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M. Beltrametti and A.J. Sommese gave a numerical criterion for a line bundle on a complex projective surface to be d-very ample. In this paper, we prove that their criterion also holds on a projective surface defined over an algebraically closed field of characteristic $p > 0$.

0. Introduction.

Let S be a nonsingular projective surface defined over an algebraically closed field k and L a nef and big line bundle on S. In this paper we study the d-very ampleness on the adjoint linear system $|K_S + L|$ of a polarized surface (S, L).

For a nonnegative integer d, L is said to be d-very ample if, given any 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ of S with length$(\mathcal{O}_Z) = d+1$, the natural restriction map $\Gamma(L) \to \Gamma(L|_Z)$ is surjective. By definition, L is 0-very ample if and only if it is generated by its global sections, and L is 1-very ample if and only if it is very ample.

When char$(k) = 0$, I. Reider's method provided us with a very useful tool in the adjunction theory for surfaces. Using the Reider's method, M. Beltrametti and A.J. Sommese [**BS**] gave a numerical criterion for a line bundle L to be d-very ample. We establish that the analogue of their criterion holds in positive characteristic. Our main result is the following theorem, which is based on N.I. Shepherd-Barron's result ([**S-B**]) on the Bogomolov instability of rank 2 locally free sheaves on surfaces.

Theorem. *Let S be a nonsingular projective surface defined over an algebraically closed field of characteristic $p > 0$. Let L be a nef line bundle on S. Assume that*

$$l := L^2 - 4d - 5 \geq 0$$

and one of the following situations holds:

(1) *S is not of general type and further not quasi-elliptic of Kodaira dimension* 1;

(2) *S is of general type with minimal model S', $p \geq 3$ and*

$$l > K_{S'}^2;$$

(3) *S is of general type with minimal model S', $p = 2$ and*

$$l > \max\{K_{S'}^2, K_{S'}^2 - 3\chi(\mathcal{O}_S) + 2\}.$$

Then either $K_S + L$ is d-very ample or there exists an effective divisor D such that $L - 2D$ is $\mathbb{Q}$-effective, D contains some 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ of S where the d-very ampleness fails and

$$L \cdot D - d - 1 \leq D^2 < \frac{L \cdot D}{2} < d + 1.$$

In Section 1, we give the definitions and some basic properties of d-very ample line bundles on projective varieties. We also recall Shepherd-Barron's result on the instability of rank 2 locally free sheaves on surfaces and Tyurin's result on a construction of locally free sheaves. And we prove the Kodaira vanishing theorem on a surface in positive characteristic. The proof of the theorem is given in Section 2.

1. Preliminaries.

Let V be an irreducible projective variety defined over an algebraically closed field. We denote its structure sheaf by $\mathcal{O}_V$ and its canonical sheaf by K_V. Line bundles are identified with linear equivalence classes of Cartier divisors. For any coherent sheaf F on V, the notation $h^i(F)$ for $i \geq 0$ denotes $\dim H^i(V, F)$, the dimension of the i-th cohomology group of F.

Let L be a line bundle on V. The restriction of L to a curve C is usually denoted by L_C. L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for every curve C on V. In this case, L is said to be *big* if $L^n > 0$, where $n = \dim V$.

We use the following standard notation from algebraic geometry:

$\sim$, the linear equivalence of line bundles;
$\chi(F) = \sum_i (-1)^i h^i(F)$, the Euler characteristic of a coherent sheaf F;
$\Gamma(L)$, the space of the global sections of L;
$|L|$, the complete linear system associated to L;
$V^{[r]}$, the Hilbert scheme of 0-dimensional subschemes $(Z, \mathcal{O}_Z)$ of V with length$(\mathcal{O}_Z) = r$.

An element of $V^{[r]}$ is called a 0-*cycle* of degree r on V. Each 0-cycle Z is defined by the ideal sheaf $I_Z \subset \mathcal{O}_V$. A 0-cycle Z' is called a *subcycle* of Z if $I_Z \subset I_{Z'}$.

