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DIFFERENTIAL OPERATORS AND C-WELLPOSEDNESS OF COMPLETE SECOND ORDER ABSTRACT CAUCHY PROBLEMS

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This paper presents a unified treatment of the complete second order Cauchy problem with differential operators as coefficient operators in $L^p(R^n)$ $(1 \le p \le \infty)$ or other function spaces. Concise criteria for strong C-wellposedness and analytic wellposedness of the Cauchy problem are obtained.

1. Introduction and preliminaries.

In this paper, we try to give a unified treatment of the (wellposed or illposed) complete second order Cauchy problem

$$(1.1) \qquad \begin{cases} u''(t) + Bu'(t) + Au(t) = 0, & t \ge 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

in the case when A, B are differential operators on some function spaces.

Since integrated semigroups, C-regularized semigroups, etc., were introduced at the end of the 80s, it has become possible for us to treat illposed first order abstract Cauchy problems (cf., e.g., [**1**, **2**, **4**, **5**, **11**, **12**, **13**, **16**, **20**, **27**] and references therein). A great deal of differential operators have been shown to generate these new types of operator families in $L^p(R^n)$ $(1 \le p \le \infty)$ or other function spaces, while very few of these operators generate the classical C_0 semigroups (i.e. strongly continuous semigroups); for example, $i\Delta$ on $L^p(R^n)$ $(1 \le p \le \infty)$ generates a strongly continuous semigroup only if $p = 2$ (cf. [**15**]). Though (1.1) may be reduced in a traditional way to a first order problem, a straightforward approach presents some advantages as stated in Fattorini [**7**] (see also Remark 3.2 in Section 3 of this paper). The authors have made a series of direct investigations on the abstract Cauchy problem of the complete second order or higher order (cf. [**17**, **18**, **21-26**]).

In this paper, following a general presentation about the strong C-wellposedness, analytic wellposedness of (1.1) in Section 2, we obtain in Section 3 a series of concise criteria for the strong C-wellposedness of (1.1), in the case of A, B being certain constant coefficient differential operators in $L^p(R^n)$ $(1 \le p \le \infty)$, $C_0(R^n)$, $UC_b(R^n)$ or $C_b(R^n)$. Then in Section 4, we show that in the space $L^p(R^n)$ $(1 \le p < \infty)$, $C_0(R^n)$ or $UC_b(R^n)$, (1.1) can be wellposed in the classical sense with its two propagators extendible analytically to the open right half plane, in the case when both A and B are strongly elliptic; meanwhile some perturbation cases are considered. Finally in Section 5, we present two examples showing possible applications.

Throughout this section, A and B are closed linear operators in a Banach space E, and C is a bounded, injective operator on E such that $A = C^{-1}AC$, $B = C^{-1}BC$.

Terminology 1.1. The Banach space $\mathbf{L}(E)$ will be all bounded linear operators from E to E. We will write $\mathcal{D}(A)$ for the domain, and $\mathcal{R}(A)$ for the image of the operator A. N denotes the positive integers, $N_0 := N \bigcup \{0\}$ and $\mathbf{C}$ denotes the complex plane. For $\theta \in (0, \pi]$,

$$\Sigma_\theta := \{z \in \mathbf{C}; \quad z \neq 0, \ |\arg z| < \theta\}.$$

For $\lambda \in \mathbf{C}$,

$$P_\lambda := \lambda^2 + \lambda B + A,$$

and

$$R_\lambda := P_\lambda^{-1}$$

if the inverse exists.

$$\rho_C(A,\ B) := \{\lambda \in \mathbf{C}; P_\lambda^{-1} \ \text{ exists}, \ \mathcal{D}(R_\lambda) \supset \mathcal{R}(C), \\ R_\lambda C \in \mathbf{L}(E) \ \text{ and } R_\lambda CA \text{ is closable}\},$$

and

$$\rho(A,\ B) := \rho_I(A,\ B),$$

where I denotes the identity operator on E.

Given a continuous and exponentially bounded $f : [0,\ \infty) \to E$, we will write the Laplace transform of f by

$$\mathcal{L}\langle f \rangle(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt,$$

for λ sufficiently large.

We will need:

Lemma 1.2 ([17, Lemma 2.3]). *Let $f_1,\ f_2 \in C([0,\ \infty),\ E)$ satisfying*

$$\|f_1(t)\|,\quad \|f_2(t)\| \le Me^{\omega t}\ \text{ for some }\ M,\ \omega > 0,$$

and let A be a closed linear operator on E satisfying that for each $\lambda > \omega$, $\mathcal{L}\langle f_1\rangle(\lambda) \in \mathcal{D}(A)$ such that

$$A\mathcal{L}\langle f_1\rangle(\lambda) = \mathcal{L}\langle f_2\rangle(\lambda) \quad \text{for } \lambda > \omega.$$

Then for each $t \ge 0$, $f_1(t) \in \mathcal{D}(A)$ and $Af_1(t) = f_2(t)$.

Definition 1.3. (1) By a solution of the Cauchy problem (1.1) we mean a map $u(\cdot) \in C^2([0,\ \infty),\ E)$ such that $u(t) \in \mathcal{D}(A)$, $u'(t) \in \mathcal{D}(B)$ for each $t \ge 0$, and $Au(\cdot)$, $Bu'(\cdot)$ are continuous, satisfying (1.1).

(2) The Cauchy problem (1.1) is called C-wellposed, if (1.1) has a solution for any $u_0 \in C(\mathcal{D}(A))$, $u_1 \in C(\mathcal{D}(A) \bigcap \mathcal{D}(B))$ and there exists a nondecreasing, positive function $M(t)$ defined in $[0,\ \infty)$ such that

$$\|u(t)\| \le M(t)\left(\|C^{-1}u_0\| + \|C^{-1}u_1\|\right), \qquad t \ge 0$$

for any solution $u(t)$ of (1.1) with $u_0,\ u_1 \in \mathcal{R}(C)$.

Definition 1.4. The pair $\{S_0(t),\ S_1(t)\}_{t\ge 0}$ of strongly-continuous families of bounded operators on E is called a strong C-propagation family for (1.1) if:

(i) C commutes with $S_0(t)$, $S_1(t)$ for each $t \ge 0$;

(ii) for each $u \in E$, $S_1(\cdot)u \in C^1([0,\ \infty),\ E)$, $S_1(t)E \subset \mathcal{D}(B)$ $(t \ge 0)$ and $BS_1(\cdot)u \in C([0,\ \infty),\ E)$;

(iii) for each $u \in E$ and $t \ge 0$, $\int_0^t S_1(s)uds \in \mathcal{D}(A)$ such that

$$A\int_0^t S_1(s)uds = Cu - S_1'(t)u - BS_1(t)u, \qquad S_1(0) = 0; \tag{1.2}$$

(iv) there exist constants M, $\omega > 0$ such that

$$\|S_0(t)\|, \quad \|BS_1(t)\|, \quad \|S_1'(t)\| \le Me^{\omega t}, \qquad t \ge 0, \tag{1.3}$$

where and in the sequel, $S_1'(t)$ denotes the operator:

$$u \mapsto \frac{d}{dt}(S_1(t)u)$$

from E to E;

(v) any solution $u(t)$ of (1.1) with initial values $u_0,\ u_1 \in \mathcal{R}(C)$ can be expressed as

$$u(t) = S_0(t)C^{-1}u_0 + S_1(t)C^{-1}u_1, \qquad t \ge 0. \tag{1.4}$$

Definition 1.5. The Cauchy problem (1.1) is called strongly C- wellposed if there exists a strong C-propagation family for (1.1).

Immediately, we know that any solution $u(t)$ of (1.1), with initial values u_0, $u_1 \in E$, is unique and

$$u(t) = C^{-1}(S_0(t)u_0 + S_1(t)u_1), \qquad t \geq 0, \tag{1.5}$$

whenever (1.1) is strongly C-wellposed. Indeed, $Cu(t)$ is also a solution of (1.1) with initial values $u(0) = Cu_0 \in \mathcal{R}(C)$, $u'(0) = Cu_1 \in \mathcal{R}(C)$, since C commutes with A, B. Hence

$$Cu(t) = S_0(t)u_0 + S_1(t)u_1, \qquad t \geq 0,$$

by (1.4). Then (1.5) follows.

Remark. When $\mathcal{D}(A) \bigcap \mathcal{D}(B)$ is dense in E and $C = I$, the definition here of strong C-wellposedness coincides with that of strong wellposedness in [**21**] (see also [**7**]). This can be seen from the following result.

Proposition 1.6. *Let the Cauchy problem* (1.1) *be strongly C-wellposed. Then:*

(i) *The Cauchy problem* (1.1) *is C-wellposed;*

(ii) *for $t \geq 0$,*

$$S_0(t)u = Cu - \int_0^t S_1(s)Auds \qquad (u \in \mathcal{D}(A)),$$
$$S_1(t)u = \int_0^t (S_0(s)u - S_1(s)Bu)ds \qquad (u \in \mathcal{D}(A) \bigcap \mathcal{D}(B));$$

(iii) $(\omega,\ \infty) \subset \rho_C(A,\ B)$ *and for $\lambda > \omega$*

$$\lambda R_\lambda Cu = \mathcal{L}\left\langle S_1'(t)u \right\rangle(\lambda), \qquad u \in E,$$
$$BR_\lambda Cu = \mathcal{L}\left\langle BS_1(t)u \right\rangle(\lambda), \qquad u \in E,$$
$$\lambda^{-1}AR_\lambda Cu = \mathcal{L}\left\langle A\int_0^t S_1(s)uds \right\rangle(\lambda), \qquad u \in E,$$
$$\lambda^{-1}R_\lambda CAu = \mathcal{L}\left\langle Cu - S_0(t)u \right\rangle(\lambda), \qquad u \in \mathcal{D}(A).$$

Proof. It is easy to verify by (1.2) that, for each $u \in \mathcal{D}(A)$,

$$v(t;\ u) := Cu - \int_0^t S_1(s)Auds$$

is a solution of (1.1) with initial values $u_0 = Cu$, $u_1 = 0$; for each $u \in \mathcal{D}(A) \bigcap \mathcal{D}(B)$

$$w(t;\ u) := \int_0^t (v(s;\ u) - S_1(s)Bu)ds$$

is a solution of (1.1) with initial values $u_0 = 0$, $u_1 = Cu$. This indicates that (1.1) has a solution for $u_0 \in C(\mathcal{D}(A))$, $u_1 \in C(\mathcal{D}(A) \bigcap \mathcal{D}(B))$, and by (1.3), (1.4), both (i) and (ii) are true.

