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COMPLETE CONICAL TYPE END IMMERSED MANIFOLDS IN $\mathbb{R}^N$

Jaime Ripoll

We introduce and describe the topology of a family of complete immersed manifolds in $\mathbb{R}^N$, having a nice behaviour at infinity, which we call conical type end manifolds. Our main result states that a complete, non compact immersed manifold in $\mathbb{R}^N$, whose $\limsup$ of the norm of the second fundamental form times the intrinsic distance of the manifold to a fixed point is strictly less than 1, as the distance goes to infinity, is a conical type end manifold. In particular, it follows that the manifold has finite topology and is properly immersed in $\mathbb{R}^N$.

1. Introduction.

In this paper we introduce and describe the topology of a family of immersed manifolds in $\mathbb{R}^N$ having a nice behaviour at infinity, which we call *conical type end manifolds*, defined as follows. Let M be a complete non compact $n-$dimensional Riemannian manifold, and let $\phi : M \to \mathbb{R}^N$ be an isometric immersion. As usual, we identify M with $\phi(M)$ and assume that $0 \notin M$.

Given $p \in M$ we denote by $N(p)$ the orthogonal projection of $p/|p|$ over $T_pM^\perp$, where T_pM is the tangent space of M at p and $T_pM^\perp$ its orthogonal complement in $\mathbb{R}^N$. Given $\alpha \geq 0$, we say that M is a $\alpha-$conical type end immersed manifold of $\mathbb{R}^N$ if

$$\delta_\alpha := \lim_{d(p,p_0)\ \to\infty} \sup d^\alpha(p,p_0)|N(p)| < 1 \tag{1}$$

where $d(p,p_0)$ is the intrinsic distance in M from p to an arbitrary but fixed point p_0 in M.

An obvious example is as follows. Setting

$$S^{N-1}(1) = \{x \in \mathbb{R}^N \mid |x| = 1\},$$

take an immersed compact manifold $V^{n-1} \subset S^{N-1}(1)$, and let M be the part of the cone over V exterior to $S^{N-1}(1)$, that is, $M = \{tx \mid t \geq 1, x \in V\}$. Since $N(p) = 0$ for all $p \in M$, it follows that M is an immersed α-conical type end manifold of $\mathbb{R}^N$ for any $\alpha \geq 0$ once one "completes" M

with some compact immersed $n-$ manifold of $\mathbb{R}^N$. Roughly speaking, this example is typical at least in the case $\alpha > 0$ in the sense that a $\alpha-$conical type end immersed manifold M, $\alpha > 0$, is uniformly asymptotic to a cone over a compact subset of the sphere (Theorem 1.1 (d)).

Our main result (Theorem 1.3) states that a complete non compact n-dimensional immersed manifold M in $\mathbb{R}^N$, $n \geq 1$, $N \geq 2$, whose norm of the second fundamental form A times the intrinsic distance d of the manifold relative to a fixed point is uniformly strictly less than 1, that is,

$$\lim_{d(p,p_0)\to\infty} \sup d(p,p_0)|A_p| < 1, \tag{2}$$

is of this type, that is, has conical type ends. In particular, it will follows that M has finite topology and is properly immersed in $\mathbb{R}^N$ (see Theorem 1.1).

K. Enomoto ([**E**]) proved, under the assumption that M is properly immersed and $n \geq 2$, that if the condition

$$\lim_{d(p,p_0)\to\infty} d^{1+\epsilon}(p,p_0)|A_p| = 0 \tag{3}$$

is satisfied for some $\epsilon > 0$ then the inversion $I(M)$ of M is a C^1 immersed manifold at the origin O of $\mathbb{R}^N$, where $I(p) := p/|p|^2$. As a direct consequence of this result one has, under the above hypothesis, that M has finite topology and the Gauss map of M is continuous at infinity in each end of M. This last conclusion fails if one assumes that condition (3) is satisfied just for $\epsilon = 0$ (for example, cones over compact manifolds immersed in spheres as defined above). Other results describing immersed manifolds in Euclidean and hyperbolic space forms, satisfying condition (3), were obtained by A. Kasue and K. Sugahara ([**KS**]).

Our contribution in comparison with Enomoto's result and also others obtained by Kasue and Sugahara is that we don't require the properness of the immersion and that our theorem gives a description of the manifolds satisfying condition (3) with $\epsilon = 0$, and more generally, condition (2), cases not yet considered.

In a recent work, H. Rosenberg ([**R**]) proved that a complete embedded minimal surface in the Euclidean 3$-$dimensional space whose second fundamental form has bounded norm is proper. It is not known if this result holds either for immersions or for higher codimensions. Of course, the assumption of the minimality of the immersion is important in Rosenberg's result. In fact, under general hypothesis, condition (2) is sharp, as shows the following simple example: The curve $\gamma(t) = e^t(1+\cos t, \sin t)$ is not proper but satisfies

$$\lim_{p\to\infty} d^{\beta}(p,p_0)|A_p| = 0,$$

for any $0 \le \beta < 1$.

We shall now give precise statements of our theorems. Recall that the asymptotic boundary $M_\infty \subset S^{N-1}(1)$ of an immersed manifold M of $\mathbb{R}^N$ is the pointwise limit of the subsets

$$(1/R)(M \cap S^{N-1}(R)) \subset S^{N-1}(1)$$

as $R \to \infty$, where $S^{N-1}(R) = \{p \,|\, |p| = R\}$.

In the next two results we give a description of the conical type end immersions:

Theorem 1.1. *Let M be a complete $n-$ dimensional α-conical type end immersed manifold in $\mathbb{R}^N$, that is, satisfying (1), $n \ge 1$, $\alpha \ge 0$. Then*

(a) *M is proper,*

(b) *there is R_0 such that M is transversal to the hyperspheres $S^{N-1}(R)$ for all $R \ge R_0$,*

(c) *M is diffeomorphic to the interior of a compact differentiable manifold with boundary,*

(d) *if $\alpha > 0$, then M converges continuously to its asymptotic boundary, that is, there are a compact $n-$dimensional differentiable manifold L, surjective immersions*

$$\sigma_R : L \to (1/R)(M \cap S^{N-1}(R)), \quad R \ge R_0$$

converging uniformly C^0, as $R \to \infty$, to a continuous surjective map $\sigma : L \to M_\infty$.