The *arithmetic genus* $p_a(D)$ of a projective scheme D of dimension 1 is defined by $p_a(D) = 1 - \chi(\mathcal{O}_D)$. Note that $p_a(D)$ is an integer. If D is an effective divisor on a nonsingular projective surface S, it can be easily seen that $p_a(D)$ satisfies the adjunction formula:

$$2\,p_a(D) - 2 = (K_S + D) \cdot D.$$

If C is a reduced curve, then the arithmetic genus $p_a(\overline{C})$ of the desingularization $\overline{C}$ of C is called the *geometric genus* of C, and denoted by $g(C)$.

The d-very ample line bundles. Let V be an irreducible projective variety defined over an algebraically closed field. Let L be a line bundle on V and W a subspace of $H^0(V, L)$. The subspace W is said to be *d-very ample* for a nonnegative integer d if, given any 0-cycle $(Z, \mathcal{O}_Z)$ of $V^{[d+1]}$, the restriction map $W \to \Gamma(\mathcal{O}_Z(L))$ is surjective.

The line bundle L is said to be *d-very ample* if $H^0(V, L)$ itself is *d-very ample.*

The following lemma is a special case of [**F**], Proposition 1.5.

Lemma 1.1 (**Clifford's theorem for Gorenstein curves**). *Let C be an irreducible and reduced Gorenstein curve and L a line bundle on C such that $h^0(L) > 0$ and $h^1(L) > 0$. Then*

$$\dim |L| \leq \frac{1}{2} \deg L.$$

Proof. If $\deg L = 0$, then $L \cong \mathcal{O}_C$, and we have $\dim |L| = h^0(L) - 1 = 0$.

Suppose that $\deg L > 0$. We now assume that $2 \dim |L| > \deg L$. Put $s := p_a(C) - h^1(L) + 1$. We have $s = \deg L - \dim |L| + 1 > 0$. Choose general points $\{P_1, \dots, P_s\} \in \mathrm{Reg}(C)$ and consider the effective divisor $D = P_1 + \cdots + P_s$, then

$$h^0(\omega_C(D)) = h^0(\omega_C) - s = h^1(L) - 1 \geq 0,$$

where ω_C is the dualizing sheaf of C.

On the other hand, since

$$h^0(L(-D)) \geq h^0(L) - s = 2h^0(L) - 2 - \deg L = 2 \dim |L| - \deg L > 0.$$

We obtain $\mathrm{Hom}(\mathcal{O}(D), L) \cong H^0(L(-D)) \neq 0$. Take a nonzero element $\varphi \in \mathrm{Hom}(\mathcal{O}(D), L)$, and we have a short exact sequence

$$0 \to \mathcal{O}(D) \xrightarrow{\varphi} L \to \mathcal{F} \to 0.$$

Then $h^1(L) \leq h^1(D)$ and

$$p_a(C) + 1 - s = h^1(L) \leq h^1(D) = h^0(\omega_C(-D)) = h^0(\omega_C) - s = p_a(C) - s.$$

This is a contradiction. □

Lemma 1.2. *If L is a d-very ample line bundle on an irreducible and reduced Gorenstein curve C, then $\deg L \geq d$. Moreover if the equality holds, then $C \cong \mathbb{P}^1$ unless $d = 0$ and $L \cong \mathcal{O}_C$.*

Proof. Note that $h^0(L) \geq d+1$ since L is d-very ample. If $h^1(L)=0$, by the Riemann-Roch theorem, we have $d+1 \leq h^0(L) = \deg L + 1 - p_a(C)$. Hence $\deg L \geq d + p_a(C) \geq d$. If $\deg L = d$, then $p_a(C)=0$, and $C \cong \mathbb{P}^1$. If $h^1(L) \neq 0$, by Lemma 1.1, we have $d \leq h^0(L) - 1 \leq \frac{1}{2}\deg L$. Hence $\deg L \geq 2\,d \geq d$. If $\deg L = d$, then $d=0$, thus we have $\deg L = 0$ and $h^0(L)=1$. In this case L is trivial. □

Therefore we have the following.

