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COVERING OF A HOLOMORPHICALLY CONVEX MANIFOLD CARRYING A POSITIVE LINE BUNDLE

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Let M be a holomorphically convex manifold and $\pi : \tilde{M} \to M$ is a holomorphic connected covering. If M carries a positive holomorphic line bundle L such that the cohomology class of $\pi^* L$ in $H^1(\tilde{M}, \mathcal{O}^*)$ vanishes, then $\tilde{M}$ is a Stein manifold.

A classical theorem of Siegel [**S**] asserts that a bounded domain in $\mathbf{C}^n$ covering a compact complex manifold is a domain of holomorphy. In [**Wa**] Watanabe showed that if a complex manifold D covering a projective manifold M and satisfies $H^1(D, \mathcal{O}^*) = 0$, then D is a Stein manifold with $H^2(D, \mathbf{Z}) = 0$, where $\mathcal{O}^*$ is the sheaf of germs of nowhere-vanishing holomorphic functions and $\mathbf{Z}$ is the additive group of integers. The purpose of this paper is to study the case where the base of covering is a holomorphically convex complex manifold carrying a positive holomorphic line bundle. Recall that a complex manifold M is holomorphically convex if, for every infinite subset S of M without limit points, there is a holomorphic function f on M which is unbounded on S. Our main result is the following:

Theorem. *Let M be a holomorphically convex complex manifold and $\pi : \tilde{M} \to M$ is a holomorphic connected covering. If M carries a positive holomorphic line bundle L such that the cohomology class of $\pi^* L$ in $H^1(\tilde{M}, \mathcal{O}^*)$ vanishes, then $\tilde{M}$ is a Stein manifold.*

It is well known that there is an isomorphism between the class of holomorphic line bundles over a complex manifold X and the cohomology group $H^1(X, \mathcal{O}^*)$ (see [**We**], Lemma 4.4, p. 101). The cohomology class of a line bundle F is defined as the class in $H^1(X, \mathcal{O}^*)$ of a holomorphic cocycle $\{f_{ij}\}$ representing F. The vanishing of the cohomology class of $\pi^* L$ in $H^1(\tilde{M}\mathcal{O}^*)$ is necessary as shown later. Since a compact complex manifold is vacuously holomorphically convex, Watanabe's theorem is a consequence of our theorem. But in his proof, it is not clear that the strongly plurisubharmonic function ϕ ([**Wa**], p. 244) does not grow to $-\infty$ near the topological boundary of $\tilde{M}$. Also as remarked by him, the condition $H^1(\tilde{M}\mathcal{O}^*) = 0$ cannot be replaced by $H^1(\tilde{M}, \mathcal{O}) = 0$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions: Consider the case $M = \tilde{M} = \mathbf{P}_1(C)$ and π is the identity mapping.

Proof of theorem. As in [**Wa**] let $\{V_i\}$ be an open covering of M such that each V_i is a local coordinate neighborhood and is biholomorphic to a connected component $\pi^{-1}(V_i)$. Choosing a suitable refinement $\{U_i\}$ of $\{V_i\}$, we can represent the line bundle L by a system of transition functions $\{f_{ij}\}$ and find a Hermitian metric $\{a_j\}$ along the fibers of L which satisfies the following conditions:

(i) Each a_j is a C^∞, real valued and positive function on U_j.

(ii) If $U_j \cap U_k = \emptyset$, then we have $a_k = |f_{jk}|^2 a_j$.

(iii) The function $-\log a_j$ is strongly plurisubharmonic on U_j i.e. the

$$(1,1)\text{-form } \; i\partial\bar{\partial}\log\frac{1}{a_j} \; \text{ is positive definite.}$$

The curvature of L is defined as $c(L) := i\partial\bar{\partial}(-\log a_j)$ on U_j. L is said to be *positive* if $c(L)$ is a positive definite $(1,1)$-form on the convex tangent bundle TM.