It is possible to prove that a surface of finite topology whose Gauss map extends continuously to the punctures is a conical type end surface. Hence, Theorem 1.1 generalizes itens (1) and (2) of Theorem 1 of [**JM**].

When $\alpha = 0$, the convergence of $(1/R)(M \cap S^{N-1}(R))$ may not satisfy (d). For instance, the asymptotic boundary of the plane curve $\gamma(t) = (t, t\cos\sqrt{\ln t})$, $t \in [1, \infty)$ is a whole strip in $S^1(1)$, as one easily sees. In this example, we even have $\lim_{d(p,p_0)\to\infty} \sup |N(p)| = 0$.

We observe that in a $\alpha-$conical type end manifold, $\alpha > 0$, the limit map $\sigma : L \to M_\infty$ may not be a topological immersion, as one sees in the example of the parboiled $z = x^2 + y^2$.

Denote by $D(R)$ the closed ball centered at the origin of $\mathbb{R}^3$ with radius R.

Proposition 1.2. *Let M be a complete α-conical type end immersed surface in $\mathbb{R}^3$, $\alpha \ge 0$. Assume that there are R_0 and V such that each connected* *component* $\gamma_R^i,\ i=1,...,k,$ *of* $(1/R)(S^2(R)\cap M)$ *satisfies*

$$\int_{\gamma_R^i} |(\gamma_R^i)''(t)|dt \le V$$

for all $R \ge R_0$, *where* t *is the arc length of* γ_R^i. *Then* $\int_{M\cap D(R)} Kdw$ *is uniformly bounded, where* K *is the Gaussian curvature of* M. *If* $\int_{M\cap D(R_n)} Kdw$ *converges as* $n\to\infty$, *we have the formula*

$$\lim_{n\to\infty}\int_{M\cap D(R_n)} Kdw = 2\pi\chi(M) - \sum_{i=1}^{k}\lim_{n\to\infty} L(\gamma_{R_n}^i). \tag{4}$$

In particular, M *has finite total curvature if and only if there exist the limits* $\lim_{R\to\infty} L(\gamma_R^i),\ i=1,...,k.$

It is interesting to notice two special cases of formula (4). One occurs when the limit curves $\gamma^1,...,\gamma^k$ of M_∞ are great circles in $S^2(1)$, this happening, for instance, when the Gauss map extends to the infinity. In this case, it follows from Theorem 1 of [**JM**] that each curve γ_R^i converges C^1 with a certain multiplicity, say m_i, to γ^i, and formula (4) yields

$$\int_M Kdw = 2\pi\left(\chi(M) - \sum_{i=1}^{k} m_i\right)$$

a very known formula in minimal surface theory, also proved in [**JM**]. The other case, in certain sense opposite to this one, occurs when γ^i are points. In this case, of course, the Gauss map does not extend to the infinity. Formula (4) gives

$$\int_M Kdw = 2\pi\chi(M).$$

Exemples are the graphs of even degree polinomials whose coefficients of higher degree are positive, where we have $\int_M Kdw = 2\pi$. Also, certain complete graphs over k multiply connected domains, where we will have $\int_M Kdw = 2\pi(1-k)$.

Related to formula (4), but in a more general context, K. Shiohama ([**S**]) proves that

$$\int_M KdM = 2\pi\chi(M) - \lim_{r\to\infty} L(r)$$

where M is a finite total curvature anullus, and $L(r)$ is the length of a geodesic circle centered at some point of M divided by r.

In the next result, we give a characterization of conical type end manifolds immersed in $\mathbb{R}^N$.

Theorem 1.3. *Let M be a complete non compact n-dimensional manifold immersed in $\mathbb{R}^N$, $n \geq 1$, whose second fundamental form A satisfies*

$$\lim_{d(p,p_0)\to\infty} \sup d^{1+2\alpha}(p,p_0)|A_p| < 1 \tag{5}$$

for some $\alpha \geq 0$. Then M is a α-conical type manifold.

2. Proof of the results.

We recall that M is a complete $n-$dimensional Riemannian manifold isometrically immersed in $\mathbb{R}^N$. We identify M with its image in $\mathbb{R}^N$ and assume that $0 \notin M$.

Given $p \in M$, we denote by $\rho(t,p)$ the flow of the gradient in M of the map $F|_M$, where $F : \mathbb{R}^N \to \mathbb{R}$ is defined by $F(p) = (1/2)|p|^2$. Since M is complete, $\rho(p,t)$ is defined for all t. We choose an arbitrary point $p_0 \in M$.

If (1) is satisfied for some $\alpha \geq 0$, then obviously there is $R_1 > 0$ such that the critical points of F are all inside the geodesic ball

$$B_{p_0}(R_1) = \{p \in M \mid d(p,p_0) < R_1\}$$

of M.

Lemma 2.1. *Let $p \in M$ be given. Setting $r(p,t) = (1/2)|\rho(p,t)|^2$, we have*

$$r'(p,t) \leq 2r(p,t) \tag{6}$$

for all t. If (1) is satisfied for some $\alpha \geq 0$, then there is a compact $K \subset M$ such that, if $\rho(p,t) \in M\backslash K$ for all $t \geq 0$ then

$$\frac{1}{2}|p|^2 e^{(1-\delta_\alpha^2)t} \leq r(p,t) \leq \frac{1}{2}|p|^2 e^{2t}, \qquad \text{if } t \geq 0 \tag{7}$$

and if $\rho(p,t) \in M\backslash K$ for all $t \leq 0$ then

$$\frac{1}{2}|p|^2 e^{2t} \leq r(p,t) \leq \frac{1}{2}|p|^2 e^{(1-\delta_\alpha^2)t}, \qquad \text{if } t \leq 0. \tag{7'}$$

Proof. We have

$$r' = \left(\frac{1}{2}\langle\rho,\rho\rangle\right)' = \langle\rho,\rho'\rangle$$

and, since

$$\rho' = \rho - |\rho|N(\rho) \tag{8}$$

we obtain

$$r' = |\rho|^2 \left(1 - |N(\rho)|^2\right) \\ = 2r \left(1 - |N(\rho)|^2\right) \leq 2r \tag{9}$$

which proves (6). Since $r'/r \leq 2$, by integration we obtain $r(p,t) \leq (1/2)|p|^2 e^{2t}$ if $t \geq 0$, and $r(p,t) \geq (1/2)|p|^2 e^{2t}$ if $t \leq 0$.

