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SUPERCUSPIDAL REPRESENTATIONS OF $GL(n)$ DISTINGUISHED BY A UNITARY SUBGROUP

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When E/F is a quadratic extension of p-adic fields, with $p \neq 2$, and H' is a unitary similitude group in $GL(n,E)$, it is shown that for every irreducible supercuspidal representation π of $GL(n,E)$ of lowest level the space of H'-invariant linear forms has dimension at most one. The analogous fact for the corresponding unitary group H also holds, so long as n is odd or E/F is ramified. When n is even and E/F is unramified, the space of H-invariant linear forms on the space of π may have dimension two.

1. Introduction.

Suppose that E/F is a quadratic extension of fields. Then one may consider matrices in $G = GL(n,E)$ which are hermitian with respect to E/F and for each such matrix there is an associated unitary group H and a group H' of unitary similitudes. We say that a representation π of G on a complex vector space V is *H-distinguished* if there exists a nonzero linear form λ on V such that $\lambda(\pi(h)v) = \lambda(v)$, for all $h \in H$ and $v \in V$. The notion of a representation being *H'-distinguished* is defined similarly. Clearly, every H'-distinguished representation is also H-distinguished.

In various examples of E/F which have been studied, there is a direct relation between the irreducible representations of $G' = GL(n,F)$ and the H'-distinguished irreducible representations of G and, moreover, a similar relation holds when the roles of G' and H' are interchanged. If E and F are global fields there is a related notion of what it means for an automorphic cuspidal representation of $GL(n,E_{\mathbb{A}})$ to be distinguished with respect to a unitary group (or a group of unitary similitudes) and it is conjectured in [**10**] (in a precise way) that these representations are essentially just the representations obtained from $GL(n,F_{\mathbb{A}})$ by quadratic base change. This conjecture is proven by means of a trace formula when $n = 2$ [**10**] and, under certain local restrictions, when $n = 3$ [**9**].

In this paper, we consider certain H-distinguished and H'-distinguished supercuspidal representations of G when E and F are finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers, when p is an odd prime and $n > 1$. It is observed in [9] that when $n > 2$ the pair (G, H) is not a Gelfand pair. In other words, there exists an irreducible admissible representation π of G such that the space $\mathrm{Hom}_H(\pi, 1)$ of H-invariant linear forms has dimension greater than one. In fact, we will show that even when $n = 2$ one may construct an irreducible supercuspidal representation of G which has a 2-dimensional space of H-invariant linear forms.

The situation simplifies when one replaces H by the larger group H' of unitary similitudes. Suppose $\iota \in E^\times$ has trace zero with respect to E/F. Let H and H' be the groups associated to the hermitian matrix $\iota\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then it is shown in [3] that $G' = GL(2, F)$ is a Gelfand subgroup of $G = GL(2, E)$. Since H' contains G', we conclude that (G, H') is also a Gelfand pair. For $n > 2$, even though H' may not be a Gelfand subgroup of G, Jacquet has conjectured that the dimension of the space of H'-invariant linear forms never exceeds one when π is irreducible, supercuspidal. We show that this is indeed true for the most basic family of supercuspidal representations. This represents a step towards proving the conjecture of [10] for automorphic representations.

To be more precise requires a few notations. Let $Z = E^\times$ be the center of G. Let K denote the maximal compact subgroup $GL(n, \mathfrak{O})$, where $\mathfrak{O}$ is the ring of integers of E. Denote by $\mathfrak{P}$ the maximal ideal of $\mathfrak{O}$ and take K_1 to be the principal congruence subgroup $1 + M(n, \mathfrak{P})$ of K. Our main result is:

Theorem 1. *Suppose r is an irreducible representation of ZK/K_1 such that the associated representation of $GL(n, \mathfrak{O}/\mathfrak{P})$ is cuspidal. Then the dimension of the space of H-invariant linear forms on the irreducible, supercuspidal representation $\mathrm{ind}_{ZK}^G(r)$ is at most one unless E/F is unramified and n is even. In the latter case, the dimension is either 0 or 2.*

In fact, the exact dimension of the space of H-invariant linear forms may be recovered from the discussion in §6. In particular, using §6 one can construct examples for which the space of invariant linear forms actually has dimension two. The proof of Theorem 1 involves a reduction to the analogous problem over the residue fields. Then, when E/F is unramified, we appeal to the fact, proved by Gow [2], that $(GL(n, q^2), U(n, q^2))$ is a Gelfand pair. The case in which E/F is ramified is similar except that the reduction of H gives rise to an orthogonal group, instead of a unitary group, over the finite fields. Though the orthogonal groups are not Gelfand subgroups of $GL(n, q)$, what is more relevant is that the Gelfand property holds for cuspidal representations. (See [4].) When n is even and E/F is ramified, a symplectic group $Sp(n, q)$ also appears and it must be shown that this does not yield extra invariant linear forms. For this, we use Klyachko's result [11] which says that an irreducible cuspidal representation of $GL(n, q)$

cannot have a symplectic model (since it has a Whittaker model).

