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VIEWING FINITE DIMENSIONAL REPRESENTATIONS THROUGH INFINITE DIMENSIONAL ONES

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Dedicated to the memory of Maurice Auslander whose questions triggered this work

We develop criteria for deciding the contravariant finiteness status of a subcategory $\mathfrak{A} \subseteq \Lambda$-mod, where Λ is a finite dimensional algebra. In particular, given a finite dimensional Λ-module X, we introduce a certain class of modules – we call them $\mathfrak{A}$-phantoms of X – which indicate whether or not X has a right $\mathfrak{A}$-approximation: We prove that X fails to have such an approximation if and only if X has infinite-dimensional $\mathfrak{A}$-phantoms. Moreover, we demonstrate that large phantoms encode a great deal of additional information about X and $\mathfrak{A}$ and that they are highly accessible, due to the fact that the class of all $\mathfrak{A}$-phantoms of X is closed under subfactors and direct limits.

1. Introduction and preliminaries.

Given a finite dimensional algebra Λ and a resolving contravariantly finite subcategory $\mathfrak{A}$ of the category Λ-mod of finitely generated left Λ-modules, the minimal right $\mathfrak{A}$-approximations of the simple left Λ-modules provide significant structural information about arbitrary objects of $\mathfrak{A}$: Indeed, if these approximations are labeled $A_1, \dots, A_n$, then a finitely generated Λ-module M belongs to $\mathfrak{A}$ if and only if M is a direct summand of a module that has a filtration with consecutive factors in $\{A_1, \dots, A_n\}$; this was proved by Auslander and Reiten [2]. (For the basic definitions, see 'Prerequisites' below.) Among other consequences, this result has an obvious homological pay-off: Namely, if all of the A_i have finite projective dimension, then the maximum of these dimensions coincides with the supremum of the projective dimensions attained on objects of $\mathfrak{A}$.

Our goal here is twofold. On one hand, we describe typical scenarios preventing the existence of right $\mathfrak{A}$-approximations of a given Λ-module X; as a result, we establish 'negative criteria' for the contravariant finiteness of subcategories $\mathfrak{A}$ of Λ-mod. On the other hand – maybe more importantly – our investigation reveals that, for non-contravariantly finite subcategories $\mathfrak{A}$ of Λ-mod, certain modules of infinite dimension over the base field take over the role which is played by the minimal right $\mathfrak{A}$-approximations in the contravariantly finite case. We call these modules '$\mathfrak{A}$-phantoms' and define them as follows: First, given a subclass $\mathfrak{C}$ of $\mathfrak{A}$ and a finitely generated left Λ-module X, we label as '$\mathfrak{C}$-approximation of X inside $\mathfrak{A}$' any homomorphism $f : A \to X$ with $A \in \mathfrak{A}$ having the property that all homomorphisms $g : C \to X$ with $C \in \mathfrak{C}$ factor through f; in other words, with the property that the induced map of functors $\operatorname{Hom}(-,A)|_{\mathfrak{C}} \xrightarrow{\operatorname{Hom}(-,f)} \operatorname{Hom}(-,X)|_{\mathfrak{C}}$ is surjective. Following common practice, we will also refer to the module A as a $\mathfrak{C}$-approximation of X inside $\mathfrak{A}$ in that case. Next we will call a Λ-module H, not necessarily in $\mathfrak{A}$, an $\mathfrak{A}$-phantom of X in case, for each finitely generated submodule H' of H, there exists a finite subclass $\mathfrak{C} = \mathfrak{C}(H')$ of $\mathfrak{A}$ with the property that H' occurs as a subfactor of *each* $\mathfrak{C}$-approximation of X inside $\mathfrak{A}$. Clearly, the class of all $\mathfrak{A}$-phantoms of X is closed under subfactors; so, in case X has a right $\mathfrak{A}$-approximation, the $\mathfrak{A}$-phantoms of X coincide with the subfactors of the *minimal* approximation. Our main result (Theorem 9) ensures existence of phantoms in general, under the mild hypothesis that $\mathfrak{A}$ be closed under finite direct sums: Namely, we prove that X has $\mathfrak{A}$-phantoms of infinite dimension over the base field if and only if X fails to have a right $\mathfrak{A}$-approximation. For an intuitive idea of the information stored in such phantoms, suppose for the moment that $X = S$ is simple; comparable to the minimal right $\mathfrak{A}$-approximation of S (in case of existence), the $\mathfrak{A}$-phantoms of S represent, in as highly compressed a form as possible, the relations characterizing those objects of $\mathfrak{A}$ which carry S in their tops. In essence, our criteria for failure of contravariant finiteness provide instructions for the uncovering of $\mathfrak{A}$-phantoms which are too big (namely, non-finitely generated) to be compatible with the existence of traditional $\mathfrak{A}$-approximations.

Our primary interest will be in the category $\mathfrak{A} = \mathcal{P}^{\infty}(\Lambda\text{-mod})$ of finitely generated Λ-modules of finite projective dimension, even though most of our results will address the situation of an arbitrary subcategory of Λ-mod which is closed under finite direct sums. Loosely speaking, the category $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ is contravariantly finite in Λ-mod when it is either very large (e.g., when Λ has finite global dimension) or very small (e.g., when Λ has vanishing finitistic dimension); in fact, by [**3**], every representation-finite subcategory of Λ-mod is contravariantly finite. There are several other classes of algebras for which the problem of whether $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ is contravariantly finite in Λ-mod is settled, e.g., for left serial algebras, it is settled in the positive [**4**]. However, in general, it is difficult to decide for an algebra Λ, given via quiver and relations for instance, whether $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ has this property. In particular, until recently there was only a single instance for which failure of contravariant finiteness of $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ had been established; this is an example of a monomial relation algebra, due to Igusa, Smalø, and Todorov [**10**], which is so closely related to the Kronecker algebra that a proof for failure of contravariant finiteness of $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ can be gleaned from the representation theory of this latter algebra. Our curiosity in this direction was triggered by a question of M. Auslander as to whether, for the monomial relation algebra with differing big and little finitistic dimensions which was exhibited by the second author in [**8**], $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ is contravariantly finite. In the meantime, Smalø and the second author have shown that contravariant finiteness of $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ forces the big and little finitistic dimensions of Λ to coincide [**9**]; hence the answer to Auslander's question is negative. However, the problem of exhibiting a simple module which fails to have a right $\mathcal{P}^{\infty}(\Lambda\text{-mod})$-approximation still proved rather intractable without a systematic theory providing direction.

The core of the paper is Section 3. The concepts introduced at the beginning of that section appear cogent in the light of the preliminary criterion for failure of contravariant finiteness presented in Section 2 and its applications to special cases. We present several examples to illustrate how phantoms mark the dividing line between contravariant finiteness and failure thereof, and to show concretely what type of information they store. In particular, we apply our techniques to the example mentioned at the outset (Section 4). These sample applications also indicate how sensitive the property of contravariant finiteness of $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ in Λ-mod is, with respect to modifications of the relations of the underlying algebra.

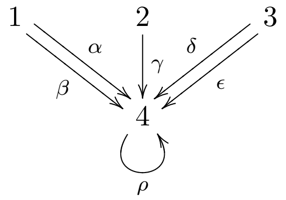
Acknowledgment: The authors would like to thank Sverre Smalø for his careful reading of a preliminary manuscript and several helpful suggestions.

Prerequisites: Throughout, Λ will denote a basic finite dimensional algebra over a field K, with a fixed set of primitive idempotents, $e_1, \dots, e_n$, and J will be the Jacobson radical of Λ. The simple left Λ-modules $\Lambda e_i/Je_i$ will be abbreviated by S_i. Moreover, Λ-mod will stand for the category of all finitely generated left Λ-modules, and Λ-Mod for the category of *all* left Λ-modules.

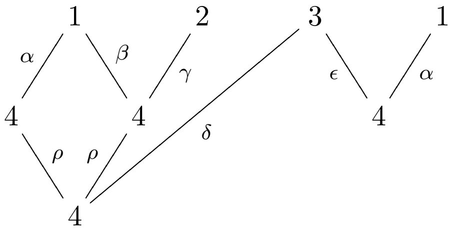
Let $\mathfrak{A}$ be a full subcategory of Λ-mod. Following Auslander/Smalø [**3**] and Auslander/Reiten [**2**], we say that a module $M \in \Lambda\text{-mod}$ has a (right) $\mathfrak{A}$-*approximation* in case there exists a homomorphism $\varphi : A \to M$ with $A \in \mathfrak{A}$ such that each homomorphism $B \to M$ with $B \in \mathfrak{A}$ factors through φ. By [**2**], existence of any $\mathfrak{A}$-approximation of M entails existence of a *minimal* $\mathfrak{A}$-approximation of M, i.e., one of minimal K-dimension, which is unique up to isomorphism. If each object in Λ-mod has an $\mathfrak{A}$-approximation, then $\mathfrak{A}$ is said to be *contravariantly finite in* Λ-mod (see [**2**, **3**]). Our favorite choice of a subcategory $\mathfrak{A} \subseteq \Lambda\text{-mod}$ will be the subcategory of all finitely generated left Λ-modules of finite projective dimension; we label it $\mathcal{P}^{\infty}(\Lambda\text{-mod})$. Finally, we call a module category $\mathfrak{A} \subseteq \Lambda\text{-mod}$ *resolving* in case $\mathfrak{A}$ is closed under extensions, as well as kernels of epimorphisms, and contains all indecomposable projective left Λ-modules. Clearly, $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ is an instance of a resolving subcategory of Λ-mod.