It folows from (1) that there is a compact $K \subset M$ such that

$$1 - |N(p)|^2 > \frac{1}{2}(1 - \delta_\alpha^2)$$

for all $p \in M \backslash K$. Therefore, if $\rho(p,t) \in M \backslash K$ either for all $t \geq 0$ or $t \leq 0$ one has, from the second equality of (9)

$$\frac{r'(p,t)}{r(p,t)} \geq 1 - \delta_\alpha^2$$

in each case. By integration we obtain the remaining inequalities (7, 7'), proving Lemma 2.1. □

Lemma 2.2. *Let us assume that (1) is satisfied for some $\alpha \geq 0$, and let R_1 be such that the critical points of F are all inside the geodesic ball $B_{p_0}(R_1)$ of M. Then*

- (i) *there is R_2 such that, given $p \in M \backslash B_{p_0}(R_2)$, then $\rho(p,t) \notin B_{p_0}(R_1)$ for all $t \geq 0$,*
- (ii) *given $p \in M \backslash B_{p_0}(R_1)$, there is $t \leq 0$ such that $\rho(p,t) \in B_{p_0}(R_1)$.*

Proof. (i) By contradiction, let us assume the opposite. Then we can get a divergent sequence $p_n \in M$ and a sequence of positive real numbers t_n such that $q_n := \rho(p_n, t_n) \in B_{p_0}(R_1)$ for all n. We have $\lim_{n\to\infty} d(p_n, q_n) = \infty$, where d is the intrinsic distance in M. But from (6) and (7'), setting $r_n(q_n, t) = (1/2)|\rho(q_n, t)|^2$, we obtain, for all n

$$d(p_n, q_n) \leq \int_{-t_n}^0 |\rho'(q_n, t)| dt \leq \int_{-\infty}^0 |\rho'(q_n, t)| dt = \int_{-\infty}^0 \sqrt{r_n'(q_n, t)} dt \\ \leq |q_n| \int_{-\infty}^0 e^{\frac{(1-\delta_\alpha^2)t}{2}} dt \leq \max_{q \in B_{R_1}(p_0)} |q| < \infty$$

contradiction! This proves (i).

(ii) Given $p \in M \backslash B_{p_0}(R_1)$, the above computations show that the length of the curve $\rho(p, -t)$, $t \geq 0$, is finite. Since M is complete, we can get a sequence $t_n \to \infty$ such that $q_n := \rho(p, t_n)$ converges to a point $q \in M$, which has to be a critical point of F. It follows that $q \in B_{p_0}(R_1)$, proving (ii). $\square$

Lemma 2.3. *Given $0 < \epsilon < 1$, let $U \subset M$ be such that $d(q, p_0)|A_q| < \epsilon$ for all $q \in U$. Assume the existence of $p \in U$ which is not a critical point of F such that $\rho(p,t) \in U$ for all $t \geq 0$. Then*

$$|N(\rho(p,t))|^2 \leq \frac{\epsilon(1-y_0)e^{2t(1-\epsilon)} + y_0 - \epsilon}{(1-y_0)e^{2t(1-\epsilon)} + y_0 - \epsilon} \tag{10}$$

for all $t \geq 0$, where $y_0 = |N(p)|^2$. In particular

$$\limsup_{t\to\infty} |N(\rho(p,t))|^2 \leq \epsilon. \tag{11}$$

Proof. Observe that

$$\left\langle \frac{q}{|q|}, \frac{N(q)}{|N(q)|} \right\rangle^2 = 1 - \sum_{i=1}^{n} \left\langle \frac{q}{|q|}, V_i \right\rangle^2 \tag{12}$$

where $\{V_1, .., V_n\}$ is an orthonormal basis of T_qM, so that the function

$$q \mapsto \left\langle \frac{q}{|q|}, \frac{N(q)}{|N(q)|} \right\rangle^2$$

is differentiable, even where $N(q) = 0$. Setting

$$y(t) = \left\langle \frac{\rho(p,t)}{|\rho(p,t)|}, \widetilde{N}(\rho(p,t)) \right\rangle^2, \tag{13}$$

where $\widetilde{N} = N/|N|$, we will prove that

$$y' \leq 2(1-y)(\epsilon - y), \tag{14}$$

for all $t \geq 0$. If $N(\rho(p,t)) = 0$ at some t then we have $y(t) = y'(t) = 0$ so that (14) is satisfied at t. If $N(\rho(p,t)) \neq 0$ then $\sqrt{y(s)}$ is differentiable at $s = t$ and, at t

$$\begin{aligned} y' &= 2\left\langle \frac{\rho}{|\rho|}, \widetilde{N} \right\rangle \left[\frac{-1}{|\rho|^2} \langle \rho, \rho' \rangle \left\langle \frac{\rho}{|\rho|}, \widetilde{N} \right\rangle + \left\langle \frac{\rho}{|\rho|}, \widetilde{N}' \right\rangle \right] \\ &= -2\left(1 - \left\langle \frac{\rho}{|\rho|}, \widetilde{N} \right\rangle^2\right) \left\langle \frac{\rho}{|\rho|}, \widetilde{N} \right\rangle^2 + 2\left\langle \frac{\rho}{|\rho|}, \widetilde{N} \right\rangle \left\langle \frac{\rho}{|\rho|}, \widetilde{N}' \right\rangle \end{aligned}$$

that is

$$y' \leq -2(1-y)y + 2\left| \left\langle \frac{\rho}{|\rho|}, \widetilde{N}' \right\rangle \right|. \tag{15}$$

