PACIFIC JOURNAL OF MATHEMATICS
Vol. 185, No. 2, 1998

FAILURE OF GLOBAL REGULARITY OF $\bar{\partial}_b$ ON A CONVEX DOMAIN WITH ONLY ONE FLAT POINT

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In this paper we exhibit a bounded domain in $\mathbf{C}^2$ with real analytic boundary which is strictly convex except at one point and for which the $\bar{\partial}_b$ operator is not analytic hypoelliptic modulo its kernel.

The importance of such an example is twofold: First it shows that the theorem of Boas and Straube on global C^∞ regularity for $\bar{\partial}_b$ on convex domains cannot be extended to the analytic case; secondly it is the first example of non analytic hypoellipticity of $\bar{\partial}_b$ on a domain with isolated weakly pseudoconvex points in the boundary.

0. Introduction.

The main result of this paper is the following theorem.

Theorem. *There exists a bounded domain Ω in $\mathbf{C}^2$, with real analytic boundary M, which is pseudoconvex, of finite type, and strictly pseudoconvex except at one point, for which the $\bar{\partial}_b$ operator is not globally analytic hypoelliptic modulo its kernel.*

To define $\bar{\partial}_b$ consider a function $f \in C^1(M)$, let F be a C^1 extension of f to a neighborhood of M and form $\bar{\partial}F$, where $\bar{\partial} = \partial_{\bar{z}_1} + \partial_{\bar{z}_2}$. If F_1 and F_2 are two extensions of the same f, then the difference of $\bar{\partial}F_1$ and $\bar{\partial}F_2$, restricted to the boundary, is a multiple of $\bar{\partial}r$, where r is a defining function for Ω; i.e., $\Omega = \{(z_1, z_2) : r(z_1, z_2) < 0\}$ and $\nabla r \neq 0$ when $r = 0$. Thus $\bar{\partial}_b$ maps functions to sections of the vector bundle given by the quotient of the one forms modulo the subspace spanned by the multiples of $\bar{\partial}r$. In local coordinates and identifying those sections with functions, $\bar{\partial}_b$ can be expressed as a complex vector field, whose real and imaginary parts, denoted by X and Y respectively, form a particular basis of the tangent space in the complex sense.

If T is another vector field, that with X and Y forms a basis of the usual tangent space, the Levi form is the real function defined by: $[X, Y] = \lambda(x)T + O(X, Y)$, where $[\ ,\ ]$ denotes the commutator and $O(\ ,\ )$ a linear combination. We say that Ω is pseudoconvex at $x_0 \in M$ if λ does not change sign in a neighborhood of x_0, strictly pseudoconvex if λ is always positive or always negative and of finite type if the Lie algebra generated by X and Y spans the tangent space at every point. Convex (resp. strictly convex) domains are always pseudoconvex (resp. strictly pseudoconvex), but the converse is not true. (Kohn and Nirenberg [**24**] gave an example of a pseudoconvex domain which is not convexifiable by a local biholomorphic change of coordinates.)

An operator L on M is called C^∞ (resp. analytic) hypoelliptic if $Lu \in C^\infty(U)$ (resp. $Lu \in C^\omega(U)$) necessarily implies $u \in C^\infty(U)$ (resp. $u \in C^\omega(U)$) for every open set U in M and every u in $\mathcal{D}'(U)$. A weaker version is global C^∞ or analytic hypoellipticity, which holds if the above condition is satisfied only for $U = M$.

$\bar\partial_b$ is never hypoelliptic in this sense and for this reason J. Kohn [**22**] has introduced the notion of hypoellipticity modulo its kernel; i.e., the requirement of the regularity of the solution of $\bar\partial_b u = f$ with $u \in$ range $\bar\partial_b^*$ locally, when f is regular. The reason for this terminology is that if $\bar\partial_b$ has closed range in $L^2(M)$ then the orthogonal complement of the kernel of $\bar\partial_b$ is equal to the range of $\bar\partial_b^*$ globally and thus, in the global case, the above condition is equivalent to $u \perp \ker\ \bar\partial_b^*$. The fundamental result of the entire theory is the following theorem of Kohn.

Theorem [**22**]. *If the range of $\bar\partial_b$ is closed in $L^2(M)$, Ω is pseudoconvex and of finite type in a neighborhood of $x_0 \in M$, then $\bar\partial_b$ is C^∞ hypoelliptic modulo its kernel there.*

For analytic hypoellipticity the problem is much more complicated, since subelliptic estimates do not necessarily imply this kind of regularity. The first counterexample was given by Baouendi and Goulaouic [**1**] for a sum of squares of vector fields which satisfy the Hörmander condition.

Two large classes of nonelliptic operators are known to be analytic hypoelliptic. The first ([**28**], [**29**]) deals with operators of principal type; i.e., operators for which the gradient of the symbol does not vanish on the characteristic variety and they are analytic hypoelliptic if they are C^∞ hypoelliptic. The second [**30**] roughly states that a (real) second order differential operator with analytic coefficients is analytic hypoelliptic if its characteristic variety is symplectic, it is subelliptic with loss of one derivative and its principal symbol vanishes of order two on the characteristic variety.

Obviously $\bar\partial_b$ does not satisfy these theorems, but both of the above theorems have "microlocal" versions ([**21**], [**28**], [**30**]); i.e., if the symbol of a (pseudo)differential operator satisfies the condition of these theorems in a conic neighborhood of the cotangent bundle then they preserve the analytic wave front set in that neighborhood. $\bar\partial_b$ is an operator of principal type and the proof of Kohn's theorem for instance shows that it is microlocally C^∞ hypoelliptic in a conic neighborhood of half of its characteristic variety and thus by Trepreau's theorem [**28**] it is microlocally analytic hypoelliptic there. If the domain has analytic boundary and is strictly pseudoconvex then the characteristic variety of $\bar\partial_b\bar\partial_b^*$ is symplectic and Kohn's proof shows that $\bar\partial_b\bar\partial_b^*$ is subelliptic with loss of one derivative on the opposite conic neighborhood and thus microlocally analytic hypoelliptic there by Treves' theorem [**30**]. This, as was first pointed out by Christ in [**6**], gives a different proof of the following theorem of Geller.

Theorem [**18**]. *If Ω is strictly pseudoconvex with real analytic boundary and $\bar\partial_b$ has closed range in $L^2(\partial\Omega)$ then $\bar\partial_b$ is analytic hypoelliptic modulo its kernel.*

It was natural to ask if it was possible to prove Geller's theorem under the hypotheses of Kohn's theorem for an analytic manifold. The first negative answer was given by Christ and Geller [**15**], who showed an unbounded domain in $\mathbf{C}^2$ which is pseudoconvex, of finite type and with real analytic boundary, for which $\bar\partial_b$ is not analytic hypoelliptic modulo its kernel. In their example the set of weakly pseudoconvex points contained a curve whose tangent was contained in the span of $\{\mathrm{Re}\,\bar\partial_b, \mathrm{Im}\,\bar\partial_b\}$. A general conjecture due to Treves [**30**] would imply that $\bar\partial_b$ is not analytic hypoelliptic modulo its kernel if M contains such a curve. The fact that existence of such a curve is sufficient in this special case was proved by Christ [**13**]; the fact that it is not necessary, again in this special case[1] , is the main result of this paper, which exhibits a domain with only one weakly pseudoconvex point for which $\bar\partial_b$ is not analytic hypoelliptic modulo its kernel. Actually in this paper a stronger result is proved, namely that for the same domain $\bar\partial_b$ is not globally analytic hypoelliptic modulo its kernel.

Global regularity is a much weaker property than regularity, as is well known.

The main result about C^∞ regularity of $\bar\partial_b$ modulo its kernel was obtained by Boas and Straube [**2**] and it states that global regularity of $\bar\partial_b$ modulo its kernel holds for domains that are convex (more generally that admit a plurisubharmonic defining function).

The first result for global analytic hypoellipticity of $\bar\partial_b$ was given by Chen [**5**], which stated that if Ω is a circular domain with a defining function r satisfying $(\star)\sum \frac{\partial r}{\partial z_k} z_k \neq 0$ then $\bar\partial_b$ is global analytic hypoelliptic modulo

[1] Treves' conjecture deals with certain curves in the *cotangent* bundle; in particular the fiber over an isolated weakly pseudoconvex point is an example of such a curve.

its kernel. ($\star$) implies the existence of a vector field T complementary to the tangent space in the complex sense, which is tangent to the orbits of the action of the torus on the boundary, and it is always satisfied if Ω is a complete Reinhardt domain. A related result which applies in a more general context was given by Christ [**12**]; a microlocal analogue would imply Chen's theorem. Other generalizations were made by Derridj and Tartakoff ([**16**] and [**17**]) and some investigators asked whether global regularity might always hold if Ω is pseudoconvex and of finite type. Again these hopes were dashed by Christ [**14**], who showed a domain pseudoconvex of finite type and with real analytic boundary, for which the Szegö projection does not preserve $C^\omega(\partial\Omega)$, which implies also a negative result for $\bar{\partial}_b$.

1. Constuction of the domain.

Lemma 1.1. *There exists a bounded domain Ω in $\mathbf{C}^2$, with real analytic boundary, which is strictly convex except at one point, for which a $\bar{\partial}_b$ operator has the form:*

$$\bar{\partial}_b = \partial_x + i\partial_y - i(x+iy)\beta((x^2+y^2),t)\partial_t \tag{1.1}$$

in local coordinates (x,y,t) given by a chart based on a neighborhood of the origin, where $\beta((x^2+y^2),t) = 6(x^2+y^2)^2 + 2t^2 + O((x^2+y^2)^2t, t^3)$.

Proof. Consider the following hypersurface S in $\mathbf{C}^2$:

$$\begin{cases} z_1 = \xi = x+iy \\ \mathrm{Re} z_2 = t \\ \mathrm{Im} z_2 = t^2 + t^2|\xi|^2 + |\xi|^6 \end{cases}$$

A first order operator L would be a tangential Cauchy-Riemann operator for S if ReL and ImL are linearly independent and L annihilates both the coordinates of the embedding. If L has the form $\bar{\partial}_\xi - i\xi\beta(|\xi|^2,t)\partial_t$ then linear independence and the equation $Lz_1 = 0$ are trivially satisfied, while the equation $Lz_2 = 0$ leads to the following equation for β:

$$i\xi(2t^2 + 6|\xi|^4) = i\xi\beta(|\xi|^2,t)(1 + i(2t + 2t|\xi|^2)). \tag{1.2}$$

This implies that

$$\beta(|\xi|^2,t) = \frac{2t^2 + 6|\xi|^4}{1 + i(2t + 2t|\xi|^2)}$$

i.e.

$$\beta(|\xi|^2,t) = 2t^2 + 6|\xi|^4 + O(t^3, t|\xi|^4).$$

The last equation implies that for the hypersurface S the $\bar{\partial}_b$ operator has the desired form, but the region delimited by S is clearly unbounded. For this reason we modify the last equation of (1.1) in the following way:

$$\mathrm{Im} z_2 \geq t^2 + t^2|\xi|^2 + |\xi|^6 + \lambda(t^2 + |\xi|^2 + |\mathrm{Im} z_2|^2)^M \tag{1.3}$$

where λ and M are constants to be determined. It is clear that if λ and M are sufficiently large, then the locus of points in $\mathbf{C}^2$ which satisfy this inequality is contained in the unit ball. We take as our Ω the connected component which contains the origin and we observe that $\partial\Omega$ is a C^ω hypersurface near the origin for which $\bar{\partial}_b$ still has the desired form if M is large enough. It remains to check that Ω is strictly convex except at the origin and that the boundary is analytic. The latter is easy since the gradient of the defining function does not vanish on the boundary. In order to prove the former we need the following lemma.

Lemma 1.2. *If $f(x) = |x|^{2M}$ and $A(x)$ is the Hessian matrix of f, then*

$$\langle A(x)w, w\rangle \geq C|x|^{2(M-1)}|w|^2, \quad w \in \mathbf{R}^n. \tag{1.4}$$

Proof. Since both sides of the inequality are homogeneous of the same degree in x it suffices to prove the statement for $|x| = 1$. Observe that, for $x \neq 0$, f is the composition of a strictly convex function, namely $g : x \to |x|^2$, with a convex and increasing function, namely $h : r \to r^M$, and thus is strictly convex. In particular we have the inequality with C strictly positive and depending continuously on x. Since the unit ball is compact we have the assertion by taking the minimum of C on $|x| = 1$. □

To finish the proof of Lemma 1.1 write $\mathrm{Im} z_2 = g + \lambda f$ and observe that g is a convex function if $|\xi|^2 < \frac{1}{3}$. (Use polar coordinates and notice that both the trace and the determinant of the Hessian matrix are positive if $|\xi|^2 < \frac{1}{3}$.) This implies that the Hessian matrix of g, as a 4×4 matrix, is nonnegative if $|\xi|^2 < \frac{1}{3}$. Outside that region the previous lemma shows that the Hessian matrix of g is dominated by the Hessian matrix of f if λ is big enough, and this implies the conclusion since the Hessian matrix of f is strictly positive except at the origin. □

2. A non linear eigenvalue problem.

In this section we study a family of differential operators A_ξ depending on a complex parameter $\xi \in \mathbf{C}$ defined by

$$A_\xi = (\partial_r + 6r^5 + 2r\xi)\left(-\partial_r - \frac{1}{r} + 6r^5 + 2r\xi\right).$$

It is important to notice that for each $\xi \in \mathbf{R}$ that operator is formally self adjoint in the space $L^2(\mathbf{R}^+, rdr)$.