In order to show (iii), we take the Laplace transform to the two sides of the first equality in (1.2) (noting (1.3)) and obtain

$$\begin{aligned} &\mathcal{L}\left\langle A\int_0^t S_1(s)uds\right\rangle(\lambda) \\ &= \lambda^{-1}Cu - \mathcal{L}\left\langle S_1'(t)u\right\rangle(\lambda) - \mathcal{L}\left\langle BS_1(t)u\right\rangle(\lambda), \quad u \in E,\ \lambda > \omega. \end{aligned} \tag{1.6}$$

Integrating by parts and using the closedness of A, B, we have

$$P_\lambda \mathcal{L}\left\langle S_1(t)u\right\rangle(\lambda) = Cu, \qquad u \in E,\ \lambda > \omega. \tag{1.7}$$

Next, we prove that for any $\lambda > \omega$, P_λ is injective. If this is not true, then there exist $v_0 \neq 0$, $\lambda_0 > \omega$ such that $P_{\lambda_0}v_0 = 0$. Clearly, $u(t) := e^{\lambda_0 t}v_0$ is a solution of (1.1) with initial values $u_0 = v_0$, $u_1 = \lambda_0 v_0$. So by (1.5)

$$e^{\lambda_0 t}Cv_0 = S_0(t)v_0 + \lambda_0 S_1(t)v_0, \qquad t \geq 0.$$

Therefore by (1.3),

$$e^{\lambda_0 t}\|Cv_0\| \leq M(1 + 2\omega^{-1})e^{\omega t}(\|v_0\| + \|\lambda_0 v_0\|), \qquad t \geq 0.$$

This is in contradiction with $\lambda_0 > \omega$. Thus R_λ exists for $\lambda > \omega$. From (1.7), we infer that

$$R_\lambda Cu = \mathcal{L}\left\langle S_1(t)u\right\rangle(\lambda), \qquad u \in E,\ \lambda > \omega. \tag{1.8}$$

This, together with (1.3), gives the first two equalities in (iii). The third equality follows immediately, with the aid of (1.2) and the identity

$$\lambda^{-1} - BR_\lambda - \lambda R_\lambda = \lambda^{-1}AR_\lambda, \qquad \lambda > \omega.$$

Finally, making use of (1.8) and the first equality in (ii) we deduce that for $u \in \mathcal{D}(A)$, $\lambda > \omega$,

$$\begin{aligned} R_\lambda CAu &= \mathcal{L}\left\langle S_1(t)Au\right\rangle(\lambda) \\ &= -\mathcal{L}\left\langle S_0'(t)u\right\rangle(\lambda) \\ &= Cu - \lambda\mathcal{L}\left\langle S_0(t)u\right\rangle(\lambda), \end{aligned}$$

by integrating by parts. This yields the last equality in (iii). The proof is then complete. □

Remark. We now pay attention to the first equality in (iii) of Proposition 1.6. When $A = 0$, it reduces to

$$(\lambda + B)^{-1}Cu = \mathcal{L}\left\langle S_1'(t)u\right\rangle(\lambda), \qquad \lambda > \omega,$$

and so one is getting a C-regularized semigroup $S_1'(t)$. When $B = 0$, it reduces to

$$\lambda(\lambda^2 + A)^{-1}Cu = \mathcal{L}\left\langle S_1'(t)u\right\rangle(\lambda), \qquad \lambda > \omega,$$

in which case, $S_1'(t)$ is a C-regularized cosine function.

Definition 1.7. The Cauchy problem (1.1) is called analytically wellposed in Σ_θ $(0 < \theta \le \frac{\pi}{2})$ if:

(i) $\mathcal{D}(A)\bigcap\mathcal{D}(B)$ is dense in E, and (1.1) is strongly I-wellposed;

(ii) both $S_0(\cdot)$ and $S_1(\cdot)$ can be extended analytically to Σ_θ, $S_1(z)E \subset \mathcal{D}(B)$ and $BS_1(z)$ is analytic in Σ_θ;

(iii) for each $\phi \in (0,\ \theta)$, $u \in E$,

$$S_0(z)u \to u, \quad BS_1(z)u \to 0, \quad S_1'(z)u \to u, \ \text{ as } \ z \to 0 \ (z \in \Sigma_\phi),$$

and there exist M_ϕ, $\omega_\phi > 0$ such that for $z \in \Sigma_\phi$,

$$\|S_0(z)\|, \quad \|BS_1(z)\|, \quad \|S_1'(z)\| \le M_\phi e^{\omega_\phi Rez}.$$

Definition 1.8. The Cauchy problem (1.1) is called analytically solvable in Σ_θ $(0 < \theta \le \frac{\pi}{2})$ if $\mathcal{D}(A)\bigcap\mathcal{D}(B)$ is dense in E, (1.1) has a unique solution $u(\cdot)$ for each $u_0 \in \mathcal{D}(A)$, $u_1 \in \mathcal{D}(A)\bigcap\mathcal{D}(B)$, and $u(\cdot)$ can be extended analytically to Σ_θ such that for each $\phi \in (0,\ \theta)$,

$$u(z) \to u(0) \ \text{ as } \ z \to 0 \ (z \in \Sigma_\phi)$$

and

$$\|u(z)\| \le M_\phi(\|Au_0\| + \|u_1\|)e^{\omega_\phi \operatorname{Re} z}, \quad z \in \Sigma_\phi$$

for some constants M_ϕ, ω_ϕ.

Definition 1.9. A linear operator B in E is called nonnegative if for each $\lambda > 0$, $\lambda \in \rho(-B)$ (the resolvent set) and

$$\sup\{\|\lambda(\lambda + B)^{-1}\|; \quad \lambda > 0\} < +\infty.$$

We define the fractional powers of a nonnegative operator in a usual way (cf. [**3**, **6**, **10**]).

Definition 1.10. A complex polynomial $p(x) = \sum_{|\beta|\le l} a_\beta x^\beta$ on R^n is called elliptic if its principal part

$$\sum_{|\beta|=l} a_\beta x^\beta = 0 \;\text{ implies }\; x = 0;$$

$p(x)$ is called strongly elliptic if

$$\operatorname{Re} \sum_{|\beta|=l} a_\beta x^\beta > 0, \quad x \in R^n \setminus \{0\}.$$

Terminology 1.11. An n-tuple of nonnegative $\beta = (\beta_1,\ \beta_2, \ldots, \beta_n)$ is called a multiindex which we sometimes denote by $\beta \in N_0^n$ and we define

$$|\beta| = \sum_{i=1}^{n} \beta_i, \quad D^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}.$$

By $\mathcal{S}(R^n)$, we denote the space of all rapidly decreasing functions on R^n with the local convex topology defined by the family of norms

$$\|f\|_m := \sup_{|\beta|\le m} \sup_{x\in R^n} (1 + |x|^2)^m |(D^\beta f)(x)|, \quad m \in N_0.$$

The Fourier transform and its inverse transform are denoted by

$$(\mathcal{F} f)(x) = \hat{f}(x) := \int_{R^n} e^{-i\langle y,\ x\rangle} f(y)dy$$

and

$$(\mathcal{F}^{-1} f)(y) := (2\pi)^{-n} \int_{R^n} e^{i\langle y,\ x\rangle} f(x)dx.$$

$\mathcal{F}L^1$ will denote the Banach algebra $\{\mathcal{F}f;\ f \in L^1\}$ under pointwise multiplication and addition with the norm

$$\|g\|_{\mathcal{F}L^1} := \|\mathcal{F}^{-1} g\|_{L^1}.$$

The space of all Fourier multipliers on $L^p(R^n)$ $(1 \le p < \infty)$ will be denoted by $\mathcal{M}_p$, which is a Banach algebra under pointwise multiplication and addition with the norm

$$\|u\|_{\mathcal{M}_p} := \sup\{\|\mathcal{F}^{-1}(u\hat{\phi})\|_{L^p}; \quad \phi \in \mathcal{S}(R^n),\ \|\phi\|_{L^p} \le 1\}.$$

We note that

$$\mathcal{F}L^1 \hookrightarrow \mathcal{M}_1 \hookrightarrow \mathcal{M}_p \ \text{ for all } \ p.$$

For more information on multipliers, we refer to [**11**, **15**, **19**].

Lemma 1.12. *Let $j,\ n \in N,\ j > \frac{n}{2}$ and $f \in C^j(R^n)$. Assume that there exist $b,\ M_f > 0$ such that, for each multiindex β with $|\beta| \le j$,*

$$|D^\beta f(x)| \le M_f(1+|x|)^{-|\beta|-b}, \qquad x \in R^n.$$

Then $f \in \mathcal{F}L^1$ and $\|f\|_{\mathcal{F}L^1} \le L_0 M_f$ for some constant L_0 independent of f.

Proof. Copying the proof of [**11**, Lemma 3.1] leads to the result as desired. □

Lemma 1.13. *Let $1 < p < \infty,\ j,\ n \in N,\ j > \frac{n}{2}$ and $f \in C^j(R^n)$. Assume that there are $a \ge 0,\ r \ge n|\frac{1}{2} - \frac{1}{p}|,\ M_f \ge 1,\ L_f > 0$ such that for each multiindex β with $|\beta| \le j,\ x \in R^n$,*

$$|D^\beta f(x)| \le L_f M_f^{|\beta|}(1+|x|)^{(a-1)|\beta|-ar}.$$

Then $f \in \mathcal{M}_p$ and there is a constant L_0 independent of f such that $\|f\|_{\mathcal{M}_p} \le L_0 L_f M_f^{n|\frac{1}{2}-\frac{1}{p}|}$.

Proof. It follows from [**19**, Theorem 1] immediately. □

Lemma 1.14. *Let $j,\ n \in N,\ j > \frac{n}{2}$ and $\{f_t\}_{t\ge 0}$ be a family of $C^j(R^n)$-functions. Assume that for each $x \in R^n,\ \beta \in N_0^n$ with $|\beta| \le j,\ t \mapsto D^\beta f_t(x)$ is continuous in $[0, \infty)$, and there exist $a > 0,\ r > \frac{n}{2}$, and a locally bounded function $M_t > 0$ such that*

$$|D^\beta f_t(x)| \le M_t^{|\beta|}(1+|x|)^{(a-1)|\beta|-ar}, \quad |\beta| \le j,\ x \in R^n\ t \ge 0.$$

Then, for each $t \ge 0,\ f_t \in \mathcal{F}L^1,\ t \mapsto f_t$ is continuous under the norm of $\mathcal{F}L^1$, and

$$\|f_t\|_{\mathcal{F}L^1} \le \text{const } M_t^{\frac{n}{2}},\ t \ge 0.$$

Proof. Proceeding similarly as in the proof of [**11**, Lemma 3.3] and applying the dominated convergence theorem leads to the desired result. □

2. General criteria.

Let $A,\ B, C$, and E be as in Section 1.

Theorem 2.1. *The Cauchy problem* (1.1) *is strongly C-wellposed if and only if the following statements hold:*

(i) $(\omega,\ \infty) \subset \rho_C(A,\ B)$ *for some* $\omega > 0$,

(ii) *for each* $i \in \{1,\ 2,\ 3\}$, *there exists a strongly continuous function* $T_i(\cdot) : [0,\ \infty) \to \mathbf{L}(E)$ *satisfying* $\|T_i(t)\| \le Me^{\omega t}$ $(t \ge 0)$ *for some* $M > 0$ *such that for* $\lambda > \omega$,

$$\lambda R_\lambda Cu = \mathcal{L}\langle T_1(t)u\rangle(\lambda), \quad \lambda^{-1}AR_\lambda Cu = \mathcal{L}\langle T_2(t)u\rangle(\lambda), \quad u \in E,$$
$$\lambda^{-1}R_\lambda CAu = \mathcal{L}\langle T_3(t)u\rangle(\lambda), \quad u \in \mathcal{D}(A).$$

Proof. The "only if" part follows from Proposition 1.6.

