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WEAK PALEY–WIENER PROPERTY FOR COMPLETELY SOLVABLE LIE GROUPS

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We prove a weak Paley–Wiener property for completely solvable Lie groups, i.e. if the group Fourier transform of a measurable, bounded and compactly supported function vanishes on a set of positive Plancherel measure then the function itself vanishes almost everywhere on the group.

1. Introduction.

Let G be a connected, simply connected, and completely solvable Lie group, with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. The equivalence classes of irreducible unitary representations $\widehat{G}$ of G is parametrized by the coadjoint orbits $\mathfrak{g}^*/G$ via the Kirillov-Bernat bijective map $K : \widehat{G} \to \mathfrak{g}^*/G$. If $\rho \in \widehat{G}$ and $\ell \in K(\rho)$, then there exists an analytic subgroup H of G and a unitary character χ of H, such that $\ell|_{\mathfrak{h}} = Id_\chi$, where $\mathfrak{h}$ is the Lie algebra of H. The induced representation $\rho = \mathrm{Ind}_H^G \chi$ is irreducible. Moreover, K is a bijection. The image on $\mathfrak{g}^*/G$ of a measure equivalent to Lebesgue measure on $\mathfrak{g}^*$ gives a Plancherel measure on $\widehat{G}$.

Let ϕ be a bounded, measurable and compactly supported function on $\mathbb{R}^n$. By the classical Paley–Wiener theorem, the Fourier transform $\widehat{\phi}$ of ϕ extends to an entire function on $\mathbb{C}^n$. Using this we can conclude that if $\widehat{\phi}$ vanishes on a set of positive Plancherel measure which is nothing but the Lebesgue measure, then $\widehat{\phi}$ vanishes on the whole of $\mathbb{R}^n$. This in turn implies that $\phi = 0$ on $\mathbb{R}^n$.

In the same spirit, for a completely solvable Lie group we will think of the following as a weak Paley–Wiener property:

Theorem. *Let G be a connected, simply connected, and completely solvable Lie group, with the unitary dual $\widehat{G}$. Let ϕ be a measurable, bounded, and compactly supported function (i.e $\phi \in L_c^\infty(G)$). Assume that there exists a subset $E \subset \widehat{G}$ with positive Plancherel measure such that $\widehat{\phi}_\rho = 0$ for all $\rho \in E$ where $\widehat{\phi}_\rho$ is the group Fourier transform of ϕ. Then $\phi = 0$ almost everywhere on G.*

In [**GG1**] we proved, the same theorem for nilpotent Lie groups, by induction on the dimension of G. To prove the above theorem, also by using induction on the dimension of G, we need a description of the dual space $\widehat{G}$ of G and an explicit Plancherel measure on $\widehat{G}$. Here, we use the results of B.N. Currey [**C**], which are generalizations of the results of L. Pukanszky [**Pu**] on nilpotent Lie groups concerning the Plancherel measure and the Plancherel formula.

2. Preliminaries.

Let G be a connected, simply connected, and completely solvable Lie group, with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. We fix a basis $\mathfrak{B} = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$, such that $\mathfrak{g}_j$ is spanned by the vectors $\{X_1, X_2 \cdots, X_j\}$, $1 \leq j \leq n$ and $\mathfrak{g}_0 = (0)$. As G is completely solvable, there exists a chain of ideals

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_i \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}$$

of $\mathfrak{g}$, such that the dimension of $\mathfrak{g}_i$ be i for all $1 \leq i \leq n$. Let $\mathfrak{B}^* = \{X_1^*, \ldots, X_n^*\}$ be the dual basis of $\mathfrak{g}^*$. We fix a Lebesgue measure dX on $\mathfrak{g}$, and a right Haar measure dg on G such that $d(\exp X) = j_G(X)dX$, where

$$j_G(X) = \left| \det \left(\frac{1 - e^{-adX}}{adX} \right) \right|.$$

Let Δ be the modular function such that for all $g' \in G$, $d(gg') = \Delta(g')dg$. Let O be a coadjoint orbit in $\mathfrak{g}^*$ and $\ell \in O$. The bilinear form $B_\ell : (X, Y) \to \ell([X, Y])$ defines a skew-symmetric and nondegenerate bilinear form on $\mathfrak{g}/\mathfrak{g}^\ell$. As the map $X \to X.\ell$ induces an isomorphism between $\mathfrak{g}/\mathfrak{g}^\ell$ and the tangent space of O at ℓ, the bilinear form B_ℓ defines a nondegenerate 2-form ω_ℓ on this tangent space. If $2k$ is the dimension of O we note that

$$\beta_O = (2\pi)^{-k}(k!)^{-1}\omega \wedge \cdots \wedge \omega \qquad (k \quad \text{times})$$

the canonical measure on O. Lemma 3.2.2. in [**DR**], says us that there exists a nonzero rational function ψ on $\mathfrak{g}^*$ such that $\psi(g.\ell) = \Delta(g)^{-1}\psi(\ell)$, $g \in G$, and $\ell \in \mathfrak{g}^*$. We fix one such ψ. There exists a unique measure m_ψ on $\mathfrak{g}^*/G$ such that

$$\int_{\mathfrak{g}^*} \phi(\ell)|\psi(\ell)|d\ell = \int_{\mathfrak{g}^*/G} \left(\int_O \phi(\ell)d\beta_O(\ell) \right) dm_\psi(O)$$

for all Borel functions ϕ on $\mathfrak{g}^*$.