Lemma 2.1. *Let*

$$\Sigma = \{\xi \in \mathbf{C}\ \exists f \in \mathcal{F},\ f \neq 0, \ \ \textit{such that}\ \ A_\xi f = 0\}$$

$$\mathcal{F} = \left\{ f \in C^2(0,\infty) : \lim_{r\to 0} f(r) = \lim_{r\to 0} rf'(r) = \lim_{r\to\infty} f(r) = \lim_{r\to\infty} rf'(r) = 0 \right\}.$$

Then Σ is discrete and nonempty.

Proof. We can explicitly write down all the solution of $A_\xi f = 0$. They are given by:

$$f_\xi(r) = c_1 e^{-\lg r + r^6 + r^2\xi} \int_0^r e^{\lg s - 2(s^6 + s^2\xi)} ds + c_2 e^{-\lg r + r^6 + r^2\xi} = c_1 f_1 + c_2 f_2$$

where lg denotes the logarithm function. Note that f_1 and f_2 are linearly independent since:

$$\lim_{r\to 0} f_1(r) = 0 \ \text{ and } \ \lim_{r\to 0} |f_2(r)| = \infty.$$

Thus the only way to get a bounded solution at $r = 0$ is to set $c_2 = 0$. We claim that for some value of $\xi \in \mathbf{C}$, f_1 is bounded also at ∞ and in fact decays exponentially. In order to prove that we need the following lemma.

Lemma 2.2. *Define $N(\xi) = \int_0^\infty e^{\lg(s) - 2(s^6 + s^2\xi)} ds$. Then $N(\xi)$ is an entire function of order $\frac{3}{2}$.*

Proof. The change of variable $s' = s^2$ gives:

$$N(\xi) = \frac{1}{2} \int_0^\infty e^{-2(s^3 + s\xi)} ds.$$

Let $-x$ be the real part of ξ. Then:

$$|N(\xi)| \leq \frac{1}{2} \int_0^\infty e^{-2(s^3 - sx)} ds.$$

It is enough to consider the case where $x \geq 1$. If $\Phi(s) = -s^3 + sx$ then $\Phi'(s) = -3s^2 + x$ and $\Phi''(s) = -6s$ thus Φ is a concave function with a maximum at $s_0 = \sqrt{\frac{x}{3}}$. We write the Taylor expansion of Φ as:

$$\Phi(s) = \Phi(s_0) + \frac{\Phi''(c_s)}{2}(s - s_0)^2 \ \ c_s \in (s, s_0) \ \text{ or } \ c_s \in (s_0, s).$$

Thus:

$$|N(\xi)| \leq \frac{1}{2} e^{2\Phi(s_0)} \int_0^\infty e^{\Phi''(c_s)(s-s_0)^2} ds.$$

We split the integral in two parts:

$$\int_0^\infty = \int_0^{s_0} + \int_{s_0}^\infty = I + II.$$

We have:

$$|I| \leq C s_0$$

trivially, while for $x \geq 1$ we have $\Phi''(c_s) < 1$ if $c_s \in (s_0, \infty)$ which implies that

$$|II| \leq \int_{s_0}^\infty e^{\Phi''(c_s)(s-s_0)^2} ds \leq \int_{s_0}^\infty e^{-(s-s_0)^2} ds \leq C.$$

Thus

$$|N(\xi)| \leq c e^{-c(s_0^3 - s_0 x)} \leq c e^{c|\xi|^{3/2}}.$$

This inequality shows that $N(\xi)$ is an holomorphic function of order less than or equal to $\frac{3}{2}$. To show that the order is exactly $\frac{3}{2}$ it suffices to find a sequence ξ_j, $|\xi_j| \to \infty$, such that:

$$|N(\xi_j)| \geq e^{c|\xi_j|^{\frac{3}{2}}}$$

for some positive constant c independent of j. If $\xi = -x$, where x is positive real number, then:

$$N(\xi) = N(-x) = \frac{1}{2} \int_0^\infty e^{-2(s^3 - sx)} ds \geq \frac{1}{2} \int_{c_1}^{c_2} e^{-2(s^3 - sx)} ds$$

where $c_1 = \frac{s_0}{2}$ and $c_2 = s_0$.

$$N(\xi) = N(-x) \geq e^{-2\left(\frac{s_0^3}{2} - \frac{s_0}{2} x\right)} \geq e^{cx\sqrt{x}} \geq e^{c|\xi|^{\frac{3}{2}}}.$$

It has long been known that every entire function of nonintegral order has infinitely many zeroes (and necessarily they are discrete). We will show that if ξ is one of these zeroes, then the corresponding function f_ξ is bounded in $[0, \infty]$. Indeed:

$$f_\xi(r) = e^{-\lg r + r^6 + r^2 \xi} \int_0^r e^{\lg s - 2(s^6 + s^2 \xi)} ds = \frac{1}{2} e^{-\lg r + r^6 + r^2 \xi} \int_0^{r^2} e^{-2(s^3 + s\xi)} ds.$$

Note that f_ξ is a continuous function of r and thus is bounded on every compact subset of $[0, \infty]$. Moreover observe that if $\xi \in \Sigma$ then:

$$\int_0^{r^2} e^{-2(s^3 + s\xi)} ds + \int_{r^2}^\infty e^{-2(s^3 + s\xi)} ds = 0;$$

thus

$$f_\xi(r) = -\frac{1}{2}e^{-\lg r + r^6 + r^2\xi}\int_{r^2}^{\infty} e^{-2(s^3+s\xi)}ds.$$

Let s_0 be as before and $r^2 > s_0 + 1$. Then

$$|f_\xi(r)| = \frac{1}{2}e^{-\lg r + r^6 - r^2 x}\int_{r^2}^{\infty} e^{-2(s^3-sx)}ds.$$

Write $\Phi(s) = \Phi(r^2) + \Phi'(c_s)(s - r^2)$ where Φ is the phase function. Then

$$|f_\xi(r)| = \frac{1}{2}e^{-\lg r - r^6 + r^2 x}\int_{r^2}^{\infty} e^{\Phi'(c_s)(s-r^2)}ds \le C.$$

Actually the last inequality implies that f_ξ decays exponentially, as does f'_ξ, and thus belongs to $\mathcal{F}$. □

Definition. Let $f : [0,\infty] \to \mathbf{C}$ be a measurable function and consider the following norms:

$$||f||^2_{H^2_\rho} = \int_0^\infty \left(|f(r)|^2\frac{1+r^{12}}{r^2} + |\partial_r f(r)|^2 + \left(\frac{1+r^{12}}{r^2}\right)^{-1}|\partial_r{}^2 f(r)|^2\right)e^{\rho r^6}rdr$$

$$||f||^2_{H^1_\rho} = \int_0^\infty \left(|f(r)|^2 + \left(\frac{1+r^{12}}{r^2}\right)^{-1}|\partial_r f(r)|^2\right)e^{\rho r^6}rdr$$

$$||f||^2_{H^0_\rho} = \int_0^\infty |f(r)|^2\left(\frac{1+r^{12}}{r^2}\right)^{-1}e^{\rho r^6}rdr$$

in which all the derivatives are taken in the sense of distributions and they are supposed to be L^2 functions locally. Define H^k_ρ $\{f : [0,\infty] \to \mathbf{C} : ||f||_{H^k_\rho} < \infty\}$.

The main result of this section is the following theorem.

Theorem 2.3. *For each $\xi \notin \Sigma$, $A_\xi : H^2_\rho \to H^0_\rho$ is a bijection provided that ρ is small enough. Moreover if K is a compact subset of the complement of Σ in $\mathbf{C}$, then there exists $r(K)$ such that A_ξ^{-1} is uniformly bounded for $\xi \in K$ if $\rho \in [0, r(K)]$.*

Before proving the theorem we need to investigate some properties of the functions which belong to H^2_ρ.

Lemma 2.4. *If $f \in H^2_\rho$ and $\rho \ge 0$ then $f \in C^1(0,\infty)$ and*

$$\lim_{r\to 0} f(r) = \lim_{r\to 0} rf'(r) = \lim_{r\to\infty} f(r) = \lim_{r\to\infty} rf'(r) = 0.$$

Proof. Notice that for any interval (a,b) that is relatively compact in $(0,\infty)$, $||f\chi_{(a,b)}||^2_{H^2_\rho}$ and $||f\chi_{(a,b)}||^2_2$ are equivalent. (Here $||\cdot||_s$ denotes the usual Sobolev norm.) Thus the first assertion follows from the usual Sobolev embedding theorem. Since $f \in H^2_\rho$ we have $\int_0^1 |\partial_r f|^2 r dr < \infty$ and $\int_0^1 \frac{1}{r^2}|f|^2 r dr < \infty$. The finiteness of the second integral implies that $\lim_{n\to\infty} \inf_{[2^{-(n+1)},2^{-n}]} |f| = 0$. Otherwise $\exists \epsilon_0 > 0$ and $\eta_k \to \infty$ such that $\inf_{[2^{-(\eta_k+1)},2^{-\eta_k}]} |f| \geq \epsilon_0$. But:

$$\int_0^1 \frac{1}{r^2}|f|^2 r dr \geq \sum_k^\infty \frac{1}{2^{\eta_k+1}} \epsilon_0^2 2^{\eta_k+1} = +\infty.$$

Now

$$\begin{aligned}
&\left| \sup_{[2^{-(n+1)},2^{-n}]} |f| - \inf_{[2^{-(n+1)},2^{-n}]} |f| \right| \\
&\leq \left| \sup_{[2^{-(n+1)},2^{-n}]} f - \inf_{[2^{-(n+1)},2^{-n}]} f \right| \\
&\leq \int_{2^{-(n+1)}}^{2^{-n}} |f'(\eta)| d\eta \\
&= \int_{2^{-(n+1)}}^{2^{-n}} (\eta)^{\frac{1}{2}} (\eta)^{-\frac{1}{2}} |f'(\eta)| d\eta \\
&\leq \int_{2^{-(n+1)}}^{2^{-n}} \frac{1}{\eta} d\eta \int_{2^{-(n+1)}}^{2^{-n}} \eta |f'(\eta)|^2 d\eta \\
&\leq C \int_{2^{-(n+1)}}^{2^{-n}} \eta |f'(\eta)|^2 d\eta \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

This implies $\limsup_{r\to 0} |f(r)| = 0 \ \Rightarrow \lim_{r\to 0} f(r) = 0$.

The other assertions are similar and are left to the reader. □

Proof of Theorem 2.3. We consider first the case $\rho = 0, \xi = 0$. Denote by $\langle \cdot,\cdot \rangle$ the scalar product in $L^2(\mathbf{R}^+, r dr)$ i.e. $\int_0^\infty f \bar{g} r dr$ and by $||\cdot||$ the corresponding norm. Using the results of the previous lemma and integrating by parts we get if $f \in H^2_0$,

$$\begin{aligned}
\langle A_0 f, f \rangle &= \int_0^\infty \left(-{\partial_r}^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} + 36 r^{10} + 24 r^4 \right) f \bar{f} r dr \\
&\geq ||\partial_r f||^2 + ||\frac{1}{r} f||^2 + c||r^5 f||^2.
\end{aligned}$$

Thus

$$\int_0^\infty |A_0 f|^2 \left(\frac{1+r^{12}}{r^2}\right)^{-1} rdr$$
$$= \int_0^\infty \left| \left(-\partial_r{}^2 f - \frac{1}{r}\partial_r f + \frac{1}{r^2} f + 24r^4 f + 36r^{10} f\right) \left(\frac{1+r^{12}}{r^2}\right)^{-\frac{1}{2}} \right|^2 rdr$$
$$\geq \int_0^\infty |\partial_r{}^2 f|^2 \left(\frac{1+r^{12}}{r^2}\right)^{-1} rdr - C|\langle A_0 f, f\rangle|$$

since the absolute value of each term except the one containing the second derivative of f, is controlled by $\langle A_0 f, f\rangle$. This implies that

$$||f||^2_{H^2_0} \leq \int_0^\infty |A_0 f|^2 \left(\frac{1+r^{12}}{r^2}\right)^{-1} rdr + C\langle A_0 f, f\rangle = ||A_0 f||^2_{H^0_0} + C\langle A_0 f, f\rangle.$$

Write $\langle A_0 f, f\rangle$ as $\int_0^\infty A_0 f (\frac{1+r^{12}}{r^2})^{\frac{1}{2}} (\frac{1+r^{12}}{r^2})^{-\frac{1}{2}} \bar{f} rdr$. Using the Schwarz inequality we get

$$||f||^2_{H^2_0} \leq ||A_0 f||^2_{H^0_0} + C||A_0 f||_{H^0_0} ||f||_{H^2_0}$$

and the usual large constant-small constant trick gives

$$||f||_{H^2_0} \leq C||A_0 f||_{H^0_0}.$$

The last inequality tells us that A_0 is injective and has closed range. We claim that A_0 is Fredholm since its cokernel is trivial (this implies also that index$(A_0) = 0$). Denote by A_0^* the adjoint of A_0 and by A_0^t the formal transpose in the space H^0_0. If g is in the kernel of A_0^* then for every smooth and compactly supported function f we have

$$0 = (A_0^* g, f)_{H^2_0} = (g, A_0 f)_{H^0_0} = (A_0^t g, f)_{H^0_0},$$

where A_0^t is taken in the sense of distributions. Thus we have $A_0^t g = 0$. A simple computation shows that

$$A_0^t = \left(\partial_r + \frac{\chi'(r)}{\chi(r)} + 6r^5\right) \left(-\partial_r - \frac{\chi'(r)}{\chi(r)} - \frac{1}{r} + 6r^5\right)$$

where $\chi(r) = (\frac{1+r^{12}}{r^2})^{-1}$. The kernel of such an operator is spanned by:

$$\psi_+(r) = \frac{1}{\chi(r)} \frac{1}{r} e^{r^6} \int_0^r s e^{-2s^6} ds$$
$$\psi_-(r) = \frac{1}{\chi(r)} \frac{1}{r} e^{r^6} \int_r^\infty s e^{-2s^6} ds.$$

We have $\psi_+(r) \geq ce^{r^6}$ and $\psi_-(r)$ is $O(1)$ as $r \to \infty$ and $\psi_+(r) \geq c\frac{1}{r^3}$ while $\psi_-(r) \geq c\frac{1}{r}$ as $r \to 0$. The asymptotic estimates imply that ψ_+ and ψ_- are linearly independent and that they cannot belong to the space H_0^0. We want to extend the invertibility of A_0 to other A_ξ, $\xi \in \mathbf{C}\backslash\Sigma$, and for this reason we need the following lemma.

