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AN INTEGRAL TRANSFORM AND LADDER REPRESENTATIONS OF $U(p, q)$

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The positive spin ladder representations of $G = U(p, q)$, which occur naturally on a Fock space $\mathcal{F}^{p,q}$, can each be realized within a space of polynomial-valued functions on the bounded realization $\mathbf{D}_{p,q}$ of G/K. This is achieved via an integral transform constructed by Mantini, 1985. An inversion formula is given for Mantini's transform. Then, natural unitary structures are obtained for the geometric realizations of the positive spin ladder representations over G/K by using the inversion formula to pull the representations back to the Fock space setting, where the unitary structures are well-known.

1. Introduction.

1.1. Statement of the problem. The ladder representations of the Lie group $U(p, q)$ may be constructed in a geometric setting as solutions to generalized massless field equations over $U(p, q)/K$. This is achieved via an integral transform devised by L.A. Mantini. The purpose of this work is to invert the integral transform, and then use the inversion formula to construct inner products for the geometric realizations of the positive-spin ladder representations.

Several key results in this paper were inspired by the work of Faraut and Koranyi (see [**5**]). In particular, we use the integrals and measures employed in [**5**] to obtain more natural, group invariant results than those of the author in [**10**]. Finally, the author would like to thank L. Mantini, E. Dunne, L. Barchini, and R. Zierau for many helpful conversations.

1.2. History. Several realizations of the oscillator representation have been given in a Fock space setting where the underlying unitary structure is well-known. For example, see [**1**], [**2**], [**8**], and [**16**]. Of interest to us is the realization given in [**2**]. Here the oscillator representation σ is realized on a Fock space $\mathcal{F}$ of functions on $\mathbf{C}^{p+q}$ that are holomorphic in the first p coordinates and antiholomorphic in the last q. The unitary structure is given by integration against a Gaussian measure. We will use this realization of σ. We may write $\mathcal{F} = \bigoplus_{n\in\mathbf{Z}} \mathcal{F}_n$, where the $\mathcal{F}_n$ are the irreducible subrepresentations of $\mathcal{F}$. The $\mathcal{F}_n$ are referred to as *ladder representations*, since the highest weights of their K-types lie on a line in the weight lattice. The realizations of the ladder representations in the Fock space setting prove to be an important tool.

We wish to realize the ladder representations in a geometric setting, where the group action is induced from the natural geometric action on sections of a vector bundle over G/K. In [7], Jakobsen and Vergne show the unitarity of a multiplier representation π_s of $U(2,2)$ on a Hilbert space $\mathcal{H}_s$ of holomorphic functions on $U(2,2)/K$ realized as the generalized upper half plane. The elements of $\mathcal{H}_s$ satisfy the mass 0, spin s equations. Also, the π_s are realizations of ladder representations, and the unitary structure for π_s is given by a Fourier transform over the boundary of the light-cone.

The work of Jakobsen and Vergne in [7] is generalized by Mantini in [11]. In [11], the positive spin ladder representations $\mathcal{F}_n$ of $U(p,q)$ are realized via an integral transform Φ_n as a subspace $\mathcal{S}_n$ of sections of a homogeneous holomorphic vector bundle $\mathbf{E}_n$ over $U(p,q)/K$ in its model as a generalized unit disk $\mathbf{D}_{p,q}$. The fiber of $\mathbf{E}_n$ over any point $\zeta \in \mathbf{D}_{p,q}$ is essentially the space $\bar{\mathcal{P}}(n, \mathbf{C}^q)$ of antiholomorphic polynomials homogeneous of degree n in q complex variables. If the real rank of $U(p,q)$ is larger than one, or if the rank of $\mathbf{E}_n$ is larger than one, these sections satisfy certain linear partial differential equations. In the $U(2,2)$ case, these equations are the massless field equations and the natural action on $\mathcal{S}_n$ is equivalent to the multiplier representation of [7]. The transform Φ_n, which is injective on $\mathcal{F}_n$, bears many similarities to the Penrose transform (see [4], [14] and [17]). We will discuss a version of Φ_n in Section 2 below. There also exists an analogous construction for the negative spin ladder representations (see [12]), but here we restrict to the positive spin case, $n \geq 1$. The transform Φ_n and its properties play a key role in this work.

1.3. Statement of results. We give an explicit inversion formula for Mantini's transform Φ_n. A natural unitary structure on $\mathcal{S}_n$ is then obtained by using the inversion formula to pull the representation back to the Fock space setting $\mathcal{F}_n$, where the unitary structure is well-known. These results, which are discussed in detail in Section 4, are as follows: For $\phi \in \mathcal{S}_n$ $(n \geq 1)$ and $z \in \mathbf{C}^{p+q}$, we obtain

$$\Phi_n^{-1}(\phi)(z) = \lim_{t \to 1^-} C \frac{d^{p+q-2}}{dt^{p+q-2}} \left[t^{\lambda-1} \int_{\mathbf{D}_{p,q}} \phi(t\zeta, z_S) e^{z_R^T \overline{\zeta z_S}} \, d\mu_\lambda(\zeta) \right],$$

and

$$\|\phi\|_n^2 = \lim_{t \to 1^-} C \frac{d^{p+q-2}}{dt^{p+q-2}} \left[t^{\lambda-1} \int_{\mathbf{D}_{p,q}} \int_{\mathbf{C}^q} \phi(t\zeta, v) \overline{\phi(\zeta, v)} e^{-|v|^2} \, dm(v) d\mu_\lambda(\zeta) \right],$$

where $r = \min(p,q)$, $\lambda = n+p+q-1$, m is Lebesgue measure, and

$$d\mu_\lambda(\zeta) = \det(I_q - \zeta^*\zeta)^{[\lambda-(p+q)]}\, dm(\zeta).$$

The number C is a constant depending on p, q, and n, given by

$$C = \frac{n!}{(\lambda-1)!}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda - \frac{pq}{r})},$$

where Γ_Ω is the Gindikin gamma function.

Observe that the inversion formula and unitary structures on $\mathcal{S}_n$ involve integration against a G-invariant measure over G/K, allowing for more natural results than those of the author's previous work (see [9] and [10]). In particular, the operator $\mathcal{L}$ first introduced in [10] is no longer needed. The kernel function and measure for our present inversion formula were inspired by the work of Faraut and Koranyi as well as that of Ørsted and Zhang (see [5] and [13], respectively).

2. The Oscillator Representation and Mantini's Transform.

We begin by describing the oscillator representation of $G = U(p,q)$ realized on a Fock space. We also introduce G/K in its bounded model as the generalized unit disk, as well as a natural action of G on polynomial-valued functions over the generalized unit disk.

2.1. Notation. The following notation is used throughout the paper.

Notation 2.1.

(a) For $r \in \mathbf{N}$, let $\mathbf{N}_0$ denote the set of nonnegative integers and let $\mathbf{N}_0^r$ denote the set of r-tuples of nonnegative integers.

(b) If $m = (m_1, m_2, \dots, m_r) \in \mathbf{N}_0^r$, then $m! := m_1!m_2!\cdots m_r!$ and $|m| := m_1 + m_2 + \cdots + m_r$. If $z \in \mathbf{C}^r$, then $z^m := z_1^{m_1} z_2^{m_2} \cdots z_r^{m_r}$.

