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SINGULAR LIMIT OF SOLUTIONS OF THE EQUATION $u_t = \Delta\left(\frac{u^m}{m}\right)$ AS $m \to 0$

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We will show that for $n = 1, 2$, as $m \to 0$ the solution $u^{(m)}$ of the fast diffusion equation $\partial u/\partial t = \Delta(u^m/m)$, $u > 0$, in $R^n \times (0,\infty)$, $u(x,0) = u_0(x) \geq 0$ in R^n, where $u_0 \in L^1(R^n) \cap L^\infty(R^n)$ will converge uniformly on every compact subset of $R^n \times (0,T)$ to the maximal solution of the equation $v_t = \Delta \log v$, $v(x,0) = u_0(x)$, where $T = \infty$ for $n = 1$ and $T = \int_{R^2} u_0 dx/4\pi$ for $n = 2$.

The degenerate parabolic equation

$$(0.1)\qquad \begin{cases} u_t = \Delta\left(\frac{u^m}{m}\right), u > 0, & \text{in } R^n \times (0,\infty) \\ u(x,0) = u_0(x) \geq 0 & \forall x \in R^n \end{cases}$$

where $m > 0$ arises in the modelling of many physical phenomenon such as the flow of gases through a porous medium or the flow of viscous fliud on a surface. When $m > 1$, the above equation is called the porous medium equation. When $0 < m < 1$, it is called the fast diffusion equation and when $m = 1$, it becomes the famous heat equation. We refer the reader to the papers by Aronson [**A**] and Peletier [**P**] for extensive reference on the above equation.

In this paper we will investigate the convergence of solution $u^{(m)}$ of the fast diffusion equation (0.1) as $m \to 0$. We will show that for $n = 1, 2$, if $u_0 \in L^1(R^n) \cap L^\infty(R^n)$, then as $m \to 0$ the solution $u^{(m)}$ of the above fast diffusion equation will converge uniformly on every compact subset of $R^n \times (0,T)$ to the maximal solution of

$$(0.2)\qquad \begin{cases} v_t = \Delta \log v, v > 0, & \text{in } R^n \times (0,T) \\ v(x,0) = u_0(x) & \forall x \in R^n \end{cases}$$

where

$$(0.3)\qquad T = \begin{cases} \infty & \text{if } n = 1 \\ \frac{1}{4\pi}\int_{R^2} u_0 dx & \text{if } n = 2. \end{cases}$$

As proved in [**ERV**] for $u_0 \in L^1(R)$ when $n = 1$ and in [**DP**], [**H**] for $u_0 \in L^1(R^2) \cap L^p(R^2)$ for some $p > 1$ when $n = 2$, there exists infinitely many solutions of (0.2). However by the results of [**ERV**] for $n = 1$ and [**DD**] for $n = 2$ there exists only a single maximal solution of (0.2) for $n = 1$, 2. So the limit $\lim_{m\to 0} u^{(m)}$ is unique.

When $n = 2$ L.F.Wu [**W1**], [**W2**] showed that (0.2) is related to the study of Ricci flow and when $n = 1$ K.G. Kurtz [**K**] showed that (0.2) is related to the limiting density of two type of particles moving against each other obeying the Boltzmann equation. Similar type of singular limits for solutions of (0.1) as $m \to \infty$ and $m \to 1$ are obtained by L.A. Caffarelli and A. Friedman [**CF2**], P.L. Lions, P.E. Souganidis and J.L. Vazquez [**LSV**] and for the p-laplacian equation which are related to the sandpile model by L.C. Evans, M. Feldman, etc. [**AEW**], [**EFG**], [**EG**].

For any domain $\Omega \subset R^n$, $T > 0$, $\phi(u) = u^m/m$ for some $0 < m < 1$ or $\phi(u) = \log u$, we say that u is a solution (respectively subsolution, supersolution) of

$$\frac{\partial u}{\partial t} = \Delta\phi(u) \tag{0.4}$$

in $\Omega \times (0, T)$ if $u \in C(\overline{\Omega} \times (0, T)) \cap C^\infty(\Omega \times (0, T))$, $u > 0$ on $\overline{\Omega} \times (0, T)$, and satisfies (0.4) (respectively $\le, \ge$) in $\Omega \times (0, T)$ in the classical sense and we say that u has initial value u_0 if the following holds

$$\lim_{t\to 0}\int_\Omega u(x, t)\eta(x)dx = \int_\Omega u_0(x)\eta(x)dx \quad \forall \eta \in C_0^\infty(\Omega).$$

