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A GENERALIZED CHERN-SIMONS FORMULA

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In the short note we will generalize the famous Chern-Simons formulae.

Let M be a C^∞ manifold, $\pi : E \to M$ a complex vector bundle of rank n, and $\Omega^p(E)$ the space of all C^∞-sections of the bundle $\Lambda^p T^*(M) \otimes E$. A connection $\mathcal{D}$ on E is defined to be a linear operator $\mathcal{D} : \Omega^0(E) \to \Omega^1(E)$ satisfying Leibnitz' rule. If $\{U_\alpha, \phi_\alpha\}$ is a trivialization of E and $e_\alpha = \{e_{\alpha,1}, \dots, e_{\alpha,n}\}$ is a local frame over U_α for E then the action of $\mathcal{D}$ on the frame can locally be represented as

$$\mathcal{D}e_\alpha = e_\alpha \, A_\alpha$$

where the matrix A_α of 1-form is called the connection matrix of $\mathcal{D}$ associated with the frame $\{e_\alpha\}$ over U_α. Let $e_\beta = \{e_{\beta,1}, \dots, e_{\beta,n}\}$ be another frame over the same U_α or another trivialization $\{U_\beta, \phi_\beta\}$ of E and A_β be the connection matrix associated with the frame $\{e_\beta\}$. Transformation property of the two connection matrices should be

$$A_\beta = G^{-1} A_\alpha G + G^{-1}\, dG \tag{1}$$

where G is transformation matrix between the two frames $\{e_\alpha\}$ and $\{e_\beta\}$, that is $e_\beta = e_\alpha \, G$, and $G \in GL(n, C)$.

Now suppose $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ are two connection operators with the connection matrices $A_\alpha^{(1)}$ and $A_\alpha^{(2)}$ over the frame $\{e_\alpha\}$. Consequently , the operator $\eta^{2,1} = \mathcal{D}^{(2)} - \mathcal{D}^{(1)}$ is given, which can be locally identifies with a matrix $\eta_\alpha^{2,1} = A_\alpha^{(2)} - A_\alpha^{(1)}$ over the frame $\{e_\alpha\}$ and transforms by the rule

$$\eta_\beta^{2,1} = G^{-1} \eta_\alpha^{2,1} G, \qquad G \in GL(n, C). \tag{2}$$

Let $\xi \in \Omega^p(E)$ and can be represented by $\xi = e_\alpha \cdot \xi_\alpha = \sum_{i=1}^n \xi_{\alpha,i} \; e_{\alpha,i}$ in terms of the local frame $\{e_\alpha\}$. We extend the action of the connection $\mathcal{D}$ to the higher differential form ξ by setting

$$\mathcal{D}(\xi) = e_\alpha \; d\xi_\alpha + \mathcal{D}e_\alpha \wedge \xi_\alpha.$$

Then the curvature operator $\mathcal{F}$ can be defined by $\mathcal{F} = \mathcal{D}^2 : \Omega^p(E) \to \Omega^{p+2}(E)$. In terms of local frame $\{e_\alpha\}$ of vector bundle E we have

$$\mathcal{F}|_{U_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha, \qquad \mathcal{F}_\alpha \equiv \mathcal{F}|_{U_\alpha}, \tag{3}$$

where $\mathcal{F}_\alpha$ is called the curvature matrix of the connection $\mathcal{D}$ associated with the frame $\{e_\alpha\}$ over U_α and it satisfies the following transformation rule:

$$\mathcal{F}_\beta = G^{-1}\mathcal{F}_\alpha G, \qquad G \in GL(n, C). \tag{4}$$

For simplicity we will sometimes write $\mathcal{F}, A, \eta^{2,1}$ instead of $\mathcal{F}_\alpha, A_\alpha, \eta^{2,1}_\alpha$, omitting the subscribe α.