Lemma 2.5. *If ξ is a complex number and $0 \leq l < 5$ then the map $T : H_0^k \to H_0^{k-1}$ given by $f \to \xi r^l f$ is compact. In particular the embedding $H_0^k \hookrightarrow H_0^{k-1}$ is compact.*

Proof. Case ($k = 1$). We have to show that if $\{f_n\} \subset H_0^1$ and $||f_n|| \leq B$ then there exists a subsequence $\{f_{n_k}\}$ such that $\xi r^l f_{n_k}$ converges in H_0^0. By assumption we know that $\int_0^\infty |f_n|^2 r dr + \int_0^\infty (\frac{1+r^{12}}{r^2})^{-1} |\partial_r f|^2 r dr < C$ which implies

$$\int_{m^{-1}}^{m} |\partial_r f_n|^2 dr \leq B(m) \ \ \forall n.$$

Using the Ascoli-Arzelá theorem we can get for each $m \geq 1$ a sequence $\{f_n^m\}$ that is a subsequence of $\{f_n^{m-1}\}$ and which converges uniformly on $[m^{-1}, m]$. A standard diagonal trick gives a subsequence of $\{f_n\}$, denoted still by $\{f_n\}$, which converges uniformly on every compact subset of $(0, \infty)$. Since $\int_0^\infty |f_n|^2 r dr < B \ \forall n$ we can find for every ϵ an M such that:

$$\int_0^{M^{-1}} |f_n|^2 \frac{r^{2+2l}\xi}{1+r^{12}} r dr + \int_M^\infty |f_n|^2 \frac{r^{2+2l}\xi}{1+r^{12}} r dr \leq \epsilon \ \forall n.$$

(Estimate the L^∞ norm of $\frac{r^{2+2l}\xi}{r^{12}+1}$ and use the inequality $\int |fg| \leq ||g||_{L^\infty} \int |f|$.) Thus if n and n' are large enough, then

$$\int_0^\infty |f_n - f_{n'}|^2 |\xi|^2 \frac{r^{2+2l}}{1+r^{12}} r dr = \int_0^{M^{-1}} + \int_{M^{-1}}^{M} + \int_M^\infty < 3\epsilon;$$

i.e., $\{f_n\}$ is Cauchy and thus converges. The proof for the case $k = 2$ is essentially the same. □

From the lemma we can conclude that for each $\xi \in \mathbf{C}$, $A_\xi = A_0 + K$ where K is a compact operator, which implies that A_ξ is Fredholm and its index is equal to the index of A_0 and thus equal to zero. In particular A_ξ is invertible if and only if it is injective. If $f \in (\ker\ A_\xi) \cap H_0^2$ then $f \in \mathcal{F}$ and thus A_ξ is injective if $\xi \notin \Sigma$ (the converse is also true and will be exploited in the next paragraph). Note that the map from the complex numbers to the space of bounded operators between H_0^2 and H_0^0, endowed with the usual norm, given by: $\mathbf{C} \ni \xi \to A_\xi \in \mathcal{B}(H_0^2, H_0^0)$ is continuous. A general principle for one parameter families of operators implies that the map $\mathbf{C}\backslash\Sigma \ni \xi \to$

$A_\xi^{-1} \in \mathcal{B}(H_0^2, H_0^0)$ is also continuous. In particular, A_ξ is uniformly bounded on compact subsets of $\mathbf{C}\backslash\Sigma$. In order to extend this result to small ρ we consider the map $T_\rho : H_0^k \to H_\rho^k$ given by $f \to e^{\rho r^6} f$. It is clear that T_ρ is a bijection, unitary if $k = 0$. Thus the map $\mathcal{G}A_\xi = T_\rho^{-1} A_\xi T_\rho : H_0^2 \to H_0^0$ is invertible if and only if $A_\xi : H_\rho^2 \to H_\rho^0$ is invertible. The difference between $\mathcal{G}A_\xi$ and A_ξ is given by:

$$\rho(c_1 r^4 + c_2 \rho r^{10} + c_2 r^5 \partial_r)$$

and thus is a bounded operator from $H_0^2 \to H_0^0$, with small norm if ρ is sufficiently small. The previous argument shows that if ρ is small enough and $\xi \notin \Sigma$ then A_ξ is invertible from H_ρ^2 to H_ρ^0.

The goal of the next lemma is to describe the behavior of A_ξ^{-1} when ξ approaches Σ.

Lemma 2.6. *Let $\xi_0 \in \Sigma$ and let Γ_0 be the boundary of a closed disk, centered at ξ_0, which does not contain any other element of Σ. Then there exist $\sigma \in N$, $\zeta_0 \in \mathbf{R}$, $\phi \in C_c^\infty(\mathbf{R}^+)$ such that if $\psi_0(r) = e^{(i\zeta_0 r^2 - r^6)}$*

$$\int_0^\infty \oint_{\Gamma_0} \phi A_\xi^{-1} (\partial_r + 6r^5 + 2r\xi) \psi_0(r) \xi^\sigma d\xi r dr \neq 0.$$

Proof. Let

$$\psi_+(r) = \frac{1}{r} e^{r^6 + r^2\xi} \int_0^r s e^{-2(s^6 + s^2\xi)} ds$$
$$\psi_-(r) = \frac{1}{r} e^{r^6 + r^2\xi} \int_r^\infty s e^{-2(s^6 + s^2\xi)} ds.$$

Note that if $\xi \notin \Sigma$ then ψ_+ and ψ_- are linearly independent since their Wronskian, evaluated at r, is given by: $\frac{1}{r}\int_0^\infty s e^{-2(s^6+s^2\xi)} ds$. Thus the general solution of $A_\xi u = h$ is given by

$$A_\xi^{-1} h(r) = c_1 f_1(r) + c_2 f_2(r) + \int_0^\infty K_\xi(r, y) h(y) dy,$$

where f_1 and f_2 are the fundamental solutions of the homogeneous equation, described in Lemma 2.1, and $K_\xi(r, y)$ is the kernel given by

$$K_\xi(r, y) = (\psi_+(r)\psi_-(y)\chi_{y>r} + \psi_+(y)\psi_-(r)\chi_{y<r}) \frac{y}{W(\xi)} = F_\xi(r, y) \frac{y}{W(\xi)},$$

where $W(\xi) = \int_0^\infty s e^{-2(s^6+s^2)} ds$. If $h \in H_\rho^0$ then $f(r) = \int_0^\infty K_\xi(r,y)h(y)dy$ is bounded. Indeed:

$$\begin{aligned} f(r) &= \int_0^\infty K_\xi(r,y)\Big(\frac{1+y^{12}}{y^2}\Big)^{-\frac{1}{2}}\Big(\frac{1+y^{12}}{y^2}\Big)^{\frac{1}{2}} h(y)dy \\ &= \frac{1}{W(\xi)}\int_r^\infty \psi_+(r)\psi_-(y)\Big(\frac{1+y^{12}}{y^2}\Big)^{-\frac{1}{2}}\Big(\frac{1+y^{12}}{y^2}\Big)^{\frac{1}{2}} h(y)ydy \\ &\quad + \frac{1}{W(\xi)}\int_0^r \psi_+(y)\psi_-(r)\Big(\frac{1+y^{12}}{y^2}\Big)^{-\frac{1}{2}}\Big(\frac{1+y^{12}}{y^2}\Big)^{\frac{1}{2}} h(y)ydy \\ &= (I(r) + II(r)) \end{aligned}$$

$$|I(r)| \leq \frac{1}{|W(\xi)|}|\psi_+(r)|\left(\int_r^\infty |\psi_-(y)|^2\Big(\frac{1+y^{12}}{y^2}\Big)ydy\right)^{\frac{1}{2}} ||h||_{H_0^0}.$$

Let us check that this quantity is bounded as $r \to 0$. We have $\psi_+(r) \approx r + O(r^2)$. The integrand is $y^{-3} + O(y^{-2})$ so $(\int_r^\infty)^{\frac{1}{2}} \approx r^{-1}$ and thus $|I|$ is bounded when r approaches zero. Now

$$|II(r)| \leq \frac{1}{|W(\xi)|}|\psi_-(r)|\left(\int_0^r |\psi_+(y)|^2\Big(\frac{1+y^{12}}{y^2}\Big)ydy\right)^{\frac{1}{2}} ||h||_{H_0^0}.$$

Reasoning as before, this quantity is bounded as r approaches zero. The proofs for the cases $r \to \infty$ and $\rho \neq 0$ are similar and rely on the behavior of our solutions as $r \to \infty$. Thus in order to get a bounded operator from H_ρ^0 to H_ρ^2 it is necessary to set $c_1 = c_2 = 0$ since f_1 and f_2 blow up at $+\infty$ and 0 respectively. (Remember that $\xi \notin \Sigma$.) Clearly $F_\xi(r,y)$ does not vanish identically on $\mathbf{R}^+ \times \mathbf{R}^+$ minus the diagonal, and for this reason it is possible to find a point $r_0 \in \mathbf{R}^+$ and a function $h \in C_c^\infty(\mathbf{R}^+)$ such that:

$$\int_0^\infty F_\xi(r_0,y)h(y)ydy \neq 0.$$

This, together with the fact that $\xi \to W(\xi)^{-1}$ has a pole at ξ_0 implies that the function $\xi \to A_\xi^{-1}h(r_0)$ has a pole at $\xi = \xi_0 \in \Sigma$ and thus if Γ_0 is a circle around ξ_0 with the property described in the statement of this lemma, then we have

$$\oint_{\Gamma_0} \frac{1}{W(\xi)} A_\xi^{-1}h(r_0)(\xi-\xi_0)^\sigma d\xi \neq 0$$

in which the factor $(\xi - \xi_0)^\sigma$ is introduced in order to get a simple pole. By the continuity of the integral, there exists $\phi \in C_c^\infty(\mathbf{R}^+)$ such that

$$\int_0^\infty \phi(r)\oint_{\Gamma_0} \frac{1}{W(\xi)} A_\xi^{-1}h(r)(\xi-\xi_0)^\sigma d\xi rdr \neq 0.$$

Let $D_{\xi_0} = (\partial_r + 6r^5 + 2r\xi_0)$. If $h \in C_c^\infty(\mathbf{R}^+)$ then a solution of $D_{\xi_0}\tilde{h} = h$ is given by

$$\tilde{h}(r) = e^{-r^6 - r^2\xi_0} \int_0^r e^{2(s^6+s^2\xi)} h(s) ds.$$

Such a solution vanishes at the origin and decays exponentially at infinity. Replacing h by $D_\xi \tilde{h}$ we have:

$$\int_0^\infty \phi(r) \oint_{\Gamma_0} \frac{1}{W(\xi)} A_\xi^{-1} D_{\xi_0} \tilde{h}(r)(\xi - \xi_0)^\sigma d\xi r dr \neq 0.$$

The difference between D_ξ and D_{ξ_0} is equal to $2r(\xi - \xi_0)$, and multiplication by such a factor has the effect of mollifying the simple pole of the integrand; for this reason we can interchange these two operators without changing the value of the double integrals. Moreover developing the factor $(\xi - \xi_0)^\sigma$ as a polynomial in ξ we have, for some $\sigma' \leq \sigma$

$$\int_0^\infty \phi(r) \oint_{\Gamma_0} \frac{1}{W(\xi)} A_\xi^{-1} D_{\xi_0} \tilde{h}(r)(\xi)^{\sigma'} d\xi r dr \neq 0.$$

Since the function $\tilde{h}$ can be approximated by linear combinations of exponentials of the kind described in the statement we have the conclusion.

For technical reasons that will became clear in the next section we need to consider a variant of the operator A_ξ, namely

$$A_{\xi,\tau} = (\partial_r + \tau(6r^5 + 2r\xi)) \left(-\partial_r - \frac{1}{r} + \tau(6r^5 + 2r\xi) \right),$$

where ξ and τ are complex numbers with the restriction $|\arg \tau| < \frac{\pi}{4}$.