But

$$2\left|\left\langle \frac{\rho}{|\rho|}, \widetilde{N}'\right\rangle\right| = 2\left|\left\langle \frac{\rho' + \rho^{\perp}}{|\rho|}, \nabla_{\rho'}\widetilde{N}\right\rangle\right| = 2\frac{|\rho'|^2}{|\rho|}\left|\left\langle \frac{\rho'}{|\rho'|}, \nabla_{\rho'/|\rho'|}\widetilde{N}\right\rangle\right|$$
$$= 2\left(1 - \left\langle \frac{\rho}{|\rho|}, \widetilde{N}\right\rangle^2\right)|\rho|\left|\left\langle \frac{\rho'}{|\rho'|}, \nabla_{\rho'/|\rho'|}\widetilde{N}\right\rangle\right|$$
$$\leq 2(1-y)d(\rho, p_0)\,|A_\rho| \leq 2\epsilon(1-y),$$

which, with (15), proves (14).

Let $z(t)$ be the solution of the differential equation

$$z' = 2(1-z)(\epsilon - z)$$

satisfying $z(0) = y_0 := y(0)$. We note that since p is not a critical point of F, $y(0) < 1$. We see that $z(t)$ is given by

$$z(t) = \frac{\epsilon(1-y_0)e^{2t(1-\epsilon)} + y_0 - \epsilon}{(1-y_0)e^{2t(1-\epsilon)} + y_0 - \epsilon} \tag{16}$$

being defined for all t if $\epsilon \leq y_0$, and for

$$t \geq \frac{1}{1-\epsilon}\ln\frac{\epsilon - y_0}{1-y_0}$$

if $y_0 < \epsilon$. Since $(\epsilon - y_0)/(1-y_0) < 1$, $z(t)$ is defined, in any case, for all $t \geq 0$. By comparison, since $y'(t) \leq z'(t)$ for all $t \geq 0$ and $y(0) = z(0)$, we obtain

$$y(t) \leq z(t),$$

for all $t \geq 0$, proving (10). Inequality (11) follows from (10), finishing the proof of Lemma 2.3. □

Lemma 2.4. *Assume that there is $p \in M$ such that the curve $t \mapsto \rho(p,t)$ satisfies*

$$\lim_{t\to-\infty}\sup d(\rho(p,0), \rho(p,t))|A_{\rho(p,t)}| < 1.$$

Then the image of the curve $t \mapsto \rho(p,t)$, $t \in (-\infty, 0]$, lies in a compact subset of M.

Proof. By contradiction, we assume the opposite. Since M is complete, the curve $t \mapsto \rho(p,t)$, $t \in (-\infty, 0]$, has infinite length. Setting $r(p,t) = (1/2)\langle \rho(p,t), \rho(p,t)\rangle$, for $t \leq 0$, we therefore have

$$\int_{-\infty}^{0}\sqrt{r'(p,t)}dt = \infty$$

since $r' = \langle \rho', \rho \rangle = \langle \rho', \rho' \rangle$. Since $\lim_{t\to-\infty} d(p, \rho(p,t)) = \infty$, it follows that $\lim_{t\to-\infty} |A_{\rho(p,t)}| = 0$ and, given $0 < \delta < 1$, there is $t_0 \leq 0$ such that

$$1 - |p||A_{\rho(p,t)}| \geq \delta$$

for all $t \leq t_0$. If $N(\rho(p,t)) \neq 0$ we can set $\widetilde{N} = N/|N|$ in a neighbourhood of t. Since

$$\rho' = \rho - \left\langle \rho, \widetilde{N} \right\rangle \widetilde{N}$$

we obtain

$$\rho'' = \rho' - \left\langle \rho, \widetilde{N}' \right\rangle \widetilde{N} - \left\langle \rho, \widetilde{N} \right\rangle \widetilde{N}'$$

and

$$\langle \rho'', \rho' \rangle = \langle \rho', \rho' \rangle - \left\langle \rho, \widetilde{N} \right\rangle \left\langle \rho', \widetilde{N}' \right\rangle$$

so that

$$\begin{aligned} \frac{r''}{r'} &= \frac{2}{\langle \rho', \rho' \rangle} \langle \rho'', \rho' \rangle = 2\left(1 - \left\langle \rho, \widetilde{N} \right\rangle \left\langle \frac{\rho'}{|\rho'|}, \nabla_{\frac{\rho'}{|\rho'|}} \widetilde{N} \right\rangle\right) \\ &\geq 2(1 - |p||A_{\rho(p,t)}|) \geq 2\delta, \end{aligned}$$

since the function $t \mapsto \rho(u,t)$ is non decreasing, for all $u \in M$. We may therefore conclude that

$$\frac{r''(p,t)}{r'(p,t)} \geq 2\delta$$

for all $t \leq t_0$ satisfying $N(\rho(p,t)) \neq 0$. If $N(\rho(p,t)) = 0$ at some point t then we have, $\rho''(p,t) = \rho'(p,t)$ so that $r''(t)/r'(t) = 2$.

It follows that $r''/r' \geq 2\delta$, for all $t \leq t_0$. By integration, we obtain

$$\ln \frac{r'(p,t_0)}{r'(p,t)} = \int_t^{t_0} \frac{r''(p,s)}{r'(p,s)} ds \geq 2\delta(t_0 - t)$$

so that

$$r'(p,t) \leq r'(p,t_0) e^{2\delta(t-t_0)}.$$

It follows that

$$\begin{aligned} \int_{-\infty}^{0} \sqrt{r'(p,t)}dt &= \int_{-\infty}^{t_0} \sqrt{r'(p,t)}dt + \int_{t_0}^{0} \sqrt{r'(p,t)}dt \leq \\ &\leq \sqrt{r'(p,t_0)} \int_{-\infty}^{t_0} e^{(1-\delta)(t-t_0)}dt + \int_{t_0}^{0} \sqrt{r'(p,t)}dt \\ &= \frac{r'(p,t_0)}{\delta} + \int_{t_0}^{0} \sqrt{r'(p,t)}dt < \infty \end{aligned}$$

contradiction! This proves Lemma 2.4. □

Proof of Theorem 1.1.