As a corollary of the proof of Theorem 1, we have:

Theorem 2. *Suppose r is an irreducible representation of ZK/K_1 such that the associated representation of $GL(n, \mathfrak{O}/\mathfrak{P})$ is cuspidal. Then the dimension of the space of H'-invariant linear forms on the irreducible, supercuspidal representation $\mathrm{ind}_{ZK}^{G}(r)$ is at most one.*

This paper is a sequel to [4] where similar results are proven for orthogonal groups in $GL(n, E)$, as well as groups of orthogonal similitudes. The authors would very much like to thank H. Jacquet for suggesting this research and for his generous advice.

2. Hermitian Matrices.

In this section, we assemble some facts about matrices in the group G which are hermitian with respect to the quadratic extension E/F.

If x is an element of E or a matrix with entries in E, let $\bar{x}$ denote the element obtained by applying the nontrivial Galois automorphism of E/F to the entries of x and if x is a square matrix with entries in E, let $x^* = {}^t\bar{x}$. Then $x \in G$ is hermitian when $x^* = x$ and we let $\mathcal{X}$ denote the space of all such matrices. The group G acts on $\mathcal{X}$ by $g \cdot x = gxg^*$. According to [8], this action has two orbits and, moreover, the orbit $\mathcal{X}_x$ of a given hermitian matrix $x \in \mathcal{X}$ is determined by the class of $\det x$ in $F^\times/NE^\times$. (Here, N denotes the norm map from $E^\times$ to $F^\times$.)

In terms of geometric algebra, determining the G-orbits in $\mathcal{X}$ is equivalent to determining the classes of nondegenerate hermitian forms on E^n up to change of basis. One may also consider the related problem of describing the classes of hermitian forms up to integral changes of basis. In other words, this is the same as describing the K-orbits in $\mathcal{X}$. We now recall Jacobowitz's solution [7] following the presentation in [5].

Fix a prime element ϖ of E such that $\varpi \in F$, if E/F is unramified, and $\varpi^2 \in F$ is E/F is ramified. In the ramified case, we also fix a unit δ in the ring of integers $\mathfrak{O}_F$ of F whose image in the residue field $\mathfrak{O}_F/\mathfrak{P}_F$ of F is not a square. In general, q will denote the order of the residue field of F. We consider sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$ of certain triples $\alpha_i = (a_i, n_i, \epsilon_i)$, such that, in general, $a_1 > \cdots > a_m$ is a decreasing sequence of integers, $n_1 + \cdots + n_m = n$ is a partition of n by positive integers, and $\epsilon_1, \ldots, \epsilon_m$ are elements of F. For each index i, we require that the triple α_i satisfy the following conditions. If E/F is unramified, $\epsilon_i = 1$. If E/F is ramified and a_i is odd, then $\epsilon_i = 1$ and n_i is even. In the remaining case in which E/F is ramified and a_i is even, we allow ϵ_i to be either 1 or δ. Having fixed E/F, we let $\mathcal{A}$ denote the set of all sequences α satisfying these requirements. For each $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}$, we let $\ell(\alpha) = m$.

For each $\alpha \in \mathcal{A}$, we now define a hermitian matrix ϖ^α such that $\{\varpi^\alpha : \alpha \in \mathcal{A}\}$ is a set of representatives for the K-orbits in $\mathcal{X}$. The matrix ϖ^α will be a direct sum $\varpi^{\alpha_1} \oplus \cdots \oplus \varpi^{\alpha_m}$ of matrices $\varpi^{\alpha_i} \in GL(n_i, E)$. If E/F is unramified, then $\varpi^{\alpha_i} = \varpi^{a_i}$, viewed as a scalar matrix in $GL(n_i, E)$. If E/F is ramified and a_i is odd then

$$\varpi^{\alpha_i} = \varpi^{a_i} \left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \oplus \cdots \oplus \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \right),$$

where the number of summands in $n_i/2$. If E/F is ramified and a_i is even, then ϖ^{α_i} is the diagonal matrix with diagonal $\varpi^{a_i}(1, \ldots, 1, \epsilon_i)$. In all cases, the matrix $\varpi^{-a_i}\varpi^{\alpha_i}$ lies in K and $\det \varpi^{\alpha_i} = \epsilon_i \varpi^{a_i n_i}$.

In what follows, we will associate $H-$invariant linear forms to certain classes of the hermitian matrices ϖ^α. In fact, it will turn out that only those ϖ^α with $\ell(\alpha) = 1$ will give rise to invariant linear forms.

3. Unitary Groups.

In general, if $x \in \mathcal{X}$ then $U(x)$ denotes the unitary group consisting of those $g \in G$ such that $gxg^* = x$. We let $GU(x)$ denote the group of all $g \in G$ such that $gxg^* = z_g x$ for some $z_g \in Z$. When $x = \varpi^\alpha$, we simplify this notation as follows: $H_\alpha = U(\varpi^\alpha)$ and $H'_\alpha = GU(\varpi^\alpha)$. In this section, we consider the image of the groups H_α in $GL(n, \mathfrak{O}/\mathfrak{P})$. To be more precise, let $k \mapsto \tilde{k}$ be the map from K to $\widetilde{G} = GL(n, \mathfrak{O}/\mathfrak{P})$ which reduces the entries of k modulo $\mathfrak{P}$. This is a homomorphism with kernel K_1 and it allows us to identify K/K_1 with the group $\widetilde{G}$. Every subgroup H of G projects to a subgroup $\widetilde{H} \approx (H \cap K)/(H \cap K_1)$ of $\widetilde{G}$. More precisely, $\widetilde{H}$ is the group obtained from $H \cap K$ by reducing entries modulo $\mathfrak{P}$.