(c) Let $\alpha \in \mathbf{N}_0^s$, $1 \le s \le r$, and let $J = (j_1, j_2, \dots, j_s) \in \mathbf{N}_0^s$ be such that $1 \le j_1 < j_2 < \cdots < j_s \le r$. Then, if $z \in \mathbf{C}^r$, $z_J := (z_{j_1}, z_{j_2}, \dots, z_{j_s})$, $|z_J|^2 := |z_{j_1}|^2 + |z_{j_2}|^2 + \cdots + |z_{j_s}|^2$, and $z_J^\alpha := z_{j_1}^{\alpha_1} z_{j_2}^{\alpha_2} \cdots z_{j_s}^{\alpha_s}$.

(d) Let $R_p := (1, 2, \dots, p)$, and $S_{p,q} := (p+1, p+2, \dots, p+q)$. When there is no danger of confusion, we will let R represent R_p and S represent $S_{p,q}$. Thus $z_R = (z_1, \dots, z_p)$, while $z_S = (z_{p+1}, \dots, z_{p+q})$.

(e) We have a partial order $\le$ on $\mathbf{N}_0^q$ given by $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i \in \{1, \dots, q\}$.

(f) For $p, q \in \mathbf{N}$, the set of $p \times q$ matrices with complex entries will be denoted by $\mathbf{C}^{p\times q}$, and the set of matrices with entries in $\mathbf{N}_0$ will be denoted by $\mathbf{N}_0^{p\times q}$. If $X \in \mathbf{C}^{p\times q}$, the conjugate transpose of X will be written as X^*.

2.2. The group $U(p,q)$ and the oscillator representation. Recall that for $p, q \in \mathbf{N}$, $U(p,q) := \{g \in GL(p+q, \mathbf{C}) \mid g^* I_{p,q} g = I_{p,q}\}$, where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

The oscillator representation of G may be realized on the Bargmann-Segal-Fock space $\mathcal{F}^{p,q}$ of functions on $\mathbf{C}^{p+q}$ given below in Definition 2.2. A complex-valued function f on $\mathbf{C}^{p+q}$ is said to be *(p,q)-holomorphic* if f is holomorphic in z_R and antiholomorphic in z_S (i.e., holomorphic in $\bar{z}_S$).

Definition 2.2. For $p, q \in \mathbf{N}$, put

$$\mathcal{F}^{p,q} := \Big\{ f : \mathbf{C}^{p+q} \to \mathbf{C} \mid f \text{ is } (p,q)\text{-holomorphic and } \int_{\mathbf{C}^{p+q}} |f(z)|^2 e^{-|z|^2} dm(z) < \infty \Big\},$$

where, in general, m represents Lebesgue measure on $\mathbf{C}^r$ scaled by $1/\pi^r$.

The integral condition in Definition 2.2 gives rise to a natural inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{F}^{p,q}$, giving $\mathcal{F}^{p,q}$ the structure of a Hilbert space.

Definition 2.3. For $f, g \in \mathcal{F}^{p,q}$, set

$$\langle f, g \rangle := \int_{\mathbf{C}^{p+q}} f(z) \overline{g(z)} e^{-|z|^2} dm(z).$$

Let $f \in \mathcal{F}^{p,q}$ and suppose

$$f(z) = \sum_{\substack{l \in \mathbf{N}_0^p \\ m \in \mathbf{N}_0^q}} a_{l,m} z_R^l \bar{z}_S^m.$$

Integration gives

$$\langle f, f \rangle = \sum_{\substack{l \in \mathbf{N}_0^p \\ m \in \mathbf{N}_0^q}} |a_{l,m}|^2 l! m!. \tag{1}$$

We now define a version σ of the oscillator representation. From the work of Blattner and Rawnsley (see [1]), we know that this representation is unitary.

Definition 2.4. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $f \in \mathcal{F}^{p,q}$, define $\sigma(g) f$ by

$$[\sigma(g) f](z) = \det(D) \int_{\mathbf{C}^{p+q}} f(g^{-1} w) e^{-\frac{1}{2}|g^{-1} w|^2} e^{w_R^* z_R + z_S^* w_S} e^{-\frac{1}{2}|w|^2} dm(w).$$

The representation σ of G on $\mathcal{F}^{p,q}$ decomposes into irreducibles, called *ladder representations*, as exhibited below. The ladder representations gained their name from the fact that the highest weights of their K-types lie equally spaced along a ray in the weight lattice.

Definition 2.5. For $n \in \mathbf{Z}$, let

$$\mathcal{F}_n^{p,q} := \{f \in \mathcal{F}^{p,q} \mid f(e^{-i\theta}z) = e^{in\theta}f(z),\ \theta \in \mathbf{R}\}.$$

Observe that $\mathcal{F}_n^{p,q}$ is a G-invariant subspace of $\mathcal{F}^{p,q}$. For $g \in G$, let $\sigma_n(g)$ denote the restriction of $\sigma(g)$ to $\mathcal{F}_n^{p,q}$.

Theorem 2.6 ([16])**.** *Let $n \in \mathbf{Z}$. The representation σ_n of G on $\mathcal{F}_n^{p,q}$ is irreducible, and there is an orthogonal direct sum decomposition*

$$\mathcal{F}^{p,q} = \bigoplus_{n \in \mathbf{Z}} \mathcal{F}_n^{p,q}.$$

2.3. A representation of G over G/K. Ultimately, we want to realize the ladder representations σ_n as polynomial-valued functions on G/K. In this section we will introduce the generalized unit disk $\mathbf{D}_{p,q}$, which is the model of G/K that we will use. Also, we construct a representation ω_n of G that acts on polynomial-valued functions over $\mathbf{D}_{p,q}$. Later, we intertwine σ_n and ω_n via an integral transform.

Definition 2.7. The generalized unit disk $\mathbf{D}_{p,q}$ is given by

$$\mathbf{D}_{p,q} := \{\zeta \in \mathbf{C}^{p\times q} \mid I_q - \zeta^*\zeta \gg 0\}.$$

Recall that $\mathbf{D}_{p,q}$ parameterizes G/K. This may be accomplished via a one-to-one correspondence between elements of $\mathbf{D}_{p,q}$ and the negative q-planes in $\mathbf{C}^{p+q}$ under the metric of signature (p,q) with matrix $I_{p,q}$, given by

$$\zeta \mapsto \ \text{col span} \ \begin{pmatrix} \zeta \\ I_q \end{pmatrix}.$$

Tracing back from the natural action of G on the negative q-planes in $\mathbf{C}^{p+q}$, one obtains the appropriate action of G on $\mathbf{D}_{p,q}$, given by

$$g.\zeta = (A\zeta + B)(C\zeta + D)^{-1}, \tag{2}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $\zeta \in \mathbf{D}_{p,q}$.

Note that $\mathbf{D}_{p,q}$ is a bounded domain in $\mathbf{C}^{p\times q}$. In fact, it is one of the four *classical domains* (cf. [15] or [6]). In the $U(1,q)$ case, $\mathbf{D}_{1,q}$ is simply the unit ball $\mathbf{B}^q$ in $\mathbf{C}^q$.

We now work toward a definition of the representation ω_n of G over $\mathbf{D}_{p,q}$. Let $\bar{\mathcal{P}}(n,\mathbf{C}^q)$ denote the set of antiholomorphic polynomials on $\mathbf{C}^q$ that are homogeneous of degree n and let $\mathcal{O}(\mathbf{D}_{p,q},\bar{\mathcal{P}}(n,\mathbf{C}^q))$ denote the set of functions holomorphic on $\mathbf{D}_{p,q}$ taking values in $\bar{\mathcal{P}}(n,\mathbf{C}^q)$. Furthermore, for $\phi\in\mathcal{O}(\mathbf{D}_{p,q},\bar{\mathcal{P}}(n,\mathbf{C}^q))$, we put

$$\phi(\zeta,v):=\phi(\zeta)(v), \tag{3}$$

where $\zeta\in\mathbf{D}_{p,q}$ and $v\in\mathbf{C}^q$.