(a) Clearly, it follows from (1) that all critical points of F are inside a geodesic ball $B_1 := B_{p_0}(R_1) \subset M$, where p_0 is a fixed point in M. We assume that R_1 is such that $K \subset B_1$, where K is the compact given in Lemma 2.1. By Lemma 2.2 (i), there is another ball $B_2 := B_{p_0}(R_2)$ such that $\rho(p,t) \in M \backslash B_1$ for all $p \in M \backslash B_2$ and for all $t \geq 0$. We assume that $R_2 > R_1$. Given a divergent sequence $\{p_n\} \subset M \backslash B_2$, we will prove that it is divergent in the space. Reasoning by contradiction, we may assume that $\{p_n\}$ converges in the space to $p \in \mathbb{R}^N$. Given n, it follows from Lemma 2.2 (ii) that there is $t_n \geq 0$ such that $q_n := \rho(p_n, -t_n) \in \partial B_2$, for all n. We choose t_n in such a way that $\rho(p_n, -t) \notin \partial B_2$ for all $0 \leq t < t_n$. We may assume, without loss of generality, that q_n converges to $q \in \partial B_2$. Furthermore, since p_n diverges in M and M is complete, $t_n \to \infty$ as $n \to \infty$. Given any divergent sequence s_m of positive real numbers, we have

$$\lim_m |\rho(q, s_m)| = \lim_m \lim_n |\rho(q_n, s_m)| \leq \lim_n |\rho(q_n, t_n)| = \lim_n |p_n| = |p| \tag{17}$$

since the function $t \mapsto |\rho(u,t)|$ is non decreasing, for all $u \in M$. On the other hand, since $\rho(q,t) \in M \backslash K$ for all $t \geq 0$, it follows from Lemma 2.1 (7) that $\lim_{t\to\infty} |\rho(q,t)| = \infty$, contradicting (17). This proves that M is properly immersed in the space and (a) is satisfied.

(b) Clearly (3) implies (b).

(c) Since M is proper and there is R_0 such that $S^{N-1}(R)$ is transversal to M for all $R \geq R_0$, $M \cap S^{N-1}(R_0)$ consists of a finite number of immersed compact connected differentiable manifolds. It follows that $M(R_0) := D(R_0) \cap M$ is a compact immersed manifold in $\mathbb{R}^N$ with boundary, where $D(R_0)$ is the ball of the space centered at the origin and with radius R_0. If L is any connected component of $M \backslash M(R_0)$ then the function $F(p) = 2|p|^2$ restricted to L has no critical points having its gradient orthogonal to the boundary N of L. We then observe that the map

$$\begin{aligned} \tau : N \times [0,\infty) &\to L \\ (p,t) \quad &\mapsto \rho(p,t) \end{aligned}$$

is a diffeomorphism. In fact: It is obviously injective and a local diffeomorphism. But it is also surjective since all critical points of F are inside $M(R_0)$ so that, by Lemma 2.2 (ii), given $p \in L \backslash N$, there is $t \geq 0$ such that $\rho(-t,p) \in N$. This proves (c).

(d) Let us assume that (1) is satisfied with $\alpha > 0$. Then, there is no loss of generality to assume that

$$\lim_{d(p,p_0)\to\infty} d^\alpha(p,p_0)|N(p)| = 0. \tag{18}$$

We will prove the existence of the limit

$$\lim_{t\to\infty}\frac{\rho(p,t)}{|\rho(p,t)|}$$

for all $p\in M\backslash B_2$ where B_2 is given in (a). Choose $p\in M\backslash B_2$. Given any unitary vector V in $\mathbb{R}^N$ it is enough to prove that the angle $\theta(p,t)$ between V and $\rho(p,t)$ has a limit for $t\to\infty$. First assume that $N=2$ and $V=(1,0)$. We can therefore write

$$\rho(p,t)=r(p,t)(\sin\theta(p,t),\cos\theta(p,t))$$

and we note that here $r(p,t)=|\rho(p,t)|$. Since $\langle\rho,\rho'\rangle=\langle\rho',\rho'\rangle=rr'$ we obtain

$$\frac{r'}{r}\left(1-\frac{r'}{r}\right)=\theta'^2 \tag{19}$$

and

$$1-\frac{r'}{r}=|N|^2 \tag{20}$$

so that

$$\lim_{t\to\infty}\frac{r'}{r}=1$$

and

$$\lim_{t\to\infty}r^{2\alpha}\left(1-\frac{r'}{r}\right)=\lim_{t\to\infty}\left(r^{\alpha}\left\langle\frac{\rho}{|\rho|},N\right\rangle\right)^2=\lim_{t\to\infty}\left(|\rho|^{\alpha}\left\langle\frac{\rho}{|\rho|},N\right\rangle\right)^2=0.$$

Therefore, from (19), we have

$$\lim_{t\to\infty}r^{\alpha}\theta'=0.$$

From Lemma 2.1 inequality (7) we have $r(p,t)\geq|p|^2e^{(1-\delta_\alpha^2)t}$ so that $\lim_{t\to\infty}e^{\alpha(1-\delta_\alpha^2)t}\theta'(p,t)=0$. It follows that there is a positive constant C such that

$$|\theta'(p,t)|\leq Ce^{-\alpha(1-\delta_\alpha^2)t},\quad t\geq 0,$$

which implies the existence of the limit $\lim_{t\to\infty}\theta(p,t)$.