For our purposes, it turns out that the unitary groups will only be relevant modulo conjugacy by elements of K. Since $kU(x)k^{-1} = U(kxk^*)$, it will suffice to consider the groups H_α, with $\alpha \in \mathcal{A}$. We remark that $U(zx) = U(x)$ when z is a scalar in $F^\times$.

The projections of the groups H_α with $\ell(\alpha) = 1$ are described by:

Proposition 1.

(a) *Suppose E/F is unramified and η is the identity matrix in G. Then $\widetilde{U(\eta)}$ is the group $U(\tilde{\eta}) \approx U(n, q^2)$ of matrices $\tilde{g} \in \widetilde{G}$ such that $\tilde{g}\tilde{g}^* = 1$.*

(b) *Suppose E/F is ramified and η is a diagonal matrix* $\operatorname{diag}(1, \ldots, 1, \epsilon) \in G$ *where $\epsilon \in \{1, \delta\}$. Then $\widetilde{U(\eta)}$ is the orthogonal group $O(\tilde{\eta})$ in $\widetilde{G}$ associated to the symmetric matrix $\tilde{\eta}$. The group $O(\tilde{\eta})$ is isomorphic to the standard orthogonal group $O(n, q)$ unless n is even and $\epsilon = \delta$.*

In the latter case, $O(\tilde{\eta})$ is isomorphic to the orthogonal group $O'(n,q)$ associated to a symmetric matrix in $GL(n,\mathbb{F}_q)$ of nonsquare discriminant.

(c) *Suppose E/F is ramified, n is even and $\eta \in G$ is the matrix which is a direct sum of $n/2$ copies of the block $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $\widetilde{U(\varpi\eta)}$ is the symplectic group $Sp(\tilde{\eta}) \approx Sp(n,q)$ in $\widetilde{G}$ associated to the skew symmetric matrix $\tilde{\eta}$.*

When $\ell(\alpha) > 1$, the essential fact about the groups $\widetilde{H}_\alpha$ is that each such group contains the unipotent radical of a (proper) parabolic subgroup of $\widetilde{G}$. Suppose $\alpha \in \mathcal{A}$ is fixed with $\ell(\alpha) > 1$ and let $n_1 + \cdots + n_m$ be the associated partition of n. Let N_α be the unipotent radical of the standard maximal opposite parabolic subgroup of G associated to the partition $(n - n_m) + n_m$ of n. In other words, N_α consists of block matrices $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \in G$ where x has dimensions $n_m \times (n - n_m)$.

Proposition 2. *For all $\alpha \in \mathcal{A}$ with $\ell(\alpha) > 1$, the group $\widetilde{N}_\alpha$ is a subgroup of $\widetilde{H}_\alpha$.*

The proofs of Propositions 1 and 2 rely on three closely related lemmas. Let L be the group of lower triangular matrices in K_1 with diagonal entries in $1 + \mathfrak{P}_F$, where $\mathfrak{P}_F$ is the maximal ideal of the ring of integers $\mathfrak{O}_F$ of F. Whenever $x \in 1 + \mathfrak{P} = (1 + \mathfrak{P})^2$, we let $\sqrt{x}$ denote the unique square root of x which lies in $1 + \mathfrak{P}$.

Lemma 1. *If $\eta \in GL(n, \mathfrak{O}_F)$ is a diagonal matrix, then $x \mapsto x\eta x^*$ defines a bijection of L onto the set of hermitian matrices in ηK_1.*

Proof. Given a diagonal matrix $\eta \in GL(n, \mathfrak{O}_F)$ and a hermitian matrix $y \in \eta K_1$, we determine the solutions $x \in L$ to the equation $x\eta x^* = y$. Equivalently, we must solve the equations:

$$x_{ij} = x_{jj}^{-1}\eta_{jj}^{-1}\left(y_{ij} - \sum_{s<j} x_{is}\eta_{ss}\bar{x}_{js}\right)$$

for the x's when $i \geq j$. When $i = j = 1$, we have $x_{11}^2 = \eta_{11}^{-1}y_{11} \in 1 + \mathfrak{P}$. Therefore, we must take $x_{11} = \sqrt{\eta_{11}^{-1}y_{11}}$. The other x_{ij}'s, with $i \geq j$, are defined by induction on $i + j$. When $i = j > 1$, we must let

$$x_{ii} = \sqrt{\eta_{ii}^{-1}y_{ii} - \eta_{ii}^{-1}\sum_{s<i} x_{is}\eta_{ss}\bar{x}_{is}}$$

and when $i > j$,

$$x_{ij} = x_{jj}^{-1}\eta_{jj}^{-1}\left(y_{ij} - \sum_{s<j} x_{is}\eta_{ss}\bar{x}_{js}\right).$$

In all cases, x_{ij} is defined in terms of other entries of x with smaller values of $i+j$. Since each of these equations has a unique solution, this must define a unique solution $x \in L$ to $x\eta x^* = y$. $\square$