The "if" part. For $t \ge 0$, define

$$S_0(t) = C - T_3(t), \qquad S_1(t)u = \int_0^t T_1(s)uds, \quad (u \in E). \tag{2.1}$$

Then for $\lambda > \omega$,

$$\mathcal{L}\langle S_0(t)u\rangle(\lambda) = \lambda^{-1}Cu - \lambda^{-1}R_\lambda CAu, \quad u \in \mathcal{D}(A), \tag{2.2}$$

$$\mathcal{L}\langle S_1(t)u\rangle(\lambda) = R_\lambda Cu, \quad \mathcal{L}\left\langle \int_0^t S_1(s)uds \right\rangle(\lambda) = \lambda^{-1}R_\lambda Cu, \quad u \in E. \tag{2.3}$$

Observe

$$\begin{aligned} BR_\lambda Cu &= \lambda^{-1}(P_\lambda - \lambda^2 - A)R_\lambda Cu \\ &= \lambda^{-1}Cu - \lambda R_\lambda Cu - \lambda^{-1}AR_\lambda Cu \\ &= \mathcal{L}\langle Cu - T_1(t)u - T_2(t)u\rangle(\lambda), \quad u \in E,\ \lambda > \omega. \end{aligned}$$

We have by (2.3) and Lemma 1.2 that

$$S_1(t)E \subset \mathcal{D}(B) \ \text{ and } \ BS_1(t) = C - T_1(t) - T_2(t), \quad t \ge 0,$$

which implies that

$$A\int_0^t S_1(s)uds = Cu - S_1'(t)u - BS_1(t)u, \qquad t \ge 0,\ u \in E.$$

Moreover, from (2.2) we get that for each $u \in \mathcal{D}(A)$, $\lambda > \omega$,

$$\mathcal{L}\langle CS_0(t)u\rangle(\lambda) = (\lambda^{-1}C - \lambda^{-1}R_\lambda CA)Cu = \mathcal{L}\langle S_0(t)Cu\rangle(\lambda),$$

and therefore

$$CS_0(t)u = S_0(t)Cu, \quad u \in \mathcal{D}(A),\ t \geq 0, \tag{2.4}$$

by the uniqueness theorem for Laplace transforms; similarly

$$CS_1(t)u = S_1(t)Cu, \quad u \in E,\ t \geq 0. \tag{2.5}$$

Next, let $u \in \mathcal{D}(A)$, $v \in \mathcal{D}(A) \bigcap \mathcal{D}(B)$. Then for $\lambda > \omega$,

$$\begin{aligned}\mathcal{L}\langle S_1(t)v\rangle(\lambda) &= R_\lambda Cv \\ &= \lambda^{-1}[(\lambda^{-1}C - \lambda^{-1}R_\lambda CA)v - R_\lambda C(Bv)] \\ &= \mathcal{L}\left\langle \int_0^t (S_0(s)v - S_1(t)Bv)ds \right\rangle(\lambda),\end{aligned}$$

and therefore

$$S_1(t)v = \int_0^t (S_0(s)v - S_1(s)Bv)ds, \quad t \geq 0; \tag{2.6}$$

similarly

$$S_0(t)u = Cu - \int_0^t S_1(s)Auds, \quad t \geq 0. \tag{2.7}$$

Thus, we can see from (2.1), (2.6) and (2.7) that

$$S_0(0)u = Cu, \quad S_0'(0)u = 0, \quad S_1(0)v = 0, \quad S_1'(0)v = Cv, \tag{2.8}$$

$$\begin{cases} S_0'(t)u = -S_1(t)Au, \quad S_0''(t)u = -T_1(t)Au, \\ S_1''(t)v = -S_1(t)Av - T_1(t)Bv, \quad t \geq 0. \end{cases} \tag{2.9}$$

Observing that for $\lambda > \omega$,

$$A(\lambda^{-1}Cu - \lambda^{-1}R_\lambda CAu) = \mathcal{L}\langle ACu - T_2(t)Au\rangle(\lambda),$$

$$\begin{aligned}AR_\lambda Cv &= \lambda^{-2}ACv - \lambda^{-1}AR_\lambda CBv - \lambda^{-2}AR_\lambda CAv \\ &= \mathcal{L}\left\langle tACv - T_2(t)Bv - \int_0^t T_2(s)Avds \right\rangle(\lambda),\end{aligned}$$

we obtain by (2.2), (2.3) and Lemma 1.2 that for $\lambda > \omega$,

$$S_0(t)u \in \mathcal{D}(A) \text{ and } \mathcal{L}\langle AS_0(t)u\rangle(\lambda) = A(\lambda^{-1}Cu - \lambda^{-1}R_\lambda CAu), \quad t \geq 0,$$
$$S_1(t)v \in \mathcal{D}(A) \text{ and } \mathcal{L}\langle AS_1(t)v\rangle(\lambda) = AR_\lambda Cv, \quad t \geq 0.$$

This together with (2.1), (2.2), (2.3), (2.9) yields, noting Lemma 1.2 again, that for $\lambda > \omega$,

$$\begin{aligned}&\mathcal{L}\langle S_0''(t)u + S_1''(t)v + AS_0(t)u + AS_1(t)v\rangle(\lambda)\\&= -\lambda R_\lambda CAu + A(\lambda^{-1}Cu - \lambda^{-1}R_\lambda CAu) - R_\lambda CAv - \lambda R_\lambda CBv + AR_\lambda Cv\\&= BR_\lambda CAu - \lambda BR_\lambda Cv\\&= \mathcal{L}\langle -BS_0'(t)u - BS_1'(t)v\rangle(\lambda).\end{aligned}$$

In conclusion,

$$t \mapsto S_0(t)u + S_1(t)v$$

is a solution of (1.1) with initial values $(Cu,\ Cv)$.

Finally, let $w(t)$ be an arbitrary solution of (1.1). Then

$$w(t) \in \mathcal{D}(A), \quad \int_0^{\frac{1}{m}} m(s+1)w'(t+s)ds \in \mathcal{D}(A)\bigcap\mathcal{D}(B), \quad t \geq 0,\ m \in N.$$

So, (2.6) holds for $v = w'(t)$ $(t \geq 0)$ by letting $m \to \infty$. From this and (2.7), we get

$$\frac{d}{ds}[S_0(t-s)w(s) + S_1(t-s)w'(s)] = 0, \quad 0 \leq s \leq t.$$

Therefore

$$Cw(t) = S_0(t)w(0) + S_1(t)w'(0), \quad t \geq 0.$$

If $w(0),\ w'(0) \in \mathcal{R}(C)$, then $C^{-1}w(0) \in \mathcal{D}(A)$ and

$$w(t) = S_0(t)C^{-1}w(0) + S_1(t)C^{-1}w'(0), \quad t \geq 0,$$

by (2.4) and (2.5). This completes the proof. □

Theorem 2.2 ([**23**]). *Let $\theta \in (0,\ \frac{\pi}{2}]$. Then the Cauchy problem (1.1) is analytically wellposed in Σ_θ if and only if $\mathcal{D}(A)\bigcap\mathcal{D}(B)$ is dense in E, and for each $\phi \in (0,\ \theta)$ there exist constants $M_\phi,\ \omega_\phi > 0$ such that $\omega_\phi + \Sigma_{\frac{\pi}{2}+\phi} \subset \rho(A,\ B)$ and*

$$\|\lambda R_\lambda\|, \quad \|\lambda^{-1}AR_\lambda\|, \quad \|\lambda^{-1}\overline{R_\lambda A}\| \leq M_\phi|\lambda|^{-1}, \quad \lambda \in \omega_\phi + \Sigma_{\frac{\pi}{2}+\phi}.$$

From the proof of [**23**, Theorem 1], as well as from [**17**, Theorem 2.5] for the uniqueness, we obtain immediately:

Theorem 2.3. *Let $\theta \in (0,\ \frac{\pi}{2}]$. Assume that $\mathcal{D}(A)\bigcap\mathcal{D}(B)$ is dense in E, and for each $\phi \in (0,\ \theta)$ there exist constants M_ϕ, $\omega_\phi > 0$ such that for $\lambda \in \omega_\phi + \Sigma_{\frac{\pi}{2}+\phi}$, $R_\lambda \in \mathbf{L}(E)$ and $\|\lambda R_\lambda\|$, $\|\lambda^{-1}AR_\lambda\| \le M_\phi|\lambda|^{-1}$. Then the Cauchy problem (1.1) is analytically solvable in Σ_θ.*

Theorem 2.4 (**Perturbation**)**.** *Let $\theta \in (0,\ \frac{\pi}{2}]$. Let A_0, B_0 be nonnegative operators such that their resolvents commute and the Cauchy problem*

$$\begin{cases} u''(t) + B_0u'(t) + A_0u(t) = 0, \quad t \ge 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

is analytically wellposed in Σ_θ. Suppose that A_1, B_1 are closed linear operators such that $\mathcal{D}(A_1) \supset \mathcal{D}(A_0^a)$, $\mathcal{D}(B_1) \supset \mathcal{D}(B_0^b)$, for some $a,\ b \in [0, 1)$. Let $A = A_0 + A_1$, $B = B_0 + B_1$. Then:

(i) *The Cauchy problem (1.1) is analytically solvable in Σ_θ;*

(ii) *the Cauchy problem (1.1) is analytically wellposed in Σ_θ,*

provided $(I + A_0)^{-a}A_1$, $(I + B_0)^{-b}B_1$ have bounded extensions on E.

Proof. Fix $\phi \in (0,\ \theta)$. By hypothesis, we have using Theorem 2.2 that there exist constants M_ϕ, $\omega_\phi > 0$ such that

$$\|\lambda R_{0\lambda}\|, \quad \|B_0R_{0\lambda}\|, \quad \|\lambda^{-1}A_0R_{0\lambda}\| \le M_\phi|\lambda|^{-1}, \tag{2.10}$$

whenever $\lambda \in \omega_\phi + \Sigma_{\frac{\pi}{2}+\phi} \subset \rho(A_0,\ B_0)$. Here $R_{0\lambda} := \lambda^2 + B_0\lambda + A_0$.

An appeal to the moment inequality yields that for $\lambda \in \omega_\phi + \Sigma_{\frac{\pi}{2}+\phi}$,

$$\begin{aligned} \|\lambda B_1R_{0\lambda}\| &\le \|B_1(I + B_0)^{-b}\|\|\lambda(I + B_0)^bR_{0\lambda}\| \\ &\le \text{const}\,|\lambda|\|(I + B_0)R_{0\lambda}\|^b\|R_{0\lambda}\|^{1-b} \\ &\le \text{const}\,|\lambda|^{-(1-b)}, \quad \text{by (2.10)}. \end{aligned}$$

Similarly, we have

$$\|A_1R_{0\lambda}\| \le \text{const}\,|\lambda|^{-2(1-a)}.$$

Thus we see that there exists $\omega'_\phi > \omega_\phi$ such that for $\lambda \in \omega'_\phi + \Sigma_{\frac{\pi}{2}+\phi}$,

$$\|\lambda B_1R_{0\lambda} + A_1R_{0\lambda}\| < \frac{1}{2},$$

and therefore R_λ exists and

$$\lambda R_\lambda = \lambda R_{0\lambda}[I + \lambda B_1R_{0\lambda} + A_1R_{0\lambda}]^{-1},$$

$$\lambda^{-1}AR_\lambda = \{\lambda^{-1}A_0 + [A_1(I+A_0)^{-1}]\lambda^{-1}(I+A_0)\}R_{0\lambda}[I+\lambda B_1R_{0\lambda}+A_1R_{0\lambda}]^{-1}.$$

Then (i) follows immediately by an application of Theorem 2.3.

When $(I+A_0)^{-a}A_1$, $(I+B_0)^{-b}B_1$ have bounded extensions on E, we have that for $\lambda \in \omega_\phi + \Sigma_{\frac{\pi}{2}+\phi}$,

$$\begin{aligned}\|\lambda\overline{R_{0\lambda}B_1}\| &\le |\lambda|\|(I+B_0)^bR_{0\lambda}\|\|\overline{(I+B_0)^{-b}B_1}\| \\ &\le \text{const } |\lambda|^{-(1-b)}, \\ \|\overline{R_{0\lambda}A_1}\| &\le \text{const } |\lambda|^{-2(1-a)}.\end{aligned}$$

Accordingly, there exists $\omega''_\phi > \omega_\phi$ such that for $\lambda \in \omega''_\phi + \Sigma_{\frac{\pi}{2}+\phi}$,

$$\lambda^{-1}R_\lambda A = [I+\lambda\overline{R_{0\lambda}B_1}+\overline{R_{0\lambda}A_1}]^{-1}\lambda^{-1}R_{0\lambda}\{A_0+(I+A_0)[(I+A_0)^{-1}A_1]\}.$$

It follows by Theorem 2.2 that the Cauchy problem is analytically wellposed in Σ_θ.