We define:

$$\Sigma_2 = \left\{ (\xi, \tau) |\arg \tau| < \frac{\pi}{4} \; \exists f \in \mathcal{F} \backslash \{0\} : A_{\xi,\tau} f = 0 \right\}$$

where $\mathcal{F}$ is the set defined previously. $\square$

Lemma 2.7. *If $(\xi, \tau) \notin \Sigma_2$ then for all small ρ, $A_{\xi,\rho} : H_\rho^2 \to H_\rho^0$ is invertible.*

Proof.

$$\begin{aligned} A_{\xi,\tau} &= (\partial_r + \tau(6r^5 + 2r\xi)) \left(-\partial_r - \frac{1}{r} + \tau(6r^5 + 2r\xi) \right) \\ &= -\partial_r{}^2 + \tau^2(6r^5 + 2r\xi)^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} - \frac{1}{r}\tau(6r^5 + 2r\xi) + \tau(30r^4 + 2\xi) \\ &- \partial_r{}^2 + \frac{1}{r^2} - \frac{1}{r}\partial_r + c\tau^2 r^{10} + p(r, \xi, \tau). \end{aligned}$$

Integration by parts gives:

$$\operatorname{Re}\langle A_{\xi,\tau}f,f\rangle \geq ||\partial_r f||^2 + \left\|\frac{1}{r}f\right\|^2 + C\operatorname{Re}\tau^2||r^5 f||^2 - C\int_0^\infty |p(r,\xi,\tau)||f|^2 r dr$$

absorbing the last term in the third and in the norm of f in H_0^0 we obtain:

$$\operatorname{Re}\langle A_{\xi,\tau}f,f\rangle \geq C\left(||\partial_r f||^2 + \left\|\frac{1}{r}f\right\|^2 + ||r^5 f||^2\right) - C||f||^2_{H_0^0}$$

and reasoning as in the proof of Theorem 2.3 we get:

$$||f||_{H_0^2} \leq C\left(||A_{\xi,\tau}f||_{H_0^0} + ||f||_{H_0^0}\right).$$

This inequality tells us that $A_{\xi,\tau}$ has closed range and finite dimensional kernel (since $H_0^2 \hookrightarrow H_0^0$ compactly). Moreover, as shown before, the cokernel of $A_{\xi,\tau}$ should be contained in the kernel of an ordinary differential operator and thus is finite dimensional: In conclusion $A_{\xi,\tau}$ is Fredholm. Fix ξ and let $\gamma : \mathbf{C} \to \operatorname{Fred}(H_0^2, H_0^0)$ $\tau \to A_{\xi,\tau}$. γ is continuous and since $\operatorname{ind} : \operatorname{Fred}(H_0^2, H_0^0) \to Z$ is continuous too and $\operatorname{ind} A_{\xi,\tau} = 0$ we have $\operatorname{ind} A_{\xi,\tau} = 0$. The solutions of $A_{\xi,\tau}g = 0$ are given by:

$$g_{\xi,\tau}(r) = c_1 e^{-\lg r + \tau(r^6 + r^2\xi)}\int_0^r e^{\lg s - 2\tau(s^6+s^2\xi)}ds + c_2 e^{-\lg r + \tau(r^6+r^2\xi)} = g_1 + g_2.$$

It easy to see that necessary and sufficient condition for $g_{\xi,\tau}$ to be in $\mathcal{F} \cap H_0^2$ is the vanishing of $\int_0^\infty s e^{-2\tau(s^6+s^2\xi)}ds$ and to set $c_2 = 0$. □

The following lemma describes the relation between Σ and Σ_2.

Lemma 2.8. $(\xi,\tau) \in \Sigma_2$ *if and only if* $\xi\tau^{\frac{2}{3}} \in \Sigma$.

Proof. The discussion of the previous lemma tells us that

$$(\xi,\tau) \in \Sigma_2 \;\text{ iff }\; \int_0^\infty s e^{-2\tau(s^6+s^2\xi)}ds = 0.$$

If we change the contour of integration $s \to s\tau^{-\frac{1}{6}}$ we get:

$$\int_0^\infty s e^{-2\tau(s^6+s^2\xi)}ds = \int_0^\infty s e^{-2(s^6+\tau^{\frac{2}{3}}s^2\xi)}ds = 0$$

and the last integral vanishes if and only if $\tau^{\frac{2}{3}}\xi$ belongs to to Σ. It will be useful also to locate the nonlinear eigenvalues. The next lemma describes a forbidden cone for them.

Lemma 2.9. *There is no* $\xi \in \Sigma$ *satisfying* $|\arg \xi| < \frac{\pi}{3}$ *or* $|\arg \xi + \pi| < \frac{\pi}{3}$.

Proof. Suppose that $\psi \in \mathcal{F}$ and satisfies $A_\xi \psi = 0$. Define $\tilde{\psi}(r) = \psi(re^{i\theta})$ [2] where $\theta = \frac{\alpha}{4}$ if $\xi = \rho e^{i\alpha}$. Then $\tilde{\psi}$ satisfies the following equation:

$$\begin{aligned}0 &= e^{-i\frac{\alpha}{2}}\left(-\partial_r{}^2 + \frac{1}{r^2} - \frac{1}{r}\partial_r\right)\tilde{\psi} + (e^{i\frac{5}{4}\alpha})^2(6r^4 + 2r\rho)^2\tilde{\psi} - e^{i\alpha}(6r^4 + 2\rho)\tilde{\psi} \\ &\quad + e^{i\alpha}(30r^4 + 2r\rho)\tilde{\psi} \\ 0 &= e^{-i\alpha}\left[e^{-\frac{3}{2}\alpha}\left(-\partial_r^2 + \frac{1}{r^2} - \frac{1}{r}\partial_r\right)\tilde{\psi} + (e^{i\frac{3}{2}\alpha})^2(6r^4 + 2r\rho)^2\tilde{\psi} + 24r^4\tilde{\psi}\right] \\ &= e^{-i\alpha}\tilde{\mathcal{L}}\tilde{\psi}.\end{aligned}$$

Integration by parts gives:

$$\mathrm{Re}\langle\tilde{\mathcal{L}}f, f\rangle = \cos(3/2\alpha)\left(||f||^2 + \left\|\frac{1}{r}f\right\|^2 + ||(6r^4 + 2r\rho)f||^2\right) + C||f||^2$$

which is strictly positive if $|\alpha| < \frac{\pi}{3}$.

3. A priori estimate.

In this section we will show how the hypothesis of analytic hypoellipticity of $\bar{\partial}_b$ modulo its kernel leads to a holomorphic extension of the solution of

$$(\star) \qquad \begin{cases}\bar{\partial}_b\bar{\partial}_b^* u = \bar{\partial}_b f \ \ f \in L^2(M) \\ u \perp \ker \bar{\partial}_b^*\end{cases}$$

on "one side" of the manifold with an estimate of its growth, if f is an analytic function. (We will formulate this concept in a more precise way.) First of all we need some definitions. We denote by $d\sigma(z_1, z_2)$ the surface measure on M. This measure is rotationally invariant with respect to the first variable, since our domain is, and it is given by a nonvanishing C^ω density times the usual Lebesgue measure in any coordinate system. Using such a measure we can consider the adjoint $\bar{\partial}_b^*$ of $\bar{\partial}_b$ in the Hilbert space $L^2(M, d\sigma)$. Kohn's theorem implies that ($\star$) has a unique solution u for every $f \in L^2(M)$ and for every open set $U \subset U'$ we have

$$||\chi_U u||_s \leq C_s(||\chi_{U'} f||_s + ||\chi_{U'} u||_0).$$

[2]Note that ψ can be extended uniquely, as a holomorphic function, to a sector of the complex plane and that extension decays exponentially on each single ray, as follows from the explicit formula for ψ.

In particular u is C^∞ if f is C^∞. Let (x, y, t, s) be the usual coordinates in $\mathbf{R}^4 \approx \mathbf{C}^2$, and consider the function

$$F_\tau(x, y, t, s) = e^{i\tau t}\psi_\tau(r(x, y)) = e^{i\tau t}e^{(x^2+y^2)\tau^{1/3}\zeta_0 - \tau(x^2+y^2)^3} \tag{3.1}$$

where τ is a positive real number. Let (U, ϕ) be the local chart near the origin described in Lemma 1.1. Since in that chart M is represented as a hypersurface with respect to the variable (x, y, t), the local expression of F_τ in that coordinate system is still given by (3.1).

Lemma 3.1. *Suppose that $\bar{\partial}_b$ is analytic hypoelliptic modulo its kernel and, for each $\tau \in \mathbf{R}^+$, denote by $\tilde{G}_\tau(x, y, t)$ the local expression in the coordinate system (U, ϕ) of the solution of*

$$\begin{cases} \bar{\partial}_b\bar{\partial}_b^* G_\tau = \bar{\partial}_b F_\tau \\ G_\tau \perp \ker \bar{\partial}_b^*. \end{cases}$$

Then $\tilde{G}_\tau$ extends to a holomorphic function of t in the "strip" $\{(x, y, t) \in \mathbf{R}^2 \times \mathbf{C} : |(x, y, t)| < c,\ 0 > \mathrm{Im}(t) > -c\}$ independent of τ in such a manner as to be continuous when $\mathrm{Im}(t) = 0$ and that satisfies

$$|\tilde{G}_\tau(x, y, t)| \leq c_1 e^{c_2|\mathrm{Im}(t)|\tau}$$

where c_1, c_2 are independent of τ.

A similar lemma for a sum of squares is proved in [**26**]. The proof of Lemma 4.2 in [**14**] applies in this case: The details are left to the reader.

Define $u_\tau(x, y, t) = e^{-i\tau t}\tilde{G}_\tau(x, y, t)$. Lemma 3.1 holds for u_τ as well. Moreover since that lemma was proved only using the estimates of $||F_\tau||_{C(\bar{W}_\delta)}$, the fact that $\bar{\partial}_b$ (and resp.$\bar{\partial}_b\bar{\partial}_b^*$) is analytic hypoelliptic (resp. locally C^∞ hypoelliptic) modulo its kernel, we have that the same conclusion is true for the derivatives of u_τ. In order to clarify the similarity between ($\star$) and the ordinary differential problem studied in the previous chapter we exploit the symmetry of our domain (to reduce the number of variables involved in the problem).

Lemma 3.2. *The operator $\bar{\partial}_b\bar{\partial}_b^*$ is rotationally invariant with respect to the first variable. Moreover $\bar{\partial}_b$ maps functions of the form $f(z_1, z_2) = e^{ik\theta_1}g(|z_1|, z_2)$ to functions of the form $e^{i(k+1)\theta_1}\tilde{g}(|z_1|, z_2)$.*

Proof. Consider $f \in C^1(M)$ and let F be a C^1 extension of f to all of Ω.

$$\bar{\partial}F = \partial_{\bar{z}_1}F d\bar{z}_1 + \partial_{\bar{z}_2}F d\bar{z}_2 \tag{3.2}$$

and

$$\bar{\partial} S = \partial_{\bar{z}_1} S d\bar{z}_1 + \partial_{\bar{z}_2} S d\bar{z}_2$$

where S is a defining function for Ω. Since ∇S does not vanish on M we have that $\partial \bar{z}_1 S$ or $\partial \bar{z}_2 S$ do not vanish simultaneously on M. Suppose that in an open set U, $\partial \bar{z}_2 S \neq 0$. Then we can express dz_2 as a linear combination of $\bar{\partial} S$ and dz_1 and using the definition of $\bar{\partial}_b$ we get for all $p \in U$

$$\bar{\partial}_b f(p) = \left(\partial_{\bar{z}_1} F + \frac{\partial_{\bar{z}_2} F \partial_{\bar{z}_1} S}{\partial_{\bar{z}_2} S} \right)_p . \tag{3.3}$$

From that expression and the fact that S is rotationally invariant with respect to the first variable it is clear that $R_\theta \bar{\partial}_b = e^{-i\theta} \bar{\partial}_b R_\theta$, where $R_{\theta_1} : (z_1, z_2) \to (e^{i\theta_1} z_1, z_2)$. Using again the rotational invariance of the measure we obtain a similar result for the adjoint, namely that $R_\theta \bar{\partial}_b^* = e^{i\theta} \bar{\partial}_b^* R_\theta$, from which the first assertion follows. To prove the second one, just observe that if f is of the form $e^{ik\theta_1} g(|z_1|, z_2)$, then it is possible to find an extension of the same form, and thus the result still follows from (3.3).