Lemma 2. *Assume n is even and E/F is ramified and let $\eta = w \oplus \cdots \oplus w$ be the block diagonal matrix in G which is the direct sum of $n/2$ copies of the block $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then for every skew hermitian matrix $y \in \eta K_1$ there exists $x \in K_1$ such that $x\eta x^* = y$.*

Proof. Let us first consider the case in which $n = 2$. Then $y = \left(\begin{smallmatrix} a & b \\ -\bar{b} & d \end{smallmatrix}\right)$, where $a, b, d \in \mathfrak{O}$, $\bar{a} = -a$, $\bar{d} = -d$ and $b \in 1 + \mathfrak{P}$. Define $b_1 \in 1 + \mathfrak{P}_F$ and $b_2 \in \mathfrak{O}_F$ by $b = b_1 + \varpi b_2$. Then

$$x = \begin{pmatrix} 1 & -\frac{a}{2} \\ \frac{-b_1+\sqrt{b_1^2+ad}}{a} & \frac{b_1+\sqrt{b_1^2+ad}}{2} - \varpi b_2 \end{pmatrix}$$

gives a matrix x with the desired properties.

Now suppose $n > 2$ is even. We regard elements of G as $\frac{n}{2} \times \frac{n}{2}$ block matrices with blocks of size 2×2. We will actually construct a matrix $x \in K_1$ which is lower diagonal as a block matrix and which also satisfies $x\eta x^* = y$. It suffices to solve the equations

$$\sum_{s \le j} x_{is} w (x_{js})^* = y_{ij},$$

where the double indices parametrize 2×2 blocks and where $i \ge j$. When $i = j = 1$, this reduces to $x_{11}w(x_{11})^* = y_{11}$. According to the $n = 2$ case, we may choose $x_{11} \in 1 + M(2, \mathfrak{P})$ which satisfies the latter equation. The remaining blocks x_{ij}, with $i \ge j$, are determined by induction on $i + j$. The diagonal blocks are given in terms of previously defined blocks by the equation:

$$x_{ii}w(x_{ii})^* = y_{ii} - \sum_{s<i} x_{is}w(x_{is})^*.$$

Again, we use the $n = 2$ to obtain a solution. (Note that this solution is not unique.) The blocks below the diagonal are defined in terms of previously defined blocks by:

$$x_{ij} = \left(y_{ij} - \sum_{s<j} x_{is}w(x_{js})^*\right)((x_{jj})^*)^{-1}w^{-1}.$$

It is elementary to check that this procedure produces a matrix x of the desired form. $\square$

Lemma 3. *If $\alpha \in \mathcal{A}$ and y is a hermitian matrix in $\varpi^\alpha K_1$, then there exists $x \in K_1$ such that $x\varpi^\alpha x^* = y$.*

Proof. Suppose $n_1 + \cdots + n_m$ is the partition of n associated to α. Then we view each element g of G as a block matrix whose ij-th block has dimensions $n_i \times n_j$. We will argue that there exists a lower triangular block matrix x such that $x\varpi^\alpha x^* = y$. To find x_{11}, we use either Lemma 1 or Lemma 2 to obtain $x_{11} \in 1 + M(n_1, \mathfrak{P})$ such that $x_{11}\varpi^{\alpha_1}x_{11}^* = y_{11}$. Then, as in the proofs of Lemmas 1 and 2, we use induction on $i+j$ to define the other blocks x_{ij} with $i \geq j$. When $i = j$, we have

$$x_{ii}\varpi^{\alpha_i}(x_{ii})^* = y_{ii} - \sum_{s<i} x_{is}\varpi^{\alpha_s}(x_{is})^*.$$

Since the right hand side lies in $\varpi^{\alpha_i}(1 + M(n_i, \mathfrak{P}))$, we may apply Lemma 1 or Lemma 2 to obtain x_{ii}. When $i > j$,

$$x_{ij} = \left(y_{ij} - \sum_{s<i} x_{is}\varpi^{\alpha_s}(x_{is})^*\right)((x_{jj})^*)^{-1}(\varpi^{\alpha_i})^{-1}.$$

It is easy to check that the matrix entries of the expression on the right hand side lie in $\mathfrak{P}$. The resulting matrix x therefore lies in K_1 and satisfies $x\varpi^\alpha x^*$. $\square$

We now turn to the proofs of the propositions:

Proof of Proposition 1. Suppose E/F is unramified and η is the identity matrix in G. Then Proposition 1(a) asserts that $\widetilde{U(\eta)} = U(\tilde{\eta})$. Suppose $k \in U(\eta)$. Reducing the equation $k\eta k^* = \eta$ modulo $\mathfrak{P}$ gives $\tilde{k}\tilde{\eta}\tilde{k}^* = \tilde{\eta}$. This shows that $\tilde{k} \in U(\tilde{\eta})$ and thus $\widetilde{U(\eta)} \subset U(\tilde{\eta})$. Conversely, if $k \in K$ and $\tilde{k} \in U(\tilde{\eta})$, then according to Lemma 1 we may choose $k_1 \in K_1$ such that $k\eta k^* = k_1\eta k_1^*$. It follows that $\tilde{k} = \tilde{k}_1^{-1}\tilde{k} \in \widetilde{U(\eta)}$ which shows that $\widetilde{U(\eta)} \supset U(\tilde{\eta})$ and proves Proposition 1(a). Parts (b) and (c) of Proposition 1 are proved similarly except that (c) uses Lemma 2 instead of Lemma 1. $\square$