The proof is then complete. □

Remark 2.5. We refer to [**8**, **25**] for related results.

3. Differential operators as coefficient operators.

Throughout this section, E is one of the Banach spaces $L^p(R^n)$ $(1 \le p \le \infty)$, $C_0(R^n)$, $C_b(R^n)$ or $UC_b(R^n)$ (the space of uniformly continuous and bounded functions). Given a complex polynomial $p(x) = \sum_{|\beta|\le l} a_\beta(ix)^\beta$ on R^n, we define

$$p(D) = \sum_{|\beta|\le l} a_\beta D^\beta = \sum_{|\beta|\le l} a_\beta \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\beta_n}$$

with

$$\mathcal{D}(p(D)) = \left\{ f\in E;\ \sum_{|\beta|\le l} a_\beta D^\beta f \in E\right\}.$$

It is easy to see that $p(D)$ is a closed operator on E and $p(D)f = \mathcal{F}^{-1}(p\hat{f})$ for all $f \in \mathcal{D}(p(D))$.

Define

$$n_E := \begin{cases} n\left|\frac{1}{2}-\frac{1}{p}\right| & \text{if } E = L^p(R^n)\ (1<p<\infty), \\ \frac{n}{2} & \text{otherwise.}\end{cases}$$

With a given $G(x) \in \mathcal{F}L^1$, we associate a bounded linear operator $\mathbf{T}\langle G(x)\rangle$ on E as follows

$$\mathbf{T}\langle G(x)\rangle f := \mathcal{F}^{-1}G * f = \mathcal{F}^{-1}(G\hat{f}), \quad \text{for all } f\in E.$$

Assuming $H(x) \in \mathcal{M}_p$ $(1 < p < \infty)$, we define

$$\mathbf{T}\langle H(x)\rangle : f \mapsto \mathcal{F}^{-1}(H\hat{f}), \quad \text{for all} \quad f \in \mathcal{S}(R^n),$$

which extends to a bounded linear operator on $L^p(R^n)$ $(1 < p < \infty)$.

By Δ, we will denote the Laplacian $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. For each $z \in \mathbf{C}$, we will write

$$\sqrt{z} := |z|^{\frac{1}{2}} e^{\frac{1}{2}\arg z}, \quad -\pi \leq \arg z < \pi,$$

so that $\operatorname{Re}\sqrt{z} \geq 0$.

Theorem 3.1. *Let $p(x)$, $q(x)$ be complex polynomials of degrees l, m respectively on R^n. Write $h = \max\{2l,\ m\}$. Assume*

$$\sup_{x\in R^n} \operatorname{Re}\left(-p(x) + \sqrt{p^2(x) - 4q(x)}\right) < \infty.$$

Let $A = q(D)$, $B = p(D)$. Then the Cauchy problem (1.1) is strongly $(I - \Delta)^{-\alpha}$-wellposed for

$$\alpha \begin{cases} \geq \frac{1}{4}\,(n_E + 1)\,h & \text{if } \ E = L^p(R^n)\ (1 < p < \infty), \\ > \frac{1}{4}\,(n_E + 1)\,h & \text{otherwise.} \end{cases}$$

If in addition, there exists $r \in (0,\ h]$ such that

$$|p^2(x) - 4q(x)| \geq C_0|x|^r, \quad |x| \geq L_0 \tag{3.1}$$

for some $C_0,\ L_0 > 0$, then the α can be improved as

$$\alpha \begin{cases} \geq \frac{1}{4}\,(n_E h + h - r) & \text{if } \ E = L^p(R^n)\ (1 < p < \infty), \\ > \frac{1}{4}\,(n_E h + h - r) & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. For $\lambda \in R$, define

$$\begin{aligned} \mathcal{D}(\widetilde{P}_\lambda) &= \{f \in E;\ \ \mathcal{F}^{-1}[(\lambda^2 + p(x)\lambda + q(x))\hat{f}] \in E\}, \\ \widetilde{P}_\lambda f &= \mathcal{F}^{-1}[(\lambda^2 + p(x)\lambda + q(x))\hat{f}] \quad \text{for all} \quad f \in \mathcal{D}(\widetilde{P}_\lambda). \end{aligned}$$

Clearly, $\widetilde{P}_\lambda$ is a closed operator on E and

$$P_\lambda \subset \widetilde{P}_\lambda, \quad \mathcal{D}(B)\bigcap \mathcal{D}(\widetilde{P}_\lambda) \subset \mathcal{D}(A), \quad \mathcal{D}(A)\bigcap \mathcal{D}(\widetilde{P}_\lambda) \subset \mathcal{D}(B). \tag{3.3}$$

Write

$$\omega := \frac{1}{2} \sup_{x\in R^n} \operatorname{Re}\left(-p(x) + \sqrt{p^2(x) - 4q(x)}\right).$$

Then $\omega < \infty$ by hypothesis. We note that for each $\lambda > \omega$,

$$(\lambda^2 + p(x)\lambda + q(x))^{-1} \in C^\infty(R^n).$$

For each $\lambda > \omega$, put

$$\mathcal{D}(\widetilde{R}_\lambda) = \{f \in E;\ \ \mathcal{F}^{-1}[(\lambda^2 + p(x)\lambda + q(x))^{-1}\hat{f}] \in E\},$$
$$\widetilde{R}_\lambda f = \mathcal{F}^{-1}[(\lambda^2 + p(x)\lambda + q(x))^{-1}\hat{f}] \quad \text{for all} \ \ f \in \mathcal{D}(\widetilde{R}_\lambda).$$

It is easy to see that

$$\widetilde{P}_\lambda \widetilde{R}_\lambda f = f \quad (f \in \mathcal{D}(\widetilde{R}_\lambda)), \qquad \widetilde{R}_\lambda \widetilde{P}_\lambda f = f \quad (f \in \mathcal{D}(\widetilde{P}_\lambda)).$$

Whence, $\widetilde{P}_\lambda$ is injective and $\widetilde{P}_\lambda^{-1} = \widetilde{R}_\lambda$ for each $\lambda > \omega$. As a consequence, $\widetilde{R}_\lambda$ is a closed operator in E for $\lambda > \omega$.

Set

$$c_\alpha(x) = (1 + |x|^2)^{-\alpha}, \quad x \in R^n,$$

$$\mu_\pm(x) = \frac{1}{2}\left(-p(x) \pm \sqrt{p^2(x) - 4q(x)}\right), \quad x \in R^n. \tag{3.4}$$

We have that for each multiindex β,

$$|D^\beta c_\alpha(x)| \le \text{const}\,(1 + |x|)^{-2\alpha - |\beta|}, \quad x \in R^n. \tag{3.5}$$

This shows by Lemma 1.12 that $c_\alpha(x) \in \mathcal{F}L^1$ when $\alpha \neq 0$. Let

$$C_\alpha = \begin{cases} I & \text{if} \ \ E = L^2(R^n), \\ \mathbf{T}\langle c_\alpha(x)\rangle & \text{otherwise.} \end{cases}$$

Then $C_\alpha = (I - \Delta)^{-\alpha}$.

Next, we set

$$\mathcal{P}(x) = \begin{pmatrix} 0 & (1 + |x|^2)^{\frac{m}{4}} \\ -(1 + |x|^2)^{-\frac{m}{4}} q(x) & -p(x) \end{pmatrix}, \qquad x \in R^n.$$

By virtue of [9, p. 169, Theorem 2], we have

$$\left\| e^{t\mathcal{P}(x)} \right\| \le \text{const}\left(1 + t + t|x|^{\frac{h}{2}}\right) e^{\omega t}, \qquad t \ge 0,\ x \in R^n. \tag{3.6}$$

In order to get a better estimate on $\|e^{t\mathcal{P}(x)}\|$ for $|x| \ge L_0$ in the case of (3.1) holding, we put

$$g_t(x) := \frac{1}{\sqrt{p^2(x) - 4q(x)}}\left(e^{t\mu_+(x)} - e^{t\mu_-(x)}\right), \quad t \ge 0,\ |x| \ge L_0,$$
$$w_t(x) := e^{t\mu_+(x)} + e^{t\mu_-(x)}, \qquad t \ge 0,\ |x| \ge L_0.$$

Clearly, for $\lambda > \omega$, $t \geq 0$,

$$\int_0^\infty e^{-\lambda t} g_t(x)dt = \left(\lambda^2 + p(x)\lambda + q(x)\right)^{-1},$$
$$\int_0^\infty e^{-\lambda t} w_t(x)dt = (2\lambda + p(x))\left(\lambda^2 + p(x)\lambda + q(x)\right)^{-1}.$$

From this and the easily verified equality

$$(\lambda - \mathcal{P}(x))^{-1} = \begin{pmatrix} (\lambda + p(x))\left(\lambda^2 + p(x)\lambda + q(x)\right)^{-1} & (1+|x|^2)^{\frac{m}{4}}\left(\lambda^2 + p(x)\lambda + q(x)\right)^{-1} \\ -(1+|x|^2)^{-\frac{m}{4}} q(x)\left(\lambda^2 + p(x)\lambda + q(x)\right)^{-1} & \lambda\left(\lambda^2 + p(x)\lambda + q(x)\right)^{-1} \end{pmatrix},$$

$$\lambda > \omega, \tag{3.7}$$

it follows, by the uniqueness theorem for Laplace transforms, that if we write

$$e^{t\mathcal{P}(x)} = \begin{pmatrix} v_{11}(x;\ t) & v_{12}(x;\ t) \\ v_{21}(x;\ t) & v_{22}(x;\ t) \end{pmatrix}, \qquad t \geq 0,\ x \in R^n, \tag{3.8}$$

then for $t \geq 0$, $|x| \geq L_0$,

$$v_{11}(x;\ t) = \frac{1}{2}\left(w_t(x) + p(x)g_t(x)\right),$$
$$v_{22}(x;\ t) = \frac{1}{2}\left(w_t(x) - p(x)g_t(x)\right),$$
$$v_{12}(x;\ t) = (1+|x|^2)^{\frac{m}{4}} g_t(x),$$
$$v_{21}(x;\ t) = -(1+|x|^2)^{-\frac{m}{4}} q(x)g_t(x).$$

Obviously,

$$|w_t(x)| \leq 2e^{\omega t}, \qquad t \geq 0,\ |x| \geq L_0,$$

and by (3.1),

$$|g_t(x)| \leq 2C_0^{-\frac{1}{2}}|x|^{-\frac{r}{2}}e^{\omega t}, \qquad t \geq 0,\ |x| \geq L_0.$$

This combined with (3.6) yields that for all $t \geq 0$, $x \in R^n$,

$$|v_{11}(x;\ t)|,\ |v_{22}(x;\ t)| \leq \text{const}\,(1+t)(1+|x|)^{l-\frac{r}{2}}e^{\omega t},$$
$$|v_{12}(x;\ t)|,\ |v_{21}(x;\ t)| \leq \text{const}\,(1+t)(1+|x|)^{\frac{m}{2}-\frac{r}{2}}e^{\omega t},$$

and therefore by (3.8),

$$\left\|e^{t\mathcal{P}(x)}\right\| \leq \text{const}\,(1+t)(1+|x|)^{\frac{1}{2}(h-r)}e^{\omega t}, \qquad t \geq 0,\ x \in R^n, \tag{3.9}$$

valid for the case of (3.1). In fact for the otherwise case, (3.9) also holds by (3.6), if we let $r = 0$ (here and in the sequel). Now, note that for each multiindex β

$$\left\| D^{\beta}\mathcal{P}(x)\right\| \le \text{const}\,(1+|x|)^{\frac{h}{2}-|\beta|}, \qquad x \in R^n. \tag{3.10}$$