We say that u is a solution (respectively subsolution, supersolution) of

$$\begin{cases} u_t = \Delta\phi(u), u > 0, & \text{in } R^n \times (0,\infty) \\ u(x, 0) = u_0(x) \ge 0 & \forall x \in R^n \end{cases}$$

if u is a solution (respectively subsolution, supersolution) of (0.4) in $R^n \times (0, T)$ with initial value u_0. We say that $\widetilde{v}$ is a maximal solution of (0.2) if $\widetilde{v} \ge v$ for any solution v of (0.2). For any $x_0 \in R^n$, $R > 0$, we let $B_R(x_0) = \{x \in R^n : |x - x_0| < R\}$ and $B_R = B_R(0)$.

The plan of the paper is as follows. In Section 1 we will recall some existence results and construct some subsolutions of (0.1). In Section 2 we will use the subsolutions obtained in Section 1 to obtain lower bound estimates for the solution of (0.1). We will also prove a Harnack type inequality for solution of (0.1) and a convergence result for the case $n = 1$, 2, under the assumption that $u_0 \in L^1(R^n) \cap L^\infty(R^n)$ and $\varepsilon(1 + |x|^2)^{-\alpha} \le u_0(x) \le M$ for some constants $\varepsilon > 0$, $M \ge 1$, and $1 < \alpha < 2$ for all $x \in R^n$. The general convergence result for $u_0 \in L^1(R^n) \cap L^\infty(R^n)$ then follows by an approximation argument. Since we are concerned with the limit $m \to 0$, we will assume $0 < m < 1$ and let $u^{(m)}$ be the solution of (0.1) for the rest of the paper.

Section 1.

In this section we will recall some existence results and comparsion principles for solutions of (0.1) in [**AB**], [**BBC**], [**DK**]. We will also construct some subsolutions of (0.1). We first start with an existence result of [**AB**], [**DK**]:

Theorem 1.1 ([**AB**], [**DK**]). *If $u_0 \in L^1(R^n)$ and $(n-2)_+/n < m < 1$, then (0.1) will have a unique solution $u^{(m)} \in L^\infty((0,\infty); L^1(R^n)) \cap C^\infty(R^n \times (0,\infty))$, $u^{(m)} > 0$ on $R^n \times (0,\infty)$ and satisfying*

$$u_t^{(m)} \le \frac{u^{(m)}}{(1-m)t} \quad \text{in } \ R^n \times (0,\infty).$$

Proof. Existence of solution to (0.1) is proved in [**DK**], [**AB**]. Let $p = u_t^{(m)}/u^{(m)}$. Then as in [**AB**], [**CF1**], [**ERV**] p will satisfy the equation

$$p_t = u^{(m)m-1}\Delta p + 2mu^{(m)m-2}\nabla u^{(m)} \cdot \nabla p - (1-m)p^2.$$

Since $1/(1-m)t$ also satisfies the above equation but with initial value ∞. By the maximum principle

$$u_t^{(m)} \le \frac{u^{(m)}}{(1-m)t}$$

and the theorem follows. □

Theorem 1.2 ([**BBC**],[**E**]). *If $(n-2)_+/n < m < 1$ and $u_1^{(m)}$, $u_2^{(m)}$, are subsolution and supersolutions of (0.1) with initial values $u_{0,1}$, $u_{0,2} \in L^1(R^n)$ respectively, then*

$$\int_{R^n} \left(u_1^{(m)}(x,t) - u_2^{(m)}(x,t)\right)_+ dx \le \int_{R^n} (u_{0,1} - u_{0,2})_+ dx \quad \forall t > 0$$

and if $u_1^{(m)}$ and $u_2^{(m)}$ are two solutions of (0.1) with initial values $u_{0,1}$, $u_{0,2} \in L^1(R^n)$ respectively, then

$$\int_{R^n} \left|u_1^{(m)}(x,t) - u_2^{(m)}(x,t)\right| dx \le \int_{R^n} |u_{0,1} - u_{0,2}| dx \quad \forall t > 0.$$

In particular if $u_{0,1} \le u_{0,2}$, then $u_2 \le u_2$.