The line bundle π^*L defined on $\tilde{M}$ has $\{f_{ij}o\pi\}$ as transition functions and $\{a_j o\pi\}$ for Hermitian metric. Since $\{\pi^{-1}(U_j)\}$ is an open covering of $\tilde{M}$, $\{f_{ij}o\pi\}$ defines an element of the cohomology class of π^*L. By hypothesis $[\pi^*L] = 0$ in $H^1(\tilde{M}, \mathcal{O}^*)$, then there an open covering Y_j of $\tilde{M}$ such that the cohomology class of $[f_{ij}o\pi] = 0$ in $H^1(\{Y_j\}, \mathcal{O}^*)$. Taking a refinment of $\{U_j\}$ and $\{Y_j\}$, we may suppose that $\{U_j\} = \{Y_j\}$. Then there is a cochain $\{f_j\}$ of $C^0(\{\pi^{-1}(U_j)\}, \mathcal{O}^*)$ such that $f_{jk}o\pi = \frac{f_k}{f_j}$ on $\pi^{-1}(U_j) \cap \pi^{-1}(U_k)$. The holomorphic section of π^*L over $\tilde{M}$ defined by $s = f_j$ on $\pi^*(U_j)$ is nowhere zero. We can define a C^∞ strongly Psh function on $\tilde{M}$ in the following way:

$$\phi(x) := -\log(a_j o\pi(x)|f_j(x)|^2) = -\log\|s(x)\|^2_{\pi^*L} \quad \text{for } \; x \in \pi^{-1}(U_j)$$

where $\|s\|_{\pi^*L}$ is the norm of s with respect to the Hermitian metric of π^*L induced by L.

Since M is holomorphically convex and L is positive, it is easy to see that M admits a complete Kähler metric g. Simply take

$$g := i\partial\bar{\partial}(\chi o\Psi) + c(L),$$

where $\Psi \in C^\infty(M)$ is an exhaustive Psh function (see [**H**], p. 117, Theorem 5.1.6) and $\chi : \mathbf{R} \to \mathbf{R}$ is a smooth function such that $\chi' > 0$, $\chi'' \geq 0$, and $\chi'(t) \to +\infty$ fast as $t \to +\infty$. Since π is a connected covering, the pull back π^*g of g by π, define a complete Kähler metric $\tilde{g}$ on $\tilde{M}$.

Now let S be an infinite subset without linit point in $\tilde{M}$. We must produce a holomorphic function f on $\tilde{M}$ such that $|f|$ is unbounded on S. Since it suffices to consider any infinite subset of S, we may assume that S is actually equal to a sequence of points $\{x_\nu\}$.

Case 1. If $\{y_\nu\}$ has a limit point y in M, then we may assume $y_\nu \to y$. Let $(U\phi)$ be a local coordinate neighborhood such that $\phi(y) = 0$, $\phi(U) = B(0, R) \subset \mathbf{C}^n$ and $\pi^{-1}(U) = \cup_j V_j$ with $V_j \cap V_k = \emptyset$ for every $j \neq k$ and $\pi_j := \pi|_{V_j} : V_j \to U$ is a biholomorphic map. Taking ν large enough, we may assume that $y_\nu \in U$ for $\nu = 1, 2, \ldots$. Since $x_\nu \in \pi^{-1}(U)$ there is a unique $j(\nu) \in \mathbf{N}$ such that $x_\nu \in V_{j(\nu)}$. If $I_\nu := \{\nu \in \mathbf{N},\ j(\nu) = j(\mu)\}$ and $\nu_k = \sup\{\nu \in I_k\}$, then $x_{\nu_k} \in V_{\nu_k}$ and $V_{j(k)} \cap V_{j(k')} = \emptyset$ if $k \neq k'$. Since the set I_k is finite, the subsequence $\{z_k := x_{\nu_k}\}$ has not limit point in $\tilde{M}$. Set $W_k := V_{j(\nu_k)}$ and $\pi_k := \pi_{j(\nu_k)}$ and consider the biholomorphic map $\phi_k = \phi o \pi_k : W \to B(0, R)$. The map ϕ_k satisfies the following properties:

(1) $\phi_k(z_k) = \phi(y_{\nu_k})\ \forall\ k \in \mathbf{N}$; and

(2) $\alpha\phi_k^* g_e \leq \tilde{g} \leq \beta\phi_k^* g_e$ on W_k where the constants α, β are independent of k and g_e is the Euclidean metric of $\mathbf{C}^n$.