Then using Leibniz's formula, we deduce by (3.9) and (3.10) that for each multiindex β

$$\begin{aligned}&\left\| D^{\beta} e^{t\mathcal{P}(x)}\right\| \\ &\le \text{const}\,(1+t)^{|\beta|+1}(1+|x|)^{(\frac{h}{2}-1)|\beta|+\frac{1}{2}(h-r)}e^{\omega t}, \quad t \ge 0,\ x \in R^n.\end{aligned}$$

This implies by (3.8) that for each multiindex β,

$$\begin{aligned}&\left|D^{\beta}v_{11}(x;\ t)\right|, \quad \left|D^{\beta}v_{22}(x;\ t)\right|, \quad \left|D^{\beta}v_{12}(x;\ t)\right| \\ &\le \text{const}\,(1+t)^{|\beta|+1}(1+|x|)^{(\frac{h}{2}-1)|\beta|+\frac{1}{2}(h-r)}e^{\omega t}, \quad t \ge 0,\ x \in R^n.\end{aligned} \tag{3.11}$$

Set

$$v_0(x;\ t) = (1+|x|^2)^{-\frac{m}{4}}v_{12}(x;\ t), \quad t \ge 0,\ x \in R^n. \tag{3.12}$$

Then combining (3.11) with (3.5) shows, by Leibniz's formula, that for each multiindex β,

$$\begin{aligned}&\left|D^{\beta}\left[v_0(x;\ t)c_{\alpha}(x)\right]\right|, \quad \left|D^{\beta}\left[v_{11}(x;\ t)c_{\alpha}(x)\right]\right|, \quad \left|D^{\beta}\left[v_{22}(x;\ t)c_{\alpha}(x)\right]\right| \\ &\le \text{const}\,(1+t)^{|\beta|+1}(1+|x|)^{(\frac{h}{2}-1)|\beta|+\frac{1}{2}(h-r)-2\alpha}e^{\omega t}, \quad t \ge 0,\ x \in R^n.\end{aligned}$$

Therefore, we deduce by virtue of Lemmas 1.13 and 1.14 that, if $\alpha \ge \frac{1}{4}\left(hn\left|\frac{1}{2}-\frac{1}{p}\right| + h - r\right)$, $1 < p < \infty$, then

$$v_0(x;\ t)c_{\alpha}(x),\ v_{11}(x;\ t)c_{\alpha}(x),\ v_{22}(x;\ t)c_{\alpha}(x) \in \mathcal{M}_p$$

and

$$\begin{aligned}&\|v_0(x;\ t)c_{\alpha}(x)\|_{\mathcal{M}_p},\ \|v_{11}(x;\ t)c_{\alpha}(x)\|_{\mathcal{M}_p},\ \|v_{22}(x;\ t)c_{\alpha}(x)\|_{\mathcal{M}_p} \\ &\le \text{const}\,(1+t)^{1+n\left|\frac{1}{2}-\frac{1}{p}\right|}e^{\omega t}, \quad t \ge 0;\end{aligned}$$

if $\alpha > \frac{1}{4}\left(\frac{1}{2}hn + h - r\right)$, then

$$v_0(x;\ t)c_{\alpha}(x),\ v_{11}(x;\ t)c_{\alpha}(x),\ v_{22}(x;\ t)c_{\alpha}(x) \in \mathcal{F}L^1,$$

being continuous in $t \in [0,\ \infty)$ under the norm of $\mathcal{F}L^1$, and

$$\begin{aligned}&\|v_0(x;\ t)c_\alpha(x)\|_{\mathcal{F}L^1},\ \|v_{11}(x;\ t)c_\alpha(x)\|_{\mathcal{F}L^1},\ \|v_{22}(x;\ t)c_\alpha(x)\|_{\mathcal{F}L^1}\\ &\leq \text{const}\,(1+t)^{1+\frac{n}{2}}e^{\omega t}, \quad t \geq 0.\end{aligned}$$

Accordingly, putting

$$\begin{gathered}V_0(t) = \mathbf{T}\langle v_0(x;\ t)c_\alpha(x)\rangle,\ V_{11}(t) = \mathbf{T}\langle v_{11}(x;\ t)c_\alpha(x)\rangle,\\ V_{22}(t) = \mathbf{T}\langle v_{22}(x;\ t)c_\alpha(x)\rangle, \quad t \geq 0,\end{gathered}$$

we have that

$$BV_0(t) = V_{11}(t) - V_{22}(t), \quad t \geq 0, \tag{3.13}$$

and

$$\|V_0(t)\|,\ \|V_{11}(t)\|,\ \|V_{22}(t)\| \leq \text{const}\,(1+t)^{1+n_E}e^{\omega t}, \quad t \geq 0; \tag{3.14}$$

moreover, when

$$E = L^1(R^n), \quad L^\infty(R^n), \quad C_0(R^n), \quad C_b(R^n), \quad \text{or } UC_b(R^n),$$

$$t \mapsto V_0(t), \quad t \mapsto V_{11}(t), \quad t \mapsto V_{22}(t) \qquad (\text{for } t \geq 0)$$

are continuous in the uniform operator topology. On the other hand, we observe that for each $t_0 \in [0,\ \infty)$, $\phi \in \mathcal{S}(R^n)$,

$$\lim_{t\to t_0} v_0(x;\ t)c_\alpha(x)\hat{\phi} = v_0(x;\ t_0)c_\alpha(x)\hat{\phi}$$

under the topology of $\mathcal{S}(R^n)$, and therefore

$$\lim_{t\to t_0} \mathcal{F}^{-1}(v_0(x;\ t)c_\alpha(x)\hat{\phi}) = \mathcal{F}^{-1}(v_0(x;\ t_0)c_\alpha(x)\hat{\phi})$$

under the topology of $\mathcal{S}(R^n)$. This indicates that

$$\lim_{t\to t_0} V_0(t)\phi = V_0(t_0)\phi$$

under the topology of $\mathcal{S}(R^n)$, and so under the norm of $L^p(R^n)$ $(1 < p < \infty)$. Thus, (3.14) and the denseness of $\mathcal{S}(R^n)$ in $L^p(R^n)$ $(1 < p < \infty)$ together yield that $V_0(\cdot)$ is strongly continuous when $E = L^p(R^n)$ $(1 < p < \infty)$. So do $V_{11}(\cdot)$ and $V_{22}(\cdot)$ by a similar argument.

Finally, define

$$J_\lambda f = \int_0^\infty e^{-\lambda t} V_0(t) f dt, \quad K_\lambda f = \int_0^\infty e^{-\lambda t} V_{22}(t) f dt, \quad \lambda > \omega + 1,\ f \in E.$$

We note by (3.7), (3.8) and (3.12) that for $\lambda > \omega + 1$, $x \in R^n$,

$$\int_0^\infty e^{-\lambda t} v_0(x;\ t) dt = (\lambda^2 + p(x)\lambda + q(x))^{-1},$$
$$\int_0^\infty e^{-\lambda t} v_{11}(x;\ t) dt = (\lambda + p(x))(\lambda^2 + p(x)\lambda + q(x))^{-1},$$
$$\int_0^\infty e^{-\lambda t} v_{22}(x;\ t) dt = \lambda(\lambda^2 + p(x)\lambda + q(x))^{-1}.$$

From this, we obtain using Fubini's theorem that for $\lambda > \omega + 1$, $\phi \in C_c^\infty(R^n)$, and

$$f \in \begin{cases} \mathcal{S}(R^n) & \text{if } E = L^p(R^n)\ (1 < p < \infty) \\ E & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
&\langle J_\lambda f,\ (\lambda^2 + p(-D)\lambda + q(-D))\phi \rangle \\
&= \int_0^\infty e^{-\lambda t} \langle \mathcal{F}^{-1}(v_0(x;\ t) c_\alpha(x) \hat{f}),\ (\lambda^2 + p(-D)\lambda + q(-D))\phi \rangle dt \\
&= \int_0^\infty e^{-\lambda t} [\mathcal{F}^{-1}(v_0(x;\ t) c_\alpha(x) \hat{f}) * (\lambda^2 + p(D)\lambda + q(D))\phi_-](0) dt \\
&= \int_0^\infty e^{-\lambda t} \mathcal{F}^{-1}(v_0(x;\ t) c_\alpha(x) \hat{f} (\lambda^2 + p(x)\lambda + q(x)) \hat{\phi}_-)(0) dt \\
&= \mathcal{F}^{-1}(c_\alpha(x) \hat{f} \hat{\phi}_-)(0) = (\mathcal{F}^{-1}(c_\alpha(x)\hat{f}) * \phi_-)(0) \\
&= \langle C_\alpha f,\ \phi \rangle, \quad \text{where } \ \phi_-(x) = \phi(-x).
\end{aligned}$$

Similarly, we get that for λ, f, ϕ as above,

$$\langle K_\lambda f,\ (\lambda^2 + p(-D)\lambda + q(-D))\phi \rangle = \langle \lambda C_\alpha f,\ \phi \rangle.$$

In conclusion, for λ, f as above,

$$\widetilde{P}_\lambda J_\lambda f = C_\alpha f, \quad \widetilde{P}_\lambda K_\lambda f = \lambda C_\alpha f.$$

Using the closedness of $\widetilde{P}_\lambda$ and the denseness of $\mathcal{S}(R^n)$ in $L^p(R^n)$, we infer that the above equalities hold for all $f \in E$ in any case. Consequently

$$J_\lambda f = \widetilde{R}_\lambda C_\alpha f, \quad K_\lambda f = \lambda \widetilde{R}_\lambda C_\alpha f, \qquad \lambda > \omega + 1,\ f \in E. \tag{3.15}$$

The first equality together with (3.13), (3.14) implies that

$$\mathcal{R}\left(\widetilde{R}_\lambda C_\alpha\right) \subset \mathcal{D}(B), \quad B\widetilde{R}_\lambda C_\alpha f = \int_0^\infty e^{-\lambda t}\left(V_{11}(t) - V_{22}(t)\right) f dt,$$
$$\lambda > \omega + 1, \ f \in E.$$

Combining this with (3.3) establishes that $\mathcal{R}\left(\widetilde{R}_\lambda C_\alpha\right) \subset \mathcal{D}(A)$. Therefore

$$\mathcal{R}\left(C_\alpha\right) \subset \mathcal{D}(R_\lambda), \quad R_\lambda C_\alpha = \widetilde{R}_\lambda C_\alpha, \qquad \lambda > \omega + 1. \tag{3.16}$$

Accordingly, we obtain by (3.15) that for $\lambda > \omega + 1, \ f \in E$,

$$BR_\lambda C_\alpha f = \int_0^\infty e^{-\lambda t}\left(V_{11}(t) - V_{22}(t)\right) f dt, \quad \lambda R_\lambda C_\alpha f = \int_0^\infty e^{-\lambda t} V_{22}(t) f dt. \tag{3.17}$$