Proposition 1.3. *Let L be a d-very ample line bundle on a nonsingular projective surface S. Then $L \cdot C \geq d$ for every effective curve C on S. Moreover if the equality holds, then $C \cong \mathbb{P}^1$ unless $d=0$ and $L_C \cong \mathcal{O}_C$.*

The Shepherd-Barron's result on the instability of locally free sheaves. Let S be a nonsingular projective surface defined over an algebraically closed field.

Definition. A rank 2 locally free sheaf E on S is said to be *unstable* (in the sense of Bogomolov) if there exists a short exact sequence

$$0 \to \mathcal{O}_S(A) \to E \to \mathcal{O}_S(B) \otimes I_Z \to 0$$

where $A, B \in \mathrm{Pic}(S)$ and I_Z is the ideal sheaf of a 0-cycle Z on S and $A-B$ satisfies

(i) $(A-B)^2 > 0$ and

(ii) $(A-B)\cdot H > 0$ for any ample divisor H on S.

Theorem 1.4 (**Shepherd-Barron [S-B]**). *Let S be a nonsingular projective surface defined over an algebraically closed field of characteristic $p>0$ and E a rank 2 locally free sheaf on S.*

(1) *Suppose that S is not of general type and $c_1(E)^2 > 4\,c_2(E)$. If S is not a quasi-elliptic surface of Kodaira dimension* 1, *then E is unstable.*

(2) *Suppose that S is of general type with minimal model S'.*

(a) *If $p \geq 3$ and $c_1(E)^2 - 4\,c_2(E) > K_{S'}^2$, then E is unstable.*

(b) *If $p=2$ and $c_1(E)^2 - 4\,c_2(E) > \max\{K_{S'}^2, K_{S'}^2 - 3\chi(\mathcal{O}_S)+2\}$, then E is unstable.*

Recall that a *quasi-elliptic surface* is a fibration $f: S \to C$ of a surface S over a nonsingular curve C, with $f_*\mathcal{O}_S = \mathcal{O}_C$, with almost all fibres rational with a cusp. By a result of Tate, it is known that such a situation can occur only if $\mathrm{char}(k) = 2$ or 3.

The Tyurin's result on a construction of locally free sheaves. Let S be a nonsingular projective surface defined over an algebraically closed field. Let L be a line bundle on S. For a 0-cycle $Z \in S^{[r]}$, consider a short exact sequence

$$0 \to L \otimes I_Z \to L \to \mathcal{O}_Z(L) \to 0.$$

Then we have a cohomology long exact sequence

$$0 \to H^0(L \otimes I_Z) \to H^0(L) \to H^0(\mathcal{O}_Z(L)) \to H^1(L \otimes I_Z) \to H^1(L) \to 0.$$

Now put

$$\delta(Z, L) := h^1(L \otimes I_Z) - h^1(L).$$

Note that the integer $\delta(Z, L)$ is nonnegative.

The cycle Z is called *L-stable* (in the sense of Tyurin) if $\delta(Z, L) > \delta(Z', L)$ for any subcycle Z' of Z.

Note that L is d-very ample if and only if $\delta(Z, L) = 0$ for all $Z \in S^{[d+1]}$.

Theorem 1.5 (**Tyurin** [**T**], **Lemma 1.2**). *Let L be a line bundle on a nonsingular projective surface S defined over an algebraically closed field and let Z be an L-stable 0-cycle of S. Then there exists an extension*

$$0 \to H^1(L \otimes I_Z) \otimes_{\mathcal{O}_S} K_S \to E(Z, L) \to L \otimes I_Z \to 0,$$

where $E(Z, L)$ is a locally free sheaf on S of rank $h^1(L \otimes I_Z) + 1$.

The Kodaira vanishing theorem. We give the Kodaira vanishing theorem on a surface in positive characteristic.