If $N>2$, we can write $\rho(p,t)$ in spherical coordinates extending V to an orthonormal basis of the space and we will obtain

$$\frac{r'}{r}\left(1-\frac{r'}{r}\right)=\theta'^2+\sum_i a_i^2$$

for certains function $a_i = a_i(p,t)$, where $\theta(p,t)$ is the angle between $\rho(p,t)$ and V. Since one still has (20), the proof continues as in the case $N = 2$.

Suppose that $M \cap S^{N-1}(R_0)$ is the image of an immersion $H : L \to \mathbb{R}^N$ where L is a $(n-1)$-dimensional differentiable manifold and R_0 is given as in (c). Given $R \geq R_0$, we define $\sigma_R : L \to (1/R)(M \cap S^N(R))$ by

$$\sigma_R(x) = (1/R)\left(\rho(H(x), \mathbb{R}) \cap S^{N-1}(R)\right).$$

By the implicit function theorem, it follows that σ_R is a surjective immersion for all $R \geq R_0$. From what we have proved above, it follows that σ_R converges as $R \to \infty$, to the map

$$\begin{aligned} \sigma : L &\to M_\infty \\ x \quad &\mapsto \lim_{t\to\infty} \frac{\rho(H(x),t)}{|\rho(H(x),t)|}. \end{aligned}$$

To prove the uniform convergence of the family $\{\sigma_R\}$, we consider the family $\{\tau_t : L \to S^N(1)\}$ given by

$$\tau_t(x) = \frac{\rho(H(x),t)}{|\rho(H(x),t)|}$$

which clearly converges to σ as $t \to \infty$. Given $\epsilon > 0$, by (18), there is a geodesic ball B_3 in M containing B_2 such that

$$|p|^\alpha |N(p)| < \epsilon$$

for all $p \in M \backslash B_3$. Let t_1 be such that $\rho(H(x),t) \in M \backslash B_3$ for all $t \geq t_1$ and for all $x \in L$. It follows that

$$|\rho(H(x),s)|^\alpha |N(\rho(H(x),s))| < \epsilon$$

so that, by Lemma 2.1 (7'),

$$|N(\rho(H(x),s))| < \frac{\epsilon}{|\rho(H(x),s)|^\alpha} < \frac{\epsilon e^{-\frac{\alpha(1-\delta_\alpha^2)s}{2}}}{{R_0}^\alpha} \tag{21}$$

for all $x \in L$ and for all $s \geq t_1$. Denoting by $\theta(x)$ the angle between $\sigma(x)$ and V and, as before, by $\theta(x,t)$ the angle between $\tau_t(x)$ and V we have

$$|\theta(x) - \theta(x,t)| \leq \int_t^\infty |\theta'(x,s)|\, ds.$$

By (19) and (21), setting $r(x,t) = |\rho(H(x),t)|$, we obtain

$$|\theta'(x,s)| = \sqrt{\frac{r'(x,s)}{r(x,s)}\left(1 - \frac{r'(x,s)}{r(x,s)}\right)} \leq \sqrt{1 - \frac{r'(x,s)}{r(x,s)}}$$
$$= |N(\rho(H(x),s))| \leq \frac{\epsilon e^{-\frac{\alpha(1-\delta_\alpha^2)s}{2}}}{R_0^\alpha}$$

$$|\theta(x) - \theta(x,t)| \leq \frac{2\epsilon e^{-\frac{\alpha(1-\delta_\alpha^2)t}{2}}}{\alpha R_0^\alpha}$$

proving that the family $\{\tau_t\}$ converges uniformly to σ as $t \to \infty$. This proves that σ is continuous.

Now, given $\epsilon > 0$, there is t_1 such that $|\tau_t(x) - \sigma(x)| < \epsilon$, for all $t \geq t_1$ and for all $x \in L$. Clearly, there is R_1 such that $|\sigma_R(x) - \sigma(x)| \leq |\tau_{t_1}(x) - \sigma(x)|$ for all $R \geq R_1$ and for all $x \in L$, implying that $\{\sigma_R\}$ also converges uniformly to σ. This concludes the proof of Theorem 1.1. □

The completness of M in Theorem 1.1 is an essential hypothesis (to apply Lemma 2.2). In this sense, it is interesting to observe the example of the cone M over an immersed but not proper curve in the sphere. Condition (1) is satisfied for all α but M is not proper.

Proof of Proposition 1.2. Applying Gauss-Bonnet theorem on $M(R) := M \cap D(R)$, where $D(R)$ is the ball of $\mathbb{R}^3$ centered at the origin with radius R, we get

$$\int_{M(R)} K dM = 2\pi\chi(M(R)) - \sum_{i=1}^{k} \int_{\gamma_i(R)} k_i(R)(t)dt \tag{22}$$

where $k_i(R)$ is the geodesic curvature of $R\gamma_i(R)$. We observe that

$$k_i(R)(t) = \langle \gamma_i(R)''(t), n(t) \rangle$$

where t is the arc lenght of $\gamma_i(R)$ and $n(t)$ the exterior conormal vector of $\partial M(R)$ at $\gamma_i(R)(t)$. We can write

(23)
$$\begin{aligned} k_i(R)(t) &= \langle \gamma_i(R)''(t), n(t) - \gamma_i(R)(t) \rangle + \langle \gamma_i(R)''(t), \gamma_i(R)(t) \rangle \\ &= \langle \gamma_i(R)''(t), n(t) - \gamma_i(R)(t) \rangle - 1/R \end{aligned}$$

since $\langle \gamma_i(R)(t), \gamma_i(R)(t) \rangle = 1$.