Proof of Proposition 2. Assume we are given $u = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \in N_\alpha \cap K$, where c is an $n_m \times (n - n_m)$ block. It suffices to show that we can choose blocks a, b and d such that the block matrix $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lies in $H_\alpha \cap K$ and $\tilde{x} = \tilde{u}$. Let $\beta = (\alpha_1, \ldots, \alpha_{m-1})$. Then $\varpi^\alpha = \left(\begin{smallmatrix} \varpi^\beta & 0 \\ 0 & \varpi^{\alpha_m} \end{smallmatrix}\right)$. The equation $x\varpi^\alpha x^* = \varpi^\alpha$ is equivalent to three equations involving blocks:

(1) $d\varpi^{\alpha_m}d^* = \varpi^{\alpha_m} - c\varpi^\beta c^*$,

(2) $a\varpi^{\beta}c^* + b\varpi^{\alpha_m}d^* = 0$,

(3) $a\varpi^{\beta}a^* + b\varpi^{\alpha_m}b^* = \varpi^{\beta}$.

Since the right hand side of the equation (1) is hermitian and lies in $\varpi^{\alpha_m}(1+M(n_m,\mathfrak{P}))$, we may apply either Lemma 1 or Lemma 2 to obtain a solution $d \in 1 + M(n_m,\mathfrak{P})$. Equation (2) may be solved for b and used to eliminate b from the equation (3). Equation (3) then becomes:

$$a^{-1}\varpi^{\beta}(a^*)^{-1} = \varpi^{\beta} + \varpi^{\beta}c^*(d^*)^{-1}(\varpi^{\alpha_m})^{-1}d^{-1}c\varpi^{\beta}.$$

The right hand side lies in $\varpi^{\beta}(1+M(n-n_m,\mathfrak{P}))$ and thus Lemma 3 gives a solution $a \in 1 + M(n-n_m,\mathfrak{P})$. Substituting a back into the expression for b obtained from equation (2), we see that the entries of b lie in $\mathfrak{P}$. Therefore we have constructed the desired matrix x. □

4. Invariant Distributions.

In this section, we construct our supercuspidal representations and decompose the space of invariant distributions (with respect to a unitary group) as a direct sum of simpler spaces. In the following section, using known results for algebraic groups over finite fields, we deduce that any nonzero summand must have dimension one. To complete the proofs of our main theorems, we must finally count the number of nonzero summands.

Fix an irreducible representation r of ZK on a complex vector space W. We assume the restriction of r to K_1 is trivial and thus there is an associated representation $\tilde{r}$ of $\widetilde{G}$ obtained by restricting r to K and using the identification $\widetilde{G} = K/K_1$. The representation $\tilde{r}$ is said to be "cuspidal" if no nonzero vector in W is fixed by the unipotent radical of any (proper) parabolic subgroup of $\widetilde{G}$. Let $\mathcal{F}$ be the space of functions $f : G \to W$ such that $f(kg) = r(k)f(g)$, for all $k \in ZK$ and $g \in G$. The group G acts on $\mathcal{F}$ by right translations. We denote by $\mathcal{F}_c$ the submodule of functions with compact support modulo the center of G. We let $\mathrm{ind}_{ZK}^{G}(r)$, or simply π, denote the representation of G on $\mathcal{F}_c$. It is well known that the cuspidality of $\tilde{r}$ implies that π is an irreducible, supercuspidal representation of G. (See (3.05) of [**12**].)

We now fix a hermitian matrix $x \in \mathcal{X}$ and let $H = U(x)$. The problem which concerns us is the determination of the space $\mathrm{Hom}_H(\mathcal{F}_c, 1) = \mathrm{Hom}_H(\pi, 1)$ of linear forms λ on $\mathcal{F}_c$ which are H-invariant in the sense that $\lambda(\pi(h)f) = \lambda(f)$, for all $h \in H$ and $f \in \mathcal{F}_c$. In the language of [**1**], this is the space of H-invariant distributions on the ℓ-sheaf $\mathcal{F}$.

If $g \in G$, we let $\mathcal{F}_g$ denote the space of $f \in \mathcal{F}$ whose support is contained in the double coset $ZKgH$. If $\mathcal{F}_{g,c} = \mathcal{F}_g \cap \mathcal{F}_c$, then, since $ZK\backslash G/H$ is a discrete space, $\mathcal{F}_c = \oplus_g \mathcal{F}_{g,c}$, where g ranges over a set of representatives for the double coset space $ZK\backslash G/H$. Furthermore, since each summand is stable under the action of H, the space $\mathrm{Hom}_H(\mathcal{F}_c, 1)$ decomposes as a direct sum of the spaces $\mathrm{Hom}_H(\mathcal{F}_{g,c}, 1)$ of H-invariant distributions on $\mathcal{F}_g$. Similarly, if $\mathcal{F}'_g$ is the space of $f \in \mathcal{F}$ with support in KgH' and $\mathcal{F}'_{g,c} = \mathcal{F}'_g \cap \mathcal{F}_c$ then $\mathrm{Hom}_{H'}(\mathcal{F}'_c, 1)$ is a direct sum of the spaces $\mathrm{Hom}_{H'}(\mathcal{F}'_{g,c}, 1)$ as g varies over a set of representatives for $K\backslash G/H'$.