Lemma 1.3. *If $T_1 > 0$, $0 < m < 1$, $\alpha > 1$, $0 < k < 1$ are constants satisfying the condition $(1-m)^{-1} < \alpha < 2(1-m)^{-1}$ and $0 < k < (4\alpha)^{-1/(\alpha(1-m)-1)}$, then the function*

$$w(x,t) = \frac{[(1-m)(T_1-t)_+]^{1/1-m}}{(k+|x|^2)^\alpha} \tag{1.1}$$

is a subsolution of

$$u_t = \Delta(u^m/m) \tag{1.2}$$

in $R^n \times (0,T_1)$, $n = 1, 2$, *with initial value* $[(1-m)T_1]^{1/1-m}(k+|x|^2)^{-\alpha}$.

Proof. Write $w(x,t) = f(t)g(x)$ where $f(t) = [(1-m)(T_1-t)_+]^{1/1-m}$ and $g(x) = g(r) = (k+|x|^2)^{-\alpha}$ where $r = |x|$. For $n = 1$ by direct computation,

$$\begin{cases}(g^m)'(r) = -2\alpha mr(k+r^2)^{-\alpha m-1} \\ (g^m)''(r) = 4\alpha m(\alpha m+1)r^2(k+r^2)^{-\alpha m-2} - 2\alpha m(k+r^2)^{-\alpha m-1}.\end{cases}$$

Hence

$$\begin{aligned}&\Delta(g^m) + mg \\ &= m(k+|x|^2)^{-\alpha}(1 + 4\alpha(\alpha m+1)(k+|x|^2)^{\alpha(1-m)-2}|x|^2 \\ &\quad - 2\alpha(k+|x|^2)^{\alpha(1-m)-1}) \\ &= m(k+|x|^2)^{-\alpha}(1 + (2\alpha + 4\alpha^2 m)(k+|x|^2)^{\alpha(1-m)-1} \\ &\quad - 4\alpha(\alpha m+1)(k+|x|^2)^{\alpha(1-m)-2}k) \\ &\geq m(k+|x|^2)^{-\alpha}(1 + (2\alpha+4\alpha^2 m)k^{\alpha(1-m)-1} - 4\alpha(\alpha m+1)k^{\alpha(1-m)-1}) \\ &\geq m(k+|x|^2)^{-\alpha}(1 - 2\alpha k^{\alpha(1-m)-1}) \geq 0.\end{aligned}$$

Similarly for $n = 2$, we have

$$\begin{aligned}&\Delta(g^m) + mg \\ &= m(k+|x|^2)^{-\alpha}(1 + 4\alpha(\alpha m+1)(k+|x|^2)^{\alpha(1-m)-2}|x|^2 \\ &\quad - 4\alpha(k+|x|^2)^{\alpha(1-m)-1}) \\ &\geq m(k+|x|^2)^{-\alpha}(1 - 4\alpha k^{\alpha(1-m)-1}) \geq 0.\end{aligned}$$

Hence

$$\Delta\left(\frac{w^m}{m}\right) - w_t = f^m \Delta\left(\frac{g^m}{m}\right) - f_t g = \frac{f^m}{m}\cdot(\Delta(g^m) + mg) \geq 0$$

Thus w is a subsolution of (0.1) with initial value $[(1-m)T_1]^{1/1-m}(k+|x|^2)^{-\alpha}$ and the lemma follows. □

As a consequence of Theorem 1.2 and Lemma 1.3 we have the the following corollary.

Corollary 1.4. *If* $u_0 \in L^1(R^n)$ *and* $u_0(x) \geq \varepsilon(k+|x|^2)^{-\alpha}$ *for some constants* $\varepsilon > 0$, $0 < k < 1$, $0 < m < 1$, *satisfying* $(1-m)^{-1} < \alpha < 2(1-m)^{-1}$ *and* $0 < k < (4\alpha)^{-1/(\alpha(1-m)-1)}$, *then the solution* $u^{(m)}$ *of* (0.1) *is bounded below by* $(\varepsilon^{1-m} - (1-m)t)_+^{1/1-m}(k+|x|^2)^{-\alpha}$.

Theorem 1.5. *If* $u_0 \in L^1(R^1)$ *for* $n = 1$ *and* $u_0 \in L^1(R^2) \cap L^p(R^2)$ *for some* $p > 1$ *for* $n = 2$, *then there exists a unique maximal solution* v *of* (0.2) *in* $R^n \times (0,T)$ *where* T *is given by* (0.3).