Let $\mathcal{M}_n \equiv C^{n^2}$ denote the vector space of $n \times n$ matrices with complex entries. A polynomial function $\tilde{P} : \mathcal{M}_n \to C$, homogeneous of degree r in the entries, is said to be invariant if $\tilde{P}(g\alpha g^{-1}) = \tilde{P}(\alpha)$, for all $\alpha \in \mathcal{M}_n$ and $g \in GL(n, C)$. A r-linear form

$$P : \overbrace{\mathcal{M}_n \times \cdots \times \mathcal{M}_n}^{r} \to C$$

is said to be invariant if the identity $P(g\alpha_1 g^{-1}, \cdots, g\alpha_r g^{-1}) = P(\alpha_1, \cdots, \alpha_r)$ holds for all $\alpha_1, \cdots, \alpha_r \in \mathcal{M}_n$ and $g \in Gl(n, C)$. An invariant r-linear form P clearly induces an invariant homogeneous polynomial $\tilde{P}$ of degree r by setting $\tilde{P}(\alpha) = P(\alpha, \cdots, \alpha)$. In fact, the converse is also true by setting

$$P(\alpha_1, \ldots, \alpha_r) = \frac{(-1)^r}{r!} \sum_{j=1}^{r} \sum_{i_1 < \cdots < i_j} (-1)^j \tilde{P}(\alpha_{i_1} + \cdots + \alpha_{i_j}). \tag{5}$$

This means that any invariant polynomial $\tilde{P}$ of degree r can be realized as the restriction of an invariant r-linear form on $\mathcal{M}_n \times \cdots \times \mathcal{M}_n$. And we have

$$P(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_r) P(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_r). \tag{6}$$

We also apply $P(\alpha_1, \ldots, \alpha_l, \alpha^{r-l})$ instead of $P(\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_r)$ if $\alpha_{l+1} = \alpha_{l+2} = \cdots = \alpha_r$.

Now, we linearly extend the action of P to $\mathcal{M}_n^{d_1} \times \cdots \times \mathcal{M}_n^{d_r}$, where $\mathcal{M}_n^{d_i}$ denote the vector space of $n \times n$ matrix with d_i-forms as entries and we make use of the same symbol P to the extended one. For example, when w_i are d_i-forms $(i = 1, 2, \ldots, r)$ we define

$$P(\omega_1\alpha_1, \ldots, \omega_r\alpha_r) = \omega_1 \wedge \cdots \wedge \omega_r P(\alpha_1, \ldots, \alpha_r),$$

where α_i are general matrices. Moreover, if W_i are matrices of d_i-forms written as $W_i = \sum_{l_i} \omega_{l_i}\alpha_{l_i}$, then set

$$
\begin{aligned}
P(W_1, \dots, W_r) &= \sum_{l_1 \dots l_r} P(\omega_{l_1}\alpha_{l_1}, \dots, , \omega_{l_r}\alpha_{l_r}) \\
&= \sum_{l_1 \dots l_r} \omega_{l_1} \wedge \dots \wedge \omega_{l_r} P(\alpha_{l_1}, \cdots, \alpha_{l_r}).
\end{aligned} \tag{7}
$$

And we have

$$
\begin{aligned}
&P(W_1, \dots, W_i, \dots, W_j, \dots, W_r) \\
&= (-1)^{(d_{i+1}+\cdots+d_{j-1})(d_i+d_j)+d_i d_j} \cdot P(W_1, \dots, W_j, \dots, W_i, \dots, W_r).
\end{aligned} \tag{8}
$$

We take a small parameter t and let $g = e^{tT}$ for a matrix T. We have

$$
P(gW_1g^{-1}, \dots, gW_rg^{-1}) = P(W_1, \dots, W_r). \tag{9}
$$

Expanding the matrix with respect to the parameter t,

$$
g = I + tT + \frac{1}{2!}t^2T^2 + \cdots,
$$

and substituting it into the equation (9) above we can get the following lemma.

Lemma. *Let P be an invariant $r-$linear form and T be a general matrix. Then the identity*

$$
\sum_i^r P(W_1, \dots, TW_i - W_iT, \dots, W_r) = 0
$$

holds.