Without loss of generality, we may assume that $\|\phi_k(z_k)\| \leq \frac{R}{9}$. Hence $B(0, \frac{R}{9}) \subset\subset B(\phi_k(z_k), \frac{R}{9}) \subset\subset B(0, R)$ and by (2) we have

$$\alpha\|\phi_k(x) - \phi_k(z_k)\| \leq d_{\tilde{g}}(x, z_k) \leq \beta\|\phi_k(x) - \phi_k(z_k)\| \quad \forall_x \in W_k.$$

Thus $X_k := \phi^{-1}(B(\phi_k(z_k), \frac{R}{4})) \subset B_{\tilde{g}}(z_k, \frac{\beta B}{4})$. Let λ be a test function in $B(0, \frac{B}{8})$ such that $\lambda = 1$ in $B(0, \frac{R}{9})$. Let $\Psi : \tilde{M} \to \mathbf{R} \cup \{-\infty\}$ be the function Ψ defined by

$$\Psi(x) = \begin{cases} \lambda(\phi_k(x)) \log\|\phi_k(x) - \phi_k(z_k)\|^{2n} & \text{if } \ x \in X_k \\ 0 & \text{if } \ x \in \tilde{M} \backslash \cup_k X_k. \end{cases}$$

Then Ψ is a smooth function on $\tilde{M} \backslash \{z_k\}_{k=1}^{\infty}$. Thanks to (2) there is a constant $K > 0$ such that

$$i\partial\bar{\partial}\Psi \geq -K\tilde{g} \ \text{ in a distributional sense on all of } \ \tilde{M}.$$

Put $\gamma_k = \sup_{X_k} \|s\|^2_{\pi^* L}$ and $\rho_k = \sup_{1 \leq j \leq k} \gamma_j$. By passing to a subsequence, we may assume that $r(z_k) := d_{\tilde{g}}(z_k, x_o) \geq k + \rho_k$ for all $k \in \mathbf{N}$ ($x_o \in \tilde{M}$ is a fixed point).

Lemma ([**N1**, Lemma 1.1]). *There is a smooth and exhaustive function* $\tau : \tilde{M} \to \mathbf{R}$ *such that*

(i) $r \leq C_1\tau \leq C_2 r$; *and*

(ii) $i\partial\bar{\partial}\tau \geq -C_3\tilde{g}$, *on* $\pi^{-1}(U)$

where the constants may depend on U.

Define a smooth section t of $\pi^* L^m$ on $\tilde{M}$ by

$$t(x) = \begin{cases} \lambda(\phi_k(x)) e^{r(z_k)} \bigotimes_1^m s(x) & \text{if } x \in X_k \\ 0 & \text{elsewhere.} \end{cases}$$

We obtain a smooth $\bar{\partial}$-form of type $(0,1)$ with values in $\pi^* L^m$ and support contained in $\cup_{k=1}^{\infty} X_k$, by defining

$$\omega = \bar{\partial} t.$$

Since $\lambda = 1$ on $B(0, \frac{R}{8})$ and s is holomorphic, we conclude that ω vanishes on $\phi_k^{-1}(B(0, \frac{R}{8})) \subset X_k$ for $k = 1, 2, 3, \ldots$. Moreover, we have $|\lambda o \phi_k| \leq$ Const on $X_k \backslash \phi_k^{-1}(B(0, \frac{R}{8}))$, and, because the map ϕ_k satisfies the property (2), we have $|\bar{\partial}(\lambda o \phi_k)| \leq$ Const on X_k, where the constants are independent of k. Therefore on the set $X_k \backslash \phi_k^{-1}(B(0, \frac{R}{8}))$, hence on all X_k, we have