Proof. For $n=1$ the theorem is proved in [**ERV**]. For $n=2$ by Theorem 4.3 of [**H**] (cf. [**DP**] Theorem 1.2) there exists a solution v of (0.2) in $R^2\times(0,T)$ satisfying the condition

$$\int_{R^2} v(x,t)dx=\int_{R^2} u_0dx-4\pi t \quad \forall 0<t\le T \tag{1.3}$$

where T is given by (0.3). By Prop 3.1 and Prop 4.1 of [**DD**], there exists a maximal solution v_1 of (0.2) in $R^2\times(0,T)$. By Theorem 1.3 of [**DP**],

$$\begin{aligned}
&\int_{R^2} v_1(x,t)dx\le\int_{R^2} u_0dx-4\pi t \quad \forall 0<t\le T\\
\Rightarrow\quad &\int_{R^2} v_1(x,t)dx\le\int_{R^2} v(x,t)dx \quad \forall 0<t\le T \quad \text{by (1.3)}\\
\Rightarrow\quad &v(x,t)=v_1(x,t) \quad \forall x\in R^2, 0<t\le T \text{ since } v\le v_1.
\end{aligned}$$

Hence v is the maximal solution solution of (0.2). □

Lemma 1.6. *There exists a constant* $0<m_1\le 1/2$ *such that the function*

$$u_1(x,t)=\frac{Am^2t^{1/1-m}}{|x|^{2/(1-m)}(|x|^m-1)^2}$$

is a subsolution of (1.2) *in* $R^2\setminus B_2\times(0,\infty)$ *for all* $0<m\le m_1$ *where* $0<A\le(2/3)^4$ *is a constant.*

Proof. Let $b=2/(1-m)$ and $g(x)=g(r)=r^{-mb}(r^m-1)^{-2m}$ where $r=|x|$. Then

$$\begin{aligned}
\Delta g&=g''+\frac{1}{r}g' \\
&=(mb)^2r^{-mb-2}(r^m-1)^{-2m}+4m^3br^{-mb+m-2}(r^m-1)^{-2m-1}\\
&\quad+4m^4r^{-mb+m-2}(r^m-1)^{-2m-1}\\
&\quad+2m^3(2m+1)r^{-mb+m-2}(r^m-1)^{-2m-2}\\
&\ge 2m^3r^{-mb+m-2}(r^m-1)^{-2m-2} \quad \forall r\ge 2\\
&\Rightarrow\quad \Delta\left(\frac{u_1^m}{m}\right)-u_{1,t}\\
&=A^mm^{2m-1}t^{m/1-m}\Delta g-Am^2(1-m)^{-1}t^{m/1-m}r^{-b}(r^m-1)^{-2}\\
&\ge 2A^mm^{2m+2}t^{m/1-m}r^{-mb+m-2}(r^m-1)^{-2m-2}\\
&\quad-Am^2(1-m)^{-1}t^{m/1-m}r^{-b}(r^m-1)^{-2}\\
&=m^2A^mt^{m/1-m}r^{-b}(r^m-1)^{-2m-2}(2m^{2m}r^m\\
&\quad-(1-m)^{-1}A^{1-m}(r^m-1)^{2m}).
\end{aligned}\tag{1.4}$$

Since $\lim_{m\to 0} m^m = 1$, there exists $0 < m_1 \le 1/2$ such that $m^m > 2/3$ for all $0 < m \le m_1$. Thus

$$\begin{aligned} &2m^{2m}r^m - (1-m)^{-1}A^{1-m}(r^m-1)^{2m} \\ &\ge 2(2/3)^2 r^m - 2A^{1/2}r^{2m^2} \\ &\ge 2(2/3)^2 r^{2m^2}(r^{m(1-2m)}-1) \ge 0 \quad \forall 0 < A \le (2/3)^4, 0 < m \le m_1, r \ge 2. \end{aligned}$$

Hence the right hand side of (1.4) is positive for all $0 < m \le m_1$, $r \ge 2$, and the lemma follows. □

By direct computation we also have the follow result:

Lemma 1.7. *For any $0 < A < 1$, $0 < m < 1$, the function*

$$u_2(x,t) = A\left(\frac{t}{|x|^2}\right)^{1/(1-m)}$$

is a subsolution of (1.2) in $R \setminus B_2 \times (0,\infty)$.