If ϕ is a 1-form and f is a function matrix then from the lemma it follows that

$$
\begin{aligned}
&\sum_i^r (-1)^{d_1+\cdots+d_{i-1}} P(W_1, \dots, \phi \wedge fW_i, \dots, W_r) \\
&= \sum_i^r \phi \wedge P(W_1, \cdots, fW_i, \dots, W_r) \\
&= \sum_i^r (-1)^{d_1+\cdots+d_i} P(W_1, \dots, W_if \wedge \phi, \dots, W_r).
\end{aligned} \tag{10}
$$

For the case of Θ being any matrix of 1-form, we can write $\Theta = \sum \theta_i f_i$, where θ_i are 1-forms and f_i function matrices. Using r linearity of P and (10), we obtain

$$\sum_{i=1}^{r}(-1)^{d_1+\cdots+d_{i-1}}[P(W_1,\ldots,\Theta\wedge W_i,\ldots,W_r) - (-1)^{d_i}P(W_1,\ldots,W_i\wedge\Theta,\ldots,W_r)] = 0. \tag{11}$$

Let $\mathcal{D}^{(i)}(i = 0,\ldots,k)$ be $k+1$ connections given for the vector bundle E and $A^{(i)}(i = 0,1,\ldots,k)$ are the connection matrices associated with the frame $\{e_\alpha\}$. We define the interpolations among them as follows

$$\begin{aligned} &A_{j_0,j_1,\cdots,j_k} = A^{(j_0)} + t_1\eta^{j_1,j_0} + \cdots + t_k\eta^{j_k,j_0}, \\ &0 \le t_1,\ldots,t_k \le 1, \qquad t_1+\cdots+t_k \le 1, \end{aligned} \tag{12}$$

where $\eta^{j_k,j_0} = A^{(j_k)} - A^{(j_0)}$. The curvature $\mathcal{F}_{j_0,j_1,\cdots,j_k}$ is

$$\mathcal{F}_{j_0,j_1,\cdots,j_k} = dA_{j_0,j_1,\cdots,j_k} + A_{j_0,j_1,\cdots,j_k}\wedge A_{j_0,j_1,\cdots,j_k}.$$

We have the Bianchi identity

$$d\mathcal{F}_{j_0,j_1,\cdots,j_k} = [\mathcal{F}_{j_0,j_1,\cdots,j_k},\ A_{j_0,j_1,\cdots,j_k}]. \tag{13}$$

Moreover we have

$$\partial/\partial t_i \mathcal{F}_{j_0,j_1,\cdots,j_k} = d\eta^{j_i,j_0} + \eta^{j_i,j_0}\wedge A_{j_0,j_1,\cdots,j_k} + A_{j_0,j_1,\cdots,j_k}\wedge\eta^{j_i,j_0}. \tag{14}$$

We now consider k-simplex set Δ_k consisting of $t_i \ge 0 (i = 1,\ldots,k)$, $\sum_{i=1}^{k} t_i \le 1$ in Euclidean space R^k. The orientation of Δ_k is fixed to be $[\partial/\partial t_1,\ldots,\partial/\partial t_k]$ and the volume element is $dt_1\wedge\cdots\wedge dt_k$. Projection of Δ_k to the boundary $\partial\Delta_k$ with $t_i = 0$ $(i = 1,\ldots,k)$ are denoted by $\Delta_{k-1}(\hat{t}_i)$. Its orientation should be $(-1)^i[\partial/\partial t_1,\ldots,\widehat{\partial/\partial t_i},\ldots,\partial/\partial t_k]$ and the volume element $dt_1\wedge\cdots\wedge d\hat{t}_i\wedge\cdots\wedge dt_k$.