Theorem 1.6. *Let S be a nonsingular projective surface defined over an algebraically closed field of characteristic $p > 0$ and L a nef and big line bundle on S. Assume that one of the following situations holds:*

(1) *S is not of general type and further not quasi-elliptic of Kodaira dimension* 1;

(2) *S is of general type with minimal model S', $p \geq 3$ and*

$$L^2 > K_{S'}^2;$$

(3) *S is of general type with minimal model S', $p = 2$ and*

$$L^2 > \max\{K_{S'}^2, K_{S'}^2 - 3\chi(\mathcal{O}_S) + 2\}.$$

Then

$$H^i(S, K_S + L) = 0 \qquad \textit{for } \ i > 0.$$

Proof (Reid [**Re**]). For $i = 2$, we consider the complete linear system $|-L|$. If $|-L| \neq \phi$, then there exists an effective divisor D such that D is linearly equivalent to $-L$. This contradicts that L is nef and big. Hence $|-L| = \phi$, and we have, by the Serre duality,

$$H^2(K_S + L) = H^0(-L)^\vee = 0.$$

If $i = 1$, we have, by the Serre duality,

$$H^1(K_S + L)^\vee \cong \mathrm{Ext}^1_{\mathcal{O}_S}(K_S + L, K_S) \cong \mathrm{Ext}^1_{\mathcal{O}_S}(L, \mathcal{O}_S).$$

Then an element $\delta \in H^1(K_S + L)^\vee$ corresponds to an extension of locally free sheaves on S

$$(\delta) \qquad 0 \to \mathcal{O}_S \to E \to L \to 0.$$

Moreover the extension (δ) splits if and only if $\delta = 0$. From the extension (δ), we see that E satisfies

$$c_1(E) = c_1(L) \quad \text{and} \quad c_2(E) = 0.$$

Since $c_1(E)^2 - 4c_2(E) = c_1(L)^2 = L^2$, it follows from Theorem 1.4 that E is unstable. Hence by definition we obtain the short exact sequence

$$0 \to \mathcal{O}_S(A) \to E \to \mathcal{O}_S(B) \otimes I_Z \to 0,$$

where $A, B \in \mathrm{Pic}(S)$ and I_Z is the ideal sheaf of an effective 0-cycle Z on S. By combining this sequence and the extension (δ), we have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{O}_S & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_S(A) & \xrightarrow{\alpha} & E & \longrightarrow & \mathcal{O}_S(B) \otimes I_Z & \longrightarrow & 0. \\
 & & & & \downarrow{\scriptstyle \beta} & & & & \\
 & & & & L & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

We see easily that the composition map $\varphi := \beta \circ \alpha : \mathcal{O}_S(A) \to E \to L$ is not a zero map. Hence $H^0(L - A) \neq 0$, and we have $|L - A| \neq \phi$. We may assume that B is effective since $B \sim L - A$. Then, since

$$(L - 2B) \cdot L = (A - B) \cdot L \geq 0,$$

we have $L^2 \geq 2B \cdot L$. On the other hand, we have

$$0 = A \cdot B + \deg Z = (L - B) \cdot B + \deg Z.$$

Hence $B^2 \geq B \cdot L \geq 0$. By the Hodge index theorem

$$(B \cdot L)^2 \geq B^2 L^2 \geq 2\,(B \cdot L)^2,$$

and we obtain $B^2 = B \cdot L = 0$. Therefore B is numerically equivalent to 0, and thus we have $B = 0$ since B is effective. Then since $L = A$ and $I_Z = \mathcal{O}_S$, the extension (δ) splits and $\delta = 0$. This completes the proof. $\square$

2. A numerical criterion for d-very ampleness.

In this section we prove the theorem of Reider-type in arbitrary characteristic, which is the main result in our paper. The theorem in Introduction is an immediate consequence of this result.

First we need the following lemma from commutative algebra.

Lemma 2.1 ([**BS**], **Lemma 1.2**). *Let R be a Noetherian local ring and I, J ideals of R with $I \subset J$. Assume that* $\operatorname{length}(R/I) < \infty$. *Then there exists a chain*

$$I = I_0 \subset I_1 \subset \cdots \subset I_r = J$$

of ideals of R with $\operatorname{length}(I_i/I_{i-1}) = 1$ *for $i = 1, \dots, r$.*

The following lemma is the slight improvement of Lemma 2.2 in [**N**].