$$\|\omega\|_{e^{-\Psi} \pi^* L^m} \leq \text{Const } e^{2r(z_k)} \rho_k^{2m},$$

where the constant is independent of k. Also as mentioned above, ω vanishes on the complement $\tilde{M} \backslash \cup_k X_k$. Therefore, if we define a singular Hermitian metric in $\pi^* L^m$ on $\tilde{M}$ by

$$\|?\|_m^2 := e^{-(3C_1 r + \Psi)} \|?\|_{\pi^* L^m}^2,$$

then $\|?\|_m$ is smooth on $\tilde{M} \backslash \{z_k\}_{k=1}^{\infty}$, and

$$\begin{aligned} \int_{\tilde{M}} \|\omega\|_m^2 dV_{\tilde{g}} &= \int_{\tilde{M}} \|\omega\|_{e^{\Psi} \pi^* L^m}^2 e^{-3C_1 r} dV_{\tilde{g}} \\ &= \sum_{k=1}^{\infty} \int_{X_k} \|\omega\|_{e^{\Psi} \pi^* L^m}^2 e^{-3C_1 r} dV_{\tilde{g}}. \end{aligned}$$

Hence

$$\int_{\tilde{M}} \|\omega\|_m^2 dV_{\tilde{g}} \leq \text{Const} \sum_{k=1}^{\infty} \rho_k^{2m} e^{2r(z_k)} e^{-3C_1 r} dV_{\tilde{g}}.$$

By the previous Lemma, we have $r \leq C_1 r$ om X_k. Therefore, since $X_k \subset B_{\tilde{g}}(z_k, \frac{\beta R}{4})$,

$$r(z_k) - \frac{\beta R}{4} \leq r(x) \leq C_1 r(x)$$

for every $x \in X_k$ and $k = 1, 2, \dots$. Thus

$$\begin{aligned}\int_{\tilde{M}} \|\omega\|_m^2 dV_{\tilde{g}} &\leq \text{Const} \sum_{k=1}^{\infty} \rho_k^{2m} \int_{X_k} e^{2r(z_k)-3r(z_k)} dV_{\tilde{g}} \\ &\leq \sum_{k=1}^{\infty} \rho_k^{2m} e^{-r(z_k)} \operatorname{vol}\left(B_{g_e}\left(0, \frac{R}{4}\right)\right) \\ &\leq \text{Const} \sum_{k=1}^{\infty} \rho_k^{2m} e^{-\rho_k} e^{-k}.\end{aligned}$$

Let $a := \lim_{k\to\infty} \rho_k$. If $a < +\infty$ then

$$\int_{\tilde{M}} \|\omega\|_m^2 dV_{\tilde{g}} \leq \text{Const} \sum_{k=1}^{\infty} e^{-k} < +\infty.$$

Now if $a = +\infty$, then

$$\int_{\tilde{M}} \|\omega\|_m^2 dV_{\tilde{g}} \leq \text{Const} \sum_{k=1}^{\infty} \rho_k^{2m} e^{-\rho_k} < +\infty.$$

Let f be a smooth $(0,1)$-form in $\tilde{M}$ with values in $\pi^* L$. By Cauchy-Schwarz inequality

$$\left(\int_{\tilde{M}} |\langle f, \omega\rangle_m| dV_{\tilde{g}}\right)^2 \leq \int_{\tilde{M}} \|\omega\|_m^2 dV_{\tilde{g}} \int_{\operatorname{supp} t(\omega)} \|f\|_m^2 dV_{\tilde{g}}.$$

There exist a constants $c > 0$ and d such that $c(L) \geq c.g$ and $\text{Ricci}(g) \geq d.g$ on U. Since $c(\pi^*(L^m)) = m\pi^*(c(L))$, for m sufficiently large we have

$$\text{Ricci}(\tilde{g}) + i\partial\bar{\partial}\Psi + i3C_1\partial\bar{\partial} + m\pi^* c(L) \geq \tilde{g} \quad \text{on} \;\; \pi^{-1}(U).$$