Definition. Let P be an invariant r-linear polynomial. The $Q-$ polynomials of differential form associated with P are defined as

$$\begin{aligned} &Q_r^{(k)}(A^{(j_0)},\ldots,A^{(j_k)};\Delta_k) \\ &= \frac{r!}{(r-k)!}\int_{\Delta_k} P(\eta^{j_1,j_0},\ldots,\eta^{j_k,j_0},\mathcal{F}^{r-k}_{j_0,j_1,\cdots,j_k})dt_1\wedge\cdots\wedge dt_k \end{aligned}$$

$$Q_r^{(0)}(A;\Delta_0) = P(\mathcal{F}^r), \quad k \le r. \tag{15}$$

From the definition one can easily show that the Q-polynomials are independent of the trivialization $\{U_\alpha, \phi_\alpha\}$ of the vector bundle E and the frame $\{e_\alpha\}$ over $\{U_\alpha\}$ due to the transformation properties (2) and (4). This means that the Q-polynomials are well-defined global differential forms on M. So we can also rewrite the definition (15) as

$$\begin{aligned} &Q_r^{(k)}(\mathcal{D}^{(j_0)}, \dots, \mathcal{D}^{(j_k)}; \Delta_k) \\ &= \frac{r!}{(r-k)!} \int_{\Delta_k} P(\eta^{j_1,j_0}, \dots, \eta^{j_k,j_0}, \mathcal{F}^{r-k}_{j_0,j_1,\cdots,j_k}) dt_1 \wedge \cdots \wedge dt_k \\ &Q_r^{(0)}(\mathcal{D}; \Delta_0) = P(\mathcal{F}^r), \quad k \leq r. \end{aligned}$$

Based upon properties of the invariant polynomial P, it can also be shown that under permutation of the connections $\mathcal{D}^{(i)}$, the Q-polynomials satisfy

$$Q_r^{(k)}(\mathcal{D}^{(0)}, \dots, \mathcal{D}^{(k)}; \Delta_k) = \epsilon^{0\cdots k}_{j_0\cdots j_k} Q_r^{(k)}(\mathcal{D}^{(j_0)}, \dots, \mathcal{D}^{(j_k)}; \Delta_k). \tag{16}$$

The most important property of the Q-polynomials is described by the following theorem:

Theorem 1. *Assume that the Q-polynomials are defined as Eq.* (15). *Then there exists a relation between k-th Q-polynomial and* $(k-1)$*-th Q-polynomials as follows:*

$$\begin{aligned} &dQ_r^{(k)}(\mathcal{D}^{(0)}, \dots, \mathcal{D}^{(k)}; \Delta_k) \\ &= \sum_{i=0}^{k} (-1)^i Q_r^{(k-1)}(\mathcal{D}^{(0)}, \dots, \hat{\mathcal{D}}^{(i)}, \dots, \mathcal{D}^{(k)}; \Delta_{k-1}(\hat{i})). \end{aligned} \tag{17}$$

Proof. We first calculate

$$\begin{aligned} &dP(\eta^{1,0}, \dots, \eta^{k,0}, \mathcal{F}^{r-k}_{0,1,\dots,k}) \\ &= \sum_{i=1}^{k} (-1)^{i-1} P(\eta^{1,0}, \dots, d\eta^{i,0}, \dots, \eta^{k,0}, \mathcal{F}^{r-k}_{0,1,\cdots,k}) \\ &\quad + (-1)^k \sum_{j=0}^{r-k-1} P(\eta^{1,0}, \dots, \eta^{k,0}, \mathcal{F}^{j}_{0,1,\cdots,k}, d\mathcal{F}_{0,1,\cdots,k}, \mathcal{F}^{r-k-1-j}_{0,1,\cdots,k}). \end{aligned} \tag{18}$$

By means of Bianchi identity (13) and the identity (11), we get

$$dP(\eta^{1,0},\dots,\eta^{k,0},\mathcal{F}^{r-k}_{0,1,\cdots,k})$$

(19)

$$= \sum_{i=0}^{k}(-1)^{i-1}P(\eta^{1,0},\dots,\partial/\partial t_i\mathcal{F}_{0,1,\cdots,k},\dots,\eta^{k,0},\mathcal{F}^{r-k}_{0,1,\cdots,k}),$$

$$= \frac{1}{r-k+1}\sum_{i=1}^{k}(-1)^{i-1}\partial/\partial t_i P(\eta^{1,0},\dots,\hat{\eta}^{i,0},\dots,\eta^{k,0},\mathcal{F}^{r-k+1}_{0,1,\cdots,k}).$$