Given a module $M \in \Lambda$-Mod, we call an element $m \in M$ a *top element* of M if $m \in M \setminus JM$ and $e_i m = m$ for some i; in this case, we also say that m is a top element *of type* e_i. In all of our examples, Λ will be a split finite dimensional algebra over K, that is, Λ will be of the form $K\Gamma/I$ where Γ is a quiver and I an admissible ideal in the path algebra $K\Gamma$. We will briefly and informally review the second author's conventions for the graphical communication of information about countably generated Λ-modules. For additional detail, see [7, 8]. (We point out that our labeled graphs are related to the module diagrams studied by Alperin [1] and Fuller [5].)

Let Γ be the quiver



and $\Lambda = K\Gamma/\langle \rho^2 \rangle$. To say that a left Λ-module M has the *layered and labeled graph* shown below,



with respect to a sequence m_1, m_2, m_3, m_4 of top elements of M which are K-linearly independent modulo JM, is to convey the following information:

- $M/JM \cong S_1^2 \oplus S_2 \oplus S_3$, the top elements m_i of M have type e_i for $i = 1, 2, 3$, and m_4 has type e_1;
- $JM/J^2M \cong S_4^3$, the three copies of S_4 modulo J^2M being generated by αm_1, βm_1, ϵm_3, such that γm_2 is congruent to a nonzero scalar multiple of βm_1 modulo J^2M, and αm_4 congruent to a nonzero scalar multiple of ϵm_3;
- $J^2M/J^3M = J^2M \cong S_4$ is generated by $\rho\alpha m_1$, and is also generated by any of the elements $\rho\beta m_1$, $\rho\gamma m_2$, or δm_3.

Finally, we will consider the following two finitistic dimensions of Λ: The *left little finitistic dimension*, $\operatorname{l\,fin\,dim} \Lambda$, which is the supremum of the finite projective dimensions attained on Λ-mod, and the *left big finitistic dimension*, $\operatorname{l\,Fin\,dim}\Lambda$, which stands for the analogous supremum attained on all of Λ-Mod.

2. A few motivating examples.

In this section, we specialize to the situation where the algebra Λ is split, i.e., we assume throughout that $\Lambda = K\Gamma/I$ is a path algebra modulo relations. We start by exhibiting an elementary sufficient condition for failure of contravariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ in Λ-mod. In our first example we will apply it to the algebra constructed by Igusa/Smalø/Todorov [10], which provided the first known instance of such failure. While this criterion is quite easy to handle, its scope is rather limited, and it will later be supplemented by a criterion of far wider applicability.

For convenience of exposition, we will often view left Λ-modules M as representations of the quiver Γ satisfying the commutativity relations dictated by I; given a path $p : e_1 \to e_2$ in $K\Gamma$, we will in that case, write $f_p : e_1M \to e_2M$ for the K-linear map corresponding to p.

Elementary Criterion 1. *Let $\Lambda = K\Gamma/I$. Suppose that e_1 and e_2 are vertices of the quiver Γ (not necessarily distinct) and $p, q \in K\Gamma \setminus I$ paths from e_1 to e_2 with $\Lambda p \cap \Lambda q = 0$ (we view p and q as elements of Λ whenever called for by the context). Moreover, suppose that*

(i) *the cyclic module $\Lambda(p, q)$ generated by the element $(p, q) \in \Lambda^2$ has finite projective dimension,*

and that one of the following two conditions is satisfied: either,

(ii) *whenever $M \in \mathcal{P}^\infty(\Lambda\text{-mod})$, then $f_p(e_1M \setminus JM) \cap f_q(e_1JM) = \varnothing$; or,*

(ii′) *whenever $M \in \mathcal{P}^\infty(\Lambda\text{-mod})$, then $\ker(f_p) \subseteq \ker(f_q)$, and $\ker(f_p) \subseteq e_1JM$.*

Then the simple module $S_1 = \Lambda e_1/Je_1$ does not have a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation.

Before we justify the criterion, we point out that the second part of Criterion 1(ii’) can often be verified without effort; we label it as follows:

(iii) Whenever M is in $\mathcal{P}^\infty(\Lambda\text{-mod})$, then $\ker(f_p) \subseteq e_1JM$.

Indeed, suppose that p is an arrow such that Λp splits off in Je_1 (this is obviously true when Λ is a monomial relation algebra). Then Condition (iii) holds if and only if $\operatorname{p\,dim}\Lambda p = \infty$. For, if $\operatorname{p\,dim}\Lambda p = \infty$ and $M \in \Lambda$-mod contains a top element of type e_1 which is annihilated by p, then Λp is isomorphic to a direct summand of $\Omega^1(M)$, which entails that $\operatorname{p\,dim} M = \infty$. If, on the other hand, $\operatorname{p\,dim}\Lambda p < \infty$, then the module $M = \Lambda/\Lambda p$ violates Condition (iii).

A readily recognizable situation in which the blanket hypothesis of the criterion, as well as conditions (i) and (ii) are satisfied is as follows: p is an arrow $e_1 \to e_2$, $q \in K\Gamma \setminus I$ a path from e_1 to e_2 of positive length which is different from p such that $Jp = qJ = 0$, and $\operatorname{p\,dim} \Lambda q < \infty$, while $\operatorname{p\,dim} \Lambda e_2/Je_2 = \infty$.

Proof of Criterion 1. We start by assuming the blanket hypothesis of the criterion and condition (i) to construct an infinite family of objects $(N_n)_{n\in\mathbb{N}}$ in $\mathcal{P}^\infty(\Lambda\text{-mod})$. Namely, for $n \in \mathbb{N}$, we let $b_1 = \cdots = b_n = e_1$, define a left Λ-module

$$N_n := \left(\bigoplus_{i=1}^{n} \Lambda b_i\right) \Big/ \left(\sum_{i=1}^{n-1} \Lambda(pb_i - qb_{i+1})\right),$$

and write $\overline{b_i}$ for the residue class of b_i in N_n. To compute the first syzygy $\Omega^1(N_n)$ of N_n, consider the projective cover $\pi : \bigoplus_{i=1}^n \Lambda b_i \to N_n$ with $\pi(b_i) = \overline{b_i}$, and set $C_i = \Lambda(pb_i - qb_{i+1})$ for $1 \le i \le n-1$. Note that $C_i \simeq \Lambda(p,q)$, whence $\operatorname{p\,dim} C_i < \infty$ by condition (i). We will conclude that $\operatorname{p\,dim} N_n < \infty$ by showing that $\ker \pi = \bigoplus_{i=1}^{n-1} C_i$. Suppose that

$$\begin{aligned}0 &= \sum_{i=1}^{n-1} \lambda_i(pb_i - qb_{i+1})\\ &= \lambda_1 pb_1 + (\lambda_2 p - \lambda_1 q)b_2 + \cdots + (\lambda_{n-1}p - \lambda_{n-2}q)b_{n-1} - \lambda_{n-1}qb_n\end{aligned}$$

for certain coefficients $\lambda_i \in \Lambda$. This implies that $\lambda_1 p = 0$, $\lambda_{n-1}q = 0$ and $\lambda_i p - \lambda_{i-1}q = 0$ for $2 \le i \le n-1$, and in view of the hypothesis that $\Lambda p \cap \Lambda q = 0$, the latter equations entail $\lambda_i p = \lambda_{i-1}q = 0$. Thus we obtain $\lambda_i(pb_i - qb_{i+1}) = 0$ for $1 \le i \le n-1$ as required.

Case I. Suppose that, in addition, condition (ii) holds, but that there nonetheless exists a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation $\varphi : A \to S_1$ for S_1. Pick $n > \operatorname{length}(A)$, define $f : N_n \to S_1$ via $f(\overline{b_1}) = e_1 + Je_1$, $f(\overline{b_i}) = 0$ for $i \ge 2$, and let $g \in \operatorname{Hom}_\Lambda(N_n, A)$ be such that $f = \varphi g$, i.e., such that the following diagram commutes:

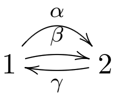
$$\begin{tikzcd} & N_n \arrow[dl, "g"'] \arrow[d, "f"] \\ A \arrow[r, "\varphi"] & S_1 \end{tikzcd} .$$

Next pick $m \ge 1$ minimal with the property that $g(\overline{b_1}), \dots, g(\overline{b_m})$ are K-linearly dependent modulo JA; such an integer m exists because $\operatorname{length}(N_n/JN_n) = n > \operatorname{length}(A)$. Say $\sum_{i=1}^m k_i g(\overline{b_i}) \in JA$ with $k_i \in K$, not all zero. Clearly, $k_m \ne 0$. Moreover, $k_1 = 0$, since $0 = f(\sum_{i=2}^m k_i\overline{b_i}) = \varphi g(\sum_{i=2}^m k_i\overline{b_i}) = -k_1\varphi g(\overline{b_1}) = -k_1(e_1 + Je_1)$. In particular, this shows that $m \ge 2$. Now set $x = \sum_{i=2}^m k_i\bar{b}_{i-1}$. We will check that $g(x) = \sum_{i=1}^{m-1} k_{i+1}g(\overline{b_i})$ again belongs to JA, a contradiction to the minimal choice of m. Indeed, $pg(x) = g(\sum_{i=2}^m k_i p\bar{b}_{i-1}) = g(\sum_{i=2}^m k_i q\overline{b_i}) = qg(\sum_{i=2}^m k_i\overline{b_i}) \in qJA$, which by condition (ii) entails that $g(x)$ is not a top element of A. This shows $g(x) \in JA$ and thus completes the argument for Case I.