Definition 2.8. For $n\geq 0$, define $J_n: U(p,q)\times\mathbf{D}_{p,q}\to GL(\bar{\mathcal{P}}(n,\mathbf{C}^q))$ by

$$J_n(g,\zeta)f(v):=\det[C\zeta+D]\;f([C\zeta+D]^*v)$$

for $g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, $\zeta\in\mathbf{D}_{p,q}$ and $v\in\mathbf{C}^q$.

Definition 2.9. Suppose $g\in G$ and $\phi\in\mathcal{O}(\mathbf{D}_{p,q},\bar{\mathcal{P}}(n,\mathbf{C}^q))$. Define $\omega_n(g)\phi$ by

$$(\omega_n(g)\phi)(\zeta,v):=J_n(g^{-1},\zeta)^{-1}\phi(g^{-1}.\zeta,v)$$

for $\zeta\in\mathbf{D}_{p,q}$ and $v\in\mathbf{C}^q$.

Observe that the representation ω_n is the natural geometric action of G on sections of the vector bundle $\mathbf{E}_n=\mathbf{D}_{p,q}\times\mathcal{O}(\mathbf{D}_{p,q},\bar{\mathcal{P}}(n,\mathbf{C}^q))$ via the factor of automorphy J_n.

2.4. The transform Φ_n. In [11], Mantini gives a geometric construction of the positive spin ladder representations σ_n via an integral transform Φ_n. In this section, we explain the transform Φ_n and some of its properties. Further details may be found in [11] and [12].

Proposition 2.10 ([11])**.** *Suppose $n\in\mathbf{N}_0$ and $f\in\mathcal{F}_n^{p,q}$. Then there is an integral transform $\Phi_n:\mathcal{F}_n^{p,q}\to\mathcal{O}(\mathbf{D}_{p,q},\bar{\mathcal{P}}(n,\mathbf{C}^q))$ given by*

$$(\Phi_n f)(\zeta,v)=\int_{\mathbf{C}^q} f(\zeta w,w)e^{v^*w}e^{-|w|^2}\,dm(w),$$

where $v\in\mathbf{C}^q$ and $(\Phi_n f)(\zeta,v)$ depends holomorphically on $\zeta\in\mathbf{D}_{p,q}$.

The transform Φ_n of Proposition 2.10 is simplified from the transform constructed by Mantini in [**11**]. Here we have used the trivialization of the vector bundle in [**11**] with the multiplier action ω_n. We have also used Davidson's model of the Fock space representation σ_n instead of the L^2-cohomology realization of Blattner and Rawnsley (see [**1**]). The transform is essentially given by restriction to a negative q-plane followed by a projection operator, giving rise to an L^2-version of the Penrose transform.

The properties of Φ_n play a key role in this work. We present the continuity and G-equivariant properties of Φ_n below.

Theorem 2.11 ([**11**]). *The mapping Φ_n is one-to-one for $n \geq 0$ and identically zero for $n < 0$. Furthermore, for all $f \in \mathcal{F}_n^{p,q}$, $g \in U(p,q)$, and $n \geq 0$, the mapping Φ_n satisfies*

$$\omega_n(g)(\Phi_n f) = \Phi_n(\sigma_n(g)f).$$

Lemma 2.12 ([**11**]). *Fix $n \in \mathbf{Z}$ and $\zeta \in \mathbf{D}_{p,q}$. The mapping of $\mathcal{F}_n^{p,q}$ into $\mathcal{F}_n^{0,q}$ given by*

$$f \mapsto \Phi_n f(\zeta, \cdot)$$

is continuous.

3. Some Technical Considerations.

We now take care of some technicalities that are necessary for the statement and proof of our main results.

3.1. A computation involving Φ_n. In order to develop an inversion formula for Φ_n, we need to be able to describe explicitly certain elements in the image of Φ_n. In this section, we develop notation and compute some image elements.

Definition 3.1. Define, for $\alpha \in \mathbf{N}_0^p$, $\beta \in \mathbf{N}_0^q$ and $z \in \mathbf{C}^{p+q}$,

$$f_{\alpha\beta}(z) := z_R^\alpha \bar{z}_S^\beta.$$

We will compute the image of $f_{\alpha\beta}$ under the transform $\Phi_{(|\beta|-|\alpha|)}$. In order to simplify our expressions, we first introduce notation.

Definition 3.2. Given $\gamma \in \mathbf{C}^{p\times q}$ and $i \in \{1,\dots,p\}$, let $\gamma_{(i)}$ denote the i-th row of γ, let $r(\gamma)$ denote the element of $\mathbf{C}^q$ obtained by taking the sum of the rows of γ, and let $c(\gamma)$ denote the element of $\mathbf{C}^p$ obtained by taking the sum of the columns of γ. Finally, let $|\gamma|$ denote the sum of the entries of γ.

Definition 3.3. Given $\alpha \in \mathbf{N}_0^p$ and $\tilde{\alpha} \in \mathbf{N}_0^q$, let

$$M^{p\times q}(\tilde{\alpha},\alpha) = \{\gamma \in \mathbf{N}_0^{p\times q} \mid c(\gamma) = \alpha \quad \text{and} \quad r(\gamma) = \tilde{\alpha}\}.$$

Notice that $M(\tilde{\alpha},\alpha)$ is nonempty only when $|\alpha| = |\tilde{\alpha}|$. When p and q are understood, let $M(\tilde{\alpha},\alpha)$ stand for $M^{p\times q}(\tilde{\alpha},\alpha)$.

Definition 3.4. If $\gamma \in \mathbf{N}_0^{p\times q}$ and $\zeta \in \mathbf{D}_{p,q}$, we write

$$\zeta^\gamma := \prod_{i=1}^{p} \zeta_{(i)}^{\gamma_{(i)}} \qquad \text{and} \qquad \gamma! := \prod_{i=1}^{p} \gamma_{(i)}!,$$

respectively.

Lemma 3.5 ([10]). *Suppose that $n \geq 0$, and that $\alpha \in \mathbf{N}_0^p$ and $\beta \in \mathbf{N}_0^q$ with $|\beta| - |\alpha| = n$. Then $f_{\alpha\beta} \in \mathcal{F}_n^{p,q}$ and*

$$\left(\Phi_n f_{\alpha\beta}\right)(\zeta, v) = \sum_{\substack{\eta \in \mathbf{N}_0^q(n) \\ \eta \leq \beta}} \sum_{\gamma \in M(\beta-\eta,\alpha)} \left(\frac{\alpha!\, \beta!}{\gamma!}\zeta^\gamma\right) \frac{\bar{v}^\eta}{\eta!},$$

where $\zeta \in \mathbf{D}_{p,q}$, $v \in \mathbf{C}^q$, and $f_{\alpha\beta}$ is as in Definition 3.1. For the order $\leq$, refer to part (e) *of 2.1.*

3.2. The action of $\mathbf{u}(p,q)$ and highest weight vectors. In order to understand how an inverse for Φ_n should behave on generic elements of $\Phi_n(\mathcal{F}_n^{p,q})$, we begin by understanding how it works on highest weight vectors of K-types. In this section we explore the K-types of σ_n and ω_n.

We first examine the K-types of $\mathcal{F}_n^{p,q}$ under σ_n (see Definition 2.4).