Using the Bochner-Kodaira-Nakano equality in Kählerian geometry [**D**], we have

$$\begin{aligned}&\left(\int_{\tilde{M}} |\langle f, \omega\rangle_m| dV_{\tilde{g}}\right)^2 \\ &\leq \int_{\tilde{M}} \|\omega\|_m^2 dV_{\tilde{g}} \left(\int_{\operatorname{supp} t(\omega)} \|\overline{\partial^*} f\|_m^2 dV_{\tilde{g}} + \int_{\operatorname{supp} t(\omega)} \|\bar{\partial} f\|_m^2 dV_{\tilde{g}}\right),\end{aligned}$$

where $\overline{\partial^*}$ is the formal adjoint of $\bar{\partial}$ acting on $(0,1)$-forms in $L^2(\tilde{M}; (\pi^* L^m, \|?\|_m))$ as an unbounded operator. Since the metric $\tilde{g}$ is complete, the space of smooth $(0,1)$-forms with compact support on $\tilde{M}$ and with values in $\pi^* L^m$ is dense in $\text{Dom}(\overline{\partial^*}) \cap \operatorname{Ker} \bar{\partial}$ with respect to the (singular) graph norm [**D**]:

$$\|f\| := \int_{\tilde{M}} \|f\|_m^2 dV_{\tilde{g}} + \int_{\tilde{M}} \|\overline{\partial^*} f\|_m^2 dV_{\tilde{g}} + \int_{\tilde{M}} \|\bar{\partial} f\|_m^2 dV_{\tilde{g}}.$$

By density, $\forall\, f \in \mathrm{Dom}(\overline{\partial^*}) \cap \mathrm{Ker}\,\bar\partial$ we have

$$\int_{\tilde M} |\langle f, \omega\rangle_m| dV_{\tilde g} \\ \le \left(\int_{\tilde M} \|\omega\|_m^2 dV_{\tilde g}\right)^{\frac12} \left(\int_{\tilde M} \|\overline{\partial^*} f\|_m^2 dV_{\tilde g} + \int_{\tilde M} \|\bar\partial f\|_m^2 dV_{\tilde g}\right)^{\frac12}.$$

Since $\bar\partial\omega = 0$, it suffices to consider the $\bar\partial$-closed forms f. The above inequality become

$$\int_{\tilde M} |\langle f, \omega\rangle_m| dV_{\tilde g} \\ \le \left(\int_{\tilde M} \|\omega\|_m^2 dV_{\tilde g}\right)^{\frac12} \left(\int_{\tilde M} \|\overline{\partial^*} f\|_m^2 dV_{\tilde g}\right)^{\frac12} \quad \forall\, f \in \mathrm{Dom}\,\overline{\partial^*}.$$

By Lemma 4.1.1 in [H], there exists a smooth section σ of $\pi^* L^m$ on $\tilde M$ such that

$$\bar\partial \sigma = \omega \quad \text{and} \quad \int_{\tilde M} \|\sigma\|_m^2 dV_{\tilde g} \le \int_{\tilde M} \|\omega\|_m^2 dV_{\tilde g}.$$

Therefore $\|\sigma\|_m^2$ is integrable on the neighborhood $\phi_k^{-1}(B(0, \frac{R}{8}))$ of z_k, on which

$$\Psi(x) = n \log \|\phi_k(x) - \phi_k(z_k)\|^2.$$

Hence

$$\begin{aligned} \infty &> \int_{\phi_k^{-1}(B(0,\frac{R}{8}))} \|\sigma\|_m^2 dV_{\tilde g} \\ &= \int_{\phi_k^{-1}(B(0,\frac{R}{8}))} \|\sigma\|_{\pi^* L^m} e^{-3C_1 r} e^{-\Psi} dV_{\tilde g} \\ &= \int_{\phi_k^{-1}(B(0,\frac{R}{8}))} \frac{\|\sigma\|_{\pi^* L^m}^2 e^{-3C_1 r}}{\|\phi_k - \phi_k(z_k)\|^{2n}} dV_{\tilde g}. \end{aligned}$$

