PACIFIC JOURNAL OF MATHEMATICS
Vol. 185, No. 2, 1998

$\mathbf{R}^2$-IRREDUCIBLE UNIVERSAL COVERING SPACES OF $\mathbf{P}^2$-IRREDUCIBLE OPEN 3-MANIFOLDS

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An irreducible open 3-manifold W is $\mathbf{R}^2$-irreducible if it contains no non-trivial planes, i.e. given any proper embedded plane Π in W some component of $W-\Pi$ must have closure an embedded halfspace $\mathbf{R}^2 \times [0,\infty)$. In this paper it is shown that if M is a connected, $\mathbf{P}^2$-irreducible, open 3-manifold such that $\pi_1(M)$ is finitely generated and the universal covering space $\widetilde{M}$ of M is $\mathbf{R}^2$-irreducible, then either $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$ or $\pi_1(M)$ is a free product of infinite cyclic groups and fundamental groups of closed, connected surfaces other than S^2 or $\mathbf{P}^2$. Given any finitely generated group G of this form, uncountably many $\mathbf{P}^2$-irreducible, open 3-manifolds M are constructed with $\pi_1(M) \cong G$ such that the universal covering space $\widetilde{M}$ is $\mathbf{R}^2$-irreducible and not homeomorphic to $\mathbf{R}^3$; the $\widetilde{M}$ are pairwise non-homeomorphic. Relations are established between these results and the conjecture that the universal covering space of any irreducible, orientable, closed 3-manifold with infinite fundamental group must be homeomorphic to $\mathbf{R}^3$.

1. Introduction.

Suppose M is a connected, $\mathbf{P}^2$-irreducible, open 3-manifold with $\pi_1(M)$ finitely generated and non-trivial. It is easy to construct examples of such M for which the universal covering space $\widetilde{M}$ is not homeomorphic to $\mathbf{R}^3$. Start with any 3-manifold N satisfying the given conditions. Let U be a **Whitehead manifold**, i.e. an irreducible, contractible, open 3-manifold which is not homeomorphic to $\mathbf{R}^3$(see e.g. [**17**], [**4**]). Choose end-proper embeddings of $[0,\infty)$ in each of N and U. (A map between manifolds is **end-proper** if pre-images of compact sets are compact; it is ∂-**proper** if the pre-image of the boundary is the boundary; it is **proper** if it has both these properties. These terms are applied to a submanifold if its inclusion map has the corresponding property.) Let X and Y be the exteriors of these rays. (The **exterior** of a submanifold is the closure of the complement of a regular neighborhood of it.) ∂X and ∂Y are each planes. We identify them to obtain a $\mathbf{P}^2$-irreducible open 3-manifold M with $\pi_1(M) \cong \pi_1(N)$. Let

$p : \widetilde{M} \to M$ be the universal covering map. Then $\widetilde{M}$, $p^{-1}(X)$, and $p^{-1}(Y)$ are $\mathbf{P}^2$-irreducible [**5**]. Each component $\widetilde{Y}$ of $p^{-1}(Y)$ has interior $\widetilde{U}$ homeomorphic to U and so contains a compact, connected subset J which does not lie in a 3-ball in $\widetilde{U}$. If $\widetilde{M}$ were homeomorphic to $\mathbf{R}^3$ then J would lie in a 3-ball B in $\widetilde{M}$. Standard general position and minimality arguments applied to ∂B and $\partial\widetilde{Y}$ would then yield a 3-ball B' in $\widetilde{U}$ containing J, a contradiction. Alternatively, one could use the Tucker Compactification Theorem [**15**] to obtain a compact polyhedron K in $\widetilde{U}$ such that some component V of $\widetilde{U} - K$ has non-finitely generated fundamental group. But this is impossible since the union of V and $\widetilde{M} - \widetilde{U}$ is a component of $\widetilde{M} - K$ whose fundamental group is isomorphic to $\pi_1(V)$.

In this example $\partial\widetilde{Y}$ is a **non-trivial plane** in $\widetilde{M}$, i.e. a proper plane Π such that no component of $\widetilde{M} - \Pi$ has closure homeomorphic to $\mathbf{R}^2 \times [0, \infty)$ with $\Pi = \mathbf{R}^2 \times \{0\}$. This paper shows that it is harder to find examples if one rules out this behavior by requiring that $\widetilde{M}$ be $\mathbf{R}^2$**-irreducible** in the sense that, in addition to being irreducible, it contains no non-trivial planes.

Define a **closed surface group** to be the fundamental group of a closed, connected 2-manifold.

Theorem 1. *Let M be a connected, $\mathbf{P}^2$-irreducible, open 3-manifold with $\pi_1(M)$ finitely generated. If the universal covering space $\widetilde{M}$ of M is $\mathbf{R}^2$-irreducible, then either*

(1) *$\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$ or*

(2) *$\pi_1(M)$ is a free product of infinite cyclic groups and infinite closed surface groups.*

The second possibility can be disjoint from the first.

Theorem 2. *Suppose G is a free product of finitely many infinite cyclic groups and infinite closed surface groups. Then there is a $\mathbf{P}^2$-irreducible open 3-manifold M such that $\pi_1(M) \cong G$ and $\widetilde{M}$ is an $\mathbf{R}^2$-irreducible Whitehead manifold. Moreover, for each given G there are uncountably many such M for which the $\widetilde{M}$ are pairwise non-homeomorphic.*

This generalizes an example of Scott and Tucker [**13**] for which G is infinite cyclic. (We remark that their example has a mistake. It is, however, easy to correct. See Section 4 for details.)

These results have a bearing on the following well-known problem.

Conjecture 1 (Universal Covering Conjecture)**.** *Let X be a closed, connected, irreducible, orientable 3-manifold with $\pi_1(X)$ infinite. Then the universal covering space $\widetilde{X}$ of X is homeomorphic to $\mathbf{R}^3$.*

Since there are only countably many homeomorphism types of closed 3-

manifolds, Theorem 2 implies that there must exist uncountably many $\mathbf{R}^2$-irreducible Whitehead manifolds $\widetilde{M}$ which cover open 3-manifolds M with $\pi_1(M) \cong G$ but cannot cover a closed 3-manifold. This generalizes a result of Tinsley and Wright [**14**] which shows that there must exist uncountably many non-$\mathbf{R}^2$-irreducible Whitehead manifolds $\widetilde{M}$ which cover open 3-manifolds M with $\pi_1(M)$ infinite cyclic but cannot cover a closed 3-manifold. Unfortunately this argument does not provide any *specific* such examples. Specific examples of non-$\mathbf{R}^2$-irreducible Whitehead manifolds $\widetilde{M}$ which cover open 3-manifolds M with $\pi_1(M)$ infinite cyclic or, more generally, a countable free group, but cannot cover a closed 3-manifold are given in [**10**] and [**11**], respectively. At the time of this writing the problem of providing specific examples of $\mathbf{R}^2$-irreducible Whitehead manifolds which non-trivially cover other open 3-manifolds but cannot cover a closed 3-manifold is still open.

One can make several conjectures related to Conjecture 1. We consider the selection below. In all of them G is assumed to be a finitely generated group of covering translations acting on a Whitehead manifold W with quotient a 3-manifold M.

Conjecture 2. *G is a free product of infinite cyclic groups and fundamental groups of ∂-irreducible Haken manifolds.*

Conjecture 3. *G is a free group or contains an infinite closed surface group.*

Conjecture 4. *If W is $\mathbf{R}^2$-irreducible, then G is a free product of infinite cyclic groups and infinite closed surface groups.*

A proper plane Π in W is **equivariant** if for each $g \in G$ either $g(\Pi) = \Pi$ or $\Pi \cap g(\Pi) = \emptyset$.