Moreover, it is plain that

$$\begin{aligned} C_\alpha^{-1} A C_\alpha f &= \mathcal{F}^{-1}\left\{\frac{1}{c_\alpha(x)}\mathcal{F}\mathcal{F}^{-1}[q(x)\mathcal{F}\mathcal{F}^{-1}\left(c_\alpha(x)\mathcal{F}f\right)]\right\} \\ &= \mathcal{F}^{-1}\{q(x)\mathcal{F}f\} = Af, \quad f \in \mathcal{D}(C_\alpha^{-1}AC_\alpha) = \mathcal{D}(A); \\ C_\alpha^{-1} B C_\alpha f &= Bf, \quad f \in \mathcal{D}(C_\alpha^{-1}BC_\alpha) = \mathcal{D}(B); \\ \widetilde{P}_\lambda A R_\lambda C_\alpha &= AC_\alpha \text{ on } \mathcal{D}(A), \\ \widetilde{P}_\lambda B R_\lambda C_\alpha &= BC_\alpha \text{ on } \mathcal{D}(B). \end{aligned}$$

This shows by (3.16) that

$$\begin{cases} R_\lambda C_\alpha A u = A R_\lambda C_\alpha u, & u \in \mathcal{D}(A), \\ R_\lambda C_\alpha B u = B R_\lambda C_\alpha u, & u \in \mathcal{D}(B), \end{cases}$$

which implies that $R_\lambda C_\alpha A$, $R_\lambda C_\alpha B$ are closable. Thus, recalling (3.17), we can apply Theorem 2.1 to obtain the desired results. □

Remark 3.2. (1) In [**5**, Chapters XIII and XIV] and [**13**], arbitrary systems of constant coefficient partial differential operators are dealt with by introducing a matrix of differential operators

$$\mathcal{A} := \left(p_{i,j}(D)\right)_{k\times k};$$

with the usual matrix reduction of (ACP_2) to (ACP_1), with

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ -q(D) & -p(D) \end{pmatrix},$$

the related theorems in [**5**, **13**] will produce a similar result as Theorem 3.1, for

$$\alpha > \begin{cases} \frac{\max\{l,\ m\}}{2} & \text{if } E = L^2(R^n), \\ \frac{1}{2}\left(\frac{n}{2}+1\right)\max\{l,\ m\} & \\ \quad +\frac{1}{2}\left(\left[\frac{n}{2}\right]+1-\frac{n}{2}\right)(\max\{l,\ m\}-1) & \text{otherwise.} \end{cases}$$

By comparison, the α in Theorem 3.1 is sharper. Moreover, there is also another advantage of Theorem 3.1. In order to illustrate this, we write

$$p(x) = \sum_{|\beta|\le l} a_\beta (ix)^\beta, \qquad q(x) = \sum_{|\beta|\le m} b_\beta (ix)^\beta.$$

From Theorem 3.1 one gets the information that the solution $u(\cdot)$ satisfies

$$t \longmapsto \sum_{|\beta|\le l} a_\beta D^\beta u'(t), \quad t \longmapsto \sum_{|\beta|\le m} b_\beta D^\beta u(t) \in C([0,\ \infty),\ E).$$

On the other hand, we note that $\mathcal{A}$ is not closed in general. Thus using the related theorems in [**5**, **13**] with the operator matrix $\mathcal{A}$ shows merely that

$$t \longmapsto \sum_{|\beta|\le l} a_\beta D^\beta u'(t) + \sum_{|\beta|\le m} b_\beta D^\beta u(t) \in C([0,\ \infty),\ E),$$

without giving the information whether

$$t \longmapsto \sum_{|\beta|\le l} a_\beta D^\beta u'(t), \quad t \longmapsto \sum_{|\beta|\le m} b_\beta D^\beta u(t) \in C([0,\ \infty),\ E).$$

(2) Let $q(x) \equiv 0$. Then Theorem 3.1 gives a result for regularized semigroups (recalling the remark after Proposition 1.6). Moreover, in the case when $p(x)$ is elliptic (corresponding to $r = h$ in (3.1)), one is getting the best possible α, that is

$$\alpha \begin{cases} \ge \frac{1}{4} n_E h & \text{if } E = L^p(R^n)\ (1 < p < \infty), \\ > \frac{1}{4} n_E h & \text{otherwise,} \end{cases}$$

in view of the original results for regularized or integrated semigroups (cf. [**11**, **12**, **27**]).

Similarly, letting $p(x) \equiv 0$ in Theorem 3.1 will yield a result for regularized cosine function; see also Remark 3.4 for related information.

Theorem 3.3. *Suppose that $p_1(x)$, $p_2(x)$, $q_1(x)$, $q_2(x)$ are real polynomials of degrees l_1, l_2, m_1, m_2 respectively on R^n. Let $A = q_1(D) + iq_2(D)$, $B = p_1(D) + ip_2(D)$. Then for any α with*

$$\alpha \begin{cases} \ge n\left|\frac{1}{2} - \frac{1}{p}\right| & \text{if } E = L^p(R^n)\ (1 < p < \infty), \\ > \frac{n}{2} & \text{otherwise,} \end{cases} \tag{3.18}$$

(1) (1.1) *is strongly* $(I-\Delta)^{-\frac{1}{2}l_2\alpha}$*-wellposed, provided either*

(i) $l_1 = 0$, $m_1 < 2l_2$, $m_2 \le l_2$, *and* $p_2(x)$ *is elliptic,*
or

(ii) $l_1 = 0$, $m_1 = 2l_2$, $m_2 \le l_2$, $p_2(x)$ *is elliptic, and* $q_1(x) \ge 0$ $(|x| \ge L_0)$ *for some* $L_0 > 0$*;*

(2) (1.1) *is strongly* $(I-\Delta)^{-\frac{1}{4}m_1\alpha}$*-wellposed, provided either*

(iii) $l_1 = 0$, $l_2 \le \frac{1}{2}m_1$, $m_2 \le \frac{1}{2}m_1$, *and* $q_1(x)$ *is strongly elliptic,*
or

(iv) $0 < l_1 < \frac{1}{2}m_1$, $l_2 < \frac{1}{2}m_1$, $m_2 < l_1 + \frac{1}{2}m_1$, *and* $p_1(x)$, $q_1(x)$ *are strongly elliptic;*

(3) (1.1) *is strongly* $(I-\Delta)^{-\frac{1}{2}l_1\alpha}$*-wellposed, provided*

(v) $l_2 \le \frac{1}{2}l_1$, $m_1 \le l_1$, $m_2 \le \frac{3}{2}l_1$, *and* $p_1(x)$ *is strongly elliptic.*

Proof. For $x \in R^n$, let

$$p(x) = p_1(x) + ip_2(x), \quad q(x) = q_1(x) + iq_2(x),$$

and let

$$r_1(x) = p_1^2(x) - p_2^2(x) - 4q_1(x), \quad r_2(x) = 2p_1(x)p_2(x) - 4q_2(x).$$

Then

$$p^2(x) - 4q(x) = r_1(x) + ir_2(x), \quad x \in R^n.$$

It can be verified that

$$\sqrt{p^2(x) - 4q(x)} = s_1(x) + is_2(x), \quad x \in R^n,$$

where

$$s_1(x) = \begin{cases} \frac{\sqrt{2}}{2}\left(r_1(x) + \sqrt{r_1^2(x) + r_2^2(x)}\right)^{\frac{1}{2}} & \text{if } r_1(x) \ge 0, \\ \frac{\sqrt{2}}{2}|r_2(x)|\left(\sqrt{r_1^2(x) + r_2^2(x)} - r_1(x)\right)^{-\frac{1}{2}} & \text{if } r_1(x) < 0, \end{cases}$$

$$s_2(x) = \begin{cases} \frac{\sqrt{2}}{2}r_2(x)\left(r_1(x) + \sqrt{r_1^2(x) + r_2^2(x)}\right)^{-\frac{1}{2}} & \text{if } r_1(x) > 0, \\ \frac{\sqrt{2}}{2}sign(r_2(x))\left(\sqrt{r_1^2(x) + r_2^2(x)} - r_1(x)\right)^{\frac{1}{2}} & \text{if } r_1(x) \le 0. \end{cases}$$

Keeping this in mind and recalling Definition 1.10, we begin the following discussion. When condition (i) or (ii) holds, we have that

$$\max\{2\deg p,\ \deg q\} = 2l_2$$

and

$$|p^2(x) - 4q(x)| \geq |r_1(x)| \geq C_0|x|^{2l_2}, \quad |x| \geq L_0,$$

for some $C_0,\ L_0 > 0$; $r_1(x) < 0$ and

$$\left(\sqrt{r_1^2(x) + r_2^2(x)} - r_1(x)\right)^{-\frac{1}{2}} \leq \text{const } |x|^{-l_2}$$

for $|x|$ sufficiently large; $\deg r_2 \leq l_2$, so that

$$s_1(x) \leq \text{const}\,(1 + |x|^{l_2}|x|^{-l_2})$$

and therefore

$$\sup_{x\in R^n} \operatorname{Re}\left(-p(x) + \sqrt{p^2(x) - 4q(x)}\right) < \infty. \tag{3.19}$$

When condition (iii) holds, we have that

$$\max\{2\deg p,\ \deg q\} = m_1$$

and

$$|p^2(x) - 4q(x)| \geq |r_1(x)| \geq C_1|x|^{m_1}, \quad |x| \geq L_1, \tag{3.20}$$

for some $C_1,\ L_1 > 0$; $r_1(x) < 0$ for $|x|$ sufficiently large,

$$\deg r_2 \leq \max\{l_2,\ m_2\} \leq \frac{1}{2}m_1,$$

so that

$$s_1(x) \leq \text{const}\,(1 + |x|^{\frac{1}{2}m_1}|x|^{-\frac{1}{2}m_1}),$$

and therefore (3.19) is satisfied.

When condition (iv) holds, we have that

$$\max\{2\deg p,\ \deg q\} = m_1$$

and (3.20) is satisfied; $r_1(x) < 0$ for $|x|$ sufficiently large,

$$\deg r_2 \leq \max\{l_1 + l_2,\ m_2\},$$

so that

$$s_1(x) \leq \text{const}\,(1 + |x|^{\max\{l_1+l_2,\ m_2\}}|x|^{-\frac{1}{2}m_1}),$$

and therefore (3.19) is satisfied noting

$$\max\{l_1 + l_2,\ m_2\} - \frac{1}{2}m_1 < l_1$$

as well as the strong ellipticity of $p_1(x)$.

Finally, let condition (v) hold. We have that

$$\max\{2\deg p,\ \deg q\} = 2l_1$$

and

$$|p^2(x) - 4q(x)| \ge |r_1(x)| \ge C_2|x|^{2l_1}, \quad |x| \ge L_2,$$

for some $C_2,\ L_2 > 0$; $r_1(x) > 0$ for $|x|$ sufficiently large,

$$\deg r_2 \le \max\{l_1 + l_2,\ m_2\}.$$

Observe

$$\begin{aligned}
&\mathrm{Re}\left(-p(x) + \sqrt{p^2(x) - 4q(x)}\right) \\
&= -p_1(x) + \frac{\sqrt{2}}{2}\left(r_1(x) + \sqrt{r_1^2(x) + r_2^2(x)}\right)^{\frac{1}{2}} \\
&= \frac{r_2^2(x) - 4(p_1^2(x)p_2^2(x) + 4p_1^2(x)q_1(x))}{\left[2p_1(x) + \sqrt{2}\left(r_1(x) + \sqrt{r_1^2(x) + r_2^2(x)}\right)^{\frac{1}{2}}\right]} \\
&\quad \cdot \frac{1}{\left[\sqrt{r_1^2(x) + r_2^2(x)} + p_1^2(x) + p_2^2(x) + 4q_1(x)\right]} \\
&\le \mathrm{const}\left(1 + |x|^{\max\{2(l_1+l_2),\ 2m_2,\ 2l_1+m_1\}}|x|^{-l_1-2l_1}\right), \quad x \in R^n.
\end{aligned}$$

We see that (3.19) is satisfied.