B.N. Currey [**C**] gave an explicit description of the measure m_ψ with the help of the coadjoint orbits $\mathfrak{g}^*/G$. We recall the theorem proved by B.N. Currey in [**C**] which is the essential tool to prove our Paley–Wiener property:

Theorem 2.1. *Let G be a connected, simply connected, and completely solvable Lie group. There exists a Zariski open subset U of $\mathfrak{g}^*$, a subset $J = \{j_1 < j_2 < \cdots < j_{2k}\}$ of $\{1, 2, \cdots, n\}$, a subset $M = \{j_{r_1} < j_{r_2} < \cdots < j_{r_a}\}$ of J, for each j in M a real valued rational function q_j (non* *vanishing on U), and real analytic P_j, $1 \leq j \leq n$ functions in the variables $w_1, w_2, \ldots, w_{2k}, \ell_1, \ell_2, \cdots, \ell_n$ such that the following hold.*

1) *If a denotes the number of elements in M, for each $\epsilon \in \{1, -1\}^a$, the set*

$$U_\epsilon = \{\ell \in U \mid \text{ sign of } \quad q_{j_{r_m}}(\ell) = \epsilon_m, 1 \leq m \leq a\}$$

is a non empty open subset in $\mathfrak{g}^*$.

2) *Define $V \subset \mathbb{R}^{2k}$ by $V = \prod R_r$, where $R_r =]0, \infty[$ if $j_r \in M$ and $R_r = \mathbb{R}$ otherwise. Let $\epsilon \in \{1, -1\}^a$ and for $v \in V$, define $\epsilon v \in \mathbb{R}^{2k}$ by $(\epsilon v)_j = \epsilon_m v_j$ if $j = j_{r_m} \in M$ and $(\epsilon v)_j = v_j$ otherwise. Then for each $\ell \in U_\epsilon$, the mapping $v \to \sum_j P_j(\epsilon v, \ell) X_j^*$ is a diffeomorphism of V with the coadjoint orbit of ℓ.*

3) *Define W_D as the subspace spanned by the vectors $\{X_i^* \mid i \notin J\}$ and W_M the subspace spanned by $\{X_j^* \mid j \in M\}$. Then the set*

$$W = \{\ell \in (W_D \oplus W_M) \cap U \mid |q_j(\ell)| = 1, j \in M\}$$

is a cross-section for the coadjoint orbits U. For each $j \in M$ the rational function q_j is of the form $q_j(\ell) = \ell_j + p_j(\ell_1, \ell_2, \cdots, \ell_{j-1})$, where p_j is a rational function.

4) *For each $\ell \in U$, let $\epsilon(\ell) \in \{1, -1\}^a$ such that $\ell \in U_{\epsilon(\ell)}$. Then the mapping $P : V \times W \to U$, defined by $P(v, \ell) = \sum_j P_j(\epsilon(\ell) v, \ell) X_j^*$, is a diffeomorphism.*

B.N. Currey [**C**] proved that m_ψ is a Plancherel measure on W.

The idea is to compute the measure $\psi(\ell) dl$ in termes of product measures on $V \times W$ and then, using Lemma 1.3 of [**C**], we can read off m_ψ as a measure on W. We have to determine coordinates for W.

If the subset M of J is empty, then $W = W_D \cap U$ and the coordinates for W are obtained by identifying W_D with $\mathbb{R}^{n-2k}$, which is the parametrization of $\mathfrak{g}^*$ in the nilpotent case. On the other hand, suppose that M is non empty, and a denotes the number of elements in M. From [**C**], for each $\epsilon \in \{1, -1\}^a$, there exists a non empty Zariski open subset U_ϵ of U and U is the disjoint union of the sets U_ϵ. Let $\epsilon \in \{1, -1\}^a$ and set $W_\epsilon = W \cap U_\epsilon$. From [**C**], we have that

$$W_\epsilon = \{\ell \in (W_D \oplus W_M) \cap U \mid \text{for each} \quad j = j_{r_m} \in M, \\ \ell_j = \epsilon_m - p_j(\ell_1, \ell_2, \cdots, \ell_{j-1})\}$$

where $j \in M$ and p_j is a rational nonsingular function on U.

Let $\epsilon \in \{1, -1\}^a$. Then from [**C**], there is a Zariski open subset Λ_ϵ of W_D and a rational function $p_\epsilon : \Lambda_\epsilon \to W_M$ such that $W_\epsilon = \text{graph}\,(p_\epsilon)$.

From [**C**], the projection of U_ϵ into W_D parallel to W_J defines a diffeomorphism π_ϵ of W_ϵ with Λ_ϵ.

Remark 2.2. If G is nilpotent, then M is empty, $U_\epsilon = U$, $p_\epsilon = 0$, and $\Lambda_\epsilon = W = U \cap W_D$ is a open subset in W_D.

Let $O_{\lambda,\epsilon}$ be the coadjoint orbit via $\pi_\epsilon^{-1}(\lambda)$ for $\lambda \in \Lambda_\epsilon$ and let $\beta_{\lambda,\epsilon}$ be the canonical measure on $O_{\lambda,\epsilon}$. Identify W_D with $\mathbb{R}^{n-2k}$ via the basis $\{X_i^* \mid i \notin J\}$ and let $d\lambda$ be the Lebesgue measure on W_D. If $W_D = (0)$ the measure $d\lambda$ is a point mass measure. This is the case for the "$ax+b$ group" (see the example, paragraph 5).

Define $\Theta_\epsilon : V \times \Lambda_\epsilon \to U_\epsilon$ by $\Theta_\epsilon(v,\lambda) = P(v, \pi_\epsilon^{-1}(\lambda))$. Then Θ_ϵ is a diffeomorphism.

From 2.8 of [**C**], for any integrable function F on $\mathfrak{g}^*/G$, we have

$$\int_{\mathfrak{g}^*/G} F(O)dm_\psi(O) = \sum_\epsilon \int_{\Lambda_\epsilon} F(O_{\lambda,\epsilon})|\psi(\pi_\epsilon^{-1}(\lambda))||Pf(\pi_\epsilon^{-1}(\lambda))|(2\pi)^{-2k}d\lambda$$

where $Pf(\pi_\epsilon^{-1}(\lambda))$ denotes the Pffafian in $\pi_\epsilon^{-1}(\lambda)$.