Theorem 3.6 ([16]). *For $n \geq 0$, the Fock space $\mathcal{F}_n^{p,q}$ decomposes under σ_n into an orthogonal direct sum of K-types*

$$\mathcal{F}_n^{p,q} = \bigoplus_{s=0}^{\infty} (H_{-s}^p \otimes H_{n+s}^q),$$

where

$$H_{-s}^p = \{f(z_R) \mid f \in \mathcal{P}(s, \mathbf{C}^p)\}$$

and

$$H_{n+s}^q = \{f(z_S) \mid f \in \overline{\mathcal{P}}(n+s, \mathbf{C}^q)\}.$$

Here $\mathcal{P}(r, \mathbf{C}^m)$ (resp. $\overline{\mathcal{P}}(r, \mathbf{C}^m)$) stands for holomorphic (resp. antiholomorphic) polynomials homogeneous of degree r on $\mathbf{C}^m$.

Recall that a basis for the Lie algebra $\mathbf{g} = \mathbf{u}(p,q)$ of $G = U(p,q)$ is given by

$$(4) \quad \begin{cases} X^1_{jk} = E_{jk} - E_{kj} & \text{if } 1 \le j < k \le p, \ (p+1) \le j < k \le (p+q) \\ X^2_{jk} = i(E_{jk} + E_{kj}) & \text{if } 1 \le j \le k \le p, \ (p+1) \le j \le k \le (p+q) \\ Y^1_{jk} = E_{jk} + E_{kj} & \text{if } 1 \le j \le p, \ (p+1) \le k \le (p+q) \\ Y^2_{jk} = i(E_{jk} - E_{kj}) & \text{if } 1 \le j \le p, \ (p+1) \le k \le (p+q). \end{cases}$$

Observe that the collection of X_{jk}'s forms a basis for $\mathbf{k}$, where $\mathbf{g} = \mathbf{k} + \mathbf{p}$ is the Cartan decomposition of $\mathbf{g}$. Taking the differential of σ, we see that $\mathbf{k}$ acts irreducibly on $H^p_{-s} \otimes H^q_{n+s}$ by

$$(5) \qquad d\sigma(X^1_{jk})f(z) = \begin{cases} z_j \frac{\partial f}{\partial z_k} - z_k \frac{\partial f}{\partial z_j} & \text{if } 1 \le j < k \le p \\ \bar{z}_j \frac{\partial f}{\partial \bar{z}_k} - \bar{z}_k \frac{\partial f}{\partial \bar{z}_j} & \text{if } (p+1) \le j < k \le (p+q), \end{cases}$$

and

(6)
$$d\sigma(X^2_{jk})f(z) = \begin{cases} -i(z_j \frac{\partial f}{\partial z_k} + z_k \frac{\partial f}{\partial z_j}) & \text{if } 1 \le j \le k \le p \\ i(\bar{z}_j \frac{\partial f}{\partial \bar{z}_k} + \bar{z}_k \frac{\partial f}{\partial \bar{z}_j} + 2\delta_{jk} f) & \text{if } (p+1) \le j \le k \le (p+q). \end{cases}$$

We extend the action of $\mathbf{k}$ to its complexification $\mathbf{k_C} = \mathbf{g}l(p, \mathbf{C}) \oplus \mathbf{g}l(q, \mathbf{C})$, where

$$d\sigma(E_{jk})f(z) = \begin{cases} -z_k \frac{\partial f}{\partial z_j} & \text{if } 1 \le j,k \le p \\ \bar{z}_j \frac{\partial f}{\partial \bar{z}_k} + \delta_{jk} f & \text{if } (p+1) \le j,k \le (p+q). \end{cases}$$

A highest weight vector for an action of $\mathbf{k_C}$ will be a weight vector whose weight is contained in $\mathbf{d}^*(p) \oplus \mathbf{d}^*(q)$ and is annhilated by $\mathbf{n}(p) \oplus \mathbf{n}(q)$, where $\mathbf{d}(r)$ and $\mathbf{n}(r)$ are the diagonal and strictly upper triangular subalgebras of $\mathbf{g}l(r, \mathbf{C})$, respectively.

Lemma 3.7. *A highest weight vector for the action of $\mathbf{k_C}$ on $H^p_{-s} \otimes H^q_{n+s}$ is given by*

$$f_{n,s}(z) = z_p^s \bar{z}_{p+1}^{n+s},$$

and every $f \in H^p_{-s} \otimes H^q_{n+s}$ of the form $f(z) = z_R^\alpha \bar{z}_S^\beta$ can be written as $f = k.f_{n,s}$, for some $k \in \mathbf{k_C}$.

Now, applying Lemma 3.5, Theorem 2.11, and Lemma 3.7, we have:

Lemma 3.8. *For $n \ge 1$, and $\zeta = (\zeta_{ij}) \in \mathbf{D}_{p,q}$,*

$$\phi_{n,s}(\zeta, v) := \Phi_n(f_{n,s})(\zeta, v) = \frac{(n+s)!}{n!} \zeta_{p1}^s \bar{v}_1^n,$$

and $\phi_{n,s}$ is a highest weight vector for $\Phi_n(\mathcal{F}^{p,q}_n)$ under the action of $\mathbf{k_C}$.

3.3. An orthogonal family of polynomials over $\mathbf{D}_{p,q}$. We review some facts about the orthogonal family of polynomials over $\mathbf{D}_{p,q}$ introduced in [9]. These polynomials prove to be important, because we can express elements in the image of Φ_n in terms of them.

Definition 3.9. Fix $p, q \in \mathbf{N}$ and suppose $\alpha \in \mathbf{N}_0^p$ and $\rho \in \mathbf{N}_0^q$ with $|\alpha| = |\rho|$. For $\zeta \in \mathbf{D}_{p,q}$ define

$$\varphi_{\rho\alpha}(\zeta) := \sum_{\gamma \in M(\rho,\alpha)} \frac{1}{\gamma!} \zeta^\gamma,$$

where $M(\rho, \alpha)$ is as in Definition 3.3.

For $\alpha \in \mathbf{N}_0^p$ and $\beta \in \mathbf{N}_0^q$ with $|\beta| - |\alpha| = n$, write $\phi_{\alpha\beta} = \Phi_n f_{\alpha\beta}$ where $f_{\alpha\beta}(z) = z_R^\alpha \bar{z}_S^\beta$. Then we may rewrite the conclusion of Lemma 3.5 as

$$\phi_{\alpha\beta}(\zeta, v) = \sum_{\substack{\eta \in \mathbf{N}_0^q(n) \\ \eta \leq \beta}} \left(\alpha!\, \beta! \varphi_{(\beta-\eta),\alpha}(\zeta)\right) \frac{\bar{v}^\eta}{\eta!}. \tag{7}$$

Lemma 3.10 ([9])**.** *The mapping of $\mathbf{D}_{p,q}$ into $\mathbf{C}$ given by*

$$\zeta \mapsto \exp\left(\sum_{i=1}^p \sum_{j=1}^q \zeta_{ij}\right)$$

possesses the series expansion

$$\exp\left(\sum_{i=1}^p \sum_{j=1}^q \zeta_{ij}\right) = \sum_{m=0}^\infty \left(\sum_{\substack{\rho \in \mathbf{N}_0^q(m) \\ \alpha \in \mathbf{N}_0^p(m)}} \varphi_{\rho\alpha}(\zeta) \right).$$