Theorem 1.8. *If v is a solution of (0.2) in $R^2 \times (0,T')$ and satisfies the condition*

$$v(x,t) \ge \frac{Ct}{(|x|\log|x|)^2} \quad \forall |x| \ge R, 0 < t \le T'$$

for some constants $C > 0$ and $R > 0$, then v is the unique maximal solution of (0.2) in $R^2 \times (0,T')$.

Proof. For any $0 < t_1 < T'$, since

$$v(x,t) \ge \frac{Ct_1}{(|x|\ \log\ |x|)^2} \quad \forall |x| \ge R, t_1 < t \le T'$$

by Prop 2.1 of [**DD**] v is the unique maximal solution of

$$\begin{cases} w_t = \Delta \log\ w, w > 0, & \text{in } R^n \times (t_1, T') \\ w(x,t_1) = v(x,t_1) & \forall x \in R^n \end{cases} \tag{1.5}$$

with $n = 2$. By Theorem 4.3 of [**H**] or Theorem 1.2 of [**DP**] there exist a solution $\overline{v}$ of (1.5) satisfying

$$\int_{R^2} \overline{v}(x,t)dx = \int_{R^2} v(x,t_1)dx - 4\pi(t-t_1) \quad \forall 0 < t_1 \le t \le T'.$$

Since v is the unique maximal solution of (1.5), $v \ge \overline{v}$. Thus

$$\int_{R^2} v(x,t)dx \ge \int_{R^2} \overline{v}(x,t)dx = \int_{R^2} v(x,t_1)dx - 4\pi(t-t_1) \quad \forall 0 < t_1 \le t \le T'. \tag{1.6}$$

We next let $R>0$ and let $\eta_R\in C_0^\infty(R^2)$ be such that $0\le\eta_R\le 1$, $\eta_R(x)\equiv 1$ for $|x|\le R$, $\eta_R(x)\equiv 0$ for $|x|\ge 2R$. Then

$$
\begin{aligned}
&\int_{R^2} v(x,t_1)dx \ge \int_{R^2} v(x,t_1)\eta_R(x)dx\\
&\Rightarrow \quad \liminf_{t_1\to 0}\int_{R^2} v(x,t_1)dx \ge \liminf_{t_1\to 0}\int_{R^2} v(x,t_1)\eta_R(x)dx\\
&=\int_{R^2} u_0(x)\eta_R(x)dx\\
&\Rightarrow \quad \liminf_{t_1\to 0}\int_{R^2} v(x,t_1)dx \ge \int_{R^2} u_0(x)dx \quad \text{as} \quad R\to\infty.
\end{aligned}
$$

Hence letting $t_1\to 0$ in (1.6) we have

$$
\int_{R^2} v(x,t)dx \ge \int_{R^2} u_0(x)dx - 4\pi t \quad \forall 0<t\le T'. \tag{1.7}
$$

By Theorem 4.3 of [**H**] and the proof of Theorem 1.5 there exists a unique maximal solution $\widetilde{v}$ of (0.2) which satisfies

$$
\int_{R^2} \widetilde{v}(x,t)dx = \int_{R^2} u_0(x)dx - 4\pi t \quad \forall 0<t\le T'. \tag{1.8}
$$

Since $\widetilde{v}\ge v$, by (1.7) and (1.8) we get

$$
\begin{aligned}
&\int_{R^2} v(x,t)dx = \int_{R^2} \widetilde{v}(x,t)dx = \int_{R^2} u_0(x)dx - 4\pi t \quad \forall 0<t\le T'\\
&\Rightarrow \quad v\equiv\widetilde{v} \qquad \text{on } R^2\times(0,T').
\end{aligned}
$$

Hence v is the unique maximal solution of (0.2). □

Theorem 1.9. *If v is a solution of (0.2) in $R\times(0,T')$ and satisfies the condition*

$$
v(x,t)\ge\frac{Ct}{|x|^2} \quad \forall |x|\ge R, 0<t\le T' \tag{1.9}
$$

for some constants $C>0$ and $R>0$, then v is the unique maximal solution of (0.2) in $R\times(0,T')$.

Proof. For any $0<t_1<T'$, since v satisfies (1.9)

$$
\log\,(1/v)\le -\log\,Ct + 2\log\,|x| \le o(|x|) \quad \forall |x|\ge R, t_1<t\le T'.
$$

By the result in Section 3 of [**ERV**] v is the unique maximal solution of (1.5) with $n=1$. By an argument similar to the proof of Theorem 1.8 we get that v is the unique maximal solution of (0.2) and the theorem follows. □

Section 2.

