } F \in \ker \gamma_{z_0}^{(\xi)} \ \forall \xi \in \overline{\mathbb{D}} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ z_0 \ \text{ fixed by } \ \alpha\} \\ &= \left\{ F : \pi_0(F) \equiv 0 \text{ and } \sum_{n\geq 1} \pi_n(F)(z_0)\xi^n = 0 \ \forall\ \xi \in \overline{\mathbb{D}} \right\} \\ &= \{F : \pi_0(F) \equiv 0 \text{ and } \pi_n(F)(z_0) = 0 \ \forall\ n \geq 1 \text{ for } z_0 \text{ fixed by } \alpha\} \\ &= \mathfrak{A}_J.\end{aligned}$$

If α is elliptic, $\mathfrak{A}_J = (0)$. When α is nonperiodic, $\mathfrak{A}_S \supsetneq (0)$ as $UD_z \in \mathfrak{A}_S$ for example. In fact, $\mathfrak{A}_S = \{F : \pi_0(F) \equiv 0 \text{ and } \pi_n(F)(0) = 0 \ \forall n \geq 1\}$. If α is periodic of period K, we show $\mathfrak{A}_S = (0)$ to complete the proof.

First, note that the Fourier series of $F \in \mathfrak{A}_\alpha$ is Cesàro summable [**P2**]. Hence,

$$\lim_{N\to\infty} \left\| \sum_{l=0}^{KN-1} \frac{1}{KN} \left(\sum_{m=0}^{l} U^m D_{\pi_m(F)} \right) - F \right\| = 0.$$

Let $F \in \mathfrak{A}_S$ and $\varepsilon > 0$ be given. We show that $\pi_l(F) \equiv 0 \ \forall\ l \geq 0$ so that $F = 0$. Choose M such that if $N \geq M$ we have

$$\left\| \sum_{l=0}^{KN-1} \frac{1}{KN} \left(\sum_{m=0}^{l} U^m D_{\pi_m(F)} \right) - F \right\| = \left\| \sum_{l=0}^{KN-1} U^l D_{(1-\frac{l}{KN})\pi_l(F)} - F \right\| < \varepsilon.$$

Then,

$$\left\| \rho_{z_0,w} \left(\sum_{l=0}^{KN-1} U^l D_{(1-\frac{l}{KN})\pi_l(F)} - F \right) \right\| < \varepsilon \ \forall\ z_0, w \in \overline{\mathbb{D}} \ \text{ by Lemma V.8.}$$

Since $F \in \mathfrak{A}_S$,

$$\left\| \rho_{z_0,w} \left(\sum_{l=0}^{KN-1} U^l D_{(1-\frac{l}{KN})\pi_l(F)} \right) \right\| < \varepsilon.$$

In particular, for $0 \leq k \leq K-1$ we have

$$\left| \sum_{l=0}^{N-1} \left(1 - \frac{Kl+k}{KN} \right) \pi_{Kl+k}(F)(z_0) w^{Kl+k} \right| < \varepsilon \ \forall\ z_0 \in \overline{\mathbb{D}}, w \in \mathbb{T}.$$

Fix $l_0 \geq 0$. Note that

$$\int_{\mathbb{T}} \left| \sum_{l=0}^{N-1} \left(1 - \frac{Kl+k}{KN} \right) \pi_{Kl+k}(F)(z_0) w^{Kl+k} \right| dm(w) < \varepsilon \ \forall\ z_0 \in \overline{\mathbb{D}}.$$

It follows, since $\int_{\mathbb{T}} z^l dm(z) = 0$ unless $l = -1$, that

$$\begin{aligned}
&\left|1 - \frac{Kl_0 + k}{KN}\right| |\pi_{Kl_0+k}(F)(z_0)| \left|\int_{\mathbb{T}} w^{-1} dm(w)\right| \\
&= \left|\int_{\mathbb{T}} \left(1 - \frac{Kl_0 + k}{KN}\right) \pi_{Kl_0+k}(F)(z_0) w^{-1} dm(w)\right| \\
&= \left|\int_{\mathbb{T}} \sum_{l=0}^{N-1} \left(1 - \frac{(Kl + k)}{KN}\right) \pi_{Kl+k}(F)(z_0) w^{K(l-l_0)-1} dm(w)\right| \\
&\leq \int_{\mathbb{T}} \left|\sum_{l=0}^{N-1} \left(1 - \frac{(Kl + k)}{KN}\right) \pi_{Kl+k}(F)(z_0) w^{Kl+k}\right| \left|w^{-Kl_0-k-1}\right| dm(w) \\
&= \int_{\mathbb{T}} \left|\sum_{l=0}^{N-1} \left(1 - \frac{(Kl + k)}{KN}\right) \pi_{Kl+k}(F)(z_0) w^{Kl+k}\right| dm(w) \\
&< \varepsilon \ \forall \ z_0 \in \overline{\mathbb{D}}.
\end{aligned}$$

Choosing $N \geq M$ large enough so that $\frac{Kl_0+k}{KN} < \frac{1}{2}$ it follows that $|\pi_{Kl_0+k}(F)(z_0)|$ is arbitrarily small $\forall \ z_0 \in \overline{\mathbb{D}}$ so that $\pi_{Kl_0+k}(F) \equiv 0$ for $0 \leq k \leq K - 1$ and hence $F = 0$. $\square$

Note added in proof (June 1997). Since this paper was submitted we have learned that a proof of Corollary III.14 has been found independently by S.C. Power [**Po2**].

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