In addition to summing to an exponential function, the collection of $\varphi_{\rho\alpha}$'s has the virtue of being an orthogonal set of functions with respect to K-invariant inner products. Consider the Hermitian inner product on holomorphic polynomials over $\mathbf{D}_{p,q}$ given by

$$(f, g)_\lambda = \int_{\mathbf{D}_{p,q}} f(\zeta)\overline{g(\zeta)}\, d\mu_\lambda(\zeta), \tag{8}$$

where $d\mu_\lambda(\zeta) = \det(I_q - \zeta^*\zeta)^{[\lambda-(p+q)]} dm(\zeta)$ for $\lambda \in \mathbf{N}, \lambda \geq (p+q)$. Also, the action of K on $\mathbf{D}_{p,q}$ as in (2) induces an action of K on the holomorphic polynomials over $\mathbf{D}_{p,q}$, given by

$$(k.f)(\zeta) = f(k^{-1}.\zeta) = f(A^{-1}\zeta D), \tag{9}$$

for $k = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K$. Observe that the inner product $(\cdot,\cdot)_\lambda$ is invariant under the action of K, since for $k \in K$, $\det(I_q - (k.\zeta)^*(k.\zeta)) = \det(I_q - \zeta^*\zeta)$, and the mapping

$$\zeta \mapsto A^{-1}\zeta D$$

is a linear automorphism of $\mathbf{D}_{p,q}$ having a determinant of modulus 1.

Theorem 3.11 ([9]). *For $\lambda \in \mathbf{N}$ and $\lambda \geq (p+q)$, the collection of $\varphi_{\rho,\alpha}$'s is an orthogonal family of holomorphic polynomials with respect to the inner product $(\cdot,\cdot)_\lambda$.*

Proof. The theorem is proven in [9] only in the case $\lambda = (p+q)$. However, the same argument is valid for any K-invariant scalar product on the holomorphic polynomials over $\mathbf{D}_{p,q}$, including the inner products $(\cdot,\cdot)_\lambda$ mentioned in the statement of the theorem. □

3.4. Norms of certain highest weight vectors. We compute the norms of highest weight vectors for the action of K given in (9) with respect to the inner products $(\cdot,\cdot)_\lambda$ of Equation (8). Those seeking further details may consult [6], [5], [13], and [9].

First, observe that in the case $G = U(p,q)$, the Gindikin gamma function is given by

$$\Gamma_\Omega(\lambda) = \prod_{j=1}^{\min(p,q)} \Gamma(\lambda + 1 - j) \tag{10}$$

for $\lambda \in \mathbf{C}$, and Γ the ordinary gamma function. For a discussion of the general form of Gindikin's gamma function, see [5] or [13].

Theorem 3.12. *For $p, q, n \in \mathbf{N}$ and $m \in \mathbf{N}_0$, we have*

$$\int_{\mathbf{D}_{p,q}} |\zeta_{p1}^m|^2 \, d\mu_\lambda(\zeta) = \frac{\Gamma_\Omega(\lambda - \frac{pq}{r})}{\Gamma_\Omega(\lambda)} \frac{(\lambda-1)!m!}{(\lambda+m-1)!},$$

where $\lambda = p + q + n - 1$ and $r = \min(p,q)$.

Proof. By differentiating the action of K on the holomorphic polynomials over $\mathbf{D}_{p,q}$ as given in (9) and extending to $\mathbf{k_C}$, we see that the monomial ζ_{p1}^m is a highest weight vector of weight $(m, 0, \ldots, 0)$ (in the notation of [5]). We let V_m denote the $\mathbf{k_C}$-module generated by ζ_{p1}^m. Recall that each of the inner products $(\cdot,\cdot)_\lambda$ of Equation (8) on V_m is invariant under the action of K, and observe that any two K-invariant inner products on V_m must be scalar multiples of one another.

According to Theorem 3.6 of [5], there exists a spherical polynomial ψ_m on V_m with

$$(\psi_m, \psi_m)_{p+q} = \frac{\Gamma_\Omega(p+q-\frac{pq}{r})}{\Gamma_\Omega(p+q)} \frac{(p+q-1)!m!}{(m+p+q-1)!} \frac{m!(p-1)!}{(m+p-1)!}, \tag{11}$$

and

$$(\psi_m, \psi_m)_\lambda = \frac{\Gamma_\Omega(\lambda-\frac{pq}{r})}{\Gamma_\Omega(\lambda)} \frac{(\lambda-1)!m!}{(\lambda+m-1)!} \frac{m!(p-1)!}{(m+p-1)!}. \tag{12}$$

By Theorem 2.4 of [9], we established that

$$(\zeta_{p1}^m, \zeta_{p1}^m)_{p+q} = \frac{\Gamma_\Omega(p+q-\frac{pq}{r})}{\Gamma_\Omega(p+q)} \frac{(p+q-1)!m!}{(m+p+q-1)!}. \tag{13}$$

Hence combining Equations (11), (12), and (13) yields

$$(\zeta_{p1}^m, \zeta_{p1}^m)_\lambda = \frac{(\psi_m, \psi_m)_\lambda}{(\psi_m, \psi_m)_{p+q}} (\zeta_{p1}^m, \zeta_{p1}^m)_{p+q} = \frac{\Gamma_\Omega(\lambda-\frac{pq}{r})}{\Gamma_\Omega(\lambda)} \frac{(\lambda-1)!m!}{(\lambda+m-1)!}.$$

□

4. Main Results.

We now relay the main results of the paper. An inversion formula for Mantini's transform Φ_n will be presented, and then we will use this inversion formula to create unitary structures for the ladder representations $\Phi_n(\mathcal{F}_n^{p,q})$ over $\mathbf{D}_{p,q}$ by pulling back to the Fock space setting, where the unitary structures are well-known.

4.1. Inversion formula.

Definition 4.1. For $n \in \mathbf{N}$ and $\phi \in \mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$, put

$$\Phi_n^{-1}(\phi)(z) = \lim_{t\to 1^-} C \frac{d^{p+q-2}}{dt^{p+q-2}} \left[t^{\lambda-1} \int_{\mathbf{D}_{p,q}} \phi(t\zeta, z_S) e^{z_R^T \overline{\zeta z_S}} \, d\mu_\lambda(\zeta) \right],$$

whenever it converges. Here $z = (z_R, z_S) \in \mathbf{C}^{p+q}$, $\lambda = n+p+q-1$, $r = \min(p,q)$,

$$d\mu_\lambda(\zeta) = \det(I_q - \zeta^*\zeta)^{[\lambda-(p+q)]} \, dm(\zeta),$$

and C is a constant depending on n, p, and q given by

$$C = \frac{n!}{(\lambda-1)!} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda-\frac{pq}{r})}.$$

We assert that Φ_n^{-1} will invert the transform Φ_n. The following results address this assertion, culminating in Theorem 4.5.

Lemma 4.2. *For* $k = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K$ *and* $\phi \in \mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ *with* ϕ *polynomial in* $\zeta \in \mathbf{D}_{p,q}$*, the mapping* Φ_n^{-1} *satisfies*

$$\Phi_n^{-1}(\omega_n(k)\phi)(z) = (\sigma_n(k)(\Phi_n^{-1}\phi))(z),$$

where $z = (z_R, z_S) \in \mathbf{C}^{p+q}$.