Set $[\rho_{\lambda,\epsilon}] = K^{-1}(O_{\lambda,\epsilon})$ for $\epsilon \in \{1,-1\}^a$ and $\lambda \in \Lambda_\epsilon$. For each nonzero rational function ψ on $\mathfrak{g}^*$ satisfying $\psi(g.\ell) = \Delta(g)^{-1}\psi(\ell)$ for $g \in G$ and $\ell \in \mathfrak{g}^*$, let $A_{\psi,\lambda,\epsilon}$, denote the semi-invariant operator of weight Δ for the irreducible representation $\rho_{\lambda,\epsilon}$ corresponding to the restriction of ψ to $O_{\lambda,\epsilon}$ (this operator is constructed in [**DR**]).

In summary: Let G be a connected, simply connected, and completely solvable Lie group. Let $\{X_1^*, X_2^*, \cdots, X_n^*\}$ be a Jordan-Hölder basis of $\mathfrak{g}^*$. Then, there is a finite collection of disjoint open subsets U_ϵ of $\mathfrak{g}^*$ and there is a subspace W_D of $\mathfrak{g}^*$ such that for each ϵ, U_ϵ is parametrized by a Zariski open subset Λ_ϵ of W_D, $\cup U_\epsilon$ is dense in $\mathfrak{g}^*$, and the complement of $\cup U_\epsilon$ has Plancherel measure zero. Let ψ be a non empty rational function on $\mathfrak{g}^*$ such that $\psi(g.\ell) = \Delta(g)^{-1}\psi(\ell)$ for $g \in G$ and $\ell \in \mathfrak{g}^*$. For each ϵ, there is a rational function $r_{\psi,\epsilon}$ on W_D such that for any smooth compactly supported function ϕ on G, the function

$$\lambda \to \operatorname{Tr}(A_{\psi,\lambda,\epsilon}^{-1/2}\rho_{\lambda,\epsilon}(\phi)A_{\psi,\lambda,\epsilon}^{-1/2})|r_{\psi,\epsilon}(\lambda)|$$

on Λ_ϵ is Lebesgue integrable. For any such ϕ we have

$$\phi(e) = \sum_\epsilon \int_{\Lambda_\epsilon} \operatorname{Tr}(A_{\psi,\lambda,\epsilon}^{-1/2}\rho_{\lambda,\epsilon}(\phi)A_{\psi,\lambda,\epsilon}^{-1/2})|r_{\psi,\epsilon}(\lambda)|d\lambda$$

where $r_{\psi,\epsilon}(\lambda) = \psi(\pi_\epsilon^{-1}(\lambda))Pf(\pi_\epsilon^{-1}(\lambda))(2\pi)^{-2k}$.

3. Group Fourier Transform.

We consider two cases:

First case: We suppose that $\mathfrak{g}^\ell \subset \mathfrak{g}_{n-1}$ for all $\ell \in W_\epsilon$ i.e. all the general position orbits are saturated with respect to $\mathfrak{g}_{n-1}$. We can choose a basis of $\mathfrak{g}$ in which the first $n-1$ vectors of the basis

$$\{X_1(\ell), \ldots, X_r(\ell), \ldots, X_m(\ell), \ldots, X_{n-1}(\ell)\}$$

for $\ell \in W_\epsilon$ depends on ℓ, the $X_i(\ell)$ are in $\mathfrak{g}_j^{\ell_j}$ for certain j with $\ell_j = \ell|_{\mathfrak{g}_j}$, and $\mathfrak{g}_j^{\ell_j} = \{X \in \mathfrak{g}_j | ad^* X.\ell_j = 0\}$. As $\mathfrak{g}^\ell \subset \mathfrak{g}_{n-1}$, the last vector of the basis does not depend on ℓ. Let

$$\mathfrak{B}_{W_\epsilon}(\ell) = \{X_1(\ell), \ldots, X_r(\ell), \ldots, X_m(\ell), \ldots, X_{n-1}(\ell), X_n\}$$

be one such basis of $\mathfrak{g}$.

Remark that the index set J_1 for G_{n-1} is equal to $J \backslash \{n, j_1\}$ and that $M_1 = \{j_{r_2}, \cdots, j_{r_{a_1}}\}$ is a subset of J_1. For each $\epsilon_1 \in \{1,-1\}^{a_1}$, the set U_{ϵ_1} is a nonempty open subset of $\mathfrak{g}_{n-1}^*$. Denote W_{D_1} the subspace spanned by $\{X_i^* \mid i \notin J_1\}$ in $\mathfrak{g}_{n-1}^*$. Then, we have $W_{D_1} = W_D \oplus \mathbb{R}X_{j_1}^*$ and W_{M_1} is the subspace spanned by $\{X_j^* \mid j \in M_1\}$.

Set $W_{\epsilon_1} = W_1 \cap U_{\epsilon_1}$ where

$$W_1 = \{\ell_1 \in (W_{D_1} \oplus W_{M_1}) \cap U_1 \mid |q_j(\ell_1)| = 1, j \in M_1\}.$$

Now, by the corresponding theory for G_{n-1} we have a Zariski open subset Λ_{ϵ_1} of W_{D_1} and a rational function $p_{\epsilon_1} : \Lambda_{\epsilon_1} \to W_{M_1}$ such that $W_{\epsilon_1} = graph(p_{\epsilon_1})$.

Remark that $a_1 = a - 1$. In fact there is a case where $a_1 = a$. If we start with any chain of ideals $0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_i \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}$, to avoid this case it suffices to choose a chain in such a manner that the chain passes through the nil-radical of $\mathfrak{g}$ when $\mathfrak{g}$ is non nilpotent. Also ϵ_1 is obtained by deleting an element from ϵ. Let Λ'_{ϵ_+} denote the projection of Λ_{ϵ_+} on $\mathfrak{g}_{n-1}^*$, and Λ'_{ϵ_-} denote the projection of Λ_{ϵ_-} on $\mathfrak{g}_{n-1}^*$.