In conclusion we have, using the uniqueness of the solution of ($\star$), that the function $u_\tau(x, y, t)$ is of the form $e^{i\theta_1} g_\tau(r, t)$, where g_τ satisfies the differential equation

$$L g_\tau = (\partial_r - ir\beta(r, t)) \psi_\tau(r), \tag{3.4}$$

where

$$L = (\partial_r - ir\beta(r, t)(i\tau + \partial_t)) \left(-\partial_r - \frac{1}{r} - ir\bar{\beta}(r, t)(i\tau + \partial_t) + \alpha(r, t) \right).$$

The factor α is given by integration by parts against the density which represents the surface measure and by the commutator $[\beta, \partial_t]$. It is not important to know the precise expression of α; it suffices to observe that it is an analytic function. In order to simplify the notation we make a change of variables

$$y = \tau^{1/6} r,$$
$$s = \tau^{1/3} t,$$

and we define

$$b(y, s) = \tau^{5/6} r\beta(r, t) \qquad v(y, s) = \tau^{-1/6} g(x, t),$$
$$a(y, s) = \tau^{-2/3} \alpha(x, s) \qquad \lambda = \tau^{2/3},$$

so that

$$b(y,s) = (6y^5 + 2ys^2) + O(\lambda^{-1/2}(1+y^5))$$

for all $0 \leq y \leq \lambda^{1/5}$ and for all s in a bounded region. Indeed, we know that

$$\begin{aligned} b(y,s) &= \lambda^{5/4}\lambda^{-1/4} y\beta(\lambda^{-1/4}y, \lambda^{-1/2}s) \\ &= 6y^5 + 2ys^2 + \lambda^{5/4} O(\lambda^{-1/4}\lambda^{-3/2}ys^3, \lambda^{-5/4}\lambda^{-1/2}y^5 s), \end{aligned}$$

and the error term for $0 \leq y \leq \lambda^{1/5}$ and for s bounded is

$$\leq O(\lambda^{-1/2}y, \lambda^{-1/2}y^5) \approx O(\lambda^{-1/2}(1+y^5)).$$

(3.4) becomes

$$Lg_\tau(y,s) = (\partial_y + b(y,s))\psi(y),$$

where

$$L = (\partial_y + b(y,s)(1 - i\lambda^{-1}\partial_s))\left(-\partial_y - \frac{1}{y} + \bar{b}(y,s)(1 - i\lambda^{-1}\partial_s) + a(y,s)\right).$$

We know that if $0 \leq y \leq c\lambda^{1/4}$ and $|\mathrm{Re}(s)| \leq c\lambda^{1/2}$ and $0 \geq \mathrm{Im}(s) \geq C$ for some constants C and c then

$$|v(y,s)| \leq Ce^{B\lambda}$$

since

$$|v(y,s)| = \lambda^{1/4}|g(x,t)| \leq Ce^{C\lambda^{-1/2}|\mathrm{Im}(s)|\lambda^{3/2}} \leq e^{B\lambda}.$$

Notice that since Lemma 3.1 was proved by using only the hypothesis of analytic hypoellipticity of $\bar{\partial}_b$ modulo its kernel, the fact that $\bar{\partial}_b\bar{\partial}_b^*$ is C^∞ hypoelliptic and the estimates of the norm of F_τ, we have that the same conclusion is true for the derivatives of v with respect to y and s. We wish to prove a stronger result.

Lemma 3.3. *Let $\tilde{\Gamma}$ be the connected component of the preimage of the circle described in Lemma 2.6 under the transformation $F\colon \mathbf{C} \ni z \to z^2$ which is completely contained in the lower half plane, and let μ be a positive real number. Then for all sufficiently large λ, for all sufficiently small ρ and $s \in \tilde{\Gamma}$ we have*

$$\int_0^{\lambda^{1/5}} |v(y,s)|^2 \frac{y^{12}+1}{y^2} e^{-\rho y^6} y\,dy \leq e^{\mu\lambda}.$$

First of all, we have to prove that the integral is finite for every finite λ. This follows from the fact that v is analytic and satisfies (3.4) which implies that $v(0,s) = 0$. Since the derivative with respect to y is bounded by $e^{B\lambda}$ we have that the integral is finite. The proof of this lemma is quite complicated and will be given in several steps. We will use the following constants: $A_0, \nu, N, \sigma_0, \sigma_1, ...\sigma_N, A, \gamma$ which depend on $B, \mu, \tilde{\zeta}_0 = (\xi_0)^{1/2}$ and the coefficients of $\bar{\partial}_b$, but not on λ. First we set $A_0 > 100(B+C_0+|\tilde{\zeta}_0|^{-1}+1)$, with $C_0 = |\tilde{\zeta}_0|/2$.

Lemma 3.4 [14]. *There exist* $0 < \nu < \mu$, $N < +\infty$, $3C_0 > \sigma_0 > \sigma_1 > \cdots > \sigma_N > 2C_0$ *such that*

$$\frac{1}{2}\mu < A_0 - (N+1)\nu < \mu$$

and

$$(s - i\sigma_j)^2(1+y)^{2/3} \notin \Sigma \ \ \forall s \in \mathbf{R},\ 0 \le j \le N,\ y \in [A_0 - (j+1)\nu, A_0 - j\nu].$$

Moreover if $|s| \le 4A$ *and if* γ *and* $A\gamma$ *are sufficiently small and* $A^2\gamma$ *is sufficiently large, then we have*

$$(s - i\sigma_j)^2[1 + y - i2\gamma(s - i\sigma_j)]^{2/3} \notin \Sigma \ 0 \le j \le N, y \in I_j,$$

where $I_j = [A_0 - (j+1)\nu, A_0 - j\nu]$.

Remark 1. In Christ's paper [14] Lemma 6.1. contains a very similar statement, but its proof does not apply directly to our case since here we are dealing with parabolas instead of lines. However we can deduce the proof of our lemma from the one in Christ's paper in the following way. Let $\Sigma^2 = \{z \in \mathbf{C}$ such that $z^2 \in \Sigma\}$. Notice that Σ^2 is discrete and that there is a forbidden cone for it (i.e., there is no $\xi \in \Sigma^2$ satisfying $|\arg \xi| < \frac{\pi}{6}$ or $|\arg \xi + \pi| < \frac{\pi}{6}$). Since these are the only facts used in the proof of Lemma 6.1 [14] we have that there exist constants $\nu, N, \sigma_0, \sigma_1, ..., \sigma_N$ as before, such that

$$(s - i\sigma_j)(1+y)^{1/3} \notin \Sigma^2 \ \forall s \in \mathbf{R},\ 0 \le j \le N,\ y \in [A_0 - (j+1)\nu, A_0 - j\nu],$$

which implies the conclusion of the lemma.

Remark 2. In the course of the proof we will denote by γ and A a pair of constants such that γ and γA are sufficiently small while γA^2 is sufficiently big. The meaning of "sufficiently small" and "sufficiently large" will be specified every time these constants occur.

Let $0 \le \sigma \le 3C_0$ and define for $s \in \mathbf{R}$

$$f_\sigma = e^{-\gamma\lambda(s-i\sigma)^2} v(y, s - i\sigma)\eta(A^{-1}s),$$

where η is a smooth function which is equal to 1 on $[-2,2]$ and is equal to 0 outside $[-4,4]$. Let

$$\begin{aligned}L_\sigma &= e^{-\gamma\lambda(s-i\sigma)^2}\circ \tilde{L}\circ e^{\gamma\lambda(s-i\sigma)^2}\\ &= (\partial_y + b(y,s-i\sigma)(1-2i\gamma(s-i\sigma)-i\lambda^{-1}\partial_s))\circ\\ &\quad\left(-\partial_y - \frac{1}{y} + \bar{b}(y,s-i\sigma)(1-2i\gamma(s-i\sigma)-i\lambda^{-1}\partial_s) + a(y,s-i\sigma)\right).\end{aligned}$$

Then

$$L_\sigma f_\sigma = \psi_\sigma + O(e^{-\lambda})$$

where

$$\psi_\sigma = e^{-\gamma\lambda(s-i\sigma)^2}[\partial_y + b(y,s)]e^{iy^2\zeta_0 - y^6}.$$

Indeed, $L_\sigma f_\sigma = \psi_\sigma$ for $0\le |s|\le 2A$. When s is larger than $2A$ the cutoff function η comes into play, but both sides of this equation are smaller than $e^{\lambda[C-\gamma A^2]}$, where C depends on the quantities fixed before γ and A are chosen. Indeed,

$$\psi_\sigma(y,s) = e^{-\gamma\lambda(s-i\sigma)^2}\cdot O(1)$$

and for $|s|\ge 2A$

$$|e^{-\gamma\lambda(s-i\sigma)^2}| \le e^{-\gamma\lambda A^2 + C\gamma A\lambda + C\lambda}.$$

Similarly,

$$|L_\sigma f_\sigma(y,s)| \le |e^{-\gamma\lambda(s-i\sigma)^2}|\sum_{k=0}^{2}|\nabla^k_{s,y}v(y,s)| \le |e^{-\gamma\lambda(s-i\sigma)^2}|e^{B\lambda},$$

and thus this function is also small if $|s|>2A$ and if γA^2 is big enough.

Denote by

$$\hat{f}_\sigma(y,\xi) = \int_R f_\sigma(y,s)e^{-is\xi}ds,$$

the Fourier transform of f_σ with respect to the second variable. The next two lemmas describe the behavior of $\hat{f}_\sigma$ at infinity and the relation between $\hat{f}_\sigma$ and $\hat{f}_{\sigma'}$. The proofs of Lemma 6.2 and Lemma 6.3 in [14] apply with only minor changes in notation.

Lemma 3.5 [14]. *If $|\xi|\le A_0\lambda$, $0\le y\le \lambda^{1/5}$ and $0\le\sigma,\ \sigma'\le 3C_0$, then for all sufficiently large λ we have*

$$\hat{f}_\sigma(y,\xi) = e^{(\sigma-\sigma')\xi}\hat{f}_{\sigma'}(y,\xi) + O(e^{-\lambda}).$$

Lemma 3.6 [14]. *For all sufficiently large λ, if $0\le y\le\lambda^{1/5}$ and $|\frac{\mathrm{Im}(\tilde{\zeta}_0)}{2}|\le \sigma\le 3C_0$, then we have*

$$\int_{|\xi|>A_0\lambda}|\hat{f}_\sigma(y,\xi)|^2 d\xi \le e^{-2\lambda}.$$

The proofs of these lemmas are based only on the bound for $v(y,s)$ and its first derivatives, and for this reason the conclusions of both lemmas apply also to $\nabla_{y,s}\hat{f}_\sigma$. Define

$$||\phi||^2_{H_\rho} = \int_0^{\lambda^{1/5}} |\phi(y)|^2 \frac{y^2}{1+y^{12}} e^{\rho y^6} y dy,$$
$$||\phi||^2_{H^*_\rho} = \int_0^{\lambda^{1/5}} |\phi(y)|^2 \frac{1+y^{12}}{y^2} e^{-\rho y^6} y dy.$$

It is necessary to show that the conclusions of the two previous lemmas hold also for $||\hat{f}_\sigma(\xi)|| = ||\hat{f}_\sigma(\cdot,\xi)||$, where $||\cdot||$ denotes for the rest of this chapter the norm in the space H^*_ρ:

$$\begin{aligned}
&\left\| \hat{f}_\sigma(y,\xi) - e^{(\sigma-\sigma')\xi} \hat{f}_{\sigma'}(y,\xi) \right\|^2 \\
&= \int_0^1 \left| \hat{f}_\sigma(y,\xi) - e^{(\sigma-\sigma')\xi} \hat{f}_{\sigma'}(y,\xi) \right|^2 \frac{1+y^{12}}{y^2} e^{-\rho y^6} y dy \\
&\quad + \int_1^{\lambda^{1/5}} \left| \hat{f}_\sigma(y,\xi) - e^{(\sigma-\sigma')\xi} \hat{f}_{\sigma'}(y,\xi) \right|^2 \frac{1+y^{12}}{y^2} e^{-\rho y^6} y dy \\
&= A + B.
\end{aligned}$$

In fact if we consider the function $y \to \hat{f}_\sigma(y,\xi) - e^{(\sigma-\sigma')\xi}\hat{f}_{\sigma'}(y,\xi)$ the mean value theorem together with the fact that $v(0,s) = 0$ gives

$$A \le C \int_0^1 \left| \partial_y \hat{f}_\sigma(c_y,\xi) - e^{(\sigma-\sigma')\xi} \partial_y \hat{f}_{\sigma'}(c_y,\xi) \right|^2 |y|^2 \frac{1}{|y|^2} dy \le e^{-2\lambda},$$
$$B \le \int_1^{\lambda^{1/5}} C e^{-2\lambda} \frac{1+y^{12}}{y^2} e^{-\rho y^6} y dy \le C' e^{-2\lambda}.$$

We show now how to obtain Lemma 3.3 from the two previous lemmas together with the inequality

$$\int_{\mathbf{R}} ||\hat{f}_{\sigma_N}(\xi)||^2 d\xi \le \int_{-A_0\lambda}^{\mu\lambda} ||\hat{f}_{\sigma_N}(\xi)||^2 d\xi + e^{\mu\lambda}, \tag{3.5}$$

which will be proved below. We know that

$$|f_0(y,s)| \le C\lambda^M \text{ if } |y| \le c\lambda^{1/4}, |s| \le c\lambda^{1/2} \text{ and } s \in \mathbf{R}$$

since for any positive N there exists M such that $||F_\tau||_{C^N} \le c\lambda^M$ and $\bar{\partial}_b\bar{\partial}_b^*$ is C^∞ hypoelliptic modulo its kernel. Thus

$$\begin{aligned}
|\hat{f}_0(y,\xi)| &= \left| \int_{\mathbf{R}} e^{-\gamma\lambda s^2} v(y,s)\eta(s) e^{is\xi} d\xi \right| \\
&\le \lambda^M \int_{\mathbf{R}} e^{-\gamma s^2} \le C\lambda^M \quad \forall \xi \in \mathbf{R},
\end{aligned}$$

and since the same can be proved for $\partial_y \hat{f}_0$ we also have (again using the mean value theorem and the fact that $\hat{f}_0(0,\xi)=0$) that

$$||\hat{f}_0(\xi)||^2 \leq C\lambda^M \quad \forall \xi \in \mathbf{R}.$$

Using Lemma 3.5 we get

$$||\hat{f}_{\sigma_N}(\xi)||^2 = e^{\sigma_N \xi}||\hat{f}_0(\xi)||^2 + O(e^{-\lambda}) \ \text{ if } \ |\xi| \leq A_0\lambda,$$

which implies that

$$||\hat{f}_{\sigma_N}(\xi)||^2 \leq Ce^{c\mu\lambda} \ \text{ if } \ -A_0\lambda \leq \xi \leq \mu\lambda.$$