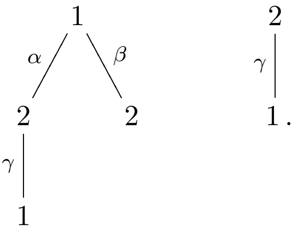
Case II. Now suppose that conditions (i) and (ii') hold. Again assume that S_1 has a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation $\varphi : A \to S_1$, and for $n > \text{length}(A)$, define $f : N_n \to S_1$ as in case I. In turn choose $g \in \text{Hom}_\Lambda(N_n, A)$ such that $f = \varphi g$. But this time, pick $m \in \mathbb{N}$ minimal with the property that $\sum_{i=1}^m k_i g(\overline{b_i}) \in JA$ and $q(\sum_{i=1}^m k_i g(\overline{b_i})) = 0$, for some scalars $k_i \in K$ which are not all zero. Such an m exists because $\ker(g)$ intersects the K-subspace of N_n generated by $\overline{b}_1, \dots, \overline{b}_n$ non-trivially. As before we obtain $k_1 = 0$ and $m \geq 2$, and again, we set $x = \sum_{i=2}^m k_i \overline{b_{i-1}}$ and compute $pg(x) = q(\sum_{i=2}^m k_i g(\overline{b_i})) = 0$. Now condition (ii') guarantees that $g(x) = \sum_{i=1}^{m-1} k_{i+1} g(\overline{b_i})$ is not a top element of A and that $qg(x) = q(\sum_{i=1}^{m-1} k_{i+1} g(\overline{b_i})) = 0$. This is, once more, incompatible with the minimal choice of m.

Example 2. [**10**] Let $\Lambda = K\Gamma/I$ be the monomial relation algebra with quiver

Γ :

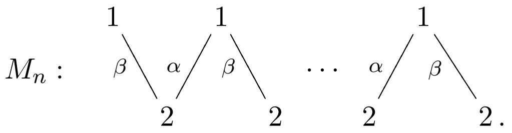


and ideal $I = \langle \alpha\gamma, \beta\gamma, \gamma\beta \rangle$ of relations. Then the indecomposable projective left Λ-modules have the following graphs:

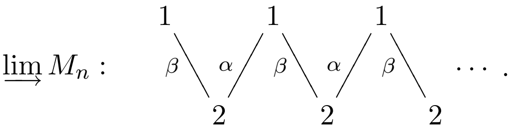


Set $p = \beta$ and $q = \alpha$, and verify the simplified versions of conditions (i) and (ii), as spelled out before the proof of Criterion 1: Clearly, $\alpha J = J\beta = 0$ and $\operatorname{p\,dim} \Lambda\alpha = 0 < \infty$, while $\operatorname{p\,dim} \Lambda e_2/Je_2 = \infty$. Thus the criterion guarantees that S_1 does not have a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation.

Note that a typical class of modules defeating attempts to find a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S_1 in Example 2 is the following class $(M_n)_{n\in\mathbb{N}}$ of strings of composition length $2n$, uniquely determined by their graphs:



(Observe that this family of modules M_n represents just a minor simplification of the family of test modules N_n exhibited in the proof of Criterion 1 for the choices $p = \beta$ and $q = \alpha$; namely $M_n \cong N_n/\Lambda\alpha\overline{b_1}$.) A stumbling block for contravariant finiteness can be more succinctly communicated via the infinite dimensional direct limit of the strings M_n:



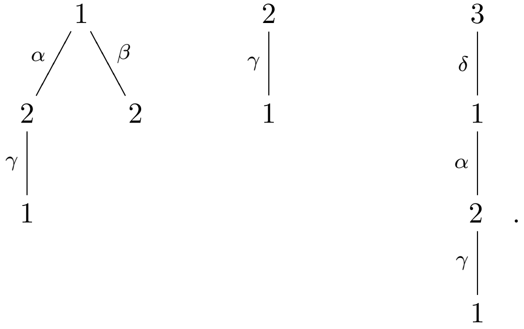
At the same time, this limit is a 'minimal' module $M \in \mathcal{P}^\infty(\Lambda\text{-Mod})$ with the property that all homomorphisms $M_n \to S_1$, $n \in \mathbb{N}$, factor through M.

By the preceding example one could be led to believe that – under the assumption that $\Lambda\beta$ is a direct summand of Je_1 having infinite projective dimension – the existence of a family of modules $(M_n)_{n\in\mathbb{N}}$ as above should imply non-existence of a $\mathcal{P}^\infty$-approximation of the simple module associated with the vertex '1'. This is not the case, however. In fact, the existence of modules $(M_n)_{n\in\mathbb{N}}$ of the indicated shape inside $\mathcal{P}^\infty(\Lambda\text{-mod})$ is only potentially troublesome. Continuing to denote the algebra of Example 2 by Λ, we next give an example of an algebra Δ with $\Lambda\text{-mod} \subseteq \Delta\text{-mod}$ such that $\Delta e_i = \Lambda e_i$ for $i = 1, 2$ and $(\Lambda\text{-mod}) \cap \mathcal{P}^\infty(\Delta\text{-mod}) = \mathcal{P}^\infty(\Lambda\text{-mod})$; in particular, all of the Λ-modules M_n belong also to $\mathcal{P}^\infty(\Delta\text{-mod})$. On the other hand, we will see that $\mathcal{P}^\infty(\Delta\text{-mod})$ *is* contravariantly finite in Δ-mod in this example.

Example 3. Let $\Delta = K\Gamma'/I'$, where Γ' is

$$\begin{tikzcd}
3 \arrow[r, "\delta"] & 1 \arrow[r, bend left=50, "\alpha"] \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\gamma"]
\end{tikzcd}$$

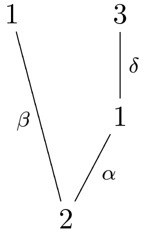
and the ideal I' is generated by monomial relations such that the indecomposable projective left Δ-modules have graphs



This time the 'zipper effect' of the previous example (where the requirement that the homomorphisms $f \in \operatorname{Hom}_\Lambda(M_n, S)$ be factorizable through a module $M \in \mathcal{P}^\infty(\Lambda\text{-mod})$ forces the K-dimension of M to grow with increasing n) can be stopped. This is due to the new projective module Λe_3.

We will prove the contravariant finiteness of $\mathcal{P}^\infty(\Delta\text{-mod})$ by explicitly describing minimal $\mathcal{P}^\infty(\Delta\text{-mod})$-approximations of the simple left Δ-modules. We claim that the following canonical epimorphism is a right $\mathcal{P}^\infty(\Delta\text{-mod})$-approximation of S_1:

$\varphi_1 : \quad A_1 \quad =$



$\longrightarrow \quad S_1$.

Note that $A_1 = \big(\Delta e_1 \oplus \Delta e_3\big)/\big(\Delta\alpha e_1 + \Delta(\beta e_1 - \alpha\delta e_3)\big)$ is the injective envelope of the simple module $S_2 = \Delta e_2/Je_2$. We again use J to denote the Jacobson radical of Δ, and if x_1 and x_3 stand for the residue classes of e_1 and e_3 in A_1, respectively, we let $\varphi_1(x_1) = e_1 + Je_1$ and $\varphi_1(x_3) = 0$.

Note first that $\Omega^1(A_1) = (\Delta e_2)^2$, whence $\operatorname{p\,dim} A_1 = 1$. Now let $f : M \to S_1$ be an epimorphism with $M \in \mathcal{P}^\infty(\Delta\text{-mod})$, $m_1 \in M$ a top element of type e_1 such that $f(m_1) = e_1 + Je_1$, and let $m_2, \dots, m_r \in \ker(f)$ be such that $m_1 + JM, \dots, m_r + JM$ form a K-basis for $e_1(M/JM)$. Then each βm_i is a nonzero element in the socle of M, since otherwise $\Omega^1(M)$ would contain a direct summand isomorphic to S_2, which is impossible in view of the fact that $\operatorname{p\,dim} S_2 = \infty$, while $\operatorname{p\,dim} M < \infty$. More strongly, this argument shows that $\beta m_1, \dots, \beta m_r$ are K-linearly independent, whence in particular $\beta\overline{m}_1$ remains nonzero in $\overline{M} = M/\left(\sum_{i=2}^{2} \Delta\beta m_i\right)$. It is enough to factor the map $\overline{f} : \overline{M} \to S_1$ induced by f through φ_1. In doing this, it is clearly harmless to assume that $\alpha\overline{m}_1 = k\beta\overline{m}_1$ for some scalar $k \in K$ which may be zero; if this is not a priori the case, we factor out the submodule $\Delta(\alpha - \beta)\overline{m}_1$ in addition. Furthermore, we may assume that the elements $m_2, \dots, m_r \in \ker(f)$ are chosen in such a way that there exists an integer s between 2 and r with the property that $\beta\overline{m}_1 = \alpha\overline{m}_i$ for $2 \le i \le s$ and $\Delta\beta\overline{m}_1 \cap \left(\sum_{i=s+1}^{r} \Delta\alpha\overline{m}_i\right) = 0$.