Consequently, using Theorem 3.1 leads to the results as required. □

Remark 3.4. Regarding the incomplete second order Cauchy problem

$$\begin{cases} u''(t) + q(D)u(t) = 0, & t \ge 0, \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3.21}$$

Theorem 3.3 (2) shows that (3.21) is strongly $(I-\Delta)^{-\frac{1}{4}\alpha\deg(\mathrm{Re}[q(x)])}$-wellposed for any α as in (3.18), if $q(x)$ is strongly elliptic, and

$$\deg(\mathrm{Im}[q(x)]) \le \frac{1}{2}\deg(\mathrm{Re}[q(x)]).$$

This will yield a larger set of initial values for the solutions of (3.21) under a weaker condition, compared with [**2**, Theorems 6.5-6.7].

4. Analytic wellposedness.

In this section, we assume that E is one of the Banach spaces $L^p(R^n)$ $(1 \leq p < \infty)$, $C_0(R^n)$ or $UC_b(R^n)$. Given a polynomial $p(x)$, $p(D)$ will be defined as in Section 3. We claim that $\mathcal{D}(P(D))$ is dense in E. Indeed, if $E = L^p(R^n)$ $(1 \leq p < \infty)$ or $C_0(R^n)$, then the Schwartz space $\mathcal{S}(R^n)$ (which is contained in $\mathcal{D}(P(D))$, is dense in E. If $E = UC_b(R^n)$, then

$$\mathcal{D}(P(D)) \supset \{J_\varepsilon * f;\ \varepsilon > 0,\ f(x) \in E\},$$

where $J_\varepsilon \in C^\infty(R^n)$ with support in $\{x \in R^n;\ |x| \leq \varepsilon\}$ satisfying

$$\int_{R^n} J_\varepsilon(x)dx = 1.$$

This implies that $\mathcal{D}(P(D))$ is dense in $UC_b(R^n)$ since

$$\lim_{\varepsilon \to 0} J_\varepsilon * f(x) = f(x)$$

uniformly in R^n, whenever $f \in UC_b(R^n)$.

We also remark that in general, $\mathcal{D}(P(D))$ is not dense in $L^\infty(R^n)$ or $C_b(R^n)$.

Theorem 4.1. *Suppose that $p(x)$, $q(x)$ are real polynomials of degrees l, m respectively on R^n such that they are strongly elliptic and $l < m < 2l$. Assume A_1, B_1 are closed linear operators on E such that $\mathcal{D}(A_1) \supset \mathcal{D}(\Delta^{\frac{am}{2}})$, $\mathcal{D}(B_1) \supset \mathcal{D}(\Delta^{\frac{bl}{2}})$, for some $a, b \in [0,\ 1)$. Let $A = q(D) + A_1$, $B = p(D) + B_1$. Then the Cauchy problem (1.1) is analytically solvable in $\Sigma_{\frac{\pi}{2}}$; furthermore, (1.1) is analytically wellposed in $\Sigma_{\frac{\pi}{2}}$ provided $(I - \Delta)^{-\frac{am}{2}} A_1$, $(I - \Delta)^{-\frac{bl}{2}} B_1$ have bounded extensions on E.*

Proof. Firstly, we write

$$A_0 = q(D), \quad B_0 = p(D), \quad R_{0\lambda} = \lambda^2 + B_0\lambda + A_0.$$

By hypothesis, there are constants L_0, $C_0 > 0$ such that

$$p(x) \geq C_0|x|^l, \quad q(x) \geq C_0|x|^m, \quad |x| > L_0.$$

Without loss of generality, we may and do assume (with $A_1 + d_1 I$, $B_1 + d_2 I$ replacing A_1, B_1 respectively for some d_1, $d_2 > 0$, if necessary) that

$$q(x) \geq C_1|x|^m, \quad p^2(x) - 4q(x) \geq C_1(1 + |x|)^{2l},$$
$$\sqrt{p^2(x) - 4q(x)} + p(x) \geq C_1(1 + |x|)^l, \qquad x \in R^n, \tag{4.1}$$

for some $C_1 > 1$. Define $\mu_\pm(x)$ as in (3.4). Since

$$\mu_+(x) = -\frac{2q(x)}{p(x) + \sqrt{p^2(x) - 4q(x)}}, \quad x \in R^n, \tag{4.2}$$

we have

$$\sigma_0 := \sup_{x \in R^n} \operatorname{Re} \mu_\pm(x) \leq 0.$$

Also, a simple calculation shows by (4.1) that for each multiindex β,

$$|D^\beta \mu_-(x)| \leq \text{const}\,(1+|x|)^{l-|\beta|}, \quad x \in R^n, \tag{4.3}$$

$$|D^\beta \mu_+(x)| \leq \text{const}\,(1+|x|)^{m-l-|\beta|}, \quad x \in R^n. \tag{4.4}$$

Now, set

$$\begin{aligned} v_0(x;\ z) &= \frac{1}{\sqrt{p^2(x) - 4q(x)}} \left(e^{\mu_+(x)z} - e^{\mu_-(x)z}\right), \quad x \in R^n,\ z \in \mathbf{C}, \\ v(x;\ z) &= p(x) v_0(x;\ z), \quad x \in R^n,\ z \in \mathbf{C}, \\ w(x;\ z) &= e^{\mu_+(x)z} + e^{\mu_-(x)z}, \quad x \in R^n,\ z \in \mathbf{C}. \end{aligned}$$

Then, (4.1) implies that for each $z \in \mathbf{C}$,

$$e^{\mu_\pm(x)z},\ v_0(x;\ z),\ v(x;\ z),\ w(x;\ z) \in C^\infty(R^n).$$

Fix $z_0 \in \Sigma_{\frac{\pi}{2}}$. Observe that for each multiindex β,

$$\begin{gathered} \left|D^\beta\left[\mu_-(x)e^{\mu_-(x)z}\right]\right|, \\ \left|D^\beta\left[\frac{p(x)\mu_-(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_-(x)z}\right]\right|, \quad \left|D^\beta\left[\frac{\mu_-(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_-(x)z}\right]\right| \\ \leq \text{const}\,(1+|x|)^{l+(l-1)|\beta|} e^{-\frac{1}{2}C_1|x|^l \operatorname{Re} z_0} \end{gathered}$$

valid for all $x \in R^n$, $z \in \mathbf{C}$ with $|z - z_0| < \frac{1}{2}\operatorname{Re} z_0$, by (4.1) and (4.3);

$$\begin{gathered} \left|D^\beta\left[\mu_+(x)e^{\mu_+(x)z}\right]\right|, \\ \left|D^\beta\left[\frac{p(x)\mu_+(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_+(x)z}\right]\right|, \quad \left|D^\beta\left[\frac{\mu_+(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_+(x)z}\right]\right| \\ \leq \text{const}\,(1+|x|)^{m-l+(m-l-1)|\beta|} e^{-C_2|x|^{m-l} \operatorname{Re} z_0} \end{gathered}$$

(where C_2 is some constant) valid for x, z as above, by (4.1), (4.2) and (4.4).

Accordingly, we can see by Lemma 1.12 that the $\mathcal{F}L^1$-valued functions

$$z \mapsto e^{\mu_\pm(x)z},\ z \mapsto \frac{p(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_\pm(x)z},\ z \mapsto \frac{1}{\sqrt{p^2(x)-4q(x)}}e^{\mu_\pm(x)z}$$

are analytic in $\Sigma_{\frac{\pi}{2}}$. Hence, letting

$$V_0(z) = \mathbf{T}\langle v_0(x;\ z)\rangle,\ V(z) = \mathbf{T}\langle v(x;\ z)\rangle,\ W(z) = \mathbf{T}\langle w(x;\ z)\rangle,\ z \in \Sigma_{\frac{\pi}{2}},$$

we know that

$$V_0(z),\ V(z),\ W(z) \text{ are analytic in } \Sigma_{\frac{\pi}{2}}, \tag{4.5}$$

$$V_0(z) \subset \mathcal{D}(B_0) \ \text{ and } \ B_0V_0(z) = V(z), \quad z \in \Sigma_{\frac{\pi}{2}}. \tag{4.6}$$

Next, we have by (4.1) and (4.2) that

$$\begin{aligned} \left|e^{\mu_-(x)z}\right| &\le e^{-C_3|x|^l \operatorname{Re} z}, \quad x \in R^n,\ z \in \Sigma_{\frac{\pi}{2}}, \\ \left|e^{\mu_+(x)z}\right| &\le e^{-C_3|x|^{m-l} \operatorname{Re} z}, \quad x \in R^n,\ z \in \Sigma_{\frac{\pi}{2}}, \end{aligned} \tag{4.7}$$

for some constant $C_3 > 0$. This combined with (4.1)-(4.4) implies that for any multiindex β with $|\beta| \ge 1$,

$$\begin{gathered} \left|D^\beta\left[e^{\mu_-(x)z}\right]\right|, \\ \left|D^\beta\left[\frac{1}{\sqrt{p^2(x)-4q(x)}}e^{\mu_-(x)z}\right]\right|, \quad \left|D^\beta\left[\frac{p(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_-(x)z}\right]\right| \\ \le \text{const} \sum_{i=1}^{|\beta|} |z|^i(1+|x|)^{li-|\beta|}e^{-C_3|x|^l \operatorname{Re} z}, \quad x \in R^n,\ z \in \Sigma_{\frac{\pi}{2}}, \\ \left|D^\beta\left[e^{\mu_+(x)z}\right]\right|, \\ \left|D^\beta\left[\frac{1}{\sqrt{p^2(x)-4q(x)}}e^{\mu_+(x)z}\right]\right|, \quad \left|D^\beta\left[\frac{p(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_+(x)z}\right]\right| \\ \le \text{const} \sum_{i=1}^{|\beta|} |z|^i(1+|x|)^{(m-l)i-|\beta|}e^{-C_3|x|^{m-l} \operatorname{Re} z}, \quad x \in R^n,\ z \in \Sigma_{\frac{\pi}{2}}. \end{gathered}$$

When

$$li - |\beta| < -\frac{n}{2},$$

we get

$$\begin{aligned}&\left\|(1+|x|)^{li-|\beta|}e^{-C_3|x|^l \operatorname{Re} z}\right\|_{L^2(R^n)}\\ &\leq \left\|(1+|x|)^{li-|\beta|}\right\|_{L^2(R^n)}\\ &\leq \text{const}, \quad z\in\Sigma_{\frac{\pi}{2}}.\end{aligned}$$

When

$$-\frac{n}{2} < li-|\beta| < 0,$$

we have

$$\begin{aligned}&\left\|(1+|x|)^{li-|\beta|}e^{-C_3|x|^l \operatorname{Re} z}\right\|_{L^2(R^n)}\\ &\leq \text{const}\,(\operatorname{Re} z)^{-i+\frac{2|\beta|-n}{2l}}\left\||x|^{li-|\beta|}e^{-C_3|x|^l}\right\|_{L^2(R^n)}\\ &\leq \text{const}\,(\operatorname{Re} z)^{-i+\frac{2|\beta|-n}{2l}}, \quad z\in\Sigma_{\frac{\pi}{2}}.\end{aligned}$$