The measure on W_{ϵ_1} is

$$d\mu_1(\pi_{\epsilon_1}^{-1}(\lambda_1)) = \sum_{\epsilon_1 \in \{1,-1\}^{a_1}} (2\pi)^{-(2k-2)} \psi_1(\pi_{\epsilon_1}^{-1}(\lambda_1)) Pf(\pi_{\epsilon_1}^{-1}(\lambda_1)) d\lambda_1$$

where $Pf(\pi_{\epsilon_1}^{-1}(\lambda_1))^2 = \det(\pi_{\epsilon_1}^{-1}(\lambda_1)([X_i, X_j])_{i,j \in J_1})$ with $\pi_{\epsilon_1}^{-1}(\lambda_1) = \pi_\epsilon^{-1}(\lambda)|_{\mathfrak{g}_{n-1}^*}$ and ψ_1 is a non empty rational function on $\mathfrak{g}_{n-1}^*$ such that we have $\psi_1(h.\ell_1) = \Delta(h)^{-1}\psi_1(\ell_1)$. Remark that, $\mathfrak{g}^{\ell_{n-1}} = \mathfrak{g}^\ell \oplus \mathbb{R}X_{j_1}$, $[X_i, X_j] \in \mathfrak{g}_{n-1}$ for i,j in J_1, and $\ell([X_{j_1}, \mathfrak{g}_{n-1}]) = 0$.

Remark 3.1. For $\ell \in W_\epsilon$, let $A(\ell) = (\ell[X_i, X_j])_{i,j \in J}$ be the skew-symmetric matrix.

$$A(\ell) = \begin{pmatrix} 0 & \cdots 0 & \cdots \ell([X_n, X_{j_1}]) \\ 0 & & * \\ \vdots & A_{n-1}(\ell) & \vdots \\ \ell([X_{j_1}, X_n]) & * & * \end{pmatrix}$$

where $A_{n-1}(\ell) = \ell([X_i, X_j])_{i,j \in J_1}$.

Then: $\det A(\ell)^{\frac{1}{2}} = |\ell([X_{j_1}, X_n])| (\det A_{n-1}(\ell)^{\frac{1}{2}})$.

That is, $Pf(\ell) = \ell([X_{j_1}, X_n]) Pf(\ell_{n-1})$ where $\ell_{n-1} = \ell|_{\mathfrak{g}_{n-1}}$.

Lemma 3.2. *We suppose that* $\mathfrak{g}^\ell \subset \mathfrak{g}_{n-1}$ *for all* $\ell \in W_\epsilon$*. Let* ψ *be a non empty rational function on* $\mathfrak{g}^*$ *such that* $\psi(x.\ell) = \Delta(x)^{-1}\psi(\ell)$ *for all* $\ell \in W_\epsilon$ *and* $x \in G$*. Then:*

i. $\psi(\ell) = \psi(\ell')$ *for* $\ell' \in \ell + \mathfrak{g}_{n-1}^\perp$.

ii. *Let* $\ell_1 \in \mathfrak{g}_{n-1}^*$ *and* $\tilde{\ell_1}$ *be an extension of* ℓ_1 *to* $\mathfrak{g}^*$*. By taking* $\psi_1(\ell_1) = \psi(\tilde{\ell_1})$ *we obtain a rational function* ψ_1 *on* $\mathfrak{g}_{n-1}^*$ *verifying* $\psi_1(h.\ell_1) = \Delta_{G_{n-1}}(h)^{-1}\psi_1(\ell_1)$ *for* $h \in G_{n-1}$ *and* $\ell_1 \in W_{\epsilon_1}$*.*

Proof. We have $G^\ell \subset G^{\ell_{n-1}}$ for $\ell \in W_\epsilon$ hence the stabilizer of $\ell_{n-1} \in \mathfrak{g}_{n-1}^*$ in G is also equal to $G^{\ell_{n-1}}$.

Let $\ell' = \ell + \gamma$ where $\gamma \in \mathfrak{g}_{n-1}^\perp$. Then $\ell' = a.\ell$ with $a \in G^{\ell_{n-1}}$, hence we have that $\psi(\ell') = \psi(a.\ell) = \Delta(a)^{-1}\psi(\ell)$. We have to verify that $\Delta(a) = 1$ if $a \in G^{\ell_{n-1}}$. But, $\Delta(a) = \Delta_{G_{n-1}}(a)$ since G_{n-1} is normal in G. Moreover, $G_{n-1}/G_{n-1}^{\ell_{n-1}}$ has an invariant measure, so we have $\Delta_{G_{n-1}}(a) = \Delta_{G_{n-1}^{\ell_{n-1}}}(a)$. It suffices to see that $G_{n-1}^{\ell_{n-1}}$ is abelian since, the orbit of ℓ_1 is of maximal dimension (see [**B2**], Chapter II). Hence $\psi(\ell') = \psi(\ell)$ which allows us to define ψ_1.

For all $h \in G_{n-1}$ and $\ell_1 \in \mathfrak{g}_{n-1}^*$ we have

$$\psi_1(h.\ell_1) = \psi(\widetilde{h.\ell_1}) = \psi(h.\tilde{\ell_1}) = \Delta_G(h)^{-1}\psi(\tilde{\ell_1}) = \Delta_{G_{n-1}}(h)^{-1}\psi_1(\ell_1).$$

□

We express the measure $d\mu_1$ on W_{ϵ_1} in terms of local coordinates on $\mathfrak{g}_{n-1}^*$. From the above remark and the Lemma we have that

$$\begin{aligned} d\mu_1 &= \sum_{\epsilon_1 \in \{1,-1\}^{a_1}} (2\pi)^{2k-2} \frac{1}{\psi_1(\pi_{\epsilon_1}^{-1}(\lambda_1))} \frac{1}{Pf(\pi_{\epsilon_1}^{-1}(\lambda_1))} d\lambda_1 \\ &= \left(\sum_{\epsilon'} (2\pi)^{2k-2} \frac{\pi_\epsilon^{-1}(\lambda)([X_{j_1}, X_n])}{Pf(\pi_\epsilon^{-1}(\lambda))} \frac{1}{\psi(\pi_\epsilon^{-1}(\lambda))} d\lambda \right) dX_{j_1}^* \end{aligned}$$

where ϵ' describes a part of $\{1,-1\}^a$.