The following result is an elementary consequence of Frobenius Reciprocity. A proof can be found in [4].

Proposition 3. *Let J be a closed subgroup of G and let $\mathcal{J}_{g,c}$ be the space of $f \in \mathcal{F}_c$ with support contained in the double coset $ZKgJ$. Then the map which sends $\lambda \in \mathrm{Hom}_{ZK\cap gJg^{-1}}(r, 1)$ to the linear form $\Lambda \in \mathrm{Hom}_J(\mathcal{J}_{g,c}, 1)$ defined by*

$$\Lambda(f) = \int_{(g^{-1}ZKg\cap J)\backslash J} \lambda(f(gh))\, dh,$$

for some invariant measure on the quotient $(g^{-1}ZKg \cap J)\backslash J$, is an isomorphism.

Applying Proposition 3 in the case $J = H$, we obtain:

$$\begin{aligned} \mathrm{Hom}_H(\pi, 1) &\approx \bigoplus_{g\in ZK\backslash G/H} \mathrm{Hom}_{K\cap gHg^{-1}}(r, 1) \\ &= \bigoplus_{g\in ZK\backslash G/H} \mathrm{Hom}_{\widetilde{U(gxg^*)}}(\tilde{r}, 1). \end{aligned} \tag{4.1}$$

The case of $J = H'$ in Proposition 3 gives:

$$\mathrm{Hom}_{H'}(\pi, 1) \approx \bigoplus_{g\in K\backslash G/H'} \mathrm{Hom}_{ZK\cap gH'g^{-1}}(r, 1). \tag{4.2}$$

We also observe that

$$\mathrm{Hom}_{ZK\cap gH'g^{-1}}(r, 1) \subset \mathrm{Hom}_{\widetilde{U(gxg^*)}}(\tilde{r}, 1). \tag{4.3}$$

We will actually show that at most one of the summands in (4.2) is nonzero and, according to (4.3) and results over finite fields, such a summand has dimension one.

5. Finite Field Results.

In this section, we study the possible summands which may occur in equation (4.1) one at a time. In other words, we consider the invariant distributions supported on a single double coset $ZKgH$.

In the previous section, g was chosen to lie in a set of representatives for one of the double coset spaces $ZK\backslash G/H$ or $K\backslash G/H'$. Let us now be somewhat more specific. If $\mathcal{X}_x$ denotes the G-orbit of the hermitian matrix x, then $gH \mapsto gxg^*$ gives an identification of G/H with $\mathcal{X}_x$. The action of K on $\mathcal{X}_x$ corresponds to the action of K on G/H by right translations. Therefore the K-orbits in $\mathcal{X}_x$ correspond to the double cosets $K\backslash G/H$. According to the discussion in §2 above, the set of all matrices ϖ^α, where $\alpha \in \mathcal{A}$ and $\det(\varpi^\alpha x^{-1}) \in NE^\times$, is a set of representatives for the K-orbits in $\mathcal{X}_x$. For each such α, we may choose $g_\alpha \in G$ such that $g_\alpha x g_\alpha^* = \varpi^\alpha$. These elements g_α form a set of representatives for the double cosets $K\backslash G/H$. In particular, every double coset $ZK\backslash G/H$ contains a representative of the form g_α.

Fix an element g_α. Note that $g_\alpha H g_\alpha^{-1} = H_\alpha$. We now consider the dimension of the space $\mathrm{Hom}_H(\mathcal{F}_{g_\alpha,c}, 1)$. According to the Proposition 3, this is the same as the dimension of the space of $(K \cap g_\alpha H g_\alpha^{-1})$-fixed vectors for r.

Lemma 4. $\mathrm{Hom}_H(\mathcal{F}_{g_\alpha,c}, 1) = 0$ *unless* $\ell(\alpha) = 1$.

Proof. Suppose $w \in W$ is fixed by $K \cap g_\alpha H g_\alpha^{-1}$. Then w is fixed by $\widetilde{H}_\alpha$. According to Proposition 2, when $\ell(\alpha) > 1$, the vector w is fixed by $\widetilde{N}_\alpha$. The cuspidality of $\tilde{r}$ now implies that either $w = 0$ or $\ell(\alpha) = 1$. □

We now consider in turn the various cases in which $\ell(\alpha) = 1$. As in section 3, various finite groups of Lie type will arise. We recall that the unitary group associated to any invertible hermitian matrix in $GL(n, q^2)$ is isomorphic to the unitary group $U(n, q^2)$ associated to the identity matrix. When n is odd, all orthogonal groups associated to symmetric invertible matrices in $GL(n, q)$ are conjugate to the standard orthogonal group $O(n, q)$. When n is even, however, there is a second conjugacy class of orthogonal groups. If we fix a nonsquare $\delta_0 \in \mathbb{F}_q$ and let $O'(n, q)$ be the orthogonal group associated to the diagonal matrix $\mathrm{diag}(1, \ldots, 1, \delta_0)$ then this gives an orthogonal group which is not conjugate to $O(n, q)$.