Lemma 2.2. *Let S be a nonsingular projective surface defined over an algebraically closed field k and L a line bundle on S such that $H^1(K_S + L) = 0$. Let Z be a 0-cycle with $\deg Z = d + 1$ where d is a nonnegative integer. Assume that $K_S + L$ is $(d-1)$-very ample and the restriction map*

$$\Gamma(K_S + L) \longrightarrow \Gamma(\mathcal{O}_Z(K_S + L))$$

is not surjective. Then there exists a rank 2 locally free sheaf E on S which is given by the short exact sequence

$$0 \to \mathcal{O}_S \to E \to I_Z(L) \to 0,$$

where I_Z is the ideal sheaf of Z.

Proof. Since $K_S + L$ is $(d-1)$-very ample and $\deg Z = d+1$, the cycle Z is $(K_S + L)$-stable in the sense of Tyurin. Thus, from Theorem 1.5, we have a locally free extension

$$0 \to H^1((K_S + L) \otimes I_Z) \otimes \mathcal{O}_S \to E \otimes (-K_S) \to L \otimes I_Z \to 0.$$

It is sufficient to prove that rank $E = 2$, i.e., $h^1((K_S + L) \otimes I_Z) = 1$. By Lemma 2.1, we can take a subcycle $Z' \subset Z$ such that $\deg Z' = d$. Then, from the $(d-1)$-very ampleness of $K_S + L$, we have

$$h^0(K_S + L) - h^0((K_S + L) \otimes I_Z) < h^0(\mathcal{O}_Z(K_S + L)) = d + 1$$

and

$$h^0(K_S + L) - h^0((K_S + L) \otimes I_{Z'}) = h^0(\mathcal{O}_{Z'}(K_S + L)) = d.$$

Since

$$\begin{aligned} h^0(K_S + L) - d - 1 &< h^0((K_S + L) \otimes I_Z) \\ &\le h^0((K_S + L) \otimes I_{Z'}) \\ &= h^0(K_S + L) - d, \end{aligned}$$

we obtain $h^0((K_S+L)\otimes I_Z) = h^0((K_S+L)\otimes I_{Z'})$. Therefore from the short exact sequence

$$0 \to (K_S+L)\otimes I_Z \to (K_S+L)\otimes I_{Z'} \to k \to 0,$$

we have

$$0 \to k \to H^1((K_S+L)\otimes I_Z) \to H^1((K_S+L)\otimes I_{Z'}) \to 0.$$

On the other hand, the $(d-1)$-very ampleness of K_S+L implies $h^1((K_S+L)\otimes I_{Z'}) = h^1(K_S+L)$. Therefore, by our assumption that $H^1(K_S+L)=0$, we have

$$\begin{aligned} h^1((K_S+L)\otimes I_Z) &= h^1((K_S+L)\otimes I_{Z'}) + 1 \\ &= h^1(K_S+L)+1 \\ &= 1. \end{aligned}$$

This completes the proof. □

Let E be a rank 2 locally free sheaf on a nonsingular projective surface S. If E is unstable, then, by definition, there exists a short exact sequence

$$0 \to \mathcal{O}_S(A) \to E \to \mathcal{O}_S(B)\otimes I_W \to 0$$

where $A, B \in \mathrm{Pic}(S)$ and I_W is the ideal sheaf of a 0-cycle W on S and $A-B$ satisfies

(i) $(A-B)^2 > 0$ and

(ii) $(A-B)\cdot H > 0$ for any ample divisor H on S.

With this notation, we have the following lemma, which plays an essential part in the Reider's method. Its proof is the same as [**BFS**], Proposition 1.4 and we give it for the sake of completeness:

Lemma 2.3. *Let S be a nonsingular projective surface defined over an algebraically closed field. Let E be a rank 2 locally free sheaf on S which is given as the following extension*

$$0 \to \mathcal{O}_S \to E \to L\otimes I_Z \to 0, \tag{2.3.1}$$

where Z is a 0-cycle with $\deg Z > 0$ and L is a line bundle on S. Assume that L is nef and big. If E is unstable, then there exists an effective divisor D containing Z such that $A \sim L-D$, $B\sim D$, $L-2D$ is $\mathbb{Q}$-effective, and

$$L\cdot D - \deg Z \le D^2 < \frac{L\cdot D}{2} < \deg Z.$$

Proof. Since E is unstable, we have a short exact sequence

$$0 \to \mathcal{O}_S(A) \to E \to \mathcal{O}_S(B)\otimes I_W \to 0, \tag{2.3.2}$$

where $A, B \in \mathrm{Pic}(S)$ and I_W is the ideal sheaf of a 0-cycle W on S and $A-B$ satisfies

(i) $(A-B)^2 > 0$

(ii) $(A-B)\cdot H > 0$ for any ample divisor H on S.