From (3.5) we conclude that

$$\int_{\mathbf{R}} ||f_{\sigma_N}||^2 \leq \int_{A_0\lambda}^{\mu\lambda} ||f_{\sigma_N}(\xi)||^2 d\xi + Ce^{\mu\lambda} \leq Ce^{\mu\lambda}.$$

The same kind of result is true for f_σ with $\frac{|\mathrm{Im}\tilde{\zeta}_0|}{2} \leq \sigma \leq 2C_0$. Indeed,

$$\int_{\mathbf{R}} ||\hat{f}_\sigma(\xi)||^2 d\xi = \int_{-\infty}^{-A_0\lambda} + \int_{-A_0\lambda}^{A_0\lambda} + \int_{A_0\lambda}^{\infty} = I + II + III.$$

The first and the third terms are easily estimated using Lemma 3.6, while

$$|II| \leq \int_{A_0\lambda}^{0} e^{(\sigma-0)\xi}||\hat{f}_0(\xi)||^2 d\xi + \int_0^{A_0\lambda} e^{(\sigma-\sigma_N)\xi}||\hat{f}_{\sigma_N}(\xi)||^2 d\xi + O(e^{-\lambda})$$

and thus the result follows from the corresponding bounds for $\hat{f}_0$ and $\hat{f}_\sigma$. This implies that

$$\int_{\mathbf{R}} \int_0^{\lambda^{1/5}} |f_0(y, s-i\sigma)|^2 \frac{y^{12}+1}{y^2} e^{-\rho y^6} y dy ds \leq Ce^{C\mu\lambda}$$

holds for all σ in the interval $\left[\frac{|\,\mathrm{Im}\,\tilde{\zeta}_0|}{2}, 2C_0\right]$. Since $f_0(y,s)$ is holomorphic for $|\mathrm{Re}(s)| < 2A$, $|\mathrm{Im}(s)| < K$ we have, using the mean value theorem, that

$$\int_0^{\lambda^{1/5}} |f_0(y,s)|^2 \frac{y^{12}+1}{y^2} e^{-\rho y^6} y dy \leq Ce^{C\mu\lambda}$$

for all $s \in \tilde{\Gamma}$ provided that A and K are sufficiently large. Since, for $|s| < 2A$, $f_0(y,s) = v(y,s)e^{\gamma\lambda s^2}$ the conclusion follows if γ is sufficiently small.

Lemma 3.7. *There exists $\delta > 0$ such that for all sufficiently large λ, $0 \leq j \leq N$, and $\lambda^{-1}\xi \in I_j$ we have*

$$||\hat{f}_{\sigma_j}(\xi)||^2 \leq e^{-\delta\lambda} \int_{|\eta| \leq A_0\lambda} ||\hat{f}_{\sigma_j}(\eta)||^2 d\eta + e^{-\delta\lambda}. \tag{3.6}$$

Let us deduce (3.5) from Lemma 3.7. For $j=0$ the lemma gives

$$\int_{\lambda I_0} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi \leq \lambda e^{-\delta\lambda} \int_{-A_0\lambda}^{A_0\lambda} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi + \lambda e^{-\delta\lambda}$$
$$\leq e^{-\delta'\lambda} \int_{[-A_0\lambda, A_0\lambda]\backslash\lambda I_0} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi + e^{-\delta'\lambda} \int_{\lambda I_0} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi + e^{-\delta'\lambda}.$$

Absorbing the second term in the left hand side we obtain, for $\delta \leq \delta'$ (we use the convention that δ could change in the course of the proof a finite number of times),

$$\int_{\lambda I_0} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi \leq e^{-\delta\lambda} \int_{[-A_0\lambda, A_0\lambda]\backslash\lambda I_0} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi + e^{-\delta\lambda}. \tag{3.7}$$

By Lemma 3.5 we know that

$$\int_{\lambda I_0} ||\hat{f}_{\sigma_1}(\xi)||^2 d\xi = \int_{\lambda I_0} e^{2(\sigma_1-\sigma_0)\xi} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi + O(e^{-\lambda}). \tag{3.8}$$

Since for $\xi \in \lambda I_0$ we have $e^{(\sigma_1-\sigma_0)\xi} \leq e^{(\sigma_1-\sigma_0)(A_0-\nu)\lambda}$, we obtain that

$$(3.8) \leq e^{2(\sigma_1-\sigma_0)(A_0-\nu)\lambda} \int_{\lambda I_0} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi + O(e^{-\lambda}), \tag{3.9}$$

and by (3.7) we get

$$(3.9) \leq e^{-\delta\lambda} e^{2(\sigma_1-\sigma_0)(A_0-\nu)\lambda} \int_{[-A_0\lambda, A_0\lambda]\backslash\lambda I_0} \|\hat{f}_{\sigma_0}(\xi)\|^2 d\xi + e^{2(\sigma_1-\sigma_0)(A_0-\nu)\lambda} e^{-\delta\lambda}. \tag{3.10}$$

Now if $\xi \in [-A_0\lambda, A_0\lambda]\backslash I_0$ we have $e^{(\sigma_1-\sigma_0)\xi} \geq e^{(\sigma_1-\sigma_0)(A_0-\nu)\lambda}$ and thus

$$(3.10) \leq e^{-\delta\lambda} \int_{[-A_0\lambda, A_0\lambda]\backslash\lambda I_0} e^{2(\sigma_1-\sigma_0)\xi} ||\hat{f}_{\sigma_0}(\xi)||^2 d\xi + e^{-\lambda}$$
$$\leq e^{-\delta\lambda} \int_{[-A_0\lambda, A_0\lambda]\backslash\lambda I_0} ||\hat{f}_{\sigma_1}(\xi)||^2 d\xi + e^{-\delta\lambda}.$$

In conclusion we have

$$\int_{\lambda^{-1}\xi \in I_0} ||\hat{f}_{\sigma_1}\xi||^2 d\xi \leq e^{-\delta\lambda} \int_{[A_0\lambda, A_0\lambda]} ||\hat{f}_{\sigma_1}(\xi)||^2 d\xi + e^{-\delta\lambda}. \tag{3.11}$$

Lemma 3.7 for $j=1$ gives

$$\int_{\lambda I_1} ||\hat{f}_{\sigma_1}\xi||^2 d\xi \leq e^{-\delta\lambda} \int_{[A_0\lambda, A_0\lambda]} ||\hat{f}_{\sigma_1}(\xi)||^2 d\xi + e^{-\delta\lambda} \tag{3.12}$$

and thus adding together (3.11) and (3.12) we obtain

$$\int_{\lambda I_1 \cup \lambda I_0} ||\hat{f}_{\sigma_1}\xi||^2 d\xi \leq e^{-\delta\lambda} \int_{[A_0\lambda, A_0\lambda]} ||\hat{f}_{\sigma_1}(\xi)||^2 d\xi + e^{-\delta\lambda}. \tag{3.13}$$

Using the same trick as before we conclude that

$$\int_{\lambda I_1 \cup \lambda I_0} ||\hat{f}_{\sigma_1}\xi||^2 d\xi \leq e^{-\delta\lambda} \int_{[A_0\lambda, A_0\lambda]\setminus\lambda I_0 \cup \lambda I_1} ||\hat{f}_{\sigma_1}(\xi)||^2 d\xi + e^{-\delta\lambda}. \tag{3.14}$$

The general case follows by induction.

Definition. Let Ω be an open set in $\mathbf{C}$, and let $f(y,s)$ be a continuous function that is holomorphic with respect to $s \in \Omega$ and such that $f(\cdot,s) \in H^k_\rho \quad \forall s \in \Omega$. Define $||f||_{\tilde{H}^k_{\rho,\Omega}} = \sup_{s\in\Omega} ||f(\cdot,s)||_{H^k_\rho}$, and denote by $\tilde{H}^k_{\rho,\Omega}$ the space of all functions for which the above norm is finite.

Lemma 3.8. *There exist $\delta > 0$ and an open set $\mathbf{C} \supset \Omega \supset \{s \in \mathbf{R} : |s| \leq A\}$ such that for all sufficiently large λ, all sufficiently small ρ, all $0 \leq j \leq N$, $\xi \in \lambda I_j$ and $\phi \in H^0_\rho(\mathbf{R}^+)$ supported in $[0, \lambda^{1/5}]$ there exist*

$$g \in \tilde{H}^2_{\rho,\Omega}$$
$$E \in \tilde{H}^0_{\rho,\Omega}$$

supported in $[0, \lambda^{1/5}]$ satisfying

$$e^{is\xi} L^*_{\sigma_j}(e^{-is\xi} g)(s,y) = \phi(y) + E(s,y) \quad \text{in} \quad \Omega \times [0, \lambda^{1/5}]$$

with the bounds

$$||g||_{\tilde{H}^2_{\rho/2,\Omega}} \leq C||\phi||_{\tilde{H}^0_{\rho,\Omega}}$$
$$||E||_{\tilde{H}^0_{\rho/2,\Omega}} \leq Ce^{-\delta\lambda}||\phi||_{\tilde{H}^0_{\rho,\Omega}}.$$

Proof. Since we want to approximate the operator L_σ with the family of differential operators studied in Chapter 2 we need to extend the coefficients of the operator L_σ to all of $[0,\infty]$. To do that define

$$\tilde{b}(y,s) = h(\lambda^{-1/5}y)b(y,s) + (1 - h(\lambda^{-1/5}y))\beta(y,s)$$

where $\beta(y,s) = 6y^5 + 2y(s - i\sigma)^2$ and h is a smooth function supported in $[0,2]$ and identically equal to 1 on $[0,1]$. Then the difference between $\tilde{b}$ (henceforth denoted by b) and β is $O(\lambda^{-1/2}(1+y^5))$ for all $y \in [0,\infty]$ and for s in a bounded region and the same is true for the first order derivatives. Similarly define $\bar{b}$ and $a(y,s) = \tilde{a} = h(\lambda^{-1/5}y)a(y,s)$. In the proof, the integral in the definition of the norm H_ρ^* should be considered over the entire positive line; this technicality does not affect the statement of the lemma since all of the functions involved are compactly supported in $[0, \lambda^{1/5}]$.

$$\mathcal{L} = e^{is\xi} L_{\sigma_j}^* e^{-is\xi} = (\partial_y + [1 - 2i\gamma(s - i\sigma) + \lambda^{-1}\xi - i\lambda^{-1}\partial_s] \circ \bar{b} + a) \\ \circ \left(-\partial_y - \frac{1}{y} + [1 - 2i\gamma(s - i\sigma) + \lambda^{-1}\xi - i\lambda^{-1}\partial_s] \circ b\right).$$

Define

$$A_{\zeta,T} = (\partial_y + T(6y^5 + 2y\zeta))\left(-\partial_y - \frac{1}{y} + T(6y^5 + 2y\zeta)\right)$$

with

$$\zeta = (s - i\sigma_j)^2 \quad \text{and} \quad T = (1 - 2i\gamma(s - i\sigma_j) + \lambda^{-1}\xi).$$

We have shown in Chapter 2 that $A_{\zeta,T}$ is invertible if and only if $\zeta T^{2/3} \notin \Sigma$. By Lemma 3.4 this is the case if $\sigma = \sigma_j,\ \xi \in \lambda I_j$, and $s \in \Omega_0$ provided that Ω_0 is a sufficiently small neighborhood of $\{s \in \mathbf{R},\ |s| \le A\}$.

Write

$$\mathcal{L} = A_{\zeta,T} + \mathcal{E}$$

where $\mathcal{E}$

$$\partial_y \circ T \circ (b - \beta) - \partial_y \circ \lambda^{-1}\partial_s \circ b + T \circ \bar{b} \circ T \circ (b - \beta) - T \circ (\bar{b} - \beta) \circ T \circ \beta \\ - T \circ \bar{b} \circ i\lambda^{-1}\partial_s \circ b + T \circ (\bar{b} - \beta) \circ \left(-\partial_y - \frac{1}{y}\right) + i\lambda^{-1}\partial_s \circ \bar{b} \circ \left(\partial_y + \frac{1}{y}\right) \\ - i\lambda^{-1}\partial_s \circ \bar{b} \circ T \circ b - \lambda^{-1}\partial_s \circ \bar{b} \circ \lambda^{-1}\partial_s \circ b - a \circ \left(\partial_y + \frac{1}{y}\right) + a \circ T \circ b \\ - a \circ i\lambda^{-1}\partial_s \circ b.$$

Note that $A_{\zeta,T}$ is invertible from $\tilde{H}^2_{\rho,\Omega_0}$ to $\tilde{H}^0_{\rho,\Omega_0}$ because $A^{-1}_{\zeta,T}$ is uniformly bounded for every s in a compact set and obviously maps holomorphic functions to holomorphic functions since the coefficients of $A_{\zeta,T}$ depend holomorphically on s.