Conjecture 5 (Special Equivariant Plane Conjecture)**.** *If G is not a free product of infinite cyclic groups and infinite closed surface groups, then W contains a non-trivial equivariant plane.*

Conjecture 6 (Equivariant Plane Conjecture)**.** *If W contains a non-trivial plane, then it contains a non-trivial equivariant plane.*

These conjectures are related as follows.

Theorem 3. $(4) \Leftarrow (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (5) \Leftarrow (4+6)$.

Theorems 1 and 3 are proven in Section 2. Theorem 2 is proven in Sections 3-7. Section 3 presents a modified version of the criterion used by Scott and

Tucker [13] for showing that a 3-manifold is $\mathbf{R}^2$-irreducible. Sections 4 and 5 treat, respectively, the special cases in which G is an infinite cyclic group and an infinite closed surface group. The constructions and notation of these special cases are used in Section 6, which treats the general case. Section 7 shows how to get uncountably many M with non-homeomorphic $\widetilde{M}$ for each group G.

2. The proofs of Theorems 1 and 3.

Lemma 2.1. *Let M be a connected, $\mathbf{P}^2$-irreducible, open 3-manifold. Let Q be a compact, connected, 3-dimensional submanifold of M such that ∂Q is incompressible in M and $\pi_1(Q)$ is not an infinite closed surface group. Let $p : \widetilde{M} \to M$ be the universal covering map and G the group of covering translations. Let $\widetilde{Q}$ be a component of $p^{-1}(Q)$. Then:*

(1) *Each component of $p^{-1}(\partial Q)$ is a plane.*

(2) *There is no component Π of $\partial\widetilde{Q}$ which is invariant under the subgroup G_0 of G consisting of those covering translations which leave $\widetilde{Q}$ invariant.*

(3) *If each component of $\partial\widetilde{Q}$ is a trivial plane, then $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$.*

Proof. (1) follows from the incompressibility of ∂Q in M.

Suppose S is a component of ∂Q and Π is a component of $p^{-1}(S)$ which is invariant under G_0. Since the restriction of p to $\widetilde{Q}$ is the universal covering space of Q and the restriction of G_0 to $\widetilde{Q}$ is the group of covering translations we have that $\pi_1(S) \to \pi_1(Q)$ is an isomorphism, contradicting our assumption on $\pi_1(Q)$. This establishes (2).

We now prove (3). Suppose that each component Π of $\partial\widetilde{Q}$ bounds an end-proper halfspace H_Π in $\widetilde{M}$. Let K_Π be the closure of the component of $\widetilde{M} - \Pi$ which does not contain int $\widetilde{Q}$.

Assume that for all such Π we have $H_\Pi = K_\Pi$. Then $\widetilde{M}$ is the union of $\widetilde{Q}$ and an open collar attached to $\partial\widetilde{Q}$, hence $\widetilde{M}$ is homeomorphic to int $\widetilde{Q}$. Since Q is Haken, the Waldhausen Compactification Theorem [16] implies that $\widetilde{Q}$ is homeomorphic to a closed 3-ball minus a closed subset of its boundary, hence int $\widetilde{Q}$ is homeomorphic to $\mathbf{R}^3$, and we are done.

Thus we may assume that for some Π we have $H_\Pi \neq K_\Pi$. Then $H_\Pi \cap K_\Pi = \Pi$ and $H_\Pi \cup K_\Pi = \widetilde{M}$. Now G_0 has an element g such that $g(\Pi) \neq \Pi$. Since $\widetilde{Q} \subseteq H_\Pi$ and $g(\widetilde{Q}) = \widetilde{Q}$ we must have $g(K_\Pi) \subseteq H_\Pi$. Since $\mathbf{R}^2 \times [0, \infty)$ is $\mathbf{R}^2$-irreducible (see e.g. [9]) it follows that K_Π is homeomorphic to $\mathbf{R}^2 \times [0, \infty)$. Thus $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$. □

Proof of Theorem 1. By passing to a covering space of M, if necessary, we may assume that $\pi_1(M)$ is indecomposable with respect to free products and is neither an infinite cyclic group nor an infinite closed surface group. Let C be the Scott compact core [**12**] of M, i.e. C is a compact, connected, 3-dimensional submanifold of M such that $\pi_1(C) \to \pi_1(M)$ is an isomorphism. The conditions on $\pi_1(M)$ imply that ∂C is incompressible in M. We thus can apply Lemma 2.1 with $Q = C$ to finish the proof. □

Proof of Theorem 3. We first show that (1) ⇒ (2) ⇒ (3) ⇒ (1). If (1) is true, then M must be non-compact; this follows from the fact that if M were closed and non-orientable, then it would be Haken and so have universal covering space homeomorphic to $\mathbf{R}^3$. Let C be the Scott compact core for M. Since M is irreducible we may assume that no component of ∂C is a 2-sphere; it follows that C is irreducible. If C is ∂-irreducible, then we are done. If C is not ∂-irreducible, then there is a finite set of compressing disks for ∂C in C which express C as a ∂-connected sum of 3-balls and ∂-irreducible Haken manifolds, thus yielding (2). Clearly (2) ⇒ (3). Suppose (3) is true and M is closed. If G is free, then M is by [**2**, Theorem 5.2] a connected sum of 2-sphere bundles over S^1, hence is not aspherical, hence W is not contractible. If G contains an infinite closed surface group, then by a result of Hass, Rubinstein, and Scott [**1**] W is homeomorphic to $\mathbf{R}^3$.

Clearly Theorem 1 and the fact that M cannot be closed and non-orientable show that (1) ⇒ (4).

We now show that (1) ⇒ (5). Let C be the Scott compact core of M. Then the assumptions on G imply that there is a set of compressing disks for ∂C in C such that some component Q of C split along this collection of disks satisfies the hypotheses of Lemma 2.1. Thus any component of the pre-image of ∂Q is an equivariant non-trivial plane.

We next show that (5) ⇒ (1). Assume M is closed. If $\pi_1(M)$ is a free product of infinite cyclic groups and infinite closed surface groups, then we apply (3) to obtain (1). If $\pi_1(M)$ is not such a group, then the existence of an equivariant plane, together with the compactness of M, implies that M is Haken, and so (1) follows by Waldhausen [**16**].

Finally we show that (4 + 6) ⇒ (1). If W is $\mathbf{R}^2$-irreducible, then (4) implies the hypothesis of (2), hence implies (1). If W is not $\mathbf{R}^2$-irreducible, then (6) implies as before that M is Haken, thus (1) holds. □

3. Nice quasi-exhaustions and $\mathbf{R}^2$-irreducibility.

We shall reformulate a criterion due to Scott and Tucker [**13**] for a $\mathbf{P}^2$-irreducible open 3-manifold to be $\mathbf{R}^2$-irreducible. A proper plane Π in an open 3-manifold W is **homotopically trivial** if for any compact subset C of W the inclusion map of Π is end-properly homotopic to a map whose image is disjoint from C.

Lemma 3.1. *Let W be an irreducible, open 3-manifold, and let Π be a proper plane in W. If Π is homotopically trivial, then Π is trivial.*

Proof. This is Lemma 4.1 of [**13**]. □

Lemma 3.2. *Let W be a connected, irreducible, open 3-manifold, and let $\{C_n\}_{n\geq 1}$, be a sequence of compact 3-dimensional submanifolds of W such that $C_n \subseteq \text{int } C_{n+1}$ and*
(1) *each C_n is irreducible,*
(2) *each ∂C_n is incompressible in $W - \text{int } C_n$,*
(3) *if D is a proper disk in C_{n+1} which is in general position with respect to ∂C_n such that ∂D is not null-homotopic in ∂C_{n+1}, then $D \cap \partial C_n$ has at least two components which are not null-homotopic in ∂C_n and bound disjoint disks in D.*

Then any proper plane in W can be end-properly homotoped off C_n for any n.