When

$$li-|\beta|\geq 0,$$

we get

$$\begin{aligned}&\left\|(1+|x|)^{li-|\beta|}e^{-C_3|x|^l \operatorname{Re} z}\right\|_{L^2(R^n)}\\ &\leq \text{const}\left\|\left(1+|x|^{li-|\beta|}\right)e^{-C_3|x|^l \operatorname{Re} z}\right\|_{L^2(R^n)}\\ &\leq \text{const}\left((\operatorname{Re} z)^{-\frac{n}{2l}}+(\operatorname{Re} z)^{-i+\frac{2|\beta|-n}{2l}}\right), \quad z\in\Sigma_{\frac{\pi}{2}}.\end{aligned}$$

Keep these observations in mind. Now, we fix $\phi\in(0,\ \frac{\pi}{2})$. Then $|z|\leq \frac{2}{\cos\phi}\operatorname{Re} z$ for $z\in\Sigma_\phi$. Take $\beta\in N_0^n$ such that $|\beta|=\left[\frac{n}{2}\right]+1$. Then

$$li-|\beta|\neq -\frac{n}{2}\ \text{ for every }\ i\in\{1,\ \dots\ ,\ |\beta|\},$$

noting $l\geq 2$ by the strong ellipticity of $p(x)$. Hence

$$\left\|D^\beta\left[e^{\mu_-(x)z}\right]\right\|_{L^2(R^n)}\leq \text{const}\,(\operatorname{Re} z)^{\frac{2|\beta|-n}{2l}}e^{\operatorname{Re} z}, \quad z\in\Sigma_\phi.$$

Moreover by (4.7),

$$\left\|e^{\mu_-(x)z}\right\|_{L^2(R^n)}\leq \text{const}\,(\operatorname{Re} z)^{-\frac{n}{2l}}, \quad z\in\Sigma_\phi.$$

Thus, an application of the classical Bernstein theorem shows that

$$\left\|e^{\mu_-(x)z}\right\|_{\mathcal{F}L^1}\leq \text{const}\ e^{\operatorname{Re} z}, \quad z\in\Sigma_\phi,$$

noting

$$-\frac{n}{2l}\left(1-\frac{n}{2|\beta|}\right)+\frac{(2|\beta|-n)}{2l}\frac{n}{2|\beta|}=0.$$

Similarly, we can obtain

$$\left\|e^{\mu_+(x)z}\right\|_{\mathcal{F}L^1}, \quad \left\|\frac{1}{\sqrt{p^2(x)-4q(x)}}e^{\mu_\pm(x)z}\right\|_{\mathcal{F}L^1},$$
$$\left\|\frac{p(x)}{\sqrt{p^2(x)-4q(x)}}e^{\mu_\pm(x)z}\right\|_{\mathcal{F}L^1} \le \text{const } e^{\operatorname{Re} z}, \quad z \in \Sigma_\phi.$$

Consequently,

$$\|V_0(z)\|,\ \|V(z)\|,\ \|W(z)\| \le \text{const } e^{\operatorname{Re} z}, \quad z \in \Sigma_\phi. \tag{4.8}$$

Pick $\lambda_0 < 0$. Then (4.1) implies by Lemma 1.12 that

$$(\lambda_0 - p^2(x))^{-1}, \quad q(x)(\lambda_0 - p^2(x))^{-1} \in \mathcal{F}L^1$$

and

$$\mathbf{T}\langle(\lambda_0 - p^2(x))^{-1}\rangle E = \mathcal{D}(B^2).$$

A simple calculation shows that for $x \in R^n$, $z \in \Sigma_{\frac{\pi}{2}}$,

$$v_0(x;\ z)(\lambda_0 - p^2(x))^{-1} = \frac{1}{2}(\lambda_0 - p^2(x))^{-1}\int_0^z [w(x;\ \eta) - v(x;\ \eta)]d\eta,$$
$$w(x;\ z)(\lambda_0 - p^2(x))^{-1} = -\frac{1}{2}p(x)(\lambda_0 - p^2(x))^{-1}\int_0^z [w(x;\ \eta) - v(x;\ \eta)]d\eta$$
$$- q(x)(\lambda_0 - p^2(x))^{-1}\int_0^z (z-\eta)[w(x;\ \eta) - v(x;\ \eta)]d\eta + 2(\lambda_0 - p^2(x))^{-1}.$$

It follows that for each $\phi \in (0,\ \frac{\pi}{2})$,

$$V_0(z)\mathbf{T}\langle(\lambda_0 - p^2(x))^{-1}\rangle \longrightarrow 0, \quad V(z)\mathbf{T}\langle(\lambda_0 - p^2(x))^{-1}\rangle \longrightarrow 0,$$
$$W(z)\mathbf{T}\langle(\lambda_0 - p^2(x))^{-1}\rangle \longrightarrow 2\mathbf{T}\langle(\lambda_0 - p^2(x))^{-1}\rangle,$$

as $z \to 0$ ($z \in \Sigma_\phi$). Thus conferring to (4.8) and the denseness of $\mathcal{D}(B_0^2)$ yields that for each $u \in E$, $\phi \in (0,\ \frac{\pi}{2})$,

$$\lim_{\substack{z\to 0\\ z\in\Sigma_\phi}} V_0(z)u = 0,\ \lim_{\substack{z\to 0\\ z\in\Sigma_\phi}} V(z)u = 0,\ \lim_{\substack{z\to 0\\ z\in\Sigma_\phi}} W(z)u = 2u. \tag{4.9}$$

Then proceeding similarly as in the proof of Theorem 3.1 and noting that for $\lambda > \sigma_0$, $x \in R^n$,

$$\int_0^\infty e^{-\lambda t} v_0(x;\, t)dt = (\lambda^2 + p(x)\lambda + q(x))^{-1},$$
$$\int_0^\infty e^{-\lambda t} v(x;\, t)dt = p(x)(\lambda^2 + p(x)\lambda + q(x))^{-1},$$
$$\int_0^\infty e^{-\lambda t} w(x;\, t)dt = (2\lambda + p(x))(\lambda^2 + p(x)\lambda + q(x))^{-1},$$

we obtain that for each $\lambda > \sigma_0$,

$$R_{0\lambda} \in \mathbf{L}(E), \quad B_0 R_{0\lambda} u = R_{0\lambda} B_0 u \ (u \in \mathcal{D}(B_0)),$$
$$A_0 R_{0\lambda} u = R_{0\lambda} A_0 u \ (u \in \mathcal{D}(A_0)),$$

$$B_0 R_{0_\lambda} u = \int_0^\infty e^{-\lambda t} V(t) u dt,$$
$$2\lambda R_\lambda u = \int_0^\infty e^{-\lambda t} (W(t) - V(t)) u dt, \quad u \in E.$$

Thus Theorem 2.1 applies (by (4.8), (4.9)) and we see that the Cauchy problem

$$\begin{cases} u''(t) + p(D)u'(t) + q(D)u(t) = 0, & t \geq 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{4.10}$$

is strongly I-wellposed with two propagators

$$S_0(t) = \frac{1}{2}(W(t) + V(t)),$$
$$S_1(t)u = V_0(t)u = \frac{1}{2}\int_0^t [W(s) - V(s)]u ds, \quad t \geq 0,\ u \in E.$$

It follows from (4.5), (4.6), (4.8), (4.9) that the Cauchy problem (4.10) is analytically wellposed in $\Sigma_{\frac{\pi}{2}}$.

Finally, (4.1) implies that $p(D)$, $q(D)$ are nonnegative operators,

$$\mathcal{D}(\Delta^{\frac{bl}{2}}) \supset \mathcal{D}((p(D))^b), \quad \mathcal{D}(\Delta^{\frac{am}{2}}) \supset \mathcal{D}((q(D))^a);$$

also

$$(I + p(D))^{-b}(I - \Delta)^{\frac{bl}{2}}, \quad (I + q(D))^{-a}(I - \Delta)^{\frac{am}{2}} \in \mathbf{L}(E).$$

Therefore, applying Theorem 2.4 establishes the results as claimed. The proof is then complete. □

5. Examples.

We first consider the damped Klein-Gordon equation in one dimension

$$\begin{cases} u_{tt} + au_{tx} - \rho u_{xx} + \gamma u = 0, \quad t \geq 0,\ x \in R \\ u(0,\ x) = \phi(x), \quad u_t(0,\ x) = \psi(x),\ \ x \in R \end{cases} \tag{5.1}$$

in $E = L^p(R)$ $(1 \leq p \leq \infty)$, $C_0(R)$, $C_b(R)$ or $UC_b(R)$, where $a \in R$, ρ, $\gamma > 0$.

Take

$$p_1(x) = 0, \quad p_2(x) = ax,$$
$$q_1(x) = \rho x^2 + \gamma, \quad q_2(x) = 0,$$
$$l_1 = 0, \quad l_2 = 1, \quad m_1 = 2, \quad m_2 = 0.$$

Then we can apply Theorem 3.3 (2) to conclude that the Cauchy problem (5.1) is strongly $(I - \Delta)^{-\frac{1}{2}\alpha}$-wellposed for

$$\alpha \begin{cases} \geq \left|\frac{1}{2} - \frac{1}{p}\right| & \text{if } \ E = L^p(R^n)\ (1 < p < \infty), \\ > \frac{1}{2} & \text{otherwise.} \end{cases}$$

Next, let $a_i(x) \in C^1(R^3)$ with $\dfrac{\partial a_i(x)}{\partial x_1}, \dfrac{\partial a_i(x)}{\partial x_2}, \dfrac{\partial a_i(x)}{\partial x_3} \in C_b(R^3)$, for each $i = 1, 2, 3$. We consider the Cauchy problem

$$\begin{cases} u_{tt} + \Delta^2 u_t + \sum_{i=1}^{3} a_i(x) \dfrac{\partial}{\partial x_i} u_t - \Delta^3 u = 0, \quad t \geq 0,\ x \in R^3 \\ u(0,\ x) = \phi(x), \quad u_t(0,\ x) = \psi(x), \quad x \in R^3, \end{cases} \tag{5.2}$$

in $L^p(R^3)$ $(1 < p < \infty)$.

Take

$$p(x) = \left(\sum_{i=1}^{3} x_i^2\right)^2, \quad l = 4,$$
$$q(x) = \left(\sum_{i=1}^{3} x_i^2\right)^3, \quad m = 6,$$
$$B_1 = \sum_{i=1}^{3} a_i(x) \frac{\partial}{\partial x_i}, \quad b = \frac{1}{2},$$
$$A_1 = 0, \quad a = 0.$$

Let $q = \frac{p}{p-1}$. Observing

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i(x)(I-\Delta)^{-1} \in \mathbf{L}(L^q(R^3)),$$

we have by a duality argument that $(I-\Delta)^{-1} \sum\limits_{i=1}^{3} a_i(x) \frac{\partial}{\partial x_i}$ has a bounded extension on $L^p(R^3)$ $(1 < p < \infty)$. Thus, Theorem 4.1 tells us that (5.2) is analytically wellposed in $\Sigma_{\frac{\pi}{2}}$, and so for ϕ, $\psi \in W^{6,p}(R^3)$, it has a unique solution

$$u(\cdot) \in C^2([0,\, \infty), \quad L^p(R^3)) \bigcap C^1([0,\, \infty), \\ W^{4,p}(R^3)) \bigcap C([0,\, \infty), \quad W^{6,p}(R^3)),$$

which can be extended analytically to $\Sigma_{\frac{\pi}{2}}$ such that for each $\phi \in (0,\ \frac{\pi}{2})$,

$$\|u(z)\|_{L^p(R^3)} \leq C_\phi e^{\omega_\phi \operatorname{Re} z}(\|u(0)\|_{L^p(R^3)} + \|u'(0)\|_{L^p(R^3)}), \quad z \in \Sigma_\phi$$

for some C_ϕ, $\omega_\phi > 0$.

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