Set $B = \sum_{i=1}^{r} \Delta\overline{m}_i$, and let $\iota : B \to \overline{M}$ be the canonical embedding. In view of the preceding adjustments, we can define a map $\sigma \in \mathrm{Hom}_\Delta(B, A_1)$ by setting $\sigma(\overline{m}_1) = x_1 + k\delta x_3$, $\sigma(\overline{m}_i) = \delta x_3$ for $2 \le i \le s$, and $\sigma(\overline{m}_i) = 0$ for $s + 1 \le i \le r$. Since A_1 is injective, σ can be extended to a homomorphism $\tau \in \mathrm{Hom}_\Delta(\overline{M}, A_1)$ which makes the lower triangle in the diagram below commute.

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] & S_1 \\
B \arrow[u, "\sigma"] \arrow[r, "\iota"] & \overline{M} \arrow[u, "\overline{f}"'] \arrow[ul, dashed, "\tau"']
\end{tikzcd} .$$

Our construction entails that the upper triangle then commutes as well, which shows that φ_1 is indeed a (minimal) $\mathcal{P}^\infty(\Delta\text{-mod})$-approximation of S_1.

It is less involved to see that the canonical maps

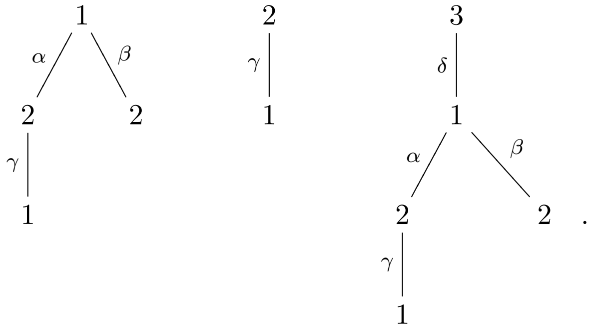
$$\varphi_2 : \begin{array}{c} 2 \\ \gamma \big| \\ 1 \end{array} \longrightarrow S_2 \qquad \text{and} \qquad \varphi_3 : \begin{array}{c} 3 \\ \delta \big| \\ 1 \end{array} \longrightarrow S_3$$

are right $\mathcal{P}^\infty(\Delta\text{-mod})$-approximations of S_2 and S_3, respectively. By [**2**], this shows that $\mathcal{P}^\infty(\Delta\text{-mod})$ is contravariantly finite in Δ-mod.

Example 3 also shows that the hypotheses of Criterion 1 cannot be simplified to the combination of conditions (i) and (iii), where (iii) is as in the remark following the statement of the criterion. Indeed, if in Example 3, we take $p = \beta$ and $q = \alpha$, then both (i) and (iii) are satisfied, but S_1 does admit a right $\mathcal{P}^\infty$-approximation.

Finally, we modify the algebra Δ of Example 3 very slightly to an algebra Ξ, with the effect that $\mathcal{P}^\infty(\Xi\text{-mod})$ again fails to be contravariantly finite. Here condition (ii) fails for any choice of p and q, but (i) and (ii') are satisfied. This sequence of modifications illustrates a phenomenon which will become more obvious in the sequel: Namely, that contravariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ in Λ-mod – as well as failure of this condition – is highly unstable.

Example 4. The quiver of Ξ is that of the algebra Δ in Example 3, but we delete one of the relations, with the effect that the K-dimension of Ξ exceeds that of Δ by 1, and the indecomposable projective left Ξ-modules take on the form



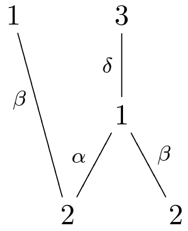
We again apply Criterion 1 with the choice $p = \beta$ and $q = \alpha$. Clearly, $\Lambda\alpha \cap \Lambda\beta = 0$. Moreover, $\operatorname{p\,dim} \Xi(\beta, \alpha) = \operatorname{p\,dim} \begin{pmatrix} 2 \\ |^{\gamma} \\ 1 \end{pmatrix} = 0$, whence condition (i) of the criterion is satisfied.

Next, we check that condition (ii') of our criterion is satisfied. Let M be in $\mathcal{P}^{\infty}(\Xi\text{-mod})$. As in Example 3, it is readily checked that, given any top element $x \in M$ of type e_1, we have $\beta x \neq 0$. So, in proving that for an *arbitrary* element $x \in M$, the vanishing of βx implies the vanishing of αx, we may assume that $x = e_1 x \in JM$ with $\beta x = 0$; since $\beta J = K\beta\delta$ and $\alpha J = K\alpha\delta$, we may moreover assume that $x \in \delta M$. Suppose that $\alpha x \neq 0$. In view of the equality $\alpha\delta J = 0$, this implies that $x \in \delta M \backslash \delta JM$, i.e., $x = \delta y$ for some top element $y \in M$ of type e_3. Let $\pi : P = \Xi e_3 \oplus Q \to M$ be a projective cover with $\pi(e_3) = y$. Then $\beta\delta e_3$ is a nonzero element of $\ker \pi = \Omega^1(M)$, and since $\Xi\beta\delta e_3 \simeq S_2$ has infinite projective dimension and is thus not a direct summand of $\Omega^1(M)$, we see that $\beta\delta e_3 \in J\Omega^1(M) \cap \beta JP = \beta\Omega^1(M)$; the last equality follows from [**7**, Lemma 1]. Thus $\beta\delta e_3 = \beta z$, where $z = e_1 z \in e_1\Omega^1(M)e_3 \subseteq e_1 JPe_3$. Since, clearly, the desired implication '$\beta u = 0 \implies \alpha u = 0$' does hold for arbitrary elements u of a projective left Ξ-module, we deduce that $\alpha\delta e_3 = \alpha z$ and conclude that $\alpha x = \alpha\pi(z) = 0$ as required.

Finally, we note that condition (ii) fails in this example. Indeed, if it would hold, it would be true for $p = \beta$ and $q = \alpha$. However,the left Ξ-module

$$M = \big(\Xi e_1 \oplus \Xi e_3\big) / \big(\Xi\alpha e_1 + \Xi(\beta e_1 - \alpha\delta e_3)\big)$$

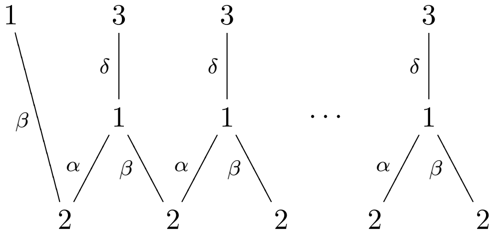
with graph



has syzygy $\Omega^1(M) = \begin{smallmatrix} 2 \\ \mid\gamma \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ \mid\gamma \\ 1 \end{smallmatrix}$, and thus $\operatorname{p\,dim} M = 1$, but if x_i is the residue class of e_i in M for $i = 1, 3$, then x_1 is a top element of type e_1 with $\beta x_1 = \alpha\delta x_3 \in \alpha JM$.

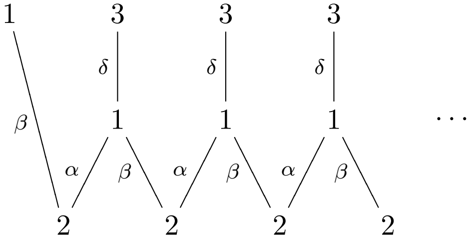
As in Example 2, we can again – in the preceding example – pin down classes of modules of finite projective dimension which are responsible for failure of contravariant finiteness of $\mathcal{P}^\infty(\Xi\text{-mod})$. For instance, there is no homomorphism $\varphi : A \to S_1$ with $A \in \mathcal{P}^\infty(\Xi\text{-mod})$ such that *all* the canonical epimorphisms from the modules

E_n :



onto S_1 can be factored through φ. Observe, however, that they can all be factored through the canonical surjection from

$E = \varinjlim E_n$:



onto S_1.

3. Relative approximations and phantoms.

If $\mathfrak{A} \subset \Lambda$-mod is a resolving contravariantly finite subcategory of Λ-mod, then, according to [2], the minimal right $\mathfrak{A}$-approximations of the simple left Λ-modules hold a substantial amount of information on arbitrary objects of

$\mathfrak{A}$. The gap in the available information when $\mathfrak{A}$ is not contravariantly finite is to be filled by direct limits of 'partial minimal approximations' as indicated informally in Section 2. (We follow the convention that 'direct limits' are colimits extending over directed index sets.)

Definitions 5. Let $\mathfrak{C} \subset \mathfrak{A}$ be full subcategories of Λ-mod such that $\mathfrak{A}$ is closed under finite direct sums, and let $\overrightarrow{\mathfrak{A}}$ be the closure of $\mathfrak{A}$ under direct limits in Λ-Mod. Moreover, let X be a finitely generated left Λ-module.

(1) A (*right*) $\mathfrak{C}$-*approximation of* X *inside* $\mathfrak{A}$ (resp. *inside* $\overrightarrow{\mathfrak{A}}$) is a homomorphism $f : A \to X$ with A in $\mathfrak{A}$ (resp. A in $\overrightarrow{\mathfrak{A}}$) such that

$$\operatorname{Hom}(-,A)|_{\mathfrak{C}} \xrightarrow{\operatorname{Hom}(-,f)} \operatorname{Hom}(-,X)|_{\mathfrak{C}} \longrightarrow 0$$

is an exact sequence of functors from $\mathfrak{C}$ to the category of abelian groups, i.e., with the property that each map $g \in \operatorname{Hom}_\Lambda(C,X)$ with C in $\mathfrak{C}$ factors through f.