Proof. Applying Definition 2.9, we obtain

$$(\omega_n(k)\phi)(\zeta, v) = \det D\ \phi(A^{-1}\zeta D, D^{-1}v) \tag{14}$$

while for $f \in \mathcal{F}^{p,q}$, Definition 2.4 gives

$$(\sigma_n(k)f)(z) = \det D\ f(A^{-1}z_R, D^{-1}z_S). \tag{15}$$

Hence, applying (14) and making the change of variable $\zeta \mapsto A\zeta D^{-1}$ gives

$$\begin{aligned}
&(\Phi_n^{-1}(\omega_n(k)\phi))(z) \\
&= \lim_{t\to 1^-} C \frac{d^{p+q-2}}{dt^{p+q-2}}\left[t^{\lambda-1}\int_{\mathbf{D}_{p,q}} \det D\ \phi(t(A^{-1}\zeta D), D^{-1}z_S)e^{z_R^T\overline{\zeta z_S}}\ d\mu_\lambda(\zeta)\right] \\
&= \lim_{t\to 1^-} C \frac{d^{p+q-2}}{dt^{p+q-2}}\left[t^{\lambda-1}\int_{\mathbf{D}_{p,q}} \det D\ \phi(t\zeta, D^{-1}z_S)e^{(A^{-1}z_R)^T\overline{\zeta\, D^{-1}z_S}}\ d\mu_\lambda(\zeta)\right] \\
&= (\sigma_n(k)(\Phi_n^{-1}\phi))(z).
\end{aligned}$$

□

Lemma 4.3. *For* $n \in \mathbf{N}$ *and* $s \in \mathbf{N}_0$*, we have*

$$(\Phi_n^{-1}\phi_{n,s})(z) = f_{n,s}(z),$$

where $z \in \mathbf{C}^{p+q}$*, and* $\phi_{n,s}$*,* $f_{n,s}$ *are as in Lemmas 3.8 and 3.7, respectively.*

Proof. First, observe that if $z = (z_R, z_S) \in \mathbf{C}^{p+q}$ and $\zeta \in \mathbf{D}_{p,q}$, since z_R is a $p \times 1$ matrix and z_S is a $q \times 1$ matrix, we have

$$z_R^T\overline{\zeta z_S} = \sum_{i=1}^{p}\sum_{j=1}^{q}(D(z_R)\bar{\zeta}D(\bar{z}_S))_{ij}, \tag{16}$$

where the $p \times p$ matrix $D(z_R)$ and the $q \times q$ matrix $D(\bar{z}_S)$ are diagonal matrices whose main diagonals are z_R and z_S, respectively. Referring to Definition 3.9, one obtains

$$\varphi_{\rho\alpha}(D(z_R)\bar{\zeta}D(\bar{z}_S)) = z_R^{\alpha}\varphi_{\rho\alpha}(\bar{\zeta})\bar{z}_S^{\rho}. \tag{17}$$

Put $\rho = (s, 0, \dots, 0)$ and $\alpha = (0, \dots, 0, s)$. Applying Lemmas 3.8 and 3.10, Theorem 3.11, Equations (16) and (17), along with Theorem 3.12 gives

$$\begin{aligned}
&\int_{\mathbf{D}_{p,q}} \phi_{n,s}(t\zeta, z_S)e^{z_R^T\overline{\zeta z_S}}\, d\mu_\lambda(\zeta) \\
&= \int_{\mathbf{D}_{p,q}} \frac{(s+n)!}{n!} t^s \zeta_{p1}^s \bar{z}_{p+1}^n e^{z_R^T\overline{\zeta z_S}}\, d\mu_\lambda(\zeta) \\
&= \int_{\mathbf{D}_{p,q}} \frac{(s+n)!}{n!} t^s \zeta_{p1}^s \bar{z}_{p+1}^n \varphi_{\rho\alpha}(D(z_R)\bar{\zeta}D(\bar{z}_S))\, d\mu_\lambda(\zeta) \\
&= \frac{(s+n)!}{s!n!} t^s z_p^s \bar{z}_{p+1}^{n+s} \int_{\mathbf{D}_{p,q}} |\zeta_{p1}^s|^2\, d\mu_\lambda(\zeta) \\
&= \frac{(s+n)!}{s!n!} t^s z_p^s \bar{z}_{p+1}^{n+s} \frac{\Gamma_\Omega(\lambda - \frac{pq}{r})}{\Gamma_\Omega(\lambda)} \frac{(\lambda-1)!s!}{(\lambda+s-1)!}.
\end{aligned}$$

Hence, with C as in Definition 4.1,

$$\begin{aligned}
(\Phi_n^{-1}\phi_{n,s})(z) &= \lim_{t\to 1^-} C\frac{d^{p+q-2}}{dt^{p+q-2}}\left[t^{\lambda-1}\int_{\mathbf{D}_{p,q}} \phi_{n,s}(t\zeta, z_S)e^{z_R^T\overline{\zeta z_S}}\, d\mu_\lambda(\zeta)\right] \\
&= \lim_{t\to 1^-} \frac{d^{p+q-2}}{dt^{p+q-2}}\left[t^{\lambda+s-1}\frac{(s+n)!}{(\lambda+s-1)!} z_p^s \bar{z}_{p+1}^{n+s}\right] \\
&= f_{n,s}(z).
\end{aligned}$$

□

We now state and prove a lemma that will be used to invoke the Fubini theorem in the proof of our main results. Recall that for $\alpha \in \mathbf{N}_0^p$ and $\beta \in \mathbf{N}_0^q$, $f_{\alpha\beta}(z) = z_R^\alpha \bar{z}_S^\beta$ and $\phi_{\alpha\beta}(\zeta, v) = \Phi(f_{\alpha\beta})(\zeta, v)$, where $z = (z_R, z_S) \in \mathbf{C}^{p+q}$.

Lemma 4.4. *Put $z = (z_R, z_S) \in \mathbf{C}^{p+q}$, $t \in (0, 1)$, $n \in \mathbf{N}$, and*

$$f = \sum_{\substack{\alpha\in\mathbf{N}_0^p, \beta\in\mathbf{N}_0^q \\ |\beta|-|\alpha|=n}} a_{\alpha\beta} f_{\alpha\beta} \in \mathcal{F}_n^{p,q}.$$

Then

$$\sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} \int_{\mathbf{D}_{p,q}} \left|a_{\alpha\beta}\phi_{\alpha\beta}(t\zeta,z_S)\det(I_q-\zeta^*\zeta)^{n-1}e^{z_R^T\overline{\zeta z_S}}\right| \, dm(\zeta)<\infty.$$

Proof. We first observe that the function $\zeta\mapsto\det(I_q-\zeta^*\zeta)^{n-1}e^{z_R\overline{\zeta z_S}}$ is bounded in modulus on $\mathbf{D}_{p,q}$, and so there is a constant c_z such that

$$\begin{aligned}&\sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} \int_{\mathbf{D}_{p,q}} \left|a_{\alpha\beta}\phi_{\alpha\beta}(t\zeta,z_S)\det(I_q-\zeta^*\zeta)^{n-1}e^{z_R^T\overline{\zeta z_S}}\right| \, dm(\zeta)\\ &\le c_z \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} |a_{\alpha\beta}|\int_{\mathbf{D}_{p,q}} |\phi_{\alpha\beta}(t\zeta,z_S)|\, dm(\zeta).\end{aligned}$$