In this section we will first prove the convergence of solution $u^{(m)}$ of (0.1) as $m\to 0$ for $n=1$, 2, under the assumption that $u_0\in L^1(R^n)\cap L^\infty(R^n)$ and $\varepsilon(1+|x|^2)^{-\alpha}\le u_0(x)\le M$ for some constants $\varepsilon>0$, $M\ge 1$, and $1<\alpha<2$ for all $x\in R^n$. The general convergence theorem then follows by an approximation argument. We will first start with a Harnack type inequality.

Lemma 2.1. *If $u^{(m)}\in L^\infty((0,\infty);L^1(R^n))\cap L^\infty(R^n\times(0,\infty))$ and $0\le u^{(m)}\le M$ for some constant M, then for any $(n-2)_+/n<m<1$, $0<t_1<t_2$, $t>0$, $R>0$, and $x_0\in R^n$, the following inequalities hold,*

$$\text{(i)}\qquad \frac{1}{|B_R|}\int_{B_R(x_0)}\frac{u^{(m)m}(x,t)}{m}dx\le\frac{u^{(m)m}(x_0,t)}{m}+C\frac{MR^2}{(1-m)t}$$

$$\text{(ii)}\qquad \frac{1}{|B_R|}\int_{t_1}^{t_2}\int_{B_R(x_0)}\frac{1-u^{(m)m}}{m}dxdt\le I_2+CMR^2$$

$$\text{(iii)}\qquad I_1\le\frac{1}{|B_R|}\int_{t_1}^{t_2}\int_{B_R(x_0)}\frac{1-u^{(m)m}}{m}dxdt+CMR^2$$

where $C>0$ is a constant independent of m and

$$I_i=\frac{(t_2-t_1)}{m}\left\{1-(1-m)u^{(m)m}(x_0,t_i)t_i^{-m/1-m}\left(\frac{t_2^{1/1-m}-t_1^{1/1-m}}{t_2-t_1}\right)\right\},\quad i=1,2. \tag{2.1}$$

Proof. We will use a modification of the argument of Lemma 6 of [**V**] and Lemma 4 of [**GH**] to prove the lemma. Let (cf. [**V**] p. 509, [**DD**] p. 653)

$$G_R(x)=\begin{cases}|x-x_0|^{2-n}-R^{2-n}+\frac{n-2}{2}R^{-n}(|x-x_0|^2-R^2) & \text{if } n\ge 3\\ \log\,(R/|x-x_0|)+\frac{1}{2}R^{-2}(|x-x_0|^2-R^2) & \text{if } n=2\\ R-|x-x_0|+\frac{R^{-1}}{2}(|x-x_0|^2-R^2) & \text{if } n=1\end{cases}$$

be the Green's function for $B_R(x_0)$. Then $G_R\ge 0$, $G_R(R)=G_R'(R)=0$, and

$$\Delta G_R=\begin{cases}n(n-2)R^{-n}-(n-2)|\partial B_1|\delta_{x_0} & \text{if } n\ge 3\\ 2R^{-2}-2\pi\delta_{x_0} & \text{if } n=2\\ R^{-1}-2\delta_{x_0} & \text{if } n=1\end{cases}$$

where δ_{x_0} is the delta mass at x_0. Then by Theorem 1.1 we have

$$
\begin{aligned}
& a_n\left(\frac{1}{|B_R|}\int_{B_R(x_0)}\frac{u^{(m)m}(x,t)}{m}dx-\frac{u^{(m)m}(x_0,t)}{m}\right)\\
&=\int_{B_R(x_0)}G_R(x)\Delta\left(\frac{u^{(m)m}}{m}\right)(x,t)dx\\
&=\int_{B_R(x_0)}G_R(x)u_t(x,t)dx \qquad (2.2)\\
&\le\frac{1}{(1-m)t}\int_{B_R(x_0)}G_R(x)u(x,t)dx\le C\frac{MR^2}{(1-m)t}
\end{aligned}
$$

where $a_1=2$, $a_2=2\pi$, and $a_n=(n-2)|\partial B_1|$ for $n\ge 3$. Thus (i) follows. Integrating (2.2) over (t_1,t_2) we get

$$
\begin{aligned}
&\frac{1}{|B_R|}\int_{t_1}^{t_2}\int_{B_R(x_0)}\frac{1-u^{(m)m}}{m}dxdt\\
&=\int_{t_1}^{t_2}\frac{1-u^{(m)m}}{m}dt-\frac{1}{a_n}\int_{B_R(x_0)}G_R(x)u(x,t)dx\bigg|_{t_1}^{t_2}.
\end{aligned}
$$