If $\mathfrak{C} = \mathfrak{A}$, a $\mathfrak{C}$-approximation of X inside $\mathfrak{A}$ will simply be called a (right) $\mathfrak{A}$-*approximation of* X, in accordance with the existing terminology.

(2) A $\mathfrak{C}$-*phantom of* X *relative to* $\mathfrak{A}$ is an object H in Λ-Mod (not necessarily in $\overrightarrow{\mathfrak{A}}$) with the following property: For every finitely generated submodule $H' \subseteq H$, there exists a finite non-empty set $\mathfrak{C}' \subset \mathfrak{C}$ such that H' is isomorphic to a subfactor of <u>each</u> $\mathfrak{C}'$-approximation of X inside $\mathfrak{A}$.

When $\mathfrak{C} = \mathfrak{A}$, we will more briefly refer to $\mathfrak{A}$-*phantoms* of X.

(3) A $\mathfrak{C}$-phantom H of X relative to $\mathfrak{A}$ is called *effective* if there exists a homomorphism $f : H \to X$ which is a $\mathfrak{C}$-approximation of X inside $\overrightarrow{\mathfrak{A}}$ (in particular, $H \in \overrightarrow{\mathfrak{A}}$ in that case).

The objects that will draw our particular attention in the sequel will be the $\mathfrak{A}$-phantoms of X. Start by noting that, given any subclass $\mathfrak{C} \subseteq \mathfrak{A}$, every $\mathfrak{C}$-phantom of X relative to $\mathfrak{A}$ is actually an $\mathfrak{A}$-phantom of X. Moreover, observe that the class of all $\mathfrak{A}$-phantoms of X is closed under subfactors and hence also under direct limits. In fact, it is readily verified that a left Λ-module H is an $\mathfrak{A}$-phantom of X if and only if H is the direct limit of a direct system of finitely generated $\mathfrak{A}$-phantoms of X. In case X fails to have an $\mathfrak{A}$-approximation, there will be a plethora of $\mathfrak{A}$-phantoms as we will shortly see. This will facilitate the construction of infinite dimensional phantoms, which in turn lie at the heart of our criteria for failure of contravariant finiteness of $\mathfrak{A}$.

The *effective* phantoms, on the other hand, are in a sense the best possible substitutes for minimal approximations in the sense of Auslander/Smalø [**3**] and Auslander/Reiten [**2**]. For example, if $X = \Lambda e/Je$ is simple, the effective $\mathfrak{A}$-phantoms of X compress information about the relations of those modules in $\mathfrak{A}$ which have a top element of type e into the tightest possible format. Compared with classical approximations, we simply renounce the requirement that this picture should fit into a finitely generated module.

Basic Observations 6. Let $\mathfrak{C}, \mathfrak{A}, \overrightarrow{\mathfrak{A}}$ and X be as in Definition 5.

(1) Since $\overrightarrow{\mathfrak{A}}$ is closed under arbitrary direct sums, it is clear that X always has $\mathfrak{C}$-approximations inside $\overrightarrow{\mathfrak{A}}$. Just add up a sufficient number of copies of each object C in $\mathfrak{C}$ to cover all homomorphisms $C \to X$. Similarly, X has a $\mathfrak{C}$-approximation inside $\mathfrak{A}$ whenever $\mathfrak{C} \subseteq \mathfrak{A}$ is a finite subclass, because $\mathrm{Hom}_\Lambda(C, X)$ has finite K-dimension for each $C \in \mathfrak{C}$.

(2) If X has phantoms relative to $\mathfrak{A}$ which have unbounded lengths, then X fails to have an $\mathfrak{A}$-approximation. In particular, this is true in case X has a non-finitely generated $\mathfrak{A}$-phantom.

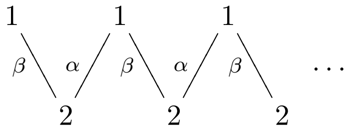
(3) If $\mathfrak{C} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$, each $\mathfrak{C}$-phantom relative to $\mathfrak{A}$ is also a $\mathfrak{D}$-phantom relative to $\mathfrak{A}$. Note the stringency of our definition in this connection: a finitely generated module H is a $\mathfrak{C}$-phantom of X relative to $\mathfrak{A}$ if and only if it is a $\mathfrak{C}'$-phantom for some *finite* subclass $\mathfrak{C}'$ of $\mathfrak{C}$.

(4) Suppose that, in addition to being closed under finite direct sums, $\mathfrak{A}$ is closed under direct summands. Then the existence of an $\mathfrak{A}$-approximation of X implies the existence of a *unique* minimal such approximation by [**2**], say $A(X)$. In that case, $A(X)$ is the only effective $\mathfrak{A}$-phantom of X, and all other $\mathfrak{A}$-phantoms of X are subfactors of $A(X)$.

7. The Examples of Section 2 Revisited. Let $X = S_1 = \Lambda e_1 / J e_1$, and $\mathfrak{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$.

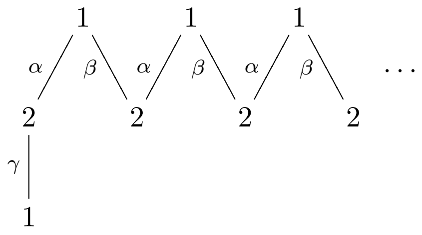
In Example 2, the module

$M = \varinjlim M_n$:



is an effective $\mathfrak{C}$-phantom of S_1 inside $\mathcal{P}^\infty(\Lambda\text{-Mod})$, where $\mathfrak{C} = \{M_n \mid n \in \mathbb{N}\}$, but

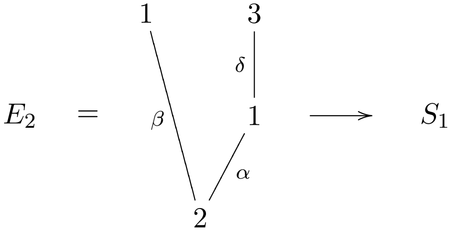
N :



is neither the source of a $\mathfrak{C}$-approximation of S_1 inside $\mathcal{P}^\infty(\Lambda\text{-Mod})$, nor a $\mathfrak{C}$-phantom of S_1 relative to $\mathcal{P}^\infty(\Lambda\text{-mod})$.

As for Example 3: Start by observing that the above graphs uniquely define left modules over the modified algebra Δ, again denoted M_n, M and N, and the class of Δ-modules $\mathfrak{C} = \{M_n \mid n \in \mathbb{N}\}$ in turn belongs to $\mathcal{P}^\infty(\Delta\text{-mod})$. Moreover, the homomorphism $f : M \to S_1 = \Delta e_1/Je_1$ which sends the top element represented by the left-most '1' in the graph of M to $e_1 + Je_1$ and sends the top elements displayed farther to the right to zero is still a $\mathfrak{C}$-approximation of S_1 inside $\mathcal{P}^\infty(\Delta\text{-mod})$. However, in the present setup, both M and N fail to be $\mathfrak{C}$-phantoms of S_1 relative to $\mathcal{P}^\infty(\Delta\text{-mod})$; indeed, as we saw earlier, S_1 has a $\mathcal{P}^\infty(\Delta\text{-mod})$-approximation in that example.

Finally, let us focus on Example 4. Viewing M_n and M as left Ξ-modules, and keeping in mind that the M_n belong to $\mathcal{P}^\infty(\Xi\text{-mod})$, we find that M is an effective $\mathfrak{C}$-phantom of S_1 relative to $\mathcal{P}^\infty(\Xi\text{-mod})$, as in Example 2. Moreover, if $\mathfrak{E} = \{E_n \mid n \in \mathbb{N}\}$ with E_n as defined after Example 4, then M is also an $\mathfrak{E}$-phantom of S_1 relative to $\mathcal{P}^\infty(\Xi\text{-mod})$, but not an effective one because the canonical epimorphism



does not factor through M. The better $\mathfrak{E}$-phantom here is $E = \varinjlim E_n$, which is actually an effective $(\mathfrak{C} \cup \mathfrak{E})$-phantom of S_1 relative to $\mathcal{P}^\infty(\Xi\text{-mod})$.

Next we prepare for a general existence result. In a nutshell: Whenever $\mathfrak{A} \subseteq \Lambda\text{-mod}$ fails to be contravariantly finite, there exist $\mathfrak{A}$-phantoms of infinite K-dimension.

Proposition 8. *Suppose that* $\mathfrak{A} \subseteq \Lambda\text{-mod}$ *is closed under finite direct sums and that* $X \in \Lambda\text{-mod}$ *does not have a* (*right*) $\mathfrak{A}$-*approximation. Then there exists a countable subclass* $\mathfrak{C}$ *of* $\mathfrak{A}$ *such that* X *fails to have a* $\mathfrak{C}$-*approximation inside* $\mathfrak{A}$.

Proof. By repeatedly applying the first of the observations under 6, we show that, for each $d \geq 1$, there exists a finite subset $\mathfrak{C}_d$ of $\mathfrak{A}$ such that X does not have a $\mathfrak{C}_d$-approximation of K-dimension $\leq d$ inside $\mathfrak{A}$.