This measure $W_{\epsilon_1} \subset \mathfrak{g}_{n-1}^*$ is a Plancherel measure on $\widehat{G_{n-1}}$, the unitary dual of G_{n-1}.

For $\ell \in W_\epsilon$, $\rho_\ell = \rho_{\lambda,\epsilon} = \mathrm{Ind}_{G_{n-1}}^G \rho_{\ell_{n-1}}$ is an induced representation of G, where $\ell_{n-1} = \ell|_{\mathfrak{g}_{n-1}}$ and $\rho_{\ell_{n-1}} = \rho_{\lambda_1,\epsilon_1}$ is a representation of G_{n-1}. Let $\mathtt{C}^\infty(G,\rho)$ be the set of $f \in \mathtt{C}^\infty(G)$ with compact support modulo G_{n-1} such that $f(hg) = (\rho_{\ell_{n-1}}(h))f(g)$ for all $h \in G_{n-1}$, $g \in G$.

For all $\phi \in \mathtt{C}_c^\infty(G)$ and $\rho_\ell \in \widehat{G}$ such that $\ell \in W_\epsilon$, the group Fourier transform is defined by

$$\widehat{\phi}_{\rho_\ell} = \int_G \phi(g)\rho_\ell(g)dg.$$

Set $\ell^t = Ad^*(\exp(-tX))\ell$. Remark that

$$\rho_{\ell^t}(g) = \rho_\ell(\exp(tX).g.\exp(-tX)).$$

Choose $X \in \mathfrak{g}\backslash\mathfrak{g}_{n-1}$. For all s, t in $\mathbb{R}$, the action of $\phi \in \mathtt{C}_c^\infty(G)$ on $f \in \mathtt{H}_{\rho_\ell}$ gives us

$$(\widehat{\phi}_{\rho_\ell} f)(\exp(tX)) = \int_G \phi(g)\rho_\ell(g)f(\exp(tX))dg.$$

As the induced representation acts by right translation on $f \in \mathtt{H}_{\rho_\ell}$, we have

$$\begin{aligned}
&(\widehat{\phi}_{\rho_\ell} f)(\exp(tX)) = \int_G \phi(g)f(\exp(tX).g)dg \\
&= \int_{\mathbb{R}}\int_{G_{n-1}} \phi(h.\exp(sX))f(\exp(tX).h.\exp(sX))dhds \\
&= \int_{\mathbb{R}}\int_{G_{n-1}} \phi(h.\exp(sX))f(\exp(tX).h.\exp(-tX).\exp(tX).\exp(sX))dhds \\
&= \int_{\mathbb{R}}\int_{G_{n-1}} \phi(h.\exp(sX))f(\exp(tX).h.\exp(-tX).\exp(t+s)X)dhds \\
&= \int_{\mathbb{R}}\int_{G_{n-1}} \phi(h.\exp(sX))\rho_{(\ell^t)_{n-1}}(h)f(\exp(t+s)X)dhds \\
&= \int_{\mathbb{R}}\int_{G_{n-1}} \phi^s(h)\rho_{(\ell^t)_{n-1}}(h)f(\exp(t+s)X)dhds \\
&= \int_{\mathbb{R}} \left(\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}}\right) f(\exp(t+s)X)ds
\end{aligned}$$

where $\phi^s(h) = \phi(h.\exp(sX))$.

For all $\alpha \in \mathbb{R}$ we set $f_\alpha(h.\exp(sX)) = e^{i\alpha s}f(h.\exp(sX))$. We have $f_\alpha \in \mathtt{H}_{\rho_\ell}$, since f is in $\mathtt{H}_{\rho_\ell}$.

Let $\ker\rho_\ell$ denote the kernal of ρ_ℓ in $C^*(G)$, the C^*- algebra of the group G. If $\phi \in \ker\rho_\ell$, then, from the above calculations, for all $f \in \mathtt{H}_{\rho_\ell}$ we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} f_\alpha(\exp(s+t)X)ds \\
&= \int_{\mathbb{R}} e^{i\alpha(s+t)}\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} f(\exp(s+t)X)ds \qquad \forall\alpha \in \mathbb{R},
\end{aligned}$$

which implies that $\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} = 0$ for all $s \in \mathbb{R}$. Conversely, for all s and t in $\mathbb{R}$, if $\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} = 0$ we have $\widehat{\phi}_{\rho_\ell} = 0$ which implies that $\phi \in \ker\rho_\ell$. We have established an equivalence

$$\phi \in \ker\rho_\ell \Longleftrightarrow \left(\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} = 0 \ \forall s, t\right).$$

Second case: If all the general position orbits are not saturated with respect to $\mathfrak{g}_{n-1}$, we can choose a basis of $\mathfrak{g}$ in such a way that the last vector of the basis X_n does not depend on ℓ and $X_n(\ell) \in \mathfrak{g}^\ell$. Let

$$\mathfrak{B}_{W_\epsilon}(\ell) = \{X_1(\ell), \ldots, X_r(\ell), \ldots, X_m(\ell), \ldots, X_{n-1}(\ell), X_n(\ell)\}$$

be one such basis of $\mathfrak{g}$ in which the $X_i(\ell)$ are in $\mathfrak{g}_j^{\ell_j}$ for certain j with $\ell_j = \ell|_{\mathfrak{g}_j}$.