Lemma 5. *Suppose* E/F *is unramified and* $\ell(\alpha) = 1$. *Then* $\varpi^\alpha \in Z$, $\widetilde{H}_\alpha \approx U(n, q^2)$ *and* $\mathrm{Hom}_H(\mathcal{F}_{g_\alpha,c}, 1)$ *has dimension at most one.*

Proof. Only the last assertion requires proof and this follows from the fact, proved in [**2**], that $(GL(n, q^2), U(n, q^2))$ is a Gelfand pair. □

Lemma 6. *Suppose* E/F *is ramified and* $\alpha = ((a, n, 1))$, *where* a *is odd and* n *is even. Then* $\widetilde{H}_\alpha \approx Sp(n, q)$ *and* $\mathrm{Hom}_H(\mathcal{F}_{g_\alpha,c}, 1) = 0$.

Proof. Every cuspidal representation of $GL(n, q)$, in particular $\tilde{r}$, has a Whittaker model. (See, for example, Appendix 3 [**6**].) But, according to [**11**], an irreducible representation cannot have both a Whittaker model and a symplectic model. To say that $\tilde{r}$ does not have a symplectic model is the same as saying that $\mathrm{Hom}_{Sp(n,q)}(\tilde{r},1)=0$. Our claim now follows from Frobenius Reciprocity. $\square$

Lemma 7. *Suppose E/F is ramified, $\ell(\alpha)=1$ and $\varpi^\alpha\in Z$. Then $\widetilde{H}_\alpha \approx O(n,q)$ and $\mathrm{Hom}_H(\mathcal{F}_{g_\alpha,c},1)$ has dimension at most one.*

Proof. Though $(GL(n,q),O(n,q))$ is not a Gelfand pair, our assertion follows from the fact, proven in [4], that if ρ is an irreducible cuspidal representation of $GL(n,q)$ then the space of $O(n,q)$-fixed vectors has dimension at most one. $\square$

Lemma 8. *Suppose E/F is ramified and $\alpha=((a,n,\delta))$, where a is even. Then $\widetilde{H}_\alpha\approx O(n,q)$, when n is odd, and $\widetilde{H}_\alpha\approx O'(n,q)$, when n is even. Furthermore, $\mathrm{Hom}_H(\mathcal{F}_{g_\alpha,c},1)$ has dimension at most one.*

Proof. Once again, as in the proof of Lemma 6, we appeal to [4] for the fact that if ρ is an irreducible cuspidal representation of $GL(n,q)$ then the space of vectors fixed by the relevant orthogonal group has dimension at most one. $\square$

6. Relevant Double Cosets.

In this section, we will complete the proofs of Theorems 1 and 2. In light of the results of §5, we will say that a matrix ϖ^α, with $\alpha\in\mathcal{A}$, is *relevant* if $\ell(\alpha)=1$ and ϖ^α is a diagonal matrix. It is only these matrices which may yield invariant linear forms. We are interested in the ZK-orbits of relevant matrices since matrices in the same orbit correspond to the same double coset in $ZK\backslash G/H$.

Lemma 9. *If E/F is unramified then there are exactly two ZK-orbits of relevant matrices and they are represented by the scalar matrices 1 and ϖ.*

Proof. When E/F is unramified, the relevant matrices are precisely the scalars ϖ^a, with $a\in\mathbb{Z}$. If $a=2b$ is even then $\varpi^a=\varpi^b(\varpi^b)^*$. Hence ϖ^a is in the ZK-orbit of 1. Similarly, if a is odd then ϖ^a is in the ZK-orbit of ϖ. So the relevant matrices lie in the orbits of 1 and ϖ. These two orbits cannot be the same. Indeed, if this were the case, we would have $\varpi=(\varpi^i k)(\varpi^i k)^*$ for some $i\in\mathbb{Z}$ and $k\in K$. Taking determinants produces a contradiction. $\square$

Lemma 10. *If E/F is ramified then there are exactly two ZK-orbits of relevant matrices and they are represented by* 1 *and* $\mathrm{diag}(1,\dots,1,\delta)$.

Proof. For simplicity, we abbreviate $\mathrm{diag}(1,\dots,1,\delta)$ as η. Since ϖ^4 is a norm, it is easy to see that every relevant matrix lies in the Z-orbit of one of the matrices: 1, η, ϖ^2, $\varpi^2\eta$. Moreover, 1 and η lie in distinct ZK-orbits since they lie in distinct G-orbits and, similarly, ϖ^2 and $\varpi^2\eta$ lie in different orbits.

Suppose -1 is a norm. Say $-1=\nu\bar{\nu}$. Then $\varpi^2=(\nu\varpi)(\nu\varpi)^*$, which shows that ϖ^2 is in the same ZK-orbit as 1. Similarly, η and $\varpi^2\eta$ lie in a common ZK-orbit.