Assuming the contrary for some $d \geq 1$, we pick a module Y_1 in $\mathfrak{A}$ with $\mathrm{Hom}_\Lambda(Y_1, X) \neq 0$ – such a module Y_1 exists by hypothesis – and let $f_1 : A_1 \to X$ be a $\{Y_1\}$-approximation of X inside $\mathfrak{A}$ such that $\dim_K A_1 \leq d$. In particular, f_1 is nonzero. Since X fails to have an $\mathfrak{A}$-approximation, there exists an object Y_2 in $\mathfrak{A}$ such that some homomorphism in $\mathrm{Hom}_\Lambda(Y_2, X)$

fails to factor through f_1. Let $f_2 : A_2 \to X$ be an $\{A_1, Y_2\}$-approximation of X inside $\mathfrak{A}$; by assumption, A_2 can be chosen to have K-dimension at most d. Inductively, our assumption thus yields a family $(A_n)_{n\geq 1}$ of objects of $\mathfrak{A}$ with $\dim_K A_n \leq d$ for all n, together with finitely generated left Λ-modules Y_n and homomorphisms $f_n : A_n \to X$ such that f_n is an $\{A_{n-1}, Y_n\}$-approximation of X inside $\mathfrak{A}$, but fails to be a $\{Y_{n+1}\}$-approximation. Accordingly, we can pick $g_n \in \mathrm{Hom}_\Lambda(A_n, A_{n+1})$ such that $f_n = f_{n+1} g_n$ and none of the g_n is an isomorphism. But since $f_1 = f_2 g_1 = f_3 g_2 g_1 = \cdots = f_{n+1} g_n \cdots g_1$ is nonzero, we deduce $g_n \cdots g_1 \neq 0$ for all n, which contradicts the Harada-Sai Lemma [6] and proves our assumption to be absurd.

Letting $\mathfrak{C}_d$ for $d \geq 1$ be as in our initial claim, the countable subset $\mathfrak{C} = \bigcup_{d\geq 1} \mathfrak{C}_d$ of $\mathfrak{A}$ is clearly as desired. □

We apply this proposition to obtain the announced existence result.

Theorem 9. *Suppose that* $\mathfrak{A} \subseteq \Lambda$-mod *is closed under finite direct sums, and let* $X \in \Lambda$-mod. *Then the following conditions are equivalent:*

(1) *X fails to have an $\mathfrak{A}$-approximation.*

(2) *There exists a countable subclass $\mathfrak{C} \subseteq \mathfrak{A}$ such that X has an effective $\mathfrak{C}$-phantom of countably infinite K-dimension relative to $\mathfrak{A}$.*

(3) *X has an $\mathfrak{A}$-phantom of infinite K-dimension.*

Proof. '(1) $\Longrightarrow$ (2)'. Assume that (1) holds. Then Proposition 8 yields a countable subclass $\mathfrak{D} = \{D_1, D_2, D_3, \dots\}$ of $\mathfrak{A}$ such that X does not have a $\mathfrak{D}$-approximation inside $\mathfrak{A}$. However, by the first of the observations under 6, there exists a $\{D_1\}$-approximation of X inside $\mathfrak{A}$, say $f_1 : A_1 \to X$.

Next we pick an $\{A_1\}$-approximation $f_2 : A_2 \to X$ of X inside $\mathfrak{A}$, together with a map $g_{1,2} \in \mathrm{Hom}_\Lambda(A_1, A_2)$ satisfying $f_1 = f_2 \circ g_{1,2}$, such that $\dim_K\big(g_{1,2}(A_1)\big)$ is as small as possible. Consequently, the following is true: Whenever $f_2' : A_2' \to X$ is an $\{A_2\}$-approximation of X inside $\mathfrak{A}$ and $g' \in \mathrm{Hom}_\Lambda(A_2, A_2')$ is such that $f_2 = f_2' \circ g'$, we have $g'(g_{1,2}(A_1)) \simeq g_{1,2}(A_1)$.

We now choose any $\{D_2, A_2\}$-approximation $f_3 : A_3 \to X$ of X inside $\mathfrak{A}$, and subsequently an $\{A_3\}$-approximation $f_4 : A_4 \to X$ inside $\mathfrak{A}$, together with a map $g_{3,4} \in \mathrm{Hom}_\Lambda(A_3, A_4)$ such that $f_3 = f_4 \circ g_{3,4}$ and $\dim_K\big(g_{3,4}(A_3)\big)$ is minimal.

Continuing along this line, we obtain a sequence of objects $(A_n)_{n\geq 1}$ and maps $f_n : A_n \to X$ such that, for $n \geq 2$, f_{2n-1} is a $\{D_n, A_{2n-2}\}$-approximation of X inside $\mathfrak{A}$ and f_{2n} is an $\{A_{2n-1}\}$-approximation which is coupled with a map $g_{2n-1,2n} \in \mathrm{Hom}_\Lambda(A_{2n-1}, A_{2n})$ such that $f_{2n-1} = f_{2n} \circ g_{2n-1,2n}$ and $\dim_K\big(g_{2n-1,2n}(A_{2n-1})\big)$ is minimal.

Set $\mathfrak{C} = \{A_1, A_2, A_3, \dots\}$ and supplement the above maps $g_{n,n+1}$ for odd n by homomorphisms $g_{n,n+1} \in \mathrm{Hom}_\Lambda(A_n, A_{n+1})$ with $f_n = f_{n+1} \circ g_{n,n+1}$ for n even. If, for $n < m$, we moreover define $g_{n,m} = g_{m-1,m} \circ \cdots \circ g_{n,n+1} : A_n \to A_m$, then $(A_n, g_{n,m})_{n,m\in\mathbb{N}, n<m}$ is an inductive system with $f_n = f_m \circ g_{n,m}$.

Set

$$A = \varinjlim A_n, \qquad\qquad f = \varinjlim f_n \in \mathrm{Hom}_\Lambda(A, X),$$

and let $h_n : A_n \to A$ be the canonical maps. Clearly, A belongs to $\overrightarrow{\mathfrak{A}}$ (see Definition 5). Moreover, each homomorphism in $\mathrm{Hom}_\Lambda(C, X)$ with $C \in \mathfrak{C}$ factors through f and, a fortiori, so does each homomorphism in $\mathrm{Hom}_\Lambda(D_n, X)$. In other words, $f : A \to X$ is a $\mathfrak{C} \cup \mathfrak{D}$-approximation of X inside $\overrightarrow{\mathfrak{A}}$.

Next we want to identify A as a $\mathfrak{C}$-phantom of X relative to $\mathfrak{A}$. Our construction entails that, for $m > 2n$, we have $g_{2n,m} \circ g_{2n-1,2n}(A_{2n-1}) \simeq g_{2n-1,2n}(A_{2n-1})$, and consequently we have $h_{2n}(U_{2n}) \simeq U_{2n}$ if we define $U_{2n} = g_{2n-1,2n}(A_{2n-1})$. Since A is the directed union of the submodules $h_{2n}(U_{2n})$, $n \in \mathbb{N}$, it suffices to show that each of the finitely generated modules U_{2n} is a $\mathfrak{C}$-phantom of X relative to $\mathfrak{A}$. For that purpose, consider the finite subset $\mathfrak{C}(U_{2n}) = \{A_{2n}\}$ of $\mathfrak{C}$, and let $f' : A' \to X$ be a $\mathfrak{C}(U_{2n})$-approximation of X inside $\mathfrak{A}$. If $g' \in \mathrm{Hom}_\Lambda(A_{2n}, A')$ is such that $f_{2n} = f' \circ g'$, our construction yields $U_{2n} \simeq g'(U_{2n}) \subseteq A'$, which shows that U_{2n} is indeed a $\mathfrak{C}$-phantom of X relative to $\mathfrak{A}$, and hence so is A. By the preceding paragraph, A is, in fact, even an effective $\mathfrak{C}$-phantom of X relative to $\mathfrak{A}$.

Finally, we note that $\dim_K A \le \aleph_0$ by construction. To prove the reverse inequality, we assume, to the contrary, that $\dim_k A < \infty$. But this means that f is a $\mathfrak{D}$-approximation of X inside $\mathfrak{A}$, which contradicts our choice of $\mathfrak{D}$ and completes the proof of '(1) $\Longrightarrow$ (2)'.

The implication '(2) $\Longrightarrow$ (3)' follows from the fact that each $\mathfrak{C}$-phantom relative to $\mathfrak{A}$, where $\mathfrak{C} \subseteq \mathfrak{A}$, is also a $\mathfrak{A}$-phantom, and '(3) $\Longrightarrow$ (1)' is a consequence of the basic observation 6(2). ☐

The following is an upgraded version of the elementary Criterion 1 for nonexistence of a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of a given simple module S. The idea underlying the proof is the same, even though we impose no restrictions on the subcategory $\mathfrak{A} \subseteq \Lambda$-mod this time. In particular, this criterion again points to a countable subclass $\mathfrak{C}$ of $\mathfrak{A}$ which obstructs the approximability of S by a *finitely generated* module of finite projective dimension. In view of the proof of Theorem 9, it can hence be used towards the explicit construction of $\mathfrak{A}$-phantoms of S. While this criterion will be instrumental in resolving the problem of contravariant finiteness in our key example (Section 4), for complex non-monomial algebras, it may still be nontrivial to verify or refute Condition (2) below. We therefore add an illlustration of how the underlying idea can still be used towards deciding questions of contravariant finiteness, even when the criterion is not readily applicable verbatim.