Proof. This is Lemma 4.2 of [**13**]. □

The precise criterion we shall use is as follows.

Lemma 3.3. *Let W be a connected, irreducible, open 3-manifold. Suppose that for each compact subset K of W there is a sequence $\{C_n\}_{n\geq 1}$ of compact 3-dimensional submanifolds such that $C_n \subseteq \text{int } C_{n+1}$ and*
(1) *each C_n is irreducible,*
(2) *each ∂C_n is incompressible in $W - \text{int } C_n$ and has positive genus,*
(3) *each $C_{n+1} - \text{int } C_n$ is irreducible, ∂-irreducible, and anannular,*
(4) *$K \subseteq C_1$.*

Then W is $\mathbf{R}^2$-irreducible.

Proof. Let D be a disk as in part (iii) of Lemma 3.2. If every component of $D \cap \partial C_n$ is null-homotopic in ∂C_n, then one can isotop D so that $D \cap C_n = \emptyset$ and hence ∂C_{n+1} is compressible in $C_{n+1} - \text{int } C_n$. If only one component α of $D \cap \partial C_n$ is not null-homotopic in ∂C_n, then $\partial D \cup \alpha$ bounds an annulus A which can be isotoped so that $A \cap \partial C_n = \alpha$, hence $C_{n+1} - \text{int } C_n$ is not anannular. If no two of the components of $D \cap \partial C_n$ which are not null-homotopic in ∂C_n bound disjoint disks in D, then these components must be nested on D. We can isotop D to remove null-homotopic components and then intermediate annuli to again get an incompressible annulus joining ∂C_{n+1} to ∂C_n. Now apply Lemma 3.2 and then Lemma 3.1. □

Let $\{C_n\}$ be a sequence of compact, connected 3-dimensional submanifolds of an irreducible, open 3-manifold W such that $C_n \subseteq \text{int } C_{n+1}$ such that $W-$ int C_n has no compact components. This will be called a **quasi-exhaustion** for W. A quasi-exhaustion for W whose union is W is an **exhaustion** for W. A quasi-exhaustion is **nice** if it satisfies conditions (1)-(3) of Lemma 3.3. Thus that lemma can be rephrased by saying that if every compact subset of W is contained in the first term of a nice quasi-exhaustion, then W is $\mathbf{R}^2$-irreducible.

We shall need some tools for constructing Whitehead manifolds with nice quasi-exhaustions. Define a compact, connected 3-manifold Y to be **nice** if it is is $\mathbf{P}^2$-irreducible, ∂-irreducible, and anannular, it contains a two-sided proper incompressible surface, and it is not a 3-ball; define it to be **excellent** if, in addition, every connected, proper, incompressible surface of zero Euler characteristic in Y is ∂-parallel. So in particular an excellent 3-manifold is anannular and atoroidal while a nice 3-manifold is anannular but may contain a non-∂-parallel incompressible torus. We note that by the torus theorem and Thurston's hyperbolization theorem a nice 3-manifold is excellent if and only if it has a hyperbolic structure.

A proper 1-manifold in a compact 3-manifold is **excellent** if its exterior is excellent; it is **poly-excellent** if the union of each non-empty subset of the set of its components is excellent.

Lemma 3.4. *Every proper* 1*-manifold in a compact, connected* 3*-manifold whose boundary contains no* 2*-spheres or projective planes is homotopic rel* ∂ *to an excellent proper* 1*-manifold.*

Proof. This is a special case of Theorem 1.1 of [7]. □

Define a k**-tangle** to be a disjoint union of k proper arcs in a 3-ball.

Lemma 3.5. *For all* $k \geq 1$ *poly-excellent* k*-tangles exist.*

Proof. This is Theorem 6.3 of [8]. □

We shall also need the following criterion for gluing together excellent 3-manifolds to get an excellent 3-manifold.

Lemma 3.6. *Let* Y *be a compact, connected* 3*-manifold. Let* S *be a compact, proper, two-sided surface in* Y*. Let* Y' *be the* 3*-manifold obtained by splitting* Y *along* S*. Let* S' *and* S'' *be the two copies of* S *which are identified to obtain* Y*. If each component of* Y' *is excellent,* S'*,* S''*, and*

$(\partial Y') - \mathrm{int}\,(S' \cup S'')$ *are incompressible in* Y', *and each component of* S *has negative Euler characteristic, then* Y *is excellent.*

Proof. This is Lemma 2.1 of [**7**]. □

4. The infinite cyclic case.

Theorem 2 of this paper was motivated by an example due to Scott and Tucker [**13**] of an open 3-manifold called M_7 which has infinite cyclic fundamental group and whose universal covering space $\widetilde{M}_7$ is a Whitehead manifold which was claimed to be $\mathbf{R}^2$-irreducible. However, on closer inspection this turns out not to be the case. We briefly describe the mistake in the construction of M_7 which allows $\widetilde{M}_7$ to have non-trivial planes and the error in the proof which allows this to go undetected. Fortunately this problem is very easy to fix, and we indicate how to do so. We then give a general procedure for building $\mathbf{P}^2$-irreducible, open 3-manifolds with infinite cyclic fundamental groups whose universal covering spaces are $\mathbf{R}^2$-irreducible Whitehead manifolds. The construction introduced here will be incorporated into that for the general case in Section 6.

The example M_7 has an exhaustion $\{C_n\}$ by genus two handlebodies. The embedding of C_n in C_{n+1} is factored through an intermediate genus two handlebody Y_n as described in Figure 8 of [**13**]. A closer examination of Figure 8(c) shows that the embedding of Y_n in C_{n+1} is actually isotopic to the standard embedding of a concentric copy of C_{n+1} in C_{n+1}. This can be seen by regarding Y_n as the result of attaching a 1-handle to a solid torus concentric with T_{n+1} and then sliding one end of the 1-handle so as to undo the Whitehead clasp shown in R_{n+1}. The result is that M_7 is homeomorphic to the monotone union of genus two handlebodies embedded as in Figure 8(b). The corresponding monotone union of the meridional disks of the lower solid tori in that figure is a proper plane whose preimage in $\widetilde{M}_7$ is an equivariant family of non-trivial planes. The error in the proof that $\widetilde{M}_7$ is $\mathbf{R}^2$-irreducible occurs in the proof of Lemma 4.8, where it is asserted that adjacent components of the link L^r_n in the handlebody V^r_{n+1} as shown in Figure 9(d) are linked. While it is true that they are linked in $\mathbf{R}^3$, it is not true that they are linked in V^r_{n+1}. There is a proper disk D in V^r_{n+1} which separates them. This can be seen as follows. Note that L^r_n is isotopic to a family of disjoint simple closed curves in ∂V^r_{n+1}. Given a component J of L^r_n, let $E \times [-1, 1]$ be a regular neighborhood of a proper disk E in V^r_{n+1} such that $E \cap L^r_n = E \cap J$ is a single transverse intersection point. Let J' be the curve in ∂V^r_{n+1} isotopic to J. Then the band sum of $E \times \{-1\}$ and $E \times \{1\}$ formed by using a band which follows the portion of J' which lies outside of $E \times (-1, 1)$ is the required disk D.