Now, applying Equation (7), the Schwarz inequality, and Theorem 3.11, we have

$$\begin{aligned}&c_z \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} |a_{\alpha\beta}|\int_{\mathbf{D}_{p,q}} |\phi_{\alpha\beta}(t\zeta,z_S)|\, dm(\zeta)\\
&\le c_z' \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} |a_{\alpha\beta}|\alpha!\beta!t^{|\alpha|}\int_{\mathbf{D}_{p,q}} \left|\sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\\eta\le\beta}}\varphi_{(\beta-\eta)\alpha}(\zeta)\right| dm(\zeta)\\
&\le c_z'' \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} |a_{\alpha\beta}|\alpha!\beta!t^{|\alpha|}\left(\int_{\mathbf{D}_{p,q}} \left|\sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\\eta\le\beta}}\varphi_{(\beta-\eta)\alpha}(\zeta)\right|^2 dm(\zeta)\right)^{\frac12}\\
&= c_z'' \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} |a_{\alpha\beta}|\alpha!\beta!t^{|\alpha|}\left(\int_{\mathbf{D}_{p,q}} \sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\\eta\le\beta}}\left|\varphi_{(\beta-\eta)\alpha}(\zeta)\right|^2 dm(\zeta)\right)^{\frac12}.\end{aligned}$$

One may calculate the integral over $\mathbf{D}_{p,q}$ by applying Theorem 2.4 of [9]. Also,

$$\sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\\eta\le\beta}}\frac{\beta!}{(\beta-\eta)!}\le\dim_{\mathbf{C}}\bar{\mathcal{P}}(n,\mathbf{C}^q)\frac{|\beta|!}{(|\beta|-n)!}.$$

Using these two results and the Schwarz inequality again gives

$$
\begin{aligned}
& c_z'' \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} |a_{\alpha\beta}|\alpha!\beta!t^{|\alpha|}\left(\int_{\mathbf{D}_{p,q}} \sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\leq\beta}} \left|\varphi_{(\beta-\eta)\alpha}(\zeta)\right|^2 \, dm(\zeta)\right)^{\frac{1}{2}} \\
&= c_z''' \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} |a_{\alpha\beta}|\alpha!\beta!t^{|\alpha|}\left(\sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\leq\beta}} \frac{|\alpha|!}{(|\alpha|+p+q-1)!}\frac{1}{(\beta-\eta)!\alpha!}\right)^{\frac{1}{2}} \\
&= c_z''' \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} |a_{\alpha\beta}|(\alpha!)^{\frac{1}{2}}(\beta!)^{\frac{1}{2}}t^{|\alpha|}\left(\sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\leq\beta}} \frac{|\alpha|!}{(|\alpha|+p+q-1)!}\frac{\beta!}{(\beta-\eta)!}\right)^{\frac{1}{2}} \\
&\leq c_z^{(4)} \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} |a_{\alpha\beta}|(\alpha!)^{\frac{1}{2}}(\beta!)^{\frac{1}{2}}\left(t^{2|\alpha|}\frac{|\alpha|!}{(|\alpha|+p+q-1)!}\frac{|\beta!|}{(|\beta|-n)!}\right)^{\frac{1}{2}} \\
&\leq c_z^{(4)} \left[\sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} |a_{\alpha\beta}|^2\alpha!\beta!\right]^{\frac{1}{2}} \\
&\qquad \times \left[\sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} t^{2|\alpha|}\frac{|\alpha|!}{(|\alpha|+p+q-1)!}\frac{|\beta|!}{(|\beta|-n)!}\right]^{\frac{1}{2}}.
\end{aligned}
$$

The last expression is finite, since the first factor is just $\|f\|_{\mathcal{F}_n^{p,q}}$, while the second factor is finite for $t<1$. □

Theorem 4.5. *Let $n\in\mathbf{N}$, $f\in\mathcal{F}_n^{p,q}$, and $\phi=\Phi_n f$. Then $(\Phi_n^{-1}\phi)(z)=f(z)$, where $z\in\mathbf{C}^{p+q}$ and Φ_n^{-1} is as in Definition 4.1.*

Proof. We have already shown in Lemma 4.3 that the theorem holds for highest weight vectors of the K-types of $\mathcal{F}_n^{p,q}$. In fact, due to Lemmas 4.2 and 3.7, we may conclude that the theorem holds for all polynomials in $\mathcal{F}_n^{p,q}$. In particular, it works for all $f_{\alpha\beta}=z_R^\alpha\bar{z}_S^\beta\in\mathcal{F}_n^{p,q}$. So suppose that f is a generic element of $\mathcal{F}_n^{p,q}$ that possesses the expansion

$$f(z)=\sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} a_{\alpha\beta}f_{\alpha\beta}(z).$$

Put $\phi = \Phi_n f$. Then by Lemma 2.12, we have

$$\phi(\zeta, v) = \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} a_{\alpha\beta}(\Phi_n f_{\alpha\beta})(\zeta, v) = \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} a_{\alpha\beta}\phi_{\alpha\beta}(\zeta, v).$$

Hence

$$(\Phi_n^{-1}\phi)(z) = \Phi_n^{-1}\left(\sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} a_{\alpha\beta}\phi_{\alpha\beta}\right)(z),$$

and by Lemma 4.4, we may apply the Fubini-Tonelli Theorem to interchange integration with summation, obtaining

$$\begin{aligned}\Phi_n^{-1}\left(\sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} a_{\alpha\beta}\phi_{\alpha\beta}\right)(z) &= \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} (\Phi_n^{-1}a_{\alpha\beta}\phi_{\alpha\beta})(z)\\ &= \sum_{\substack{\alpha\in\mathbf{N}_0^p,\beta\in\mathbf{N}_0^q\\|\beta|-|\alpha|=n}} a_{\alpha\beta}f_{\alpha\beta}(z)\\ &= f(z).\end{aligned}$$

□

4.2. Unitary structures.

Definition 4.6. For $n \in \mathbf{N}$ and $\phi \in \mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$, put

$$\|\phi\|_n^2 = \lim_{t\to 1^-} C\frac{d^{p+q-2}}{dt^{p+q-2}}\left[t^{\lambda-1}\int_{\mathbf{D}_{p,q}}\int_{\mathbf{C}^q} \phi(t\zeta, v)\overline{\phi(\zeta, v)}e^{-|v|^2}\, dm(v)d\mu_\lambda(\zeta)\right],$$

whenever it converges. Here $r = \min(p,q)$ and $\lambda = n+p+q-1$. The constant C and the measure $d\mu_\lambda$ are as in Definition 4.1.

We will show that $\|\cdot\|_n$ is a unitary norm on $\Phi_n(\mathcal{F}_n^{p,q})$. Hence the mapping Φ_n is a unitary isomorphism of the σ_n and ω_n actions.

Lemma 4.7. *For $n \in \mathbf{N}$, let $\alpha \in \mathbf{N}_0^p$ and $\beta \in \mathbf{N}_0^q$ be such that $|\beta|-|\alpha| = n$. Also, let $\lambda = n+p+q-1$ and $r = \min(p,q)$. Put*

$$C_\alpha = C\frac{(|\alpha|+\lambda-1)!}{(|\alpha|+n)!},$$

where C is as in Definition 4.1. Then

$$\int_{\mathbf{D}_{p,q}} \sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\le\beta}} \frac{\alpha!\beta!}{\eta!}|\varphi_{(\beta-\eta)\alpha}(\zeta)|^2 \det(I_q-\zeta^*\zeta)^\lambda\, d\mu(\zeta) = \frac{1}{C_\alpha}.$$