Performing integration over $\triangle_k$ and multiplying by $(r-k+1)$ in (19), we obtain

(20)

$$(r-k+1)d\int_{\triangle_k}P(\eta^{1,0},\dots,\eta^{k,0},\mathcal{F}^{r-k}_{0,1,\cdots,k})dt_1\wedge\cdots\wedge dt_k$$

$$= \sum_{i=1}^{k}(-1)^{i-1}\int_{\triangle_k}\frac{\partial}{\partial t_i}P(\eta^{1,0},\dots,\hat{\eta}^{i,0},\dots,\eta^{k,0},\mathcal{F}^{r-k+1}_{0,1,\cdots,k})dt_1\wedge\cdots\wedge dt_k$$

$$= \sum_{i=1}^{k}\int_{\triangle_k}d_t\{P(\eta^{1,0},\dots,\hat{\eta}^{i,0},\dots,\eta^{k,0},\mathcal{F}^{r-k+1}_{0,1,\cdots,k})dt_1\wedge\cdots\wedge\widehat{dt_i}\wedge\cdots\wedge dt_k\},$$

where d_t is the exterior differential operator with respect to the parameters $\{t_i\}$. Then by Stokes' theorem, the right-hand side of the last identity above is nothing but

(21)

$$\sum_{i=1}^{k}\int_{\partial\triangle_k}P(\eta^{1,0},\dots,\hat{\eta}^{i,0},\dots,\eta^{k,0},\mathcal{F}^{r-k+1}_{0,1,\cdots,k})dt_1\wedge\cdots\wedge\widehat{dt_i}\wedge\cdots\wedge dt_k$$

$$= \sum_{i=1}^{k}\int_{\triangle_{k-1}(\hat{t}_i)}(-1)^iP(\eta^{1,0},\dots,\hat{\eta}^{i,0},\dots,\eta^{k,0},\mathcal{F}^{r-k+1}_{0,1,\cdots,k})dt_1\wedge\cdots\wedge\widehat{dt_i}\wedge\cdots\wedge dt_k$$

$$+ \sum_{i=1}^{k}\int_{\triangle_k\cap\{t_1+\cdots+t_k=1\}}P(\eta^{1,0},\dots,\hat{\eta}^{i,0},\dots,\eta^{k,0},\mathcal{F}^{r-k+1}_{0,1,\cdots,k})dt_1\wedge\cdots\wedge\widehat{dt_i}\wedge\cdots\wedge dt_k$$

where orientation of the integral region $\triangle_k\cap\{t_1+\cdots+t_k=1\}$ induced by $\triangle_k$ should be $[\partial/\partial t_2\cdots\partial/\partial t_k]$. Changing parameters $\{t\}$ to $\{s\}$ as follows:

$$(22)\quad t_1 = 1-s_1-s_2-\cdots-s_{k-1},\quad t_2 = s_1,\dots,\quad t_{i-1} = s_{i-2},$$
$$t_{i+1} = s_i,\dots,\quad t_k = s_{k-1},\quad 0\le s_1,\dots,s_{k-1}\le 1.$$

It is easy to prove that the curvature 2-forms $\mathcal{F}_{0,1,\cdots,k}$ will become $\mathcal{F}_{1,2\cdots,k-1}$ under the change. The volume element $dt_1\wedge\cdots\wedge\widehat{dt_i}\wedge\cdots\wedge dt_k$ is transformed into $(-1)^{i-1}ds_1\wedge\cdots\wedge ds_{k-1}$, and the integral region $\triangle_k\cap\{t_1+\cdots+t_k=1\}$ is transformed into $\triangle_{k-1}(s)$ consisting of $\{s\}$ with $s_1\geq 0,\ldots,s_{k-1}\geq 0$ and $s_1+\cdots+s_{k-1}\leq 1$. Moreover, the parameter transformation (22) reserves the orientations $\triangle_k\cap\{t_1+\cdots+t_k=1\}$ and $\triangle_{k-1}(s)$, i.e., the orientation $[\partial/\partial t_2,\ldots,\partial/\partial t_k]$ is transformed into $[\partial/\partial s_1,\ldots,\partial/\partial s_{k-1}]$, which is just the proper orientation of $\triangle_{k-1}(s)$ as is mentioned above. Therefore, the last summation on the right-hand side of Eq. (21) is equal to