The remainder of Scott and Tucker's proof is correct, and the problem just described can easily be corrected as follows. Replace the Whitehead clasp shown in the portion R_{n+1} of Figure 8(c) by the true lover's tangle (Figure 1 on page 79 of [6]). This will induce a similar replacement in Figures 9(c,d). It follows from Proposition 4.1 of [6] that this tangle is excellent. It then follows from Lemma 3.1 of [6] that the exterior of the new link L_n^r in V_{n+1}^r is irreducible and ∂-irreducible. Hence Lemma 4.8 of [13] now holds.

We now describe our general procedure for constructing $\mathbf{R}^2$-irreducible open 3-manifolds which are infinite cyclic covering spaces.

Let $P_n = D_n \times [0,1]$, where D_n is the disk of radius n. We call P_n a **pillbox**. Identify $D_n \times \{0\}$ with $D_n \times \{1\}$ to obtain a solid torus Q_n. Let R_n be a solid torus and H_n a 1-handle $D \times [0,1]$ joining $\partial D_n \times (0,1)$ to ∂R_n. Let $V_n = P_n \cup H_n \cup R_n$ and $M_n = Q_n \cup H_n \cup R_n$. We call V_n an **eyebolt**. We embed M_n in the interior of M_{n+1} as follows.

We choose a collection of arcs $\theta_0, \theta_1, \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon$ in P_{n+1} which satisfy certain conditions described below. θ_0, θ_1, and α_0 meet in a common endpoint in int P_{n+1} but are otherwise disjoint. The other endpoints of θ_0 and α_0 lie in (int $D_{n+1}) \times \{0\}$; that of θ_1 lies in (int $D_{n+1}) \times \{1\}$. We let $\theta = \theta_0 \cup \theta_1$. All the other arcs are proper arcs in P_{n+1} which are disjoint from each other and from $\theta \cup \alpha_0$. γ_1, β_2, and δ_2 run from (int $D_{n+1}) \times \{0\}$ to itself. γ_2, β_1, and δ_1 run from (int $D_{n+1}) \times \{1\}$ to itself. α_1 runs from (int $D_{n+1}) \times \{0\}$ to (int $D_{n+1}) \times \{1\}$. α_2 runs from (int $D_{n+1}) \times \{1\}$ to int $(P_{n+1} \cap H_{n+1})$. ε runs from int $(P_{n+1} \cap H_{n+1})$ to itself. We denote the image in Q_{n+1} of an arc by the same symbol, relying on the context to distinguish an arc in P_{n+1} from its image in Q_{n+1}. We require that θ be a simple closed curve in Q_{n+1} and that $\alpha_0 \cup \beta_1 \cup \gamma_1 \cup \delta_1 \cup \alpha_1 \cup \beta_2 \cup \gamma_2 \cup \delta_2 \cup \alpha_2$ is an arc consisting of subarcs which occur in the given order. We require that any union of these arcs which contains α_0 and at least one other arc has excellent exterior in P_{n+1}, and that the same is true for any union of these arcs which contains neither θ_0, θ_1, nor α_0. This can be achieved as follows. Note that the exterior of α_0 in P_{n+1} is a 3-ball B. Choose a poly-excellent 11-tangle in B and then slide its endpoints so that exactly two of the arcs meet a regular neighborhood of α_0. Extend them to meet α_0 in the desired configuration.

Next let κ_1, κ_2, and κ_3 be product arcs in H_{n+1} joining (int $D) \times \{0\}$ to (int $D) \times \{1\}$. Let $R_n \subseteq$ int R_{n+1} be any null-homotopic embedding. Let λ_1 and λ_2 be disjoint proper arcs in $R_{n+1} -$ int R_n with λ_1 joining int $(H_{n+1} \cap R_{n+1})$ to itself and λ_2 joining int $(H_{n+1} \cap R_{n+1})$ to ∂R_n. We require $\lambda_1 \cup \lambda_2$ to be excellent in $R_{n+1} -$ int R_n. We also require that these arcs, together with ε, fit into an arc whose subarcs form the sequence $\kappa_1, \lambda_1, \kappa_2, \varepsilon, \kappa_3, \lambda_2$ and that κ_1 meets α_2 in a common endpoint.

Now we embed P_n in P_{n+1} as a regular neighborhood of the arc θ so that the two disks of $P_n \cap (D_{n+1} \times \{0,1\})$ are identified to give an embedding of Q_n in Q_{n+1}. Note that these embeddings are not consistent with the product structures. From the discussion above we have an arc ω in $M_{n+1} - \text{int}\,(Q_n \cup R_n)$ running from ∂Q_n to ∂R_n. We embed H_n as a regular neighborhood of ω. We change notation slightly by now letting α_0 be the old α_0 minus its intersection with the interior of Q_n.

We let M be the direct limit of the M_n and let $p : \widetilde{M} \to M$ be the universal covering map. $p^{-1}(Q_n) = p^{-1}(P_n)$ is the union of pillboxes $P_{n,j} = D_n \times [j, j+1]$ meeting along the $D_n \times \{j\}$ to form $D_n \times \mathbf{R}$. Note that this embedding is not the product embedding. $p^{-1}(R_n)$ is a disjoint union of solid tori $R_{n,j}$. $p^{-1}(H_n)$ is a disjoint union of 1-handles $H_{n,j}$ joining $\partial D_n \times (j, j+1)$ to $\partial R_{n,j}$; these are regular neighborhoods of lifts ω_j of ω. $p^{-1}(M_n) = p^{-1}(V_n)$ is the union of $p^{-1}(P_n)$, $p^{-1}(H_n)$, and $p^{-1}(R_n)$. It is the union of eyebolts $V_{n,j} = P_{n,j} \cup H_{n,j} \cup R_{n,j}$ meeting along the $D_n \times \{j\}$. $\widetilde{M}$ is the nested union of the $p^{-1}(M_n)$.

Let $\Sigma_n^m = \cup_{j=-m}^m V_{n,j}$ and $\Lambda_n^m = P_{n,-(m+1)} \cup P_{n,m+1}$. Let $\Phi_1^m = \emptyset$, and, for $n \geq 2$, let $\Phi_n^m = \cup_{j=m+2}^{m+n} (P_{n,-j} \cup P_{n,j})$. Note that Λ_n^m and Φ_n^m (for $n \geq 2$) are each disjoint unions of two 3-balls, $\Lambda_n^m \cap \Sigma_n^m$ is a pair of disjoint disks, and (for $n \geq 2$) so is $\Lambda_n^m \cap \Phi_n^m$. Define $C_n^m = \Sigma_n^m \cup \Lambda_n^m \cup \Phi_n^m$.

Lemma 4.1. *$\{C_m^m\}$ is an exhaustion for $\widetilde{M}$. Each C^m is a nice quasi-exhaustion.*

Proof. $C_n^m \subseteq \text{int}\, C_{n+1}^m$, and $C_n^m \subseteq C_n^{m+1}$. A given compact subset K of $\widetilde{M}$ lies in some $p^{-1}(M_n)$ and thus in a finite union of $V_{n,j}$ and hence in some $\Sigma_n^m \subseteq C_n^m \subseteq C_q^q$, where $q = \max\{m, n\}$. Thus $\{C_m^m\}$ is an exhaustion for $\widetilde{M}$.

C_n^m is a cube with $2m+1$ handles. Let $Y = C_{n+1}^m - \text{int}\, C_n^m$. We will show that Y is excellent by successive applications of Lemma 3.6.