Then from the exact sequence (2.3.1) we obtain the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_S(A) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & E & \xrightarrow{\sigma} & L \otimes I_Z & \longrightarrow & 0. \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_S(B) \otimes I_W & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

The composition map $\mathcal{O}_S(A) \to E \xrightarrow{\sigma} L \otimes I_Z$ is not a zero map. Indeed, if it were, then $\mathcal{O}_S(A) \subseteq \operatorname{Ker}(\sigma) = \mathcal{O}_S$, and $-A$ is an effective divisor. Since $L \sim B + A$ and L is nef, we have

$$(A - B) \cdot H = (2A - L) \cdot H \leq 0$$

for any ample divisor H on S. This contradicts the instability of E.

Claim. $h^0(\mathcal{O}_S(-A)) = 0$.

Indeed, since L is nef, we have

$$(A + B) \cdot H = L \cdot H \geq 0$$

for any ample divisor H on S. On the other hand, the instability of E implies

$$(A - B) \cdot H > 0.$$

Then we obtain $A \cdot H > 0$, and thus $h^0(\mathcal{O}_S(-A)) = 0$.

From the exact sequence (2.3.2), we see that $H^0(E \otimes \mathcal{O}_S(-A)) \neq 0$. Tensoring the exact sequence (2.3.1) with $\mathcal{O}_S(-A)$, we have an exact sequence

$$0 \to \mathcal{O}_S(-A) \to E \otimes \mathcal{O}_S(-A) \to \mathcal{O}_S(B) \otimes I_Z \to 0.$$

Since $h^0(\mathcal{O}_S(-A)) = 0$, we obtain

$$0 \to H^0(E \otimes \mathcal{O}_S(-A)) \to H^0(\mathcal{O}_S(B) \otimes I_Z) \qquad \text{(exact)}.$$

Hence we have

$$H^0(\mathcal{O}_S(B) \otimes I_Z) \neq 0.$$

Therefore we can take $D \in |B|$, an effective divisor containing Z. Then we obtain $D \sim B$ and $A \sim L - D$ since $L \sim B + A$.

Now, since L is nef and big, we have $0 < (A - B)^2 \cdot L^2 \leq ((A - B) \cdot L)^2$ by the Hodge index theorem. Then we have $(L - 2D) \cdot L = (A - D) \cdot L = (A - B) \cdot L > 0$, and thus $L^2 > 2L \cdot D$.

From the diagram above, we have

$$\deg Z = c_2(E) = A \cdot B + \deg W = (L - D) \cdot D + \deg W$$

and, since W is effective, we obtain

$$L \cdot D - \deg Z \leq D^2.$$

By the Hodge index theorem and $2L \cdot D < L^2$, we obtain

$$2(L \cdot D) \cdot (D^2) < L^2 \cdot D^2 \leq (L \cdot D)^2$$

and thus

$$2D^2 < L \cdot D.$$

Then we have $2D^2 < L \cdot D \leq D^2 + \deg Z$, and $D^2 < \deg Z$. Hence we obtain

$$L \cdot D \leq D^2 + \deg Z < 2 \deg Z.$$

Finally, since $L - 2D \sim A - B$, we see from [**H**], Chapter V, Corollary 1.8 that $L - 2D$ is $\mathbb{Q}$-effective. □

The following is the main result in this paper.