Now suppose -1 is not a norm. Choose $\nu\in\mathfrak{O}^\times$ so that $\nu\bar{\nu}\delta=-1$ and choose $a,b\in\mathfrak{O}_F^\times$ so that $a^2+b^2=-1$. If n is even, let ξ be the block diagonal matrix which is a direct sum of $n/2$ copies of the block $\left(\begin{smallmatrix} a & b\\ b & -a\end{smallmatrix}\right)$. If n is odd, let ξ be the sum of $(n-1)/2$ such blocks together with the 1×1 block ν. In either case, $\xi\xi^*=-1$. Thus $(\varpi\xi)(\varpi\xi)^*=\varpi^2$, which shows that 1 and ϖ^2 lie in the same ZK-orbit. When n is odd, η commutes with ξ and thus $(\varpi\xi)\eta(\varpi\xi)^*=\varpi^2\eta$. Hence, η and $\varpi^2\eta$ lie in the same ZK-orbit when n is odd. Now assume n is even. Let η be a sum of $(n-2)/2$ copies of the block $\left(\begin{smallmatrix} a & b\\ b & -a\end{smallmatrix}\right)$ and one copy of $\left(\begin{smallmatrix} 0 & \nu\\ \nu^{-1} & 0\end{smallmatrix}\right)$. Then $\zeta\eta\zeta^*=-\eta$. Thus $(\varpi\zeta)\eta(\varpi\zeta)^*=\varpi^2\eta$. Again we conclude that η and $\varpi^2\eta$ lie in the same ZK-orbit. Our claim now follows. $\square$

A double coset in $ZK\backslash G/H$ is said to be *relevant* if it has a representative g such that gxg^* is a relevant matrix. It follows from equation (4.1) and the results of §5 that

$$\mathrm{Hom}_H(\pi,1)\approx\oplus\mathrm{Hom}_{\widetilde{U(gxg^*)}}(\tilde{r},1), \tag{6.1}$$

where the sum is over the relevant double cosets $ZKgH$. We remark that only those relevant matrices ϖ^α with $\det(\varpi^\alpha x^{-1})\in NE^\times$ correspond to relevant double cosets.

Suppose E/F is unramified. Then there are four cases to consider.

Case 1. n is even; $\det x\in NE^\times$.

The relevant matrices 1 and ϖ (from Lemma 9) both correspond to relevant double cosets since the determinants of these scalar matrices are norms. The unitary groups associated to these matrices are identical (not merely isomorphic). Therefore the two summands in equation (6.1) are identical. The dimension of $\mathrm{Hom}_H(\pi,1)$ is therefore either 0 or 2. It is 2 precisely when $\tilde{r}$ has vectors fixed by the standard unitary group in $\widetilde{G}$.

Let us now consider $\mathrm{Hom}_{H'}(\pi,1)$ in Case 1. There exist elements $g,g'\in G$ such that $gxg^*=1$ and $g'x(g')^*=\varpi$. The double cosets in $K\backslash G/H'$ corresponding to 1 and ϖ are KgH' and $Kg'H'$, respectively. Since

$(g^{-1}g')x(g^{-1}g')^* = \varpi g^{-1}(g^{-1})^* = \varpi x$, we see that $g^{-1}g' \in H'$. Therefore, the double cosets KgH' and $Kg'H'$ are identical. It follows from equation (4.2) and §5 that

$$\mathrm{Hom}_{H'}(\pi, 1) \approx \mathrm{Hom}_{ZK\cap gH'g^{-1}}(r, 1).$$

Applying equation (4.3) and Lemma 5, we deduce that $\mathrm{Hom}_{H'}(\pi, 1)$ has dimension at most one.

Case 2. n is even; $\det x \notin NE^\times$.

In this case, there are no relevant double cosets in $ZK\backslash G/H$. Therefore, we must have $\mathrm{Hom}_H(\pi, 1) = 0$ and, consequently, $\mathrm{Hom}_{H'}(\pi, 1) = 0$.

Case 3. n is odd; $\det x \in NE^\times$.

The only relevant double coset in $ZK\backslash G/H$ is the double coset corresponding to the relevant matrix 1. Therefore $\mathrm{Hom}_H(\pi, 1)$ is isomorphic to the space of vectors in W which are fixed by the standard unitary group in $\widetilde{G}$. By Lemma 5, this space has dimension at most one. It follows that $\mathrm{Hom}_{H'}(\pi, 1)$ also has dimension at most one.

Case 4. n is odd; $\det x \notin NE^\times$.

Again, there is one relevant double coset in $ZK\backslash G/H$. It is associated to the scalar matrix ϖ. Once again, $\mathrm{Hom}_H(\pi, 1)$ is isomorphic to the space of vectors in W fixed by the standard unitary group in $\widetilde{G}$. Both $\mathrm{Hom}_H(\pi, 1)$ and $\mathrm{Hom}_{H'}(\pi, 1)$ must have dimension at most one.

Finally, we consider the case where E/F is ramified. In general, there is one relevant double coset in $ZK\backslash G/H$. If $\det x \in NE^\times$, this is the double coset associated to 1 and we apply Lemma 7 to deduce that the dimensions of $\mathrm{Hom}_H(\pi, 1)$ and $\mathrm{Hom}_{H'}(\pi, 1)$ are at most one. When $\det x \notin NE^\times$ the relevant double coset is associated to $\mathrm{diag}(1, \ldots, 1, \delta)$. According to Lemma 8, the dimensions of $\mathrm{Hom}_H(\pi, 1)$ and $\mathrm{Hom}_{H'}(\pi, 1)$ are at most one.

We have now demonstrated Theorems 1 and 2 in every possible case.

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