Consider a $P_{n+1,j}$ contained in C_{n+1}^m. If $|j| < m$, then it meets C_n^m in a regular neighborhood of the union of the j^{th} copies of all the arcs in P_{n+1}. Thus $Y \cap P_{n+1,j}$ is excellent, and Lemma 3.6 implies that the union of these $Y \cap P_{n+1,j}$ is excellent. For $|j| \geq m$ some care must be taken so that one is always gluing excellent 3-manifolds along surfaces of the appropriate type. Note that $Y \cap (P_{n+1,m} \cup P_{n+1,m+1} \cup \cdots \cup P_{n+1,m+n-1} \cup P_{n+1,m+n})$ is equal to the exterior of the m^{th} copy of all the arcs but β_1 and δ_1 in $P_{n+1,m}$ together with the exterior of the $(m+1)^{st}$ copy of β_2, δ_2, and θ in $P_{n+1,m+1}$, the exterior of the j^{th} copy of θ in $P_{n+1,j}$ for $m+1 < j < m+n$, and the 3-ball $P_{n+1,m+n}$. This space is homeomorphic to the exterior of the m^{th} copy of all the arcs but β_1, δ_1, and θ_1 in $P_{n+1,m+1}$ together with the exterior of the $(m+1)^{st}$ copy of β_2 and δ_2 in $P_{n+1,m+1}$, and the 3-ball consisting of the union of the $P_{n+1,j}$ for which $m+1 < j \leq m+n$. This can be seen by taking the arc consisting of the m^{th} copy of θ_1 and the j^{th} copy of θ for $m < j < m+n$ and retracting it onto the endpoint in which it meets the rest of the graph. This space is then excellent by Lemma 3.6. Similar remarks apply for $j \leq -m$, so these spaces can be added on to get that $Y \cap \cup_{j=-(m+n)}^{m+n} P_{n+1,j}$ is excellent.

We fill in the remainder of Y by adding the exteriors of the j^{th} copies of κ_1, κ_2, and κ_3 in $H_{n+1,j}$ and $\lambda_1 \cup \lambda_2$ in $R_{n+1,j} - \text{int } R_{n,j}$ for $|j| \leq m$. Since the first of these spaces is a product the union of the two spaces is homeomorphic to the second space, and Lemma 3.6 applies to complete the proof that Y is excellent.

It remains to show that each ∂C_n^m is incompressible in $\widetilde{M} - \text{int } C_n^m$. Since each $C_{n+s+1}^m - \text{int } C_{n+s}^m$ is ∂-irreducible we have that ∂C_n^m is incompressible in $C_{n+q}^m - \text{int } C_n^m$ for each $q \geq 1$. $p^{-1}(M_{n+q})$ is the union of C_{n+q}^m and the closure of $p^{-1}(M_{n+q}) - C_{n+q}^m$. These two sets meet in a collection of disjoint disks. It follows that ∂C_n^m is incompressible in $p^{-1}(M_{n+q}) - \text{int } C_n^m$. Since $\widetilde{M}$ is the nested union of the $p^{-1}(M_{n+q})$ over all $q \geq 1$ we have the desired result. □

5. The surface group case.

Let F be a closed, connected surface other than S^2 or $\mathbf{P}^2$. Let $n \geq 1$. Regard F as being obtained from a $2k$-gon E, $k \geq 2$, by identifying sides s_i and s_i', $1 \leq i \leq k$. This induces an identification of the lateral sides $S_i = s_i \times [-n, n]$ and $S_i' = s_i' \times [-n, n]$ of the **prism** $P_n = E \times [-n, n]$ which yields $Q_n = F \times [-n, n]$. Let R_n be a solid torus and H_n a 1-handle $D \times [0, 1]$. Let $V_n = P_n \cup H_n \cup R_n$, where $H_n \cap R_n = D \times \{1\}$ is a disk in ∂R_n, and $H_n \cap P_n = D \times \{0\}$ is a disk in $(\text{int } E) \times \{1\}$. We again call V_n an **eyebolt**. It is a solid torus whose image under the identification is $M_n = Q_n \cup H_n \cup R_n$, a space homeomorphic to the ∂-connected sum of $F \times [-n, n]$ and a solid torus.

We define an open 3-manifold M by specifying an embedding of M_n in the interior of M_{n+1} and letting M be the direct limit. The inclusion $[-n, n] \subseteq [-(n+1), n+1]$ induces $P_n \subseteq P_{n+1}$ and hence $Q_n \subseteq Q_{n+1}$. We let $R_n \subseteq \text{int } R_{n+1}$ be any null-homotopic embedding. Again the interesting part of the embedding will be that of H_n in M_{n+1}. It will be the regular neighborhood of a certain arc ω in $M_{n+1} - \text{int}\,(Q_n \cup R_n)$ joining ∂Q_n to ∂R_n.

The arc ω is the union of $4k+7$ arcs any two of which are either disjoint or have one common endpoint. The $4k+2$ arcs $\alpha_0, \alpha_i, \beta_i, \gamma_i, \delta_i$, $1 \leq i \leq k$, and ε lie in $E \times [n, n+1]$ and are identified with their images in Q_{n+1}; the three arcs κ_1, κ_2, and κ_3 lie in H_{n+1}, and the two arcs λ_1 and λ_2 lie in R_{n+1}. These arcs will have special properties to be described later. We first describe their combinatorics. The arcs in P_{n+1} are all proper arcs in $E \times [n, n+1]$. α_0 runs from $(\text{int } E) \times \{n\}$ to int S_1. For $1 \leq i < k$, α_i runs from int S_i to int S_{i+1}. α_k runs from int S'_k to int $(P_{n+1} \cap H_{n+1})$. For $1 \leq i \leq k$, β_i and δ_i each run from int S'_i to itself, while γ_i runs from int S_i to itself. These arcs are chosen so that under the identification their endpoints match up in such a way as to give an arc which follows the sequence $\alpha_0, \beta_1, \gamma_1, \delta_1, \alpha_1, \ldots, \beta_k, \gamma_k, \delta_k, \alpha_k$. We require ε to run from int $(P_{n+1} \cap H_{n+1})$ to itself. κ_1, κ_2, and κ_3 are product arcs in H_{n+1} lying in $(\text{int } D) \times [0, 1]$. λ_1 and λ_2 are proper arcs in $R_{n+1} - \text{int } R_n$, with λ_1 running from int $(H_{n+1} \cap R_{n+1})$ to itself and λ_2 running from int $(H_{n+1} \cap R_{n+1})$ to ∂R_n. These arcs are chosen so as to fit together into the sequence $\kappa_1, \lambda_1, \kappa_2, \varepsilon, \kappa_3, \lambda_2$ with the endpoint of κ_1 other than $\kappa_1 \cap \lambda_1$ being the same as the endpoint of α_k other than $\alpha_k \cap \delta_k$. This gives ω.

We now describe the special properties required of these arcs. We require that $\alpha_0 \cup \beta_1 \cup \gamma_1 \cup \delta_1 \cup \alpha_1 \cup \cdots \cup \beta_k \cup \gamma_k \cup \delta_k \cup \varepsilon$ be a poly-excellent $(4k+2)$-tangle in $E \times [n, n+1]$ and $\lambda_1 \cup \lambda_2$ to be an excellent 1-manifold in $R_{n+1} - \text{int } R_n$.

We now consider the universal covering map $p : \widetilde{M} \to M$. Our goal is to construct a sequence $\{C^m\}$ of nice quasi-exhaustions whose diagonal $\{C^m_m\}$ is an exhaustion for $\widetilde{M}$.