Lemma 3.3. *Assume that $\mathfrak{g}^\ell \not\subset \mathfrak{g}_{n-1}$ for all $\ell \in W_\epsilon$. Let ψ be a non empty rational function on $\mathfrak{g}^*$ such that $\psi(x.\ell) = \Delta(x)^{-1}\psi(\ell)$ for all $\ell \in W_\epsilon$ and $x \in G$. Let $\ell_1 \in \mathfrak{g}_{n-1}^*$ and $\tilde{\ell_1}$ be an extension of ℓ_1 to $\mathfrak{g}^*$. By letting $\psi_1(\ell_1) = \psi(\tilde{\ell_1})$ we obtain a rational function ψ_1 on $\mathfrak{g}_{n-1}^*$ satisfying $\psi_1(h.\ell_1) = \Delta(h)^{-1}\psi_1(\ell_1)$ for all $h \in G_{n-1}$.*

Proof. For all $\ell \in \mathfrak{g}^*$ and $\alpha \in \mathbb{R}$ we have $\ell_\alpha = \ell + \alpha X_n^*$ and $\mathfrak{g}^* = \mathfrak{g}_{n-1}^* \oplus \mathbb{R}X_n^*$. For all $h \in G_{n-1}$, we have $h.\ell_\alpha = h.\ell + \alpha X_n^*$ since $G.X_n^* = X_n^*$. By choosing $\alpha = 0$, we have $\ell_0 = \ell + 0X_n^*$ and $h.\ell_0 = h.\ell$. Hence, $\psi_1(\ell_1) = \psi(\tilde{\ell_1})$ and

$$\psi_1(h.\ell_1) = \psi(h.\ell_1) = \Delta(h)^{-1}\psi(\ell_1) = \Delta_{G_{n-1}}(h)^{-1}\psi_1(\ell_1).$$

□

Remark that the set of indices J_1 for G_{n-1} is equal to J. In this case as $\mathfrak{g}^\ell = \mathfrak{g}^{\ell_{n-1}} + \mathbb{R}X_n$ we have $W_D = W_{D_1} + \mathbb{R}X_n$, where W_{D_1} is the subspace of $\mathfrak{g}_{n-1}^*$ corresponding to W_D in $\mathfrak{g}^*$. Moreover, $\Lambda_\epsilon = \Lambda_{\epsilon_1} + \mathbb{R}X^*$. The Plancherel measure over $\widehat{G}$ can be written as;

$$\begin{aligned} d\mu(\ell) &= \sum_\epsilon (2\pi)^{2k} \frac{1}{\psi(\pi_\epsilon^{-1}(\lambda))} \frac{1}{Pf(\pi_\epsilon^{-1}(\lambda))} dX_1^* \cdots dX_{n-2k-1}^* dX_n^* \\ &= \left(\sum_{\epsilon_1} (2\pi)^{2k} \frac{1}{\psi_1(\pi_{\epsilon_1}^{-1}(\lambda_1))} \frac{1}{Pf(\pi_{\epsilon_1}^{-1}(\lambda_1))} dX_1^* \cdots dX_{n-2k-1}^* \right) dX_n^* \\ &= d\mu_1 \times dX_n^*. \end{aligned}$$

For $\ell = \pi_\epsilon^{-1}(\lambda) \in W_\epsilon$, and $\alpha \in \mathbb{R}$ we let $\ell_\alpha = \ell + \alpha X^*$. Hence, $\ell_\alpha(X) = \ell(X) + \alpha$ and $\rho_{\ell_\alpha} = \rho_\ell \otimes \chi_\alpha$ with $\chi_\alpha(h.\exp(sX)) = e^{i\alpha s}$ for all $h \in G_{n-1}$.

The restriction of ρ_{ℓ_α} to G_{n-1} is irreducible and equivalent to $\rho_{\ell_{n-1}}$ for all $\alpha \in \mathbb{R}$. For all ξ, $\eta \in \mathtt{H}_{\rho_\ell} = \mathtt{H}_{\rho_{\ell_\alpha}}$ we have

$$\begin{aligned}
\langle \widehat{\phi}_{\rho_{\ell_\alpha}} \xi, \eta \rangle &= \int_G \langle \rho_{\ell_\alpha}(g)\xi, \eta\rangle \phi(g) dg \\
&= \int_G \langle \rho_\ell \otimes \chi_\alpha(g)\xi, \eta\rangle \phi(g) dg \\
&= \int_{\mathbb{R}} \int_{G_{n-1}} \langle \rho_\ell \otimes \chi_\alpha(\exp(sX).h)\xi, \eta\rangle \phi(\exp(sX).h) dh ds \\
&= \int_{\mathbb{R}} \int_{G_{n-1}} \langle e^{i\alpha s} \rho_\ell(\exp(sX)) \rho_{\ell_{n-1}}(h)\xi, \eta\rangle \phi(\exp(sX).h) dh ds \\
&= \int_{\mathbb{R}} e^{i\alpha s} \langle \rho_\ell(\exp(sX)) \widehat{\phi}^s_{\rho_{\ell_{n-1}}} \xi, \eta\rangle ds
\end{aligned}$$

where $\phi^s(h) = \phi(\exp(sX).h)$. Hence we have expressed $\widehat{\phi}_{\rho_{\ell_\alpha}}$ with the help of $\widehat{\phi}^s_{\rho_{\ell_{n-1}}}$.

4. Weak Paley–Wiener Property.

Theorem 4.1. *Let G be a connected, simply connected, and completely solvable Lie group with the unitairy dual $\widehat{G}$, and let ϕ be a bounded, measurable and compactly supported function (i.e. $\phi \in L^\infty_c(G)$). Assume that there is a subset $E \subset \widehat{G}$ with positive Plancherel measure such that $\widehat{\phi}_\rho = 0$ for all $\rho \in E$, where $\widehat{\phi}_\rho$ is the group Fourier transform of ϕ. Then $\phi = 0$ almost every where on G.*

Proof. We proceed by induction on the dimension n of G. The result is true if the dimension of G is one, since $G \cong \mathbb{R}$. Assume that the result is true for all groups of dimension $n-1$. We can assume that E is contained in W_ϵ (it suffices to take E as the finite union of $E \cap W_\epsilon$).

First case: $\mathfrak{g}^\ell \subset \mathfrak{g}_{n-1}$ for all $\ell \in W_\epsilon$. Let $\phi \in \mathtt{C}^\infty_c(G)$. By hypothesis, for all ρ_ℓ, such that $\ell \in E$ we have $0 = \widehat{\phi}_{\rho_\ell}$; we will show that $\phi = 0$ almost every where on G.