Theorem 2.4. *Let S be a nonsingular projective surface defined over an algebraically closed field of characteristic $p \geq 0$. Let L be a nef line bundle on S. Assume that*

$$l := L^2 - 4d - 5 \geq 0$$

for a nonnegative integer d and one of the following situations holds:

(a) $p = 0$;

(b) $p > 0$ *and S is neither of general type nor quasi-elliptic of Kodaira dimension* 1;

(c) $p \geq 3$, *S is of general type with minimal model S', and*

$$l > K_{S'}^2;$$

(d) $p = 2$, *S is of general type with minimal model S', and*

$$l > \max\{K_{S'}^2, K_{S'}^2 - 3\chi(\mathcal{O}_S) + 2\}.$$

Then either $K_S + L$ is d-very ample or there exists an effective divisor D such that $L - 2D$ is $\mathbb{Q}$-effective, D contains some effective 0-cycle Z of $\deg Z \leq d + 1$ *for which the d-very ampleness fails, and*

$$L \cdot D - \deg Z \leq D^2 < \frac{L \cdot D}{2} < \deg Z. \tag{2.4.1}$$

Proof. We may assume that $K_S + L$ is $(d-1)$-very ample. Suppose that $K_S + L$ is not d-very ample. Then there exists a 0-cycle Z of $\deg Z = d+1$ for which the d-very ampleness fails.

We have $H^1(K_S + L) = 0$. This follows from the Kodaira vanishing theorem if $p = 0$ and from Theorem 1.6 if $p > 0$ since $L^2 > l$.

From Lemma 2.2, we obtain a rank 2 locally free sheaf E on S which is given by the short exact sequence

$$0 \to \mathcal{O}_S \to E \to L \otimes I_Z \to 0.$$

Moreover we have

$$c_1(E)^2 - 4c_2(E) = L^2 - 4\deg Z = L^2 - 4(d+1) > l.$$

By Theorem 1.4 and our assumption, we see that E is unstable. Therefore Lemma 2.3 implies that there exists an effective divisor D such that D contains Z and satisfies

$$L \cdot D - \deg Z \leq D^2 < \frac{L \cdot D}{2} < \deg Z.$$

This completes the proof. □

It follows from Theorem 2.4 that most of the results on d-very ample line bundles on surfaces of Kodaira dimension ≤ 0 also hold in positive characteristic (for example, see [**D**], [**T1**], [**T2**]).

Corollary 2.5. *Let $L_1, \ldots, L_t$ be ample line bundles on a nonsingular projective surface S defined over an algebraically closed field of characteristic $p > 0$. Assume that S is not of general type. Then $K_S + L_1 + \cdots + L_t$ is d-very ample for every $t \geq d+3$ unless if S is quasi-elliptic of Kodaira dimension* 1. *Furthermore $K_S + L_1 + \cdots + L_{d+2}$ is d-very ample if $d \geq 2$ or $L_i^2 \geq 2$ for some i.*

Proof. The first part follows immediately from Theorem 2.4.

We prove the second part. Put $L := L_1 + \cdots + L_{d+2}$. Since $L_i^2 \geq 2$ for some i, we have $L_i \cdot L_j \geq 2$ for some i, j by the Hodge index theorem. Hence we have $L^2 > (d+2)^2$, and thus $L^2 \geq 4d+5$.

If $K_S + L$ is not d-very ample, then from the theorem above there exists an effective divisor D satisfying the numerical condition

$$L \cdot D - d - 1 \leq D^2 < \frac{L \cdot D}{2} < d.$$

Put $m := \min_i \{L_i \cdot D\}$. Then we have $L \cdot D \geq m(d+2)$, and

$$\frac{m(d+2)}{2} \leq \frac{L \cdot D}{2} < d.$$

Thus we obtain $m = 1$. We see that, for some i,

$$2 < L_i^2 \cdot D^2 \leq (L_i \cdot D)^2 = 1$$

by the Hodge index theorem. This is a contradiction. Therefore $K_S + L$ is d-very ample. □

Remark 2.6. Corollary 2.5 is concerned with Fujita's conjectures. This conjecture is true in some cases if the characteristic of the base field is zero. No counter-example is known in positive characteristic cases. For details, we refer to [**Ei**] and [**L**].

From Theorem 2.4, we easily obtain the results on the k-very ampleness of pluricanonical linear systems on a canonical model of a surface of general type. For details, we refer to [**Ek**], [**S-B**] and [**N**].

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