Criterion 10. *Suppose that Λ is a split finite dimensional algebra and $\mathfrak{A}$ a full subcategory of* Λ-mod*. Moreover, let $e_1, \dots, e_m$ be pairwise orthogonal* *primitive idempotents of* Λ, *and* $p_1, \dots, p_m, q_1, \dots, q_m \in J$ *with* $p_i = p_i e_i$ *and* $q_i = q_i e_i$ *such that the following conditions are satisfied:*

(1) *For each* $n \in \mathbb{N}$, *there is a module* $M_n \in \mathfrak{A}$, *together with a sequence* $x_{n1}, \dots, x_{n,mn}$ *of* mn *top elements of* M_n *which are* K*-linearly independent modulo* JM_n *such that* $0 \neq p_{r(i)} x_{ni} = q_{r(i+1)} x_{n,i+1}$ *for* $1 \leq i < mn$, *where* $r(i) \in \{1, \dots, m\}$ *is congruent to* i *modulo* m.

(2) *For any object* C *in* $\mathfrak{A}$, *the following are true:*

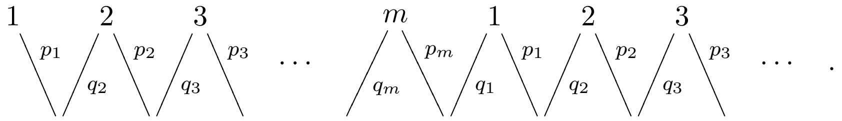
- (i) *if* $x \in C$ *is a top element of type* e_1 *then* $p_1 x \neq 0$;
- (ii) *if* $y, z \in C$ *with* $0 \neq p_{r(i)} y = q_{r(i+1)} z$, *then* $p_{r(i+1)} z \neq 0$.

Then $S_1 = \Lambda e_1 / J e_1$ *does not have an* $\{M_n \mid n \in \mathbb{N}\}$*-approximation inside* $\mathfrak{A}$. *In particular,* $\mathfrak{A}$ *is not contravariantly finite in* Λ-mod *in that case.*

Proof. Assume to the contrary that $f : A \to S_1$ is an $\{M_n \mid n \in \mathbb{N}\}$-approximation of S_1, and choose $n \in \mathbb{N}$ such that $\dim_K A < mn$. Fixing n, we will briefly write x_i for x_{ni}, $1 \leq i \leq mn$. Consider the homomorphism $g : M_n \to S_1$, defined by $g(x_1) = e_1 + Je_1$ and $g(x_i) = 0$ for $2 \leq i \leq mn$; this definition is meaningful because x_1 is a top element of M_n of type e_1. Choose $h : M_n \to A$ such that $g = fh$. Moreover, note that, due to the linear dependence of the elements $h(x_1), \dots, h(x_{mn})$ of A, there exists a natural number t, together with scalars $k_t, \dots, k_n$ such that $k_t \neq 0$ whereas $0 = p_{r(t)} \sum_{i=t}^{mn} k_i h(x_i) = p_{r(t)} z$, where $z = \sum_{i=t}^{mn} h(e_{r(t)} k_i x_i)$.

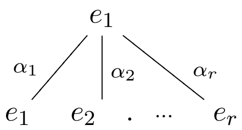
Choose t minimal with the property that the next-to-last equation is satisfied for some nonzero scalar k_t and some scalars $k_{t+1}, \dots, k_{mn}$. We claim that $t > 1$. Indeed, if $t = 1$, then $f(z) = g(\sum e_1 k_i x_i) = k_1 g(x_1) \neq 0$ and hence z is a top element of A of type e_1. By (2)(i) this implies that $p_1 z \neq 0$, a contradiction. Now set $y = \sum_{i=t}^{mn} h(e_{r(t-1)} k_i x_{i-1})$. By the minimal choice of t, we obtain $p_{r(t-1)} y \neq 0$. Using condition (1) and the choice of the $r(i)$, we further compute that $p_{r(t-1)} y = q_{r(t)} z$, and – invoking condition (2)(ii) – we conclude $p_{r(t)} z \neq 0$. But this is incompatible with our choice of t. ☐

As is backed up by the proof of Criterion 10, the hypotheses of this criterion entail the existence of an infinite dimensional phantom of S_1 relative to $\mathfrak{A}$, a graph of which contains a subgraph



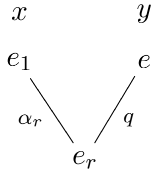
As mentioned earlier, the criterion works well for $\mathfrak{A} = \mathcal{P}^{\infty}(\Lambda\text{-mod})$ when Λ is a monomial relation algebra. One of the reasons for this can be found in the following observation which shows how easy it is to get a chain of $\mathcal{P}^{\infty}(\Lambda\text{-mod})$-phantoms started in the monomial situation.

Remark 11. Let $\Lambda = K\Gamma/I$ be a monomial relation algebra, and suppose that the simple module $S_1 = \Lambda e_1/Je_1$ has infinite projective dimension. If $\alpha_1, \dots, \alpha_r$ are arrows $\alpha_j : e_1 \to e_j$ ending in distinct vertices $e_1, \dots, e_r$, such that $\operatorname{p\,dim} \Lambda\alpha_j = \infty$ for $1 \leq j \leq r$, then

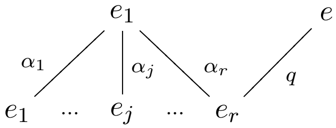


is a subgraph of the graph of a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom C of S_1. More precisely, there exists a top element $c \in C$ of type e_1 such that $\alpha_i c \neq 0$ for $1 \leq i \leq r$.

If, moreover, there exists a module $M \in \mathcal{P}^\infty(\Lambda\text{-mod})$ with a graph containing a subgraph of the form



where $x, y \in M$ are top elements of types e_1 and e respectively and q denotes a path in $K\Gamma\backslash I$, then there exists a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_1 whose graph contains a subgraph of the form



with respect to suitable top elements.

Proof. The second statement clearly follows from the first. To justify the first, we start by noting that $\bigoplus_{1 \leq j \leq r} \Lambda\alpha_j$ is a direct summand of Je_1, due to the fact that Λ is a monomial relation algebra. Let $C \in \mathcal{P}^\infty(\Lambda\text{-mod})$ have a top element c of type e_1. To see that $\alpha_j c \neq 0$ for $1 \leq j \leq r$, consider a projective cover

$$\pi : P = \Lambda x_0 \oplus \bigoplus_{i \in I} \Lambda x_i \to C$$

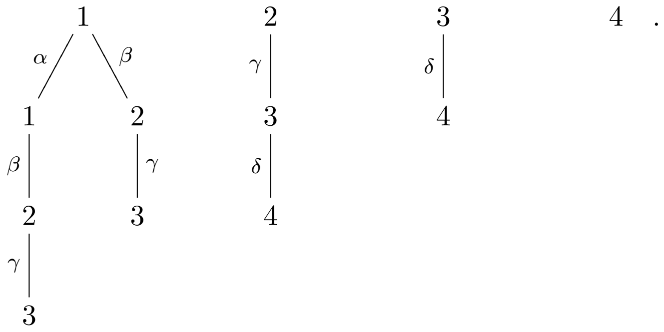
such that $x_0 = e_1$ and $\pi(x_0) = c$. If we had $\alpha_j c = 0$, we would conclude that $\Lambda\alpha_j x_0$ is a direct summand of $\ker \pi = \Omega^1(C) \leq JP$, because $\Lambda\alpha_j x_0$ is a direct summand of JP, which is incompatible with our setup. $\square$

While most of our applications demonstrate the use of phantoms towards a proof that $\mathcal{P}^\infty(\Lambda\text{-mod})$ fails to be contravariantly finite, phantoms may also be helpful in finding $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations.

Example 12 [**2**, Example on p. 137]. Let $\Lambda = K\Gamma/I$ be based on the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[r, "\gamma"] & 3 \arrow[r, "\delta"] & 4 \end{tikzcd}$$

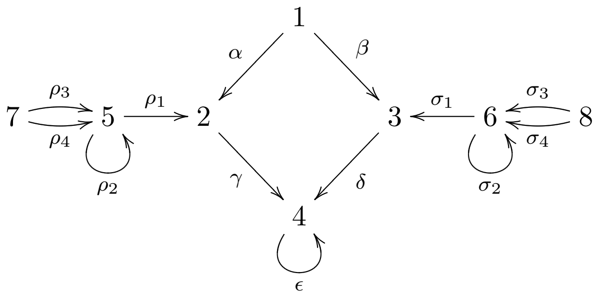
such that the Λe_i have graphs:



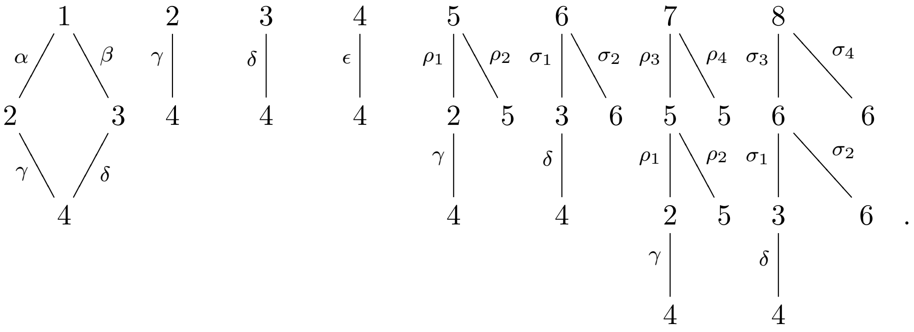
Clearly, S_1 is the only simple left Λ-module of infinite projective dimension. By Remark 11, S_1 has a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom A_1, with graph $\begin{smallmatrix}1\\ \mid\,\alpha\\ 1\end{smallmatrix}$, and it is readily checked that there is no object in $\mathcal{P}^\infty(\Lambda\text{-mod})$ having a submodule with graph $\begin{smallmatrix}1&&e\\ \alpha\diagdown&&\diagup\\ &1&\end{smallmatrix}$. One deduces that $A_1 \to S_1$ is a (minimal) $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation of S_1.