However $\|\sigma(x)\|_{\pi^* L^m}^2 e^{-3C_1 r(x)}$ is smooth while $\|\phi_k - \phi_k(z_k)\|^{-2n}$ is not locally integrable at z_k, since $\tilde M$ has real dimension $2n$ and $\phi_k(z_k) = 0$. It follows that

$$\sigma(z_k) = 0$$

for $k = 1, 2, 3, \ldots$. If f is the holomorphic function defined on $\tilde M$ in the following way:

$$f := \frac{\sigma}{\otimes_1^m s},$$

then $f(z_k) = e^{r(z_k)} \to \infty$ as $k \to \infty$. It follows that f is unbounded on S.

Case 2. If $\{y_\nu\}$ has no limit point in M, then there exists a holomorphic function f on M such that $|f(y_\nu)|$ is unbounded. Hence we may take $\pi^* f$ as our desired holomorphic function on $\tilde{M}$.

Since $\tilde{M}$ supports a smooth strongly plurisubharmonic function, then $\tilde{M}$ is a Stein manifold.

As mentioned in the introduction, the condition $[\pi^* L] = 0$ in $H^1(\tilde{M}, \mathcal{O}^*)$ is necessary. We give here an example $\tilde{M}$ of a covering of a smooth projective manifold M which is not holomorphically convex and for every positive line bundle L over M, the cohomology class of $\pi^* L$ is not zero in $H^1(\tilde{M}, \mathcal{O}^*)$. An example of M such that $\tilde{M}$ is not holomorphically convex is given in [**N2**], Ex. 4, 5, p. 451. We will use it to verify the nonvanishing of the cohomology class. For the sake of complements, we reproduce the construction.

Suppose a and b are complex numbers such that the elements $(1, 0)$, $(0, 1)$, $(a, 0)$ and $(0, b)$ form a basis for $\mathbf{C}^2$ over $\mathbf{R}$. Let $\Gamma \subset \mathbf{C}^2$ be the lattice generated by these elements over $\mathbf{Z}$, and let $\tilde{\Gamma}$ be the subgroup of Γ generated by $(1, 0), (0, 1), (a, b)$ over $\mathbf{Z}$. Then $\mathbf{C}^2/\Gamma$ is an Abelian variety which is biholomorphic to a product of 1-dimensional tori, and the map

$$\tilde{M} = \mathbf{C}^2/\tilde{\Gamma} \longrightarrow M = \mathbf{C}^2/\Gamma$$

is a covering map. If $1, a$ and b are linearly independent over $\mathbf{Z}$, then the only holomorphic functions on $\tilde{M}$ are the constants.

Now let L be a positive holomorphic line bundle over M and suppose that the cohomology class of $\pi^* L$ vanishes. Let s be the holomorphic section of $\pi^* L$ without zeros as above. Since $\mathcal{O}(\tilde{M}) = \mathbf{C}$, every holomorphic section h of $\pi^* L$ over $\tilde{M}$ can be written as $h = c.s$ where $c \in \mathbf{C}$. By Corollary 4.3 in [**N2**], there exists a positive integer p and a holomorphic section σ of $\pi^* L^p$ on $\tilde{M}$ such that $\|\sigma\|$ is unbounded on any infinite subset without limit point in $\tilde{M}$. Therefore $\sigma = c. \otimes_1^p s$ which implies $\lim_{x \to \partial \tilde{M}} \|s\| = +\infty$. Hence the strongly Psh function $\phi = -\log \|s\|^2$ is bounded from above on $\tilde{M}$. Let $\pi_1 : \mathbf{C}^2 \to \tilde{M}$ be the universal covering, and define a strongly Psh function θ on $\mathbf{C}^2$ by

$$\theta := \phi o \pi_1.$$

Since $\mathbf{C}^2$ is parabolic, the only Psh function bounded from above are the constants. This implies that $\|s\| =$ constante which contradicts the growth of s. Hence $[\pi^* L] \neq 0$ in $H^1(\tilde{M}\mathcal{O}^*)$ for every positive line bundle L on M.

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