The universal covering space $\widetilde{F}$ of F is tesselated by copies E_j of E. We fix one such copy E_1. We inductively define an exhaustion $\{F_m\}$ for $\widetilde{F}$ as follows. $F_1 = E_1$. F_{m+1} is the union of F_m and all those E_j which meet it. Each F_m is a disk (which we call a **star**). The **inner corona** I_m of F_m is the annulus $F_{m+1} - \text{int } F_m$. Each vertex on ∂F_m lies in either one or two of those E_j contained in F_m. Each E_j in I_m meets meets F_m in either an edge or a vertex; in both cases it meets exactly two adjacent E_ℓ of I_m, and each of these intersections is an edge. For $n \geq 2$ we define the **outer n-corona** O^m_n to be the annulus $F_{m+n} - \text{int } F_{m+1}$; we define $O^m_1 = \emptyset$. Let σ_2 be a proper arc in F_2 consisting of three edges of the polygons in F_2. Inductively define a proper arc σ_{m+1} in F_{m+1} by adjoining to σ_m two arcs spanning I_m which are edges of polygons in I_m. Thus each σ_m is an edge path in F_m splitting it into two unions of polygons F'_m and F''_m.

We now consider the structure of $\widetilde{M}$. For $n \geq 1$, $p^{-1}(Q_n) = p^{-1}(P_n)$ is the union of prisms $P_{n,j} = E_j \times [-n, n]$ meeting along their lateral sides to form $\widetilde{F} \times [-n, n]$. $p^{-1}(R_n)$ is a disjoint union of solid tori $R_{n,j}$. $p^{-1}(H_n)$ is a disjoint union of 1-handles $H_{n,j}$ running from $E_j \times \{n\}$ to $\partial R_{n,j}$; these are regular neighborhoods of lifts ω_j of ω. Now $p^{-1}(M_n) = p^{-1}(V_n)$ is the union of $p^{-1}(P_n)$, $p^{-1}(H_n)$, and $p^{-1}(R_n)$. It can be expressed as the union of the eyebolts $V_{n,j} = P_{n,j} \cup H_{n,j} \cup R_{n,j}$ meeting along the lateral sides of the $P_{n,j}$. Finally $\widetilde{M}$ is the nested union of the $p^{-1}(M_n)$.

Let Σ^m_n be the union of those $V_{n,j}$ such that E_j is in the star F_m. Let Λ^m_n be the union of those $P_{n,j}$ such that E_j is in the inner corona I_m. Let Φ^m_n

be the union of those $P_{n,j}$ such that E_j is in the outer n-corona O_n^m. Note that Λ_n^m and Φ_n^m (for $n \geq 2$) are solid tori, $\Lambda_n^m \cap \Sigma_n^m$ is an annulus which goes around Λ_n^m once longitudinally and consists of those lateral sides of the prisms in Σ_n^m which lie on $\partial\Sigma_n^m$, and (for $n \geq 2$) $\Lambda_n^m \cap \Phi_n^m$ is an annulus which goes around each of these solid tori once longitudinally. We now define $C_n^m = \Sigma_n^m \cup \Lambda_n^m \cup \Phi_n^m$.

Lemma 5.1. *$\{C_m^m\}$ is an exhaustion for $\widetilde{M}$. Each C^m is a nice quasi-exhaustion.*

Proof. Note that $C_n^m \subseteq \text{int}\, C_{n+1}^m$, and $C_n^m \subseteq C_n^{m+1}$. Suppose K is some compact subset of $\widetilde{M}$. Then K lies in some $p^{-1}(M_n)$ and thus in a finite union of $V_{n,j}$ and hence in some $\Sigma_n^m \subseteq C_n^m \subseteq C_q^q$, where $q = \max\{m, n\}$. Thus $\{C_m^m\}$ is an exhaustion for $\widetilde{M}$.

Each C_n^m is a cube with handles, so is irreducible. The number of handles is at least one, so ∂C_n^m has positive genus. Let $Y = C_{n+1}^m - \text{int}\, C_n^m$. We will prove that Y is excellent by successive applications of Lemma 3.6. Let $P_{n+1,j}^+$ and $P_{n+1,j}^-$ denote, respectively, $E_j \times [n, n+1]$ and $E_j \times [-(n+1), -n]$.

Consider a $P_{n+1,j}$ contained in Σ_{n+1}^m. It meets C_n^m in $P_{n,j}$ together with regular neighborhoods of certain arcs in $P_{n+1,j}^+$. These arcs consist at least of the j^{th} copies of the α_i, the γ_i, and ε which are part of the lift ω_j of ω. If another prism $P_{n+1,\ell}$ in Σ_{n+1}^m meets $P_{n+1,j}$ in a common lateral side, then either ω_j or ω_ℓ will meet this side; in the latter case this contributes a β_i and δ_i to the subsystem of arcs in $P_{n+1,j}^+$. Since the full system of arcs was chosen to be poly-excellent this subsystem of arcs is excellent and so has excellent exterior $Y \cap P_{n+1,j}^+$. Let U' be the union of those $Y \cap P_{n+1,j}^+$ such that $E_j \subseteq F_m'$. This space can be built up inductively by gluing on one $Y \cap P_{n+1,j}^+$ at a time, with the gluing being done along either a disk with two holes (when $P_{n+1,j}$ is glued along one lateral side) or a disk with four holes (when $P_{n+1,j}$ is glued along two adjacent lateral sides). No component of the complement of this surface in the boundary of either manifold is a disk, hence this surface is incompressible in each manifold. It follows that U' is excellent. Similar remarks apply to the space U'' associated with F_m''.

Next consider a $P_{n+1,j}^+$ contained in Λ_{n+1}^m. If $E_j \subseteq F_{m+1}$ and meets F_m' in an edge of $E_\ell \subseteq F_m'$, then either ω_ℓ misses $P_{n+1,j}^+$ or meets it in copies of β_i and δ_i. Thus enlarging U' by adding $Y \cap P_{n+1,j}^+$ either adds a 3-ball along a disk in its boundary, giving a space homeomorphic to U' or gives a new excellent 3-manifold. We adjoin all such $Y \cap P_{n+1,j}^+$ to U'. Then we consider those E_j which meet F_m' in a vertex. Then $P_{n+1,j}^+ = Y \cap P_{n+1,j}^+$, and one can successively adjoin these 3-balls along disks in their boundaries. We denote the enlargement of U' from all these additions again by U'. Similar remarks apply to U''.

Now $(F'_{m+n+1} - \text{int } F'_{m+1}) \times [n, n+1]$ is a 3-ball which meets U' in a disk, so we adjoin it to U' to get a new U' homeomorphic to the old one. We then adjoin the 3-ball $(f'_{m+n+1} - \text{int } F'_{m+n}) \times [-n, n] \cup F'_{m+n+1} \times [-(n+1), -n]$ which meets this space along a disk to obtain our final U'. The same construction gives U''.

Now U' and U'' are each excellent. $U' \cap U''$ is an annulus with a positive number of disks removed from its interior corresponding to its intersection with arcs passing from $F'_m \times [n, n+1]$ to $F''_m \times [n, n+1]$. No component of the complement of this surface in $\partial U'$ or in $\partial U''$ is a disk; this corresponds to the fact that $F'_m \times \{n\}$, $F''_m \times \{n\}$, $F'_m \times \{n+1\}$, and $F''_m \times \{n+1\}$ each meet some ω_j. Thus this surface is incompressible in both U' and U'', so $U' \cup U''$ is excellent.

Finally we add on the $Y \cap (H_{n+1,j} \cup R_{n+1,j})$ for $E_j \subseteq F_m$ to $U' \cup U''$ to conclude that Y is excellent.