For non-monomial relation algebras, one often needs to slightly vary the idea of Criterion 10. We illustrate the construction of phantoms in such a non-monomial situation.

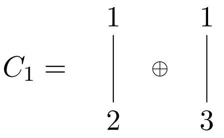
Example 13. Let $\Lambda = K\Gamma/I$, where Γ is the quiver



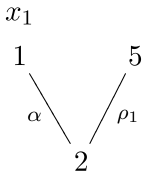
and $I \subset K\Gamma$ is the unique ideal containing $\gamma\alpha - \delta\beta$ and having the property that the indecomposable projective left Λ-modules have the graphs



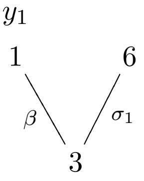
We will see that $S_1 = \Lambda e_1/Je_1$ does not have a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation by constructing $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantoms of infinite K-dimension. We start by observing that for each module M in $\mathcal{P}^\infty(\Lambda\text{-mod})$ with top element m of type e_1, either $\alpha m \neq 0$ or $\beta m \neq 0$. Thus the module



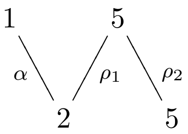
of finite projective dimension is an (effective) add(C_1)-phantom of S_1 inside $\mathcal{P}^\infty(\Lambda\text{-mod})$; a fortiori, C_1 is a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_1. Since there exist modules in $\mathcal{P}^\infty(\Lambda\text{-mod})$ having subgraphs



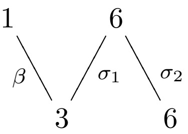
resp.



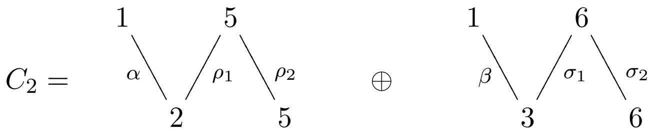
where x_1 and y_1 stand for top elements, namely



resp.



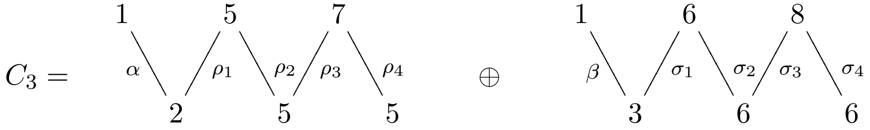
and since each $M \in \mathcal{P}^\infty(\Lambda\text{-mod})$ with a top element m of type e_5 (resp. e_6) satisfies $\rho_2 m \neq 0$ (resp. $\sigma_2 m \neq 0$), the module



is an add(C_2)-phantom of S_1 inside $\mathcal{P}^\infty(\Lambda\text{-mod})$.

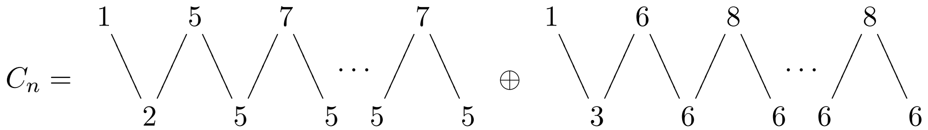
In the next step, we observe that $\mathcal{P}^\infty(\Lambda\text{-mod})$ contains objects with subgraphs

where again x_1 and x_2, resp. y_1 and y_2, denote top elements, and since each module $M \in \mathcal{P}^\infty(\Lambda\text{-mod})$ with top element m of type e_7 (resp. e_8) satisfies $\rho_4 m \neq 0$ (resp. $\sigma_4 m \neq 0$) the module



is an effective add(C_3)-phantom of S_1 inside $\mathcal{P}^\infty(\Lambda\text{-mod})$. A fortiori, C_3 is a $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom inside $\mathcal{P}^\infty(\Lambda\text{-mod})$.

Proceeding in this fashion, we obtain modules $C_n \in \mathcal{P}^\infty(\Lambda\text{-mod})$ of length $4n$, namely



all of which are $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantoms of S_1. This yields the $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom $\varinjlim C_n$ of infinite K-dimension, and shows that S_1 fails to have a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation inside $\mathcal{P}^\infty(\Lambda\text{-mod})$.

Problem 14. Characterize the simple modules over monomial relation algebras which fail to have right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations in terms of their infinite dimensional phantoms.

4. A less elementary example.

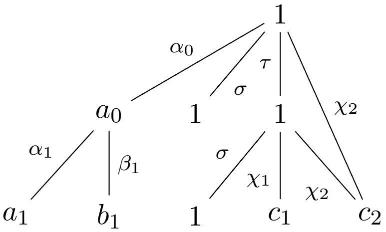
We apply Criterion 10 to a less elementary example which, in fact, motivated a major portion of this article. Namely, for the finite dimensional monomial relation algebra Λ of [8] with $\operatorname{l\,fin\,dim}\Lambda < \operatorname{l\,Fin\,dim}\Lambda$, we exhibit a simple left Λ-module which fails to have a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation.

Example 15. We refer the reader to [8, p. 378] for a definition of $\Lambda = K\Gamma/I$. We will apply Criterion 10 to show that the simple module $S_2 = \Lambda e_2/Je_2$ fails to have a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation. For that purpose, we let $\mathfrak{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$, set $m = 1$, and focus on the single primitive idempotent e_2. Moreover, we make the choices $p = \gamma_1$ and $q = \gamma_2 + \tau\gamma_2$, let $n \in \mathbb{N}$, and set $x_i = e_2$ for $i = 1, \dots, n$.

First we exhibit modules M_n as in part (1) of the criterion. Namely, we define

$$M_n = \left(\bigoplus_{i=1}^{n} \Lambda x_i\right) \Big/ \left(\sum_{i=1}^{n-1} \Lambda z_i\right),$$

where $z_i = px_i - qx_{i+1}$ for $1 \le i \le n-1$. Observe that $M_n \in \mathcal{P}^\infty(\Lambda\text{-mod})$ for each n. Indeed, the sum $\sum_{i=1}^{n-1} \Lambda z_i$ is direct and can be seen to have finite projective dimension as follows: The graph of Λz_i relative to the top element z_i is

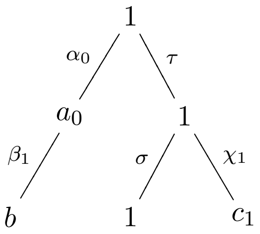


whence the graphical method of [7, Section 5] yields $\Omega^1(\Lambda z_i) = \begin{pmatrix} c_1 \\ \mid \chi_1' \\ c_1 \end{pmatrix} \oplus \begin{pmatrix} c_2 \\ \mid \chi_2' \\ c_2 \end{pmatrix}$. Thus

$$\Omega^1\left(\sum_{i=1}^{n-1} \Lambda z_i\right) \cong \big(\Lambda e(c_1)\big)^{n-1} \oplus \big(\Lambda e(c_2)\big)^{n-1}$$

is projective as required.

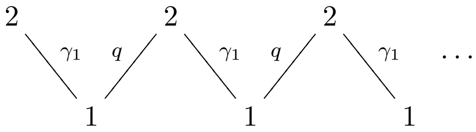
To check condition (2)(i) of Criterion 10, suppose that C belongs to $\mathcal{P}^\infty(\Lambda\text{-mod})$ and has a top element x of type e_2. Then $px = \gamma_1 x \neq 0$, since otherwise $\Omega^1(C)$ would have a direct summand isomorphic to the left ideal $\Lambda\gamma_1$. But this left ideal has infinite projective dimension, as can again be checked with the aid of its graph



and the method of [7]. This shows that condition (2)(i) is indeed met.

Finally, let us check that condition (2)(ii) of Criterion 10 is satisfied. Again let $C \in \mathcal{P}^\infty(\Lambda\text{-mod})$, and suppose that $y, z \in C$ are such that $0 \neq py = qz$, where p and q are as above. From the fact that $qz = \gamma_2 z + \tau\gamma_2 z$ does not vanish, we deduce $\gamma_2 z \neq 0$, which in turn implies that $e_2 z$ is a top element of type e_2 of C; this implication is an immediate consequence of the fact that the vertex e_2 is a source of Γ. Consequently, the preceding paragraph yields $pz \neq 0$ as required. Thus Criterion 10 applies to complete the proof that S_2 does not have a right $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation.

An infinite dimensional $\mathcal{P}^\infty(\Lambda\text{-mod})$-phantom of S_2 resulting from the preceding argument can be visualized as follows:



where again $q = \gamma_2 + \tau\gamma_2$.

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