From the boundedness of the total curvature of $\gamma_i(R)(t)$ in the space and from (3), we get

$$\lim_{R\to\infty}\int_{\gamma_i(R)} |\langle \gamma_i(R)''(t), n(t)-\gamma_i(R)(t)\rangle|\, dt = 0 \tag{24}$$

since $|n(t)-\gamma_i(R)(t)| \to 0$ as $R\to\infty$. Since M has finite topology, using (22), (23) and (24) we therefore obtain

$$\lim_{R\to\infty}\int_{M(R)} K dM = 2\pi\chi(M) - \sum_{i=1}^{k}\lim_{R\to\infty} L(\gamma_i(R))$$

proving Proposition 1.2. □

Proof of Theorem 1.3. We will prove that the set $Z\subset M$ of the critical points of F is bounded. By contradiction, let us assume the opposite. We begin by showing the existence of a geodesic ball B of M such that $Z\backslash B$ is a submanifold of M of dimension zero, that is $Z\backslash B$ is constituted of isolated points of M. Setting $R_0 = |p_0|$, we have $2d(p,p_0)\geq |p|$ for all $p\in M\backslash B_{p_0}(R_0)$. Let B, closed, be chosen such that

$$d(p,p_0)\,|A_p| \leq \frac{1+\delta_\alpha}{2} < 1 \tag{25}$$

and $d(p,p_0)\geq 2(1+\delta_\alpha)/(1-\delta_\alpha)$ for all $p\in M\backslash B$. Given $p\in Z\backslash B$, let $B_p(R)$ be a geodesic normal ball of M centered at p such that $B_p(R)\subset Z\backslash B$ and let $V_1,...,V_n$ be vector fields on $B_p(R)$ which constitute an orthonormal basis of the tangent space of M at any point of $B_p(R)$. Define $f_i : B_p(R)\to\mathbb{R}$ by

$$f_i(q) = \left\langle \frac{q}{|q|}, V_i\right\rangle.$$

Then $Z\cap B_p(R) = \cap_{i=1}^n f_i^{-1}(0)$. Given $q\in f_i^{-1}(0)$, we have

$$\begin{aligned} V_j(f_i)(q) &= \frac{1}{|q|}\langle V_j, V_i\rangle + V_j\left(\frac{1}{|q|}\right)\langle q, V_i\rangle + \left\langle \frac{q}{|q|}, \nabla_{V_j}V_i\right\rangle \\ &= \frac{1}{|q|}\delta_{ij} + \left\langle \frac{q}{|q|}, \nabla_{V_j}V_i\right\rangle \end{aligned}$$

so that

$$\begin{aligned} d(q,p_0)|V_i(f_i)(q)| &\geq 1 - d(q,p_0)|A_q| \\ &\geq \frac{1-\delta_\alpha}{2} \end{aligned}$$

and

$$|V_j(f_i)(q)| \leq \frac{1-\delta_\alpha}{4} \quad i \neq j.$$

It follows that the gradients $gradf_1, ..., gradf_n$ are linearly independent, which shows that $Z \cap B_p(R)$ is the transversal intersection of n submanifolds of M with dimension $n-1$, proving our assertion.

Let $q \in Z \backslash B$ be critical point of F. Since q is isolated, there is $p \in M$, $p \neq q$, such that the $\alpha-$limit (with respect to the vector field gradient of F in M) of p is q, that is,

$$\lim_{t\to\infty} \rho(p,t) = q.$$

By choosing p close enough to q, we may assume that $\rho(p,t) \in M\backslash B$ for all $t \geq 0$. Setting

$$y(t) = \left\langle \frac{\rho(p,t)}{|\rho(p,t)|}, N(\rho(p,t)) \right\rangle^2 \tag{26}$$

we have $\lim_{t\to\infty} y(t) = 1$. On another hand, it follows from (25) and Lemma 2.3 with $\epsilon = (1+\delta_a)/2$ the existence of t_0 such that $y(t) \leq \epsilon < 1$ for all $t \geq t_0$, contradiction!

Therefore, there is a geodesic ball $B_{p_0}(R_1)$ containing all the critical points of F. We claim that there is $R_2 > R_1$ such that if $p \in M\backslash B_{p_0}(R_2)$ then $\rho(p,t) \notin B_{p_0}(R_1)$ for all $t \geq 0$. In fact: By contradiction, assume the existence of a divergent sequence p_n in M and a divergent sequence t_n of positive real numbers such that $q_n := \rho(p_n,t_n) \to \partial B_{p_0}(R_1)$. We may assume that $\rho(p_n,t) \in M/B_{p_0}(R_1)$, for all $0 \leq t < t_n$. Without loss of generality, we may assume that q_n converges to a point $q_0 \in \partial B_{p_0}(R_1)$. It follows that the curve $t \mapsto \rho(q_0,t)$, $t \leq 0$, is divergent in M and satisfies, by hypothesis, $\lim_{t\to-\infty} \sup d(\rho,p_0)|A_\rho| < 1$, contradicting Lemma 2.4. This proves our claim. □

Since M is complete, we have

$$\lim_{t\to\infty} d(\rho(p,t),p_0) = \infty \tag{27}$$

for all $p \in M\backslash B_{p_0}(R_2)$. Let us prove that

$$\lim_{t\to\infty} \sup |N(\rho(p,t))| < 1 \tag{28}$$

for all $p \in M\backslash B_{p_0}(R_2)$.

Let $R_3 \geq R_2$ be such that (25) is satisfied for all $p \in M\backslash B_{p_0}(R_3)$. Using, as above, Lemma 2.4, we can assure the existence of R_4 such that $\rho(p,t) \in$

$M\backslash B_{p_0}(R_3)$, for all $p \in M\backslash B_{p_0}(R_4)$ and for all $t \geq 0$. Now, given $p \in M\backslash B_{p_0}(R_2)$, by (27), there is t_1 such that $\rho(p,t_1) \in M\backslash B_{p_0}(R_4)$. Therefore, by Lemma 2.3, we obtain

$$\lim_{t\to\infty} \sup |N(\rho(p,t))| = \lim_{t\to\infty} \sup N(\rho(\rho(p,t_1),t))| \leq \frac{1+\delta_a}{2} < 1$$

proving (28).