It remains to show that each ∂C^m_n is incompressible in $\widetilde{M} - \text{int } C^m_n$. First note that since each $C^m_{n+s+1} - \text{int } C^m_{n+s}$ is ∂-irreducible we must have that ∂C^m_n is incompressible in $C^m_{n+q} - \text{int } C^m_n$ for each $q \geq 1$. Now consider the set

$$\widetilde{M}_{n+q} = p^{-1}(M_{n+q}) \cup \left(\widetilde{F} \times [-(n+q+1), -(n+q)]\right).$$

It can be obtained from C^m_{n+q} as follows. First add the solid tori $R_{n+q,j} \cup H_{n+q,j}$ in $p^{-1}(M_{n+q})$ for which $E_j \subseteq F_{m+q+n}$; these meet C^m_{n+q} in disks. Then add

$$(F_{m+q+n} \times [-(n+q+1), -(n+q)]) \cup \left(\widetilde{F} - (\text{int } F_{m+q+n}) \times [-(n+q+1), n+q]\right).$$

This is a space homeomorphic to $\mathbf{R}^2 \times [0,1]$ which meets C^m_{n+q} in the disk

$$(F_{m+q+n} \times \{-(n+q)\}) \cup ((\partial F_{m+q+n}) \times [-(n+q), n+q]).$$

Lastly add all the remaining solid tori $R_{n+q,j} \cup H_{n+q,j}$, where $E_j \subseteq \widetilde{F} - \text{int } F_{m+q+n}$; these do not meet C^m_{n+q}. This description shows that $C^m_{n+q} \cap (\widetilde{M}_{n+q} - \text{int } C^m_{n+q})$ consists of (finitely many) disjoint disks, and therefore ∂C^m_n is incompressible in $\widetilde{M}_{n+q} - \text{int } C^m_n$. Finally since $\widetilde{M}$ is the nested union of the $\widetilde{M}_{n+q}$ over all $q \geq 1$ we have that ∂C^m_n is incompressible in $\widetilde{M} - \text{int } C^m_n$. □

6. The general case.

Suppose $G_1, \ldots, G_k$ are infinite cyclic groups and infinite closed surface groups. For $i = 1, \ldots, k$ let P^i_n be a pillbox or prism, as appropriate, with quotient Q_n^i a solid torus or product I-bundle over a closed surface, respectively. We let H_n^i be a 1-handle attached to P_n^i as before. We let R_n be a common solid torus to which we attach the other ends of all the H_n^i. The union of the Q_n^i and H_n^i with R_n is called M_n. As before we choose arcs in the P_{n+1}^i, H_{n+1}^i, and R_{n+1} and use them to define an embedding of M_n into the interior of M_{n+1}.

The choice of arcs in $R_{n+1} - \text{int } R_n$, as well as the embedding $R_n \subseteq \text{int } R_{n+1}$, requires some discussion, since we will want this family λ of arcs to be poly-excellent. Choose a poly-excellent $(2k+2)$-tangle λ^+ in a 3-ball B, with components λ_t^i, $1 \leq i \leq k+1$, $t = 1, 2$. Construct a graph in B by sliding one endpoint of each λ_2^i, $1 \leq i \leq k$, so that it lies on int λ_2^{k+1}. Thus these λ_2^i now join ∂B to distinct points on int λ_2^{k+1}; all the other λ_t^i still join ∂B to itself. Now choose disjoint disks E_1 and E_2 in ∂B such that E_t meets the graph in $\partial \lambda_t^{k+1} \cap \text{int } E_t$. Glue E_1 to E_2 so that B becomes a solid torus R_{n+1} and $\lambda_1^{k+1} \cup \lambda_2^{k+1}$ becomes a simple closed curve. The regular neighborhood of this simple closed curve is our embedding of R_n in the interior of R_{n+1}. Clearly R_n is null-homotopic in R_{n+1}. By Lemma 3.6 its exterior is excellent as is the exterior of the union of R_n with any of the λ_t^i, $1 \leq i \leq k$, $t = 1, 2$.

Let $p : \widetilde{M} \to M$ be the universal covering map. Then $p^{-1}(R_n)$ consists of disjoint solid tori whose union separates $p^{-1}(M_n)$ into components with closures $L_n^{i,\mu}$, where $L_n^{i,\mu}$ is a component of $p^{-1}(Q_n^i \cup H_n^i)$. Let $Z_n^{i,\mu}$ be the union of $L_n^{i,\mu}$ and all those components of $p^{-1}(R_n)$ which meet it. Then $Z^{i,\mu} = \cup_{n \geq 1} Z_n^{i,\mu}$ is an open subset of $\widetilde{M}$ which has a family $\{C^{i,\mu,m}\}$ of quasi-exhaustions as previously described. We will develop from these families an appropriate family $\{C^m\}$ of quasi-exhaustions of $\widetilde{M}$.

We start by choosing a component $\widehat{R}_1$ of $p^{-1}(R_1)$. For each n there is then a unique component $\widehat{R}_n$ of $p^{-1}(R_n)$ which contains $\widehat{R}_1$. We define C_n^1 to be the union of $\widehat{R}_n$ and the (finitely many) $C_n^{i,\mu,1}$ which contain it by. Suppose C_n^m has been defined and that it is the union of the $C_n^{i,\mu,m}$ for which $C_n^m \cap L_n^{i,\mu} \neq \emptyset$. We define C_n^{m+1} in two steps. We first take the union C' of all the $C_n^{i,\mu,m+1}$ such that $C_n^{i,\mu,m} \subseteq C_n^m$. This is just the union of the n^{th} elements of the $(m+1)^{st}$ quasi-exhaustions for those $Z^{i,\mu}$ such that $\{i, \mu\}$ is in the current index set. The second step is to enlarge the index set by adding those $\{i, \nu\}$ for which $C' \cap L_n^{i,\nu} \neq \emptyset$ and then adjoin the $C_n^{i,\nu,m+1}$ to C' in order to obtain C_n^{m+1}. One can observe that the $L_n^{i,\mu}$ and $p^{-1}(R_n)$ give $p^{-1}(M_n)$ a tree-like structure and that the passage from C_n^m to C_n^{m+1} goes out further along this tree.

Lemma 6.1. *$\{C_m^m\}$ is an exhaustion for $\widetilde{M}$. C^m is a nice quasi-exhaustion.*

Proof. Again we have $C_n^m \subseteq \text{int } C_{n+1}^m$ and $C_n^m \subseteq C_n^{m+1}$ with the result that $\{C_m^m\}$ is an exhaustion for $\widetilde{M}$.

As regards the excellence of $C_{n+1}^m - \text{int } C_n^m$ we note that the only thing new takes place in those components of $p^{-1}(R_{n+1})$ contained in C_{n+1}^m. Instead of two arcs λ_1 and λ_2 as before we have λ_1^i and λ_2^i as i ranges over some non-empty subset of $\{1, \ldots, k\}$. We then apply the poly-excellence of the full set of λ_t^i.