Proof. As usual, let $f_{\alpha\beta}(z) = z_R^\alpha \bar{z}_S^\beta$, and let $\phi_{\alpha\beta} = \Phi_n f_{\alpha\beta}$. By Theorem 4.5, we have $\|f_{\alpha\beta}\|^2_{\mathcal{F}_n^{p,q}} = \|\Phi_n^{-1}\phi_{\alpha\beta}\|^2_{\mathcal{F}_n^{p,q}}$. Using Theorem 3.11 and the fact that $\phi_{\alpha\beta}$ is homogeneous polynomial of degree $|\alpha|$ in $\zeta \in \mathbf{D}_{p,q}$, we compute

$$\begin{aligned}
&\|\Phi_n^{-1}\phi_{\alpha\beta}\|^2_{\mathcal{F}_n^{p,q}} \\
&= \int_{\mathbf{C}^{p+q}} |\Phi_n^{-1}\phi_{\alpha\beta}(z)|^2 e^{-|z|^2}\, dm(z) \\
&= \int_{\mathbf{C}^{p+q}} \left| C_\alpha \int_{\mathbf{D}_{p,q}} \phi_{\alpha\beta}(\zeta, z_S) \sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\leq\beta}} z_R^\alpha \varphi_{(\beta-\eta)\alpha}(\bar{\zeta}) \bar{z}_S^{\beta-\eta}\, d\mu_\lambda(\zeta) \right|^2 e^{-|z|^2}\, dm(z) \\
&= \int_{\mathbf{C}^{p+q}} z_R^\alpha z_S^\beta \bar{z}_R^\alpha \bar{z}_S^\beta e^{-|z|^2}\, dm(z) \\
&\qquad \times \left(C_\alpha \int_{\mathbf{D}_{p,q}} \sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\leq\beta}} \frac{\alpha!\beta!}{\eta!} \varphi_{(\beta-\eta)\alpha}(\zeta)\varphi_{(\beta-\eta)\alpha}(\bar{\zeta})\, d\mu_\lambda(\zeta) \right)^2 \\
&= \alpha!\beta! \left(C_\alpha \int_{\mathbf{D}_{p,q}} \sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\leq\beta}} \frac{\alpha!\beta!}{\eta!} \varphi_{(\beta-\eta)\alpha}(\zeta)\varphi_{(\beta-\eta)\alpha}(\bar{\zeta})\, d\mu_\lambda(\zeta) \right)^2 .
\end{aligned}$$

On the other hand, using (1) we compute that $\|\Phi_n^{-1}\phi_{\alpha\beta}\|^2_{\mathcal{F}_n^{p,q}} = \|f_{\alpha\beta}\|^2_{\mathcal{F}_n^{p,q}} = \alpha!\beta!$. The conclusion of the lemma follows immediately. □

Theorem 4.8. *For $n \in \mathbf{N}$, put $f \in \mathcal{F}_n^{p,q}$ and $\phi = \Phi_n f$. Then $\|\phi\|_n^2 = \|f\|^2_{\mathcal{F}_n^{p,q}}$.*

Proof. Fix $t \in [0, 1)$ and put

$$f(z) = \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} a_{\alpha\beta} f_{\alpha\beta} \quad \text{with} \quad \phi = \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} a_{\alpha\beta}\phi_{\alpha\beta}.$$

Expanding into Taylor series and applying (1) for the integral over $\mathbf{C}^q$, along with (7) gives

$$\begin{aligned}
&\int_{\mathbf{D}_{p,q}} \int_{\mathbf{C}^q} \phi(t\zeta, v)\overline{\phi(\zeta, v)} e^{-|v|^2} \det(I_q - \zeta^*\zeta)^\lambda\, dm(v)\, d\mu(\zeta) \\
&= \int_{\mathbf{D}_{p,q}} \int_{\mathbf{C}^q} \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} \sum_{\substack{\tau\in\mathbf{N}_0^p\\ \rho\in\mathbf{N}_0^q\\ |\rho|-|\tau|=n}} a_{\alpha\beta}\bar{a}_{\tau\rho}\phi_{\alpha\beta}(t\zeta, v)\overline{\phi_{\tau\rho}(\zeta, v)} e^{-|v|^2}\, dm(v)\, d\mu_\lambda(\zeta)
\end{aligned}$$

$$= \int_{\mathbf{D}_{p,q}} \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} \sum_{\substack{\tau\in\mathbf{N}_0^p\\ \rho\in\mathbf{N}_0^q\\ |\rho|-|\tau|=n}} a_{\alpha\beta}\bar{a}_{\tau\rho}(\alpha!\beta!)^2 \times \left(\sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\le\beta,\rho}} \frac{1}{\eta!}\varphi_{(\beta-\eta)\alpha}(t\zeta)\varphi_{(\rho-\eta)\tau}(\bar{\zeta}) \right) \, d\mu_\lambda(\zeta).$$

Using the fact that $t \in [0,1)$ we may use the dominated convergence theorem to interchange integration over $\mathbf{D}_{p,q}$ and summation in the last line above. We then apply Theorem 3.11 and Lemma 4.7 to obtain

$$\begin{aligned}
&\int_{\mathbf{D}_{p,q}} \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} \sum_{\substack{\tau\in\mathbf{N}_0^p\\ \rho\in\mathbf{N}_0^q\\ |\rho|-|\tau|=n}} a_{\alpha\beta}\bar{a}_{\tau\rho}(\alpha!\beta!)^2 \\
&\qquad\qquad \times \left(\sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\le\beta,\rho}} \frac{1}{\eta!}\varphi_{(\beta-\eta)\alpha}(t\zeta)\varphi_{(\rho-\eta)\tau}(\bar{\zeta}) \right) \, d\mu_\lambda(\zeta) \\
&= \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} \alpha!\beta!|a_{\alpha\beta}|^2 t^{|\alpha|} \int_{\mathbf{D}_{p,q}} \sum_{\substack{\eta\in\mathbf{N}_0^q(n)\\ \eta\le\beta}} \frac{\alpha!\beta!}{\eta!}\varphi_{(\beta-\eta)\alpha}(\zeta)\varphi_{(\beta-\eta)\alpha}(\bar{\zeta}) \, d\mu_\lambda(\zeta) \\
&= \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} \alpha!\beta!|a_{\alpha\beta}|^2 t^{|\alpha|} \frac{1}{C_\alpha}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{t\to1^-} C\frac{d^{p+q-2}}{dt^{p+q-2}} \left[t^{\lambda-1} \int_{\mathbf{D}_{p,q}} \int_{\mathbf{C}^q} \phi(t\zeta,v)\overline{\phi(\zeta,v)}e^{-|v|^2} \, dm(v) d\mu_\lambda(\zeta) \right] \\
&= \lim_{t\to1^-} C\frac{d^{p+q-2}}{dt^{p+q-2}} t^{\lambda-1} \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} \alpha!\beta!|a_{\alpha\beta}|^2 t^{|\alpha|} \frac{1}{C_\alpha} \\
&= \sum_{\substack{\alpha\in\mathbf{N}_0^p\\ \beta\in\mathbf{N}_0^q\\ |\beta|-|\alpha|=n}} \alpha!\beta!|a_{\alpha\beta}|^2 \\
&= \|f\|^2_{\mathcal{F}_n^{p,q}}.
\end{aligned}$$

□

Let $(\!(\cdot,\cdot)\!)_n$ be the inner product on $\Phi_n(\mathcal{F}_n^{p,q})$ induced by polarizing the norm given in Definition 4.6. We have the following corollary.

Corollary 4.9. *For $p, q, n \in \mathbf{N}$,*

(a) *The space $\Phi_n(\mathcal{F}_n^{p,q})$ endowed with the inner product $(\!(\cdot,\cdot)\!)_n$ is a Hilbert space.*

(b) *The representation ω_n of G on $\Phi_n(\mathcal{F}_n^{p,q})$ is unitary with respect to $(\!(\cdot,\cdot)\!)_n$.*

(c) *The representations ω_n and σ_n are unitarily equivalent via Φ_n.*

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