(23)
$$\sum_{i=1}^{k}(-1)^{i-1}\int_{\triangle_{k-1}(s)}P(\eta^{1,0},\ldots,\hat{\eta}^{i,0},\ldots,\eta^{k,0},\mathcal{F}_{1,2,\ldots,k-1}^{r-k+1})ds_1\wedge\cdots\wedge ds_{k-1}.$$

On the other hand, from the properties of invariant polynomial P it follows that

$$\begin{aligned}\sum_{i=1}^{k}(-1)^{i-1}&P(\eta^{1,0},\ldots,\hat{\eta}^{i,0},\ldots,\eta^{k,0},\mathcal{F}_{1,\cdots,k-1}^{r-k+1})\\&=P(\eta^{2,1},\ldots,\eta^{k,1},\mathcal{F}_{1,2,\ldots,k-1}^{r-k+1}).\end{aligned}\tag{24}$$

Thus, by means of these formulas, we obtain

(25)
$$\begin{aligned}&(r-k+1)d\int_{\triangle_k}P(\eta^{1,0},\ldots,\eta^{k,0},\mathcal{F}_{0,1,\cdots,k}^{r-k})dt_1\wedge\cdots\wedge dt_k\\&=\sum_{i=1}^{k}(-1)^{i}\int_{\triangle_{k-1}(\hat{t}_i)}P(\eta^{1,0},\ldots,\hat{\eta}^{i,0},\ldots,\eta^{k,0},\mathcal{F}_{0,1,\cdots,k}^{r-k+1})dt_1\wedge\cdots\wedge\widehat{d}t_i\wedge\cdots\wedge dt_k\\&\quad+\int_{\triangle_{k-1(s)}}P(\eta^{2,1},\ldots,\eta^{k,1},\mathcal{F}_{1,\cdots,k-1}^{r-k+1})ds_1\wedge\cdots\wedge ds_{k-1}.\end{aligned}$$

Multiplying both side by the factor $r!/(r-k+1)!$ and recalling the definition of Q-polynomials, we get the relation (17). This completes the proof of the theorem.

Let us now illustrate the meaning of the Theorem 1 by considering some simple cases. We start with the case of k=1. The interpolation (12) and the definition (15) tell us

$$A_{0,1}=A^{(0)}+t_1\eta^{1,0},\qquad\eta^{1,0}=A^{(1)}-A^{(0)},$$

and

$$\left.\begin{aligned} &Q_r^{(0)}(\mathcal{D}^{(0)};\Delta_0) = P(\mathcal{F}_{0,1}^r), \qquad Q_r^{(0)}(\mathcal{D}^{(1)};\Delta_0) = P(\mathcal{F}_1^r), \\ &Q_r^{(1)}(\mathcal{D}^{(0)};\mathcal{D}^{(1)},\Delta_1) = r\int_0^1 P(A^{(1)} - A^{(0)}, \mathcal{F}_0^{r-1}) \end{aligned}\right\}.$$

Then the Theorem 1 gives

$$P(\Omega_1^r) - P(\Omega_0^r) = dQ_r^{(2)}(\mathcal{D}^{(0)},\mathcal{D}^{(1)};\Delta_1),$$

which is just the famous Chern-Simons formulae [1].