The incompressibility of ∂C_n^m in $\widetilde{M} - \text{int } C_n^m$ follows as before. We first note that ∂C_n^m is incompressible in $C_{n+q}^m - \text{int } C_n^m$ for each $q \geq 1$. Now define $\widetilde{M}_{n+q}$ to be the union of $p^{-1}(M_{n+q})$ and, for each of the surface group factors G_i of G, the copy $\widetilde{F}^{i,\mu} \times [-(n+q+1), -(n+q)]$ of $\widetilde{F}^i \times [-(n+q+1), -(n+q)]$ contained in $Z^{i,\mu}$, where $\widetilde{F}^i$ is the universal covering space of the surface F^i with $\pi_1(F^i) \cong G_i$. Then the exterior of C_{n+q}^m in $\widetilde{M}_{n+q}$ meets it in a collection of disjoint disks, from which it follows that ∂C_n^m is incompressible in $\widetilde{M}_{n+q} - \text{int } C_n^m$, thus is incompressible in $\widetilde{M} - \text{int } C_n^m$. □

7. Uncountably many examples.

We now describe how to get uncountably many examples for a given group G. We will use a trick introduced in [8]. Let $\{X_{n,s}\}$ be a family of exteriors of non-trivial knots in S^3 indexed by $n \geq 2$ and $s \in \{0, 1\}$; they are chosen to be anannular, atoroidal, and pairwise non-homeomorphic. (One such family is that of non-trivial, non-trefoil twist knots.) One chooses a function $\varphi(n)$ with values in $\{0, 1\}$, i.e. a sequence of 0's and 1's indexed by n, and constructs a 3-manifold $M[\varphi]$ by embedding $X_{n,\varphi(n)}$ in $M_n - \text{int } M_{n-1}$ so that $\partial X_{n,\varphi(n)}$ in incompressible in $M_n - \text{int } M_{n-1}$ (but is compressible in M_n). The idea is to do this in such a way that for "large" compact sets C in $\widetilde{M}[\varphi]$ one has components of $p^{-1}(X_{n,\varphi(n)})$ which lie in $\widetilde{M} - C$ and have incompressible boundary in $\widetilde{M} - C$ for "large" values of n; moreover, every knot exterior having these properties should be homeomorphic to some $X_{n,\varphi(n)}$. Thus if $\widetilde{M}[\varphi]$ and $\widetilde{M}[\psi]$ are homeomorphic one must have $\varphi(n) = \psi(n)$ for "large" n. One then notes that there are uncountably many functions which are pairwise inequivalent under this relation.

We proceed to the details. First assume φ is fixed, so we can write $s = \varphi(n)$. The most innocuous place to embed $X_{n,s}$ is in $R_n - \text{int } R_{n-1}$ since this space is common to all our constructions. Recall that this space contains arcs λ_1, λ_2 or, if G is a non-trivial free product, arcs λ_1^i, λ_2^i, $1 \leq i \leq k$; call this collection of arcs λ. We wish $X_{n,s}$ to lie in the complement of λ in such a way that it is poly-excellent in $R_n - \text{int}\,(R_{n-1} \cup X_{n,s})$. We revise the construction of λ from Section 6 as follows. Let B_0 and B_1 be 3-balls. Choose disjoint disks D_r and D'_r in ∂B_r. Let ζ_r be a simple closed curve in $\partial B_r - (D_r \cup D'_r)$ which separates D_r from D'_r. Let A_r and A'_r be the annuli into which ζ_r splits the annulus $\partial B_r - \text{int}\,(D_r \cup D'_r)$, with the notation chosen so that $A_r \cap D_r = \emptyset$. Let τ_r be a poly-excellent $(4k+4)$-tangle in B_r which is the union of $(2k+2)$-tangles ρ_r and ρ'_r satisfying the following conditions. Each component of ρ_0 runs from int D_0 to int A'_0. Each component of ρ_1 runs from int A'_1 to int D'_1. Each component of ρ'_0 runs from int D'_1 to itself. Each component of ρ'_1 runs from int D'_1 to int D_1. We then glue A'_0 to A'_1 and D'_0 to D'_1 so as to obtain a space homeomorphic to a 3-ball minus the interior of an unknotted solid torus contained in the interior of the 3-ball. The 2-sphere boundary component is $D_0 \cup D_1$; the torus boundary component is $A_0 \cup A_1$. The gluing is done so that the endpoints of the arcs match up to give a system λ^+ of $2k+2$ arcs. Each arc in this system consists of an arc of ρ_0 followed by an arc of ρ_1 followed by an arc of ρ'_0 followed by an arc of ρ'_1. We then glue $X_{n,s}$ to this space along their torus boundaries so as to obtain a 3-ball B. We then apply the construction of Section 6 to λ^+ to get a poly-excellent system λ of arcs in $R_n - \text{int}\, R_{n-1}$. It is easily seen that this 3-manifold is nice and that $\partial X_{n,s}$ is, up to isotopy, the unique incompressible non-∂-parallel torus in it; $\partial X_{n,s}$ is also, up to isotopy, the unique incompressible torus in the exterior K_σ of any non-empty union σ of components of λ.

Lemma 7.1. *If $\widetilde{M}[\varphi]$ and $\widetilde{M}[\psi]$ are homeomorphic then there is an index N such that $\varphi(n) = \psi(n)$ for all $n \geq N$.*

Proof. Consider $\widetilde{M}$. $Y = C^m_n - \text{int}\, C^m_{n-1}$ contains copies of K_σ for various choices of σ. The closure of the complement in Y of these copies consists of excellent 3-manifolds which meet the copies along incompressible planar surfaces. It follows that the various copies of $\partial X_{n,s}$ in Y are, up to isotopy and for $n \geq 3$, the unique incompressible tori in Y. The incompressibility of ∂C^m_n in $\widetilde{M} - \text{int}\, C^m_n$ implies that these tori are also incompressible in $\widetilde{M} - \text{int}\, C^m_{n-1}$.

Suppose T is an incompressible torus in $\widetilde{M} - \text{int}\, C^m_{n-1}$. Then T lies in $\widetilde{M}_{n+q}$ for some $q \geq 0$. The exterior of C^m_{n+q} in $\widetilde{M}_{n+q}$ consists of disjoint contractible spaces to which disjoint 1-handles have been attached. It meets C^m_{n+q} in a set of disjoint disks. It follows that T can be isotoped into $C^m_{n+q} - \text{int}\, C^m_{n-1}$. Since ∂C^m_{n+u} for $1 \leq u < q$ is not a torus it is easily seen that T can be isotoped into some $C^m_v - \text{int}\, C^m_{v-1}$ and thus is isotopic to some copy of $\partial X_{v,\varphi(v)}$. Thus any knot exterior X incompressibly embedded in $\widetilde{M} - \text{int}\, C^m_{n-1}$ is homeomorphic to some $X_{v,\varphi(v)}$.

Now consider two different functions φ and ψ. We will show that if $\widetilde{M}[\varphi]$ and $\widetilde{M}[\psi]$ are homeomorphic then there is an N such that $\varphi(n) = \psi(n)$ for all $n \geq N$. Suppose $h : \widetilde{M}[\varphi] \to \widetilde{M}[\psi]$ is a homeomorphism. Distinguish the various submanifolds arising in the construction of these two manifolds by appending $[\varphi]$ and $[\psi]$, respectively. For $n \geq 2$ there are incompressibly embedded copies $\widetilde{X}_{n,\varphi(n)}$ of $X_{n,\varphi(n)}$ in $\widetilde{M}[\varphi] - \text{int}\ C_1^1[\varphi]$. There is an index ℓ such that $h(C_1^1[\varphi]) \subseteq \text{int}\ C_\ell^\ell[\psi]$. By construction $\cup_{n\geq 2}\widetilde{X}_{n,\varphi(n)}$ is end-proper in $\widetilde{M}[\varphi]$, so there is an index N such that for all $n \geq N$ we have $h(\widetilde{X}_{n,\varphi(n)}) \subseteq \widetilde{M}[\psi] - \text{int}\ C_\ell^\ell[\psi]$. Since $h(\partial \widetilde{X}_{n,\varphi(n)})$ is incompressible in $\widetilde{M}[\psi] - \text{int}\ h(C_1^1[\psi])$ it is incompressible in the smaller set $\widetilde{M}[\psi] - \text{int}\ C_\ell^\ell[\psi]$. Thus it is homeomorphic to $X_{v,\psi(v)}$ for some $v > \ell$. Since the knot exteriors are pairwise non-homeomorphic we must have $n = v$ and $\varphi(n) = \psi(v) = \psi(n)$. □

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Received February 11, 1997 and revised July 26, 1997. Research at MSRI is supported in part by NSF grant DMS-9022140.

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