If Ω is a bounded set then we have that

$$\partial_y, \frac{1}{y}, b, \beta \ \text{ are bounded from } \ \tilde{H}^k_{\rho,\Omega} \ \text{ to } \ \tilde{H}^{k-1}_{\rho,\Omega}$$

while $(b-\beta)$ and α map the same spaces with bounds given by $O(\lambda^{-1/2})$ and $O(\lambda^{-1})$, respectively. Thus the only terms to worry about are the ones that contain ∂_s. To control these terms, let Λ be a large constant to be chosen later and define

$$\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{N+1}$$

by

$$\begin{aligned}\Omega_0 &= \{|\mathrm{Re}(s)| < A + c,\ |\mathrm{Im}(s)| < c\}\\ \Omega_1 &= \{|\mathrm{Re}(s)| < A + c - \epsilon,\ |\mathrm{Im}(s)| < c - \epsilon\}\\ &\ \vdots\\ \Omega_{N+1} &= \{|\mathrm{Re}(s)| < A + \epsilon,\ |\mathrm{Im}(s)| < \epsilon\}\end{aligned}$$

where c is a positive constant independent of λ and we chose ϵ and N such that $c-(N+1)\epsilon = \epsilon$ and $N > c'\Lambda^{-1}\lambda$. The other $\Omega_j,\ 0 \le j \le N$ are selected so that

$$d(\Omega_{j+1}, \partial\Omega_j) \ge \Lambda\lambda^{-1}\ \forall j \ge 1$$

being $d(U, V)$ the usual distance between the sets U and V. Using the Cauchy integral formula we have that ∂_s is a bounded operator from $\tilde{H}^k_{\rho,\Omega_j}$ to $\tilde{H}^k_{\rho,\Omega_{j+1}}$ with a bound given by $cd(\Omega_{j+1}, \partial\Omega_j)^{-1}$ and the same is true for ${\partial_s}^2$ with a bound of $cd(\Omega_{j+1}, \partial\Omega_j)^{-2}$. Using the property of our domains we can conclude that

$$\begin{aligned}||\partial_s||_{\tilde{H}^k_{\rho,\Omega_j}, \tilde{H}^k_{\rho,\Omega_{j+1}}} &\le c\Lambda^{-1} \qquad \text{if } \ j \ge 1\\ &\le c\lambda^{-1/2} \quad \text{if } \ j = 0\end{aligned}$$

where $||T||_{A,B}$ denotes the norm of the operator T acting between the space A and B. By the above estimates we have that

$$\begin{aligned}||\mathcal{E}||_{\tilde{H}^k_{\rho\Omega_j}, \tilde{H}^k_{\rho\Omega_{j+1}}} &\le c\Lambda^{-1} \qquad \text{if } \ j \ge 1\\ &\le c\lambda^{-1/2} \quad \text{if } \ j = 0.\end{aligned}$$

We can try to solve the equation $\mathcal{L}f = \phi$ by using a Neumann series. Indeed, define

$$f = \sum_{j=0}^{N} (-1)^j (A^{-1}_{\zeta,T} \circ \mathcal{E})^j A^{-1}_{\zeta,T} \phi.$$

Then

$$\begin{aligned}\mathcal{L}f &= (A_{\zeta,T} + \mathcal{E}) \sum_{j=0}^{N} (-1)^j (A^{-1}_{\zeta,T} \circ \mathcal{E})^j A^{-1}_{\zeta,T} \phi\\ &= \sum_{j=0}^{N} (-1)^j (\mathcal{E} \circ A^{-1}_{\zeta,T})^j \phi\\ &\quad + \sum_{j=0}^{N} (-1)^j (\mathcal{E} \circ A^{-1}_{\zeta,T})^{j+1} \phi\\ &= \phi + (-1)^N (\mathcal{E} \circ A^{-1}_{\zeta,T})^{N+1} \phi.\end{aligned}$$

The above estimates imply that

$$||(\mathcal{E}\circ A_{\zeta,T}^{-1})^j\phi||_{\tilde{H}^0_{\rho,\Omega_j}} \le c^j\lambda^{-1/2}\Lambda^{1-j}$$

and

$$||A_{\zeta,T}^{-1}(\mathcal{E}\circ A_{\zeta,T}^{-1})^j\phi||_{\tilde{H}^2_{\rho,\Omega_j}} \le c^j\Lambda^{1-j}\lambda^{-1/2}.$$

Since $N \approx c\lambda\Lambda^{-1}$ if Λ is large enough it follows that

$$||\mathcal{L}f-\phi||_{\tilde{H}^0_{\rho,\Omega_{N+1}}} \le c(c/\lambda)^N\lambda^{-1/2} \le ce^{-\delta\lambda}$$

and

$$||f||_{\tilde{H}^2_{\rho,\Omega_N}} \le \sum_{j=0}^{N} ||(A_{\zeta,T}^{-1}\circ\mathcal{E})^j A_{\zeta,T}^{-1}\phi||_{\tilde{H}^0_{\rho,\Omega_0}} \le C||\phi||_{\tilde{H}^0_{\rho,\Omega_0}}.$$

The function f so constructed satisfies all the requirements of the lemma except for the one about the support. For this reason we replace f by $\eta(\lambda^{-1/5}y)f(y) = \eta_\lambda(y)f(y) = g(y)$, where η is a smooth function supported in $[0,1]$ and identically equal to 1 on $[0,1/2]$. We have that

$$\begin{aligned}&\int_{\mathbf{R}^+} |\mathcal{L}(\eta(\lambda^{-1/5}y)f(y)-f(y))|^2\frac{y^2}{1+y^{12}}e^{\rho y^6/2}ydy\\ &\le c\int_{1/2\lambda^{1/5}}^{\infty} |\mathcal{L}f(y)|^2\frac{y^2}{1+y^{12}}ye^{\rho y^6/2} + \int_{1/2\lambda^{1/5}}^{\infty} |[\mathcal{L},\eta_\lambda]f(y)|^2\frac{y^2}{1+y^{12}}ye^{\rho y^6/2}.\end{aligned}$$

The first term is easily estimated since

$$|I| \le e^{-1/2\rho\lambda^{6/5}}\int_{1/2\lambda^{1/5}}^{\infty} |\mathcal{L}f|^2\frac{y^2}{1+y^{12}}ye^{\rho y^6} \le ce^{-\lambda}.$$

Since $f \in \tilde{H}^2_{\rho,\Omega}$ we have that f and its first derivative with respect to y decay exponentially as e^{-cy^6} and thus also the second term can be estimated using the same trick. Changing the coefficient of $\mathcal{L}$ modifies the operator, but when $0 \le y \le \lambda^{1/5}$ the modified coefficients are equal to the original ones and thus we have the conclusion.

Proof of Lemma 3.7. Let $\xi \in \lambda I_j$, $\sigma = \sigma_j$ and $\phi_\xi \in H^0_\rho$ supported in $[0,\lambda^{1/5}]$ be such that

$$||\hat{f}_\sigma(\xi)|| = \langle e^{-is\xi}\phi_\xi(y), \hat{f}_\sigma\rangle + O(e^{-\lambda}).$$

In fact if ϕ_ξ is given by

$$\phi_\xi(y) = \frac{\hat{f}_\sigma(y,\xi)e^{-\rho y^6}\frac{y^2}{1+y^{12}}}{||\hat{f}_\sigma(\xi)||}$$

then $\phi_\xi \in H^0_\rho$, $||\phi_\xi||_{H^0_\rho} = 1$ and

$$\begin{aligned}||\hat{f}_\sigma(\xi)|| &= \int_0^{\lambda^{1/5}} \phi(y,\xi)\hat{f}_\sigma(y,\xi)ydy \\ &= \int_0^{\lambda^{1/5}} \int_{\mathbf{R}} \phi(y,\xi)f_\sigma(y,s)e^{-is\xi}dsydy = \int_{\mathbf{R}} F_\xi(s)ds\end{aligned}$$

and thus we just need to show that

$$\int_{-\infty}^{-A} F_\xi + \int_A^\infty F_\xi = O(e^{-\lambda}).$$

This is the case since $|f_\sigma(y,s)| = O(e^{-\lambda})$ when γA^2 is sufficiently large.

If g and E satisfy the conclusions of Lemma 3.8 then

$$\begin{aligned}\langle f_\sigma, e^{-is\xi}\phi(y)\rangle &= \langle f_\sigma, L^*_\sigma(e^{-is\xi}g) - e^{-is\xi}E\rangle \\ &= \langle L_\sigma f_\sigma, e^{-is\xi}g\rangle - \langle f_\sigma, e^{-is\sigma}\rangle\end{aligned}$$

modulo boundary terms given by integration by parts. We will check that these terms are negligible (i.e., they decay exponentially or they vanish). Indeed,

$$\begin{aligned}&\int_{-A}^{A}\int_0^{\lambda^{1/5}} L^*_\sigma(e^{-is\xi}g)f_\sigma ydyds \\ &= \int_{-A}^{A}\int_0^{\lambda^{1/5}} (\partial_y + (T - i\lambda^{-1}\partial_s)\circ b + a) \\ &\qquad \cdot \left(-\partial_y - \frac{1}{y} + (T - i\lambda^{-1}\partial_s)\circ b\right)(e^{-is\xi}g)f_\sigma ydyds\end{aligned}$$

where $T = (1 - 2i\gamma(s - i\sigma))$. The boundary term given by integration with respect to y is

$$\int_{-A}^{A} \left(-\partial_y - \frac{1}{y} + (T - i\lambda^{-1}\partial_s)\circ b\right)(e^{-is\xi}g)f_\sigma y\Big|_{y\to 0}^{y\to\lambda^{1/5}} ds.$$

Case $y \to 0$:

Since $g \in H^2_{\rho/2,\Omega}$ Lemma 2.4 implies that

$$\lim_{y\to 0} g(y,s) = \lim_{y\to 0} y\partial_y g(y,s) = \lim_{y\to 0} y\partial_s g(y,s) = 0$$

uniformly in $s \in \Omega$. Moreover we have already proved that $f_\sigma(y)$ is bounded near zero by $e^{C\lambda}$ and thus this term is zero.

Case $y \to \lambda^{1/5}$:

Again the fact that $g \in \tilde{H}^2_{\rho/2,\Omega}$ implies that g and its first partial derivatives decay exponentially like e^{-cy^6} as y tends to infinity. This follows from Lemma 2.4 for the derivative with respect to y, while for the derivative with respect to s it comes from the fact that $g(y,s)$ is bounded in Ω and holomorphic with respect to s. Since $|f_\sigma| < e^{(C+B)\lambda}$ we have that

$$|f(\lambda^{1/5},s)\nabla_{s,y} g(\lambda^{1/5},s)| + |f(\lambda^{1/5},s)g(\lambda^{1/5},s)| \le e^{(C+B)\lambda - \lambda^{6/5}} \approx e^{-\lambda}.$$

The boundary term given by integration by parts with respect to s is

$$\int_0^{\lambda^{1/5}} \lambda^{-1} b\circ \left(-\partial_y - \frac{1}{y} + (T - i\lambda^{-1}\partial_s)\circ b\right)(e^{-is\xi}g) f_\sigma \Big|_{-A}^{A} y dy.$$

We know that for every $0 \le y \le \lambda^{1/5}$

$$|f_\sigma(y, \pm A)| \le e^{\lambda[B+C-\gamma A^2]}$$

and since

$$\left| b\circ \left(\partial_y + \frac{1}{y} + \lambda^{-1}\partial_s\right)(e^{-is\xi}) g\right|$$

is bounded uniformly in y and s we also have that this boundary term is small provided that γA^2 is large. The other boundary terms can be estimated in a similar way. Using the Cauchy-Schwarz inequality and the result of the previous lemma we have

$$\begin{aligned}
|\langle f_\sigma, e^{-is\xi}E\rangle|^2 &\le \int_{-A}^{A} ||f_\sigma(s)||^2 \cdot ||E(\cdot,s)||^2_{H^0_\rho} ds \\
&\le Ce^{-\delta\lambda} \int_{-A}^{A} ||f_\sigma(s)||^2 ds \\
&\le Ce^{-\delta\lambda} \int_{\mathbf{R}} ||f_\sigma(s)||^2 ds \\
&\le Ce^{-\delta\lambda} \int_{\mathbf{R}} ||\hat{f}_\sigma(\xi)||^2 d\xi.
\end{aligned}$$

We only need to show that

$$|\langle L_\sigma f_\sigma, e^{-is\xi} g\rangle| \le e^{-\delta\lambda}.$$

To prove this deform the contour of integration by

$$s \to s - ih(s),$$

where h is a smooth function with the property

$$h(s) = \begin{cases} c_0 & \text{if } |s| < \frac{A}{2} \\ 0 \leq h(s) \leq c_0 & \forall s \\ 0 & \text{if } |s| \geq A. \end{cases}$$

We have that for $|s| < \frac{A}{2}$

$$|e^{-i(s-ih(s))\xi}| \approx e^{c_0(A_0-(j+1)\nu)\lambda} \leq ce^{-\delta\lambda}$$

where δ is a positive constant independent of j and $|e^{-i(s-ih(s))}| \leq 1$ for all other values of s. Since for all $|s| < A$ $|L_\sigma f_\sigma| \leq e^{C\lambda(\gamma+\gamma A-\gamma s^2)}$ we have

$$|\langle L_\sigma f_\sigma, g\rangle| \leq \int_{-A}^{A}\int_0^{\lambda^{1/5}} |e^{-is\xi}|e^{C\lambda(\gamma+\gamma A-\gamma s^2)}|g(y,s)|ydyds \tag{3.15}$$

and using the fact that $\int_0^{\lambda^{1/5}} |g(y,s)|ydy$ is uniformly bounded for every s in Ω (because $g \in H^2_{\rho/2,\Omega}$) we obtain:

$$\begin{aligned}(3.15) \leq C\int_{-A/2}^{A/2} e^{C\lambda(\gamma+\gamma A-\gamma s^2)}e^{-\delta\lambda}ds + C\int_{-A}^{-A/2} e^{C\lambda(\gamma+\gamma A-\gamma s^2)} \cdot 1ds \\ + C\int_{A/2}^{A} e^{C\lambda(\gamma+\gamma A-\gamma s^2)} \cdot 1ds\end{aligned}$$

from which it easy to conclude.