In the case of k=2, the interpolation connection in (12) becomes

$$A_{0,1,2} = A^{(0)} + t_1\eta^{1,0} + t_2\eta^{2,0},$$

and the Theorem 1 reads

$$\begin{aligned} &dQ_r^{(2)}(\mathcal{D}^{(0)},\mathcal{D}^{(1)},\mathcal{D}^{(2)};\Delta_2) \\ &= Q_r^{(1)}(\mathcal{D}^{(1)},\mathcal{D}^{(2)};\Delta_1) - Q_r^{(1)}(\mathcal{D}^{(0)},\mathcal{D}^{(2)};\Delta_1) + Q_r^{(1)}(\mathcal{D}^{(0)},\mathcal{D}^{(1)};\Delta_1) \end{aligned}$$

where $Q_r^{(1)}$ are defined as above and

$$Q_r^{(2)}(\mathcal{D}^{(0)},\mathcal{D}^{(1)},\mathcal{D}^{(2)};\Delta_2) = r(r-1)\int_0^1 dt_1 \int_0^{1-t_1} P(\eta^{1,0},\eta^{2,0},\mathcal{F}_{0,1,2}^{r-2})dt_2.$$

Consider the integral of the $Q-$polynomials $Q_r^{(k-1)}(\mathcal{D}^{(0)},\mathcal{D}^{(1)},\ldots,\mathcal{D}^{(k-1)};\Delta_{k-1})$ over M^{2r-k+1}, which is a $(2r-k+1)$ dimensional sub-manifold of C^∞ manifold M. That is

$$\begin{aligned} &\tilde{Q}_r^{(k-1)}(\mathcal{D}^{(0)},\mathcal{D}^{(1)},\ldots,\mathcal{D}^{(k-1)};\Delta_{k-1}) \\ &= \int_{M^{2r-k+1}} Q_r^{(k-1)}(\mathcal{D}^{(0)},\mathcal{D}^{(1)},\ldots,\mathcal{D}^{(k-1)};\Delta_{k-1}). \end{aligned}$$

The integral give a mapping $\tilde{Q}_r^{(k-1)}$ from $(\mathcal{D}^{(0)},\ldots,\mathcal{D}^{(k-1)})$ to C, which can be regarded as a $(k-1)$-th co-chain on the affine space of all connections on the vector bundle E. If we introduce a operator δ, which is defined by the following formulae

$$\begin{aligned} &(\delta\tilde{Q}_r^{(k-1)})(\mathcal{D}^{(0)},\ldots,\mathcal{D}^{(k)};\Delta_k) \\ &= \sum_{i=0}^{k}(-1)^i Q_r^{(k-1)}(\mathcal{D}^{(0)},\ldots,\hat{\mathcal{D}}^{(i)},\ldots,\mathcal{D}^{(k)};\Delta_{k-1}(\hat{t}^i)), \end{aligned}$$

then it is easy to prove that the operator is a boundary one, $\delta^2 = 0$. So the Theorem 1 can be rewritten as:

Theorem 2.

$$\int_{M^{2r-k+1}} dQ_r^{(k)}(\mathcal{D}^{(0)}, \dots, \mathcal{D}^{(k)}; \Delta_k) = (\delta \tilde{Q}_r^{(k-1)})(\mathcal{D}^{(0)}, \dots, \mathcal{D}^{(k)}; \Delta_k).$$

We note that the Theorems (1) and (2) are given for the complex vector bundle. This is due to simplicity. In fact, the theorems are also true for real vector bundle, or more general principal bundles [**2**].

For the case of principal bundles, if the submanifold M^{2r-k+1} is without boundary, $\partial M^{2r-k+1} = 0$, it follows that

$$\delta \tilde{Q}_r^{(k-1)} = 0.$$

This implies that the result given in the note should be concerned with the cohomology groups of the moduli spaces $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is affine space of all connections subjected to some conditions on principal bundles and $\mathcal{G}$ is gauge groups. In a coming paper we will discuss the problem.

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References

[1] S.S. Chern, *Complex Manifolds Without Potential Theory*, Spinger, New York, 1979.

[2] S.S. Chern and J. Simons, Ann. Math., **99** (1974), 48.

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