Notice that for all $\epsilon_1 \in \{-1, 1\}^{a_1}$, the associated set Λ_{ϵ_1} corresponds to two sets Λ_{ϵ_+} and Λ_{ϵ_-}, $\epsilon_\pm \in \{-1, 1\}^a$ in W_D. If Λ'_{ϵ_+} and Λ'_{ϵ_-} are the projections of Λ_{ϵ_+} and Λ_{ϵ_-} on $\mathfrak{g}^*_{n-1}$ such that $\Lambda'_{\epsilon_1} = (\exp \mathbb{R}X).\Lambda'_{\epsilon_+} \cup (\exp \mathbb{R}X).\Lambda'_{\epsilon_-}$ and $T_\ell = \{\exp tX.\ell_{n-1} \mid t \in \mathbb{R}\}$ are contained in the projection of Λ_{ϵ_+} or in Λ_{ϵ_-}, Λ'_{ϵ_1} is a Zariski open set in Λ_{ϵ_1}.

From paragraph 3 we have that

$$\phi \in \ker \rho_\ell \Longleftrightarrow \left(\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} = 0 \ \forall s, t \right).$$

By hypothesis, $\widehat{\phi_{\rho_\ell}} = 0$ for all $\ell \in E$ and from the above equivalence we have

$$\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} = 0$$

for all s, t in $\mathbb{R}$. This relation tells us that a set A contained in $\Lambda_{\epsilon_+} \cup \Lambda_{\epsilon_-}$ has positive Plancherel measure if and only if the set $\cup_{\rho_\ell \in A} T_\ell$ has positive Plancherel measure in Λ_{ϵ_1}.

In applying this remark to the set E, we obtain

$$\widehat{\phi}^s_{\rho_{(\ell^t)_{n-1}}} = 0$$

for all $\rho_{\ell_{n-1}}$ in $E' \subset \widehat{G}_{n-1}$ with positive Plancherel measure.

By the induction hypothesis $\phi^s = 0$ almost everywhere on G_{n-1}, which implies that $\phi = 0$ almost everywhere on G by using Fubini's theorem.

Second case: $\mathfrak{g}^\ell \not\subset \mathfrak{g}_{n-1}$ for all $\ell \in W_\epsilon$. Let $\phi \in C_c^\infty(G)$. By hypothesis, for all ρ_ℓ, such that $\ell \in E$ we have $\widehat{\phi}_{\rho_\ell} = 0$; let us show that $\widehat{\phi}_{\rho_\ell} = 0$ for all $\ell \in W_\epsilon$.

Let $\ell \in E$. For all $\alpha \in \mathbb{R}$ we have

$$\langle \widehat{\phi}_{\rho_{\ell_\alpha}} \xi, \eta \rangle = \int_{\mathbb{R}} e^{i\alpha s} \langle \rho_\ell(\exp(sX)) \widehat{\phi}^s_{\rho_{\ell_{n-1}}} \xi, \eta \rangle ds;$$

hence

$$\widehat{\phi}_{\rho_{\ell_\alpha}} = \int_{\mathbb{R}} e^{i\alpha s} \rho_\ell(\exp(sX)) \widehat{\phi}^s_{\rho_{\ell_{n-1}}} ds.$$

Set

$$\Psi(s) = \rho_\ell(\exp(sX)) \widehat{\phi}^s_{\rho_{\ell_{n-1}}}.$$

Hence

$$\begin{aligned} \widehat{\phi}_{\rho_{\ell_\alpha}} &= \int_{\mathbb{R}} \Psi(s) e^{i\alpha s} ds \\ &= \widehat{\Psi}(\alpha). \end{aligned}$$

By hypothesis, for all $\ell \in E$ we have $\widehat{\phi}_{\rho_\ell} = 0$. The above calculation tells us that there exists a set $E' \subset E$ with positive Plancherel measure such that $\widehat{\Psi}(\alpha) = 0$ for α belonging to a set of reals with positive Lebesgue measure and $\ell \in E'$. Hence $\Psi = 0$ almost everywhere; consequently we have $\Psi(s) = 0$ for almost every $s \in \mathbb{R}$. Hence

$$0 = \widehat{\phi}_{\rho_{\ell_\alpha}} = \int_{\mathbb{R}} e^{i\alpha s} \rho_\ell(\exp(sX)) \widehat{\phi}^s_{\rho_{\ell_{n-1}}} ds$$

for all α in $\mathbb{R}$, which implies that $\widehat{\phi}^s_{\rho_{\ell_{n-1}}} = 0$ for all ℓ_{n-1} in E_1 (path of E on $\mathfrak{g}^*_{n-1}$) with positive Plancherel measure on $\widehat{G_{n-1}}$. By using the induction hypothesis $\widehat{\phi}_{\rho_{\ell_{n-1}}} = 0$ for almost all $\ell_{n-1} \in W'$ (path of W_ϵ on $\mathfrak{g}^*_{n-1}$). Hence, $0 = \widehat{\phi}_{\rho_\ell}$ for almost all $\ell \in W_\epsilon$ (from the above calculation of $\widehat{\phi}_{\rho_{\ell_\alpha}}$).

Hence $\widehat{\phi}_\rho = 0$ for almost all ρ relating with the Plancherel measure. By the Plancherel formula for completely solvable Lie groups, we have

$$\phi(e) = \sum_\epsilon \int_{\Lambda_\epsilon} Tr(A^{-1/2}_{\psi,\lambda,\epsilon}\rho_{\lambda,\epsilon}(\phi)A^{-1/2}_{\psi,\lambda,\epsilon})|r_{\psi,\epsilon}(\lambda)|d\lambda$$

which implies that $\phi = 0$.