We prove now that (3) is satisfied for $\alpha = 0$. Let R_4 be as above and set

$$d = \max \{|N(p)|,\ p \in \partial B_{p_0}(R_4)\}.$$

If $d \leq (3+\delta_a)/4 < 1$, then we have

$$|N(p)| \leq (3+\delta_a)/4 < 1, \tag{29}$$

for all $p \in M\backslash B_{p_0}(R_4)$. In fact: given $p \in M\backslash B_{p_0}(R_4)$, there is $p_1 \in \partial B_{p_0}(R_4)$ such that $p = \rho(p_1,t_1)$, for some $t_1 \geq 0$. Since $\rho(p,t) \in B_{R_3}(p_0)$, $t \geq 0$, if $y(t)$ is defined as in (26) with p_1 instead of p, we obtain, from the proof of Lemma 2.3, $y(t) \leq z(t)$ for all $t \geq 0$, where $z(t)$ is given by (16) with $\epsilon = (1+\delta_\alpha)/2$. This proves (29).

If $d > (3+\delta_a)/4$, we set

$$T = \frac{1}{1-\delta_a} \ln \frac{d - \frac{1+\delta_a}{2}}{1-d} \tag{30}$$

and, since $T \geq 0$,

$$K := \{\rho(p,t) \,|\, p \in B_{p_0}(R_4), \quad 0 \leq t \leq T\}$$

is well defined. We then have $|N(q)|^2 < (3+\delta_\alpha)/4$ if $q \in M\backslash K$. In fact: Given $q \in M\backslash K$, there is $p \in \partial B_{p_0}(R_4)$ such tha $\rho(p,t_1) = q$, for some $t_1 \geq T$. By (16) and (30), we obtain, using again the proof of Lemma 2.3, $y(t_1) \leq z(t_1) \leq (3+\delta_\alpha)/4$, that is,

$$|N(q)|^2 = \left\langle \frac{q}{|q|}, N(q) \right\rangle = \left\langle \frac{\rho(p,t_1)}{|\rho(p,t_1)|}, N(\rho(p,t_1)) \right\rangle = y(t_1) \leq \frac{3+\delta_\alpha}{4}.$$

This proves that M is a 0-conical type end manifold.

Now, let us prove that M is a $\alpha-$conical type immersed manifold. Since it is a $0-$conical type end manifold, we can take $\delta > 0$ and R_6 such that

$$0 < \left\langle \frac{\rho(p,t)}{|\rho(p,t)|}, N(\rho(p,t)) \right\rangle^2 < \delta$$

for all $t \geq 0$ and for all $p \in M \backslash B_{p_0}(R_6)$. Given $p \in M \backslash B_{p_0}(R_6)$, set

$$f(t) = \left\langle \frac{\rho(p,t)}{|\rho(p,t)|^{\beta}}, N(\rho(p,t)) \right\rangle^2$$

$\beta = 1 - \alpha$, and let us prove that $\lim_{t\to\infty} \sup f(t) < 1$. Assuming the opposite, there is a sequence $t_n \to \infty$ such that $\lim_n f(t_n) = 1$. Setting

$$a(t) = d^{2\alpha+1}(\rho(p,t), p_0) \left| \left\langle \frac{\rho'}{|\rho'|}, \nabla_{\frac{\rho'}{|\rho'|}} N \right\rangle \right|,$$

since $\lim_{t\to\infty} \sup a(t) < 1$, there is t_0 and $c < 1$ such that $a_0/(\delta|\beta|) < c$, for all $t \geq t_0$, where $a_0 = a(t_0)$. Let t_1 be such that $a(t_1) \leq a(t_0)$ for all $t \geq t_1$, and set $t_2 = \max\{t_0, t_1\}$. Then, for all $t \geq t_2$, we obtain

$$\begin{aligned} f' &= -2\frac{\beta}{|\rho|^2} \langle \rho, \rho' \rangle \left\langle \frac{\rho}{|\rho|^{\beta}}, N \right\rangle + 2 \left\langle \frac{\rho}{|\rho|^{\beta}}, N \right\rangle \left\langle \frac{\rho}{|\rho|^{\beta}}, N' \right\rangle \\ &= -2\beta \left(1 - \left\langle \frac{\rho}{|\rho|^{\beta}}, N \right\rangle^2\right) \left\langle \frac{\rho}{|\rho|^{\beta}}, N \right\rangle^2 + 2 \left\langle \frac{\rho}{|\rho|^{\beta}}, N \right\rangle \left\langle \frac{\rho}{|\rho|^{\beta}}, N' \right\rangle \\ &= -2\beta \left(1 - \left\langle \frac{\rho}{|\rho|^{\beta}}, N \right\rangle^2\right) f + |\rho|^{2\alpha} \left\langle \frac{\rho}{|\rho|}, N \right\rangle \left\langle \frac{\rho}{|\rho|}, N' \right\rangle \\ &\leq -2|\beta|\delta f + 2d^{2\alpha+1}(\rho, p_0) \left| \left\langle \frac{\rho'}{|\rho'|}, \nabla_{\frac{\rho'}{|\rho'|}} N \right\rangle \right| \leq -2|\beta|\delta f + 2a_0 \end{aligned}$$

where we have used that $\rho' = \rho - \langle \rho, N \rangle N$. Let $g(t)$ be the solution of the differential equation

$$z' = -2|\beta|\delta z + 2a_0 \tag{31}$$

satisfying the condition $g(t_2) = f(t_2) = f_2$. By comparison, we obtain $f(t) \leq g(t)$, for all $t \geq t_2$. Equation (31) can be easily integrated providing

$$g(t) = \frac{a_0}{\delta|\beta|} - \frac{\frac{a_0}{\delta|\beta|} - f_2}{e^{-2\delta|\beta|}} e^{-2\delta|\beta| t},$$

which shows that $f(t) \leq a_0/\delta|\beta|$ for all $t \geq t_2$, contradiction!

It is therefore proved that $\lim_{t\to\infty} \sup f(t) < 1$. From this, as already made above for the case $\alpha = 0$, one can conclude that (1) is satisfied, proving that M is a α-conical type end manifold, concluding the proof of Theorem 1.3.

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Universidade Federal do Rio Grande do Sul
Av. Bento Gonçalve 9500
91540-000 Porto Alegre RS
Brazil
E-mail address: ripoll@athena.mat.ufrgs.br