4. Conclusion.

In this section we are going to show how the conclusion of the previous chapter, namely the bound for the solution of our differential equation, leads to a contradiction. This would imply that the hypothesis on which that bound was built is actually false; i.e., the operator $\bar{\partial}_b$ is not analytic hypoelliptic modulo its kernel. Let $\tilde{\Gamma}$ be the connected component of the preimage of the circle Γ described in Lemma 2.6 under the transformation $F\colon \mathbf{C} \to \mathbf{C},\ z \to z^2$ which is completely contained in the open lower half plane. Such a choice is possible if the radius of Γ is taken to be sufficiently small. Fix a parameterization $\gamma\colon [a,b] \to \mathbf{C},\ \theta \to \gamma(\theta)$ of $\tilde{\Gamma}$ which is a real analytic function of θ. We know that our solution $v(y,s)$ satisfies the differential equation

$$Lv(y,s) = (\partial_y + b(y,s))\psi(y),$$

where

$$\psi(y) = \exp(-y^6 + iy^2\zeta_0)$$

and

$$L = (\partial_y + b(y,s)(1 - i\lambda^{-1}\partial_s)) \left(-\partial_y - \frac{1}{y} + \bar{b}(y,s)(1 - i\lambda^{-1}\partial_s) + a(y,s)\right).$$

L acts on functions that are holomorphic with respect to s, and when it is restricted to $\mathbf{R}^+ \times \tilde{\Gamma}$ it takes the form

$$L = \left(\partial_y + b(y,\theta)\left(1 - i\lambda^{-1}\frac{1}{\gamma'(\theta)}\partial_\theta\right)\right) \cdot \left(-\partial_y - \frac{1}{y} + \bar{b}(y,\theta)\left(1 - i\lambda^{-1}\frac{1}{\gamma'(\theta)}\partial_\theta\right) + a(y,\theta)\right)$$

since $\partial_s = \frac{1}{\gamma'(\theta)}\partial_\theta$. L seen as a differential operator in y and θ, has a formal adjoint L^* given by the relation

$$\int_0^\infty \int_{\tilde{\Gamma}} Lfgd\theta ydy = \int_0^\infty \int_{\tilde{\Gamma}} fL^*gd\theta ydy$$

where one of the two functions is supposed to be compactly supported. We remark that by

$$\int_{\tilde{\Gamma}} hd\theta$$

we mean

$$\int_a^b h(\gamma(\theta))d\theta$$

while

$$\oint_{\tilde{\Gamma}} h(s)ds = \int_{\tilde{\Gamma}} h(\theta)\gamma'(\theta)d\theta$$

is the usual contour integral of complex analysis which is independent of the choice of the parameterization. The goal of the next lemma is to show that the family of differential operators studied in Chapter 2 is a good approximation of the operator L. Since its proof is essentially the same as that of Lemma 3.8 it will not be given.

Lemma 4.1. *Given $\phi \in C_c^\infty(\mathbf{R}^+)$ there exist δ and $C < \infty$ such that for all sufficiently large λ and sufficiently small ρ, there exists $f\colon \mathbf{R}^+ \times \tilde{\Gamma} \to \mathbf{C}$ supported in the interval $[0, \lambda^{1/5}]$ such that*

(4.1)
$$\int_0^\infty \int_{\tilde{\Gamma}} |L^* f(y,\theta) - \gamma(\theta)^{2\sigma+1}\gamma'(\theta)\phi(y)|^2 d\theta e^{\rho y^6} y dy \le e^{-\delta\lambda}$$

(4.2)
$$\int_0^\infty \int_{\tilde{\Gamma}} |f(y,\theta)|^2 d\theta e^{\rho y^6} y dy \le C$$

(4.3)
$$\int_0^\infty \int_{\tilde{\Gamma}} \left| f(y,\theta) - \gamma(\theta)^{2\sigma+1}\gamma'(\theta) A^{-1}_{\gamma(\theta)^2}\phi(y)\right|^2 d\theta e^{\rho y^6} y dy \le C\lambda^{-1}$$

where σ is as in Lemma 2.6.

Let ϕ be the function of Lemma 2.6 and consider the double integral

$$\omega = \int_0^\infty \oint v(y,s) s^{2\sigma+1}\phi(y) ds y dy.$$

Since the hypothesis of analytic hypoellipticity of $\bar{\partial}_b$ implies that v is holomorphic in a region containing the curve $\tilde{\Gamma}$, this integral must be zero. On the other hand, the same hypothesis leads to the conclusion that such an integral cannot vanish if λ is sufficiently large as will be shown below. In fact we have that

$$\begin{aligned}\omega = &\int_0^\infty \int_{\tilde{\Gamma}} v(y,\theta)\gamma(\theta)^{2\sigma+1}\gamma'(\theta)\phi(y) d\theta y dy \\ &+ \int_0^\infty \int_{\tilde{\Gamma}} v(y,\theta)(-L^* f(y,\theta) + L^* f(y,\theta)) d\theta y dy\end{aligned}$$

thus

$$\begin{aligned}\omega = &\int_0^\infty \int_{\tilde{\Gamma}} v(y,\theta) L^* f(y,\theta) d\theta y dy \\ &+ \int_0^\infty \int_{\tilde{\Gamma}} v(y,\theta)(\gamma(\theta)^{2\sigma+1}\gamma'(\theta)\phi(y) - L^* f(y,\theta)) d\theta y dy.\end{aligned}$$

Select a ρ for which Lemma 4.1 holds and choose μ in Lemma 3.4 which is strictly smaller than the δ of Lemma 4.1. Using the Schwarz inequality and (4.1) we get

$$\omega = \int_0^\infty \int_{\tilde{\Gamma}} f(y,\theta) L v(y,\theta) d\theta y dy + O(e^{-c\lambda})$$

for some positive c which implies that

$$\omega = \int_0^\infty \int_{\tilde{\Gamma}} f(y,\theta)(\partial_y + b(y,\theta))\psi(y)d\theta y dy + O(e^{-c\lambda}).$$

Since $|b(y,s) - 6y^5 - 2ys^2| \approx O(\lambda^{-1/2}(1+y^5))$ and $|\psi(y)|$ decays exponentially as e^{-y^6} we have that

$$\omega = \int_0^\infty \int_{\tilde{\Gamma}} f(y,\theta)(\partial_y + 6y^5 + 2y\gamma(\theta)^2)\psi(y)d\theta y dy + O(\lambda^{-1/2}).$$

If we define $D_{\gamma(\theta)} = \partial_y + 6y^5 + 2y\gamma(\theta)^2$ then, using the Cauchy-Schwarz inequality, (4.3) and the fact that A_s is self adjoint we obtain

$$\begin{aligned}
\omega &= \int_0^\infty \int_{\tilde{\Gamma}} \gamma(\theta)^{2\sigma+1}\gamma(\theta)' A^{-1}_{\gamma(\theta)^2}\phi(y) D_{\gamma(\theta)}\psi(y)d\theta y dy + O(\lambda^{-1/2}) \\
&= \int_0^\infty \oint_{\tilde{\Gamma}} s^{2\sigma+1} A^{-1}_{s^2}\phi(y) D_{s^2}\psi(y) ds y dy + O(\lambda^{-1/2}) \\
&= \int_0^\infty \oint_{\tilde{\Gamma}} s^{2\sigma+1} \phi(y) A^{-1}_{s^2}(y) D_{s^2}\psi(y) ds y dy + O(\lambda^{-1/2}) \\
&= c_0 + O(\lambda^{-1/2})
\end{aligned}$$

where $c_0 \neq 0$ by Lemma 2.6.

Acknowledgements. This paper contains part of the results of the Ph.D. dissertation of the author, who is grateful to his Ph.D. advisor Prof. Michael Christ.

References

[1] M.S. Baouendi and C. Goulaouic, *Nonanalytic-hypoellipticity for some degenerate elliptic operators*, Bulletin of American Mathematical Society, **78** (1972), 483-486.

[2] H.P. Boas, S.-C. Chen and E.J. Straube, *Exact regularity of the Bergman and Szegö projections on domains with partial transverse symmetries*, Manuscripta Mathematica, **62** (1988), 467-475.

[3] H.P. Boas and E.J. Straube, *Sobolev estimates for the complex Green operator on a class of weakly pseudoconvex boundaries*, Communications in Partial Differential Equations, **16**(**10**) (1991), 1573-1582.

[4] J.M. Bony, *Equivalence des diverses notions de spectre singulier analytique*, Exposé III, Séminaire Goulaouic-Schwartz, 1976-7.

[5] S.-C. Chen, *Real analytic regularity of the Szegö projection on circular domains*, Pacific J. of Math., **148** (1991), 225-235.

[6] M. Christ, *Remarks on analytic hypoellipticity of $\bar{\partial}_b$*, Proceedings of Symposium in honor of R.C. Gunning and J.J. Kohn.

[7] ———, *Analytic hypoellipticity breaks down for weakly pseudoconvex Reinhardt domains*, International Mathematics Research Notices, **3** (1991), 31-40.

[8] ———, *Precise analysis of $\bar{\partial}_b$ and $\bar{\partial}$ on domains of finite type in C^2*, Proceedings of the International Congress of Mathematicians, Kyoto 1990, (1991), 859-877.

[9] ———, *Analytic hypoellipticity, representations of nilpotent groups, and a nonlinear eigenvalue problem*, Duke Mathematical Journal, **72**(**3**) (1993), 595-639.

[10] ———, *A family of degenerate differential operators*, The Journal of Geometric Analysis, **3**(**6**) (1993), 579-597.

[11] ———, *Examples of analytic non-hypoellipticity of $\bar{\partial}_b$*, Communications in Partial Differental Equations, **19**(**5&6**) (1994), 911-941.

[12] ———, *Global analytic hypoellipticity in the presence of symmetry*, Mathematical Research Letters, **1** (1994), 559-563.

[13] ———, *A necessary condition for analytic hypoellipticity*, Mathematical Research Letters, **1** (1994), 241-248.

[14] ———, *The Szegö projection need not to preserve global regularity*, to appear in Annals of Mathematics.

[15] M. Christ and D. Geller, *Counterexamples to analytic hypoellipticity for domains of finite type*, Annals of Mathematics, **135** (1992), 551-566.

[16] M. Derridj and D. Tartakoff, *Global analyticity for $\Box_b$ on three dimensional pseudoconvex CR manifolds*, Communications in Partial Differential Equations, **18**(**11**) (1993), 1847-1868.

[17] ———, *Microlocal analyticity for the canonical solution of $\bar{\partial}_b$ on some rigid weakly pseudoconvex hypersurfaces in $\mathbf{C}^2$*, Communications in Partial Diff. Equations, **20**(**9-10**) (1995), 1647-1667.

[18] D. Geller, *Analytic Pseudodifferential Operators for the Heisenberg Group and Local Solvability*, Mathematical Notes, **37**, Princeton University Press, Princeton N.J., 1990.

[19] N. Hanges and A.A. Himonas, *Non-analytic hypoellipticity in the presence of symplecticity*, preprint, 1995.

[20] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Mathematica, **119** (1967), 147-171.

[21] ———, *The Analysis of Linear Partial Differential Operators* I, Springer-Verlag, Berlin, 1983.

[22] J.J. Kohn, *Estimates for $\bar{\partial}_b$ on pseudoconvex CR manifolds*, Proceedings of Symposia in Pure Mathematics, **43** (1985), 207-217.

[23] ———, *The range of Cauchy-Riemann operator*, Duke Mathematical Journal, **43** (1986), 525-545.

[24] J.J. Kohn and L. Nirenberg, *A pseudoconvex domain not admitting a holomorphic support function*, Mathematische Annalen, **201** (1973), 265-268.

[25] G. Metivier, *Une classe d'opérateurs non hypoelliptiques analytiques*, Indiana Mathematical Journal, **29** (1980), 823-860.

[26] O.A. Oleĭnik, *On the analyticity of solutions of partial differential equations and systems*, Asterisque, **2-3** (1973), 272-285.

[27] D. Tartakoff, *The local real analyticity of solutions to $\Box_b$ and $\bar{\partial}$-Neumann problem*, Acta Mathematica, **145** (1980), 177-204.

[28] J.M. Trepreau, *Sur l'hypoellipticité analytique microlocale des opérateurs de type principal*, Communications in Partial Differential Equations, **9** (1984), 1119-1146.

[29] F. Treves, *Analytic-hypoelliptic partial differential equations of principal type*, Communications in Pure and Applied Mathematics, **24** (1971), 537-570.

[30] ———, *Analytic hypoellipticity of a class of pseudodifferential operators with double characteristics and application to the $\bar{\partial}$-Neumann problem*, Communications in Partial Differential Equations, **3** (1978), 475-642.

[31] ———, *An Introduction to Pseudodifferential and Fourier Integral Operators*, Plenum Press, New York, 1980.

Received January 30, 1997.

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