Now, we consider $\phi \in L^\infty_c(G)$. Let $\{f_n\}_n$ be an apporoximate identity in $C^\infty_c(G)$. For all integers n, $f_n * \phi \in C^\infty_c(G)$. Let $\rho \in E$. If $\widehat{\phi}_\rho$ vanishes, then $(\widehat{f_n * \phi})_\rho$ also vanishes. Hence by what precedes, $f_n * \phi = 0$ (for all integers n). But, $(f_n * \phi)_{n\in N}$ converges to ϕ in $L^1(G)$, which implies that $\phi = 0$ almost everywhere on G. □

5. Example: The $ax + b$ Group.

Consider the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}.$$

We use the notation

$$(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

The Matrix multiplication gives:

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + b_1)$$

and the inverse

$$(a, b)^{-1} = (a^{-1}, -ba^{-1}).$$

Let $H = (1, b)$ be the derived group of G which is identified with $\mathbb{R}$. Let $y \in \mathbb{R}$, χ_y the character of H defined by $\chi_y((1, b)) = e^{iby}$.

Remark that $(a, b) = (1, b)(a, 0)$. Let $\rho_y = \mathrm{Ind}^G_H\chi_y$ be the induced representation of G. This representation is realized in the space $L^2(\mathbb{R})$. Recall that for all $y > 0$, ρ_y is equivalent to ρ_1 and we denote by ρ_+ the class of the representation ρ_1. If $y < 0$, ρ_y is equivalent to ρ_{-1}; we denote by ρ_- the equivalence class of this representation.

The Lie algebra $\mathfrak{g}$ of G is the set of matrices

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, (x, y) \in \mathbb{R}^2 \right\}.$$

In the basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we have $[X, Y] = Y$. With the basis X and Y we have

$$Ad(a, b) = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix}.$$

Also in the dual basis $\{X^*, Y^*\}$

$$Ad^*(a,b) = \begin{pmatrix} 1 & ba^{-1} \\ 0 & a^{-1} \end{pmatrix}.$$

For $\ell = \alpha X^* + \beta Y^* \in \mathfrak{g}^*$ the orbits of G in $\mathfrak{g}^*$ are the upper half plane $\beta > 0$, the lower half plane $\beta < 0$ and the points $(\alpha, 0)$.

Let $\mathfrak{B} = \{X, Y\}$ be the basis of $\mathfrak{g}$ defined above, and $\mathfrak{B}^* = \{X^*, Y^*\}$ the dual basis of $\mathfrak{g}^*$. There exists a set $J = \{j_1, j_2\} \subseteq \{1, 2\}$ and $M = \{j_2\}$ a subset of J, so that $V \subset \mathbb{R}^2$, $V =]0, \infty[\times \mathbb{R}$. We have $W_D = (0)$ and W_M is spanned by the vector $\{X^*_{j_2} \mid j_2 \in M\}$.

The Zariski open sets U_+ and U_- are the half planes of $\mathfrak{g}^*$ defined above and $U = U_+ \cup U_-$. Here, $a = 1$ and $\epsilon \in \{1, -1\}$.

Since there are only two orbits, the set

$$W = \{\ell \in W_M \cap U \mid |q_{j_2}(\ell)| = 1, j_2 \in M\}$$

is a union of two points in $\mathfrak{g}^*$. We have $W_+ = W \cap U_+$ and $W_- = W \cap U_-$. Let $\epsilon \in \{1, -1\}$. In this case the Zariski open set is $\Lambda_\epsilon = \Lambda_+$ or $\Lambda_\epsilon = \Lambda_-$ of W_D, which reduces to a point.

In this particular case we can prove weak Paley–Wiener property by direct calculations.

Let $\phi \in \mathtt{C}^\infty_c(G)$, $f \in L^2(\mathbb{R}^*_+)$ and $(t, 0) \in \mathbb{R}^*_+$: then

$$\begin{aligned}
(\widehat{\phi}_{\rho_\ell} f)(t) &= \int_G \phi((a,b)) \rho_\ell((a,b)) f(t) a^{-2} da db \\
&= \int_G \phi((a,b)) f((a,b)^{-1}(t,0)) a^{-2} da db \\
&= \int_{\mathbb{R}^*_+} \int_{\mathbb{R}} \phi((a,b)) f((a^{-1}t, -ba^{-1})) a^{-2} da db \\
&= \int_{\mathbb{R}^*_+} \int_{\mathbb{R}} \phi((a,b)) f((a^{-1}t, 0)(1, -bt^{-1})) a^{-2} da db \\
&= \int_{\mathbb{R}^*_+} \left(\int_{\mathbb{R}} \phi((a,b)) \chi_y((1, -bt^{-1})) db \right) f((a^{-1}t, 0)) a^{-2} da \\
&= \int_{\mathbb{R}^*_+} \left(\int_{\mathbb{R}} \phi^a(b) e^{-ibyt^{-1}} db \right) f((a^{-1}t, 0)) a^{-2} da \\
&= \int_{\mathbb{R}^*_+} \widehat{\phi}^a_{\chi_{yt^{-1}}} f((a^{-1}t, 0)) a^{-2} da,
\end{aligned}$$

where $\phi^a(b) = \phi((a,b))$.

Remark that $\phi^a \in \mathtt{C}^\infty_c(\mathbb{R})$. By hypothesis we have $\widehat{\phi}_{\rho_\ell} = 0$ for all $\ell \in E$. The above calculation implies that for all $a > 0$ we have $\widehat{\phi}^a_{\chi_{yt^{-1}}} = 0$ for almost all $t > 0$ and for fixed y.

As $\phi^a \in C_c^\infty(\mathbb{R})$, $\widehat{\phi}^a_{\chi_{yt^{-1}}}$ extends as an entire function over $\mathbb{C}$. $\widehat{\phi}^a_{\chi_{yt^{-1}}}$ vanishes on a set in which the Plancherel measure $d\mu_1$ is positive hence by the classical Paley–Wiener theorem, we can conclude that $\phi^a = 0$, and then $\phi = 0$ almost everywhere on G.

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