By Theorem 1.1,

$$
\frac{u(x,t_2)}{t_2^{1/1-m}}\le\frac{u(x,t)}{t^{1/1-m}}\le\frac{u(x,t_1)}{t_1^{1/1-m}}\quad\forall t_1\le t\le t_2.
$$

Hence

$$
\int_{t_1}^{t_2}\frac{1-u^{(m)m}}{m}dt\le\int_{t_1}^{t_2}\frac{1-u^{(m)m}(x_0,t_2)(t/t_2)^{m/1-m}}{m}dt\le I_2
$$

where I_2 is given by (2.1). Since $G_R\ge 0$ and

$$
\int_{B_R(x_0)}G_R(x)u(x,t)dx\le CMR^2\quad\forall t>0
$$

(ii) follows. Similarly

$$
\int_{t_1}^{t_2}\frac{1-u^{(m)m}}{m}dt\ge I_1
$$

where I_1 is given by (2.1) and (iii) follows. □

Lemma 2.2. *Let $u^{(m)}\in L^\infty((0,\infty);L^1(R^n))\cap L^\infty(R^n\times(0,\infty))$ be the solution of (0.1). If there exists a constant $M\ge 1$ such that $0\le u^{(m)}\le M$ for all $0<m<1$ and*

$$
\liminf_{m\to 0}u^{(m)}(x_0,t_0)>0 \qquad (2.3)
$$

for some $x_0 \in R^n$, $n = 1, 2$, $0 < t_0 < T$, $T > 0$. *Then for any* $R > 0$, $0 < t_1 < t_2 < t_0$, *there exists* $C > 0$ *and* $0 < m_0 < 1$ *depending on* t_1, t_2, $t_0 - t_2$, *and* $R > 0$ *such that*

$$u^{(m)}(x,t) \ge C \quad \forall 0 < m \le m_0, x \in B_R, t_1 \le t \le t_2. \tag{2.4}$$

Proof. Without loss of generality we may assume that $R > |x_0|$. By (2.3) there exists $0 < m_1 < 1$ and $\delta > 0$ such that

$$u^{(m)}(x_0,t_0) \ge \delta \quad \forall 0 < m \le m_1.$$

Since $B_1(y) \subset B_{2R+1}(x_0)$ for any $y \in B_R$, by Lemma 2.1 we have for any $y \in B_R$, $0 < t_1 \le t \le t_2 < t_0$,

$$\begin{aligned}
&\frac{(t_0-t)}{m}\left\{1-(1-m)u^{(m)m}(y,t)t^{-m/1-m}\left(\frac{t_0^{1/1-m}-t^{1/1-m}}{t_0-t}\right)\right\} \\
&\le \frac{1}{|B_1|}\int_t^{t_0}\int_{B_1(y)}\frac{1-u^{(m)m}}{m}dxds + CMR^2 \\
&\le \frac{1}{|B_1|}\int_t^{t_0}\int_{B_1(y)\cap\{u^{(m)}\le 1\}}\frac{1-u^{(m)m}}{m}dxds + CMR^2 \\
&\le \frac{(2R+1)^n}{|B_{2R+1}|}\int_t^{t_0}\int_{B_{2R+1}(x_0)\cap\{u^{(m)}\le 1\}}\frac{1-u^{(m)m}}{m}dxds + CMR^2 \\
&\le \frac{(2R+1)^n}{|B_{2R+1}|}\int_t^{t_0}\int_{B_{2R+1}(x_0)}\frac{1-u^{(m)m}}{m}dxds \\
&\quad + \frac{(2R+1)^n}{|B_{2R+1}|}\int_t^{t_0}\int_{B_{2R+1}(x_0)\cap\{u^{(m)}>1\}}\frac{u^{(m)m}-1}{m}dxds + CMR^2 \\
&\le (2R+1)^n\frac{(t_0-t)}{m}\left\{1-(1-m)u^{(m)m}(x_0,t_0)t_0^{-m/1-m}\right. \\
&\qquad\qquad \left.\cdot\left(\frac{t_0^{1/1-m}-t^{1/1-m}}{t_0-t}\right)\right\} \\
&\quad + \frac{(2R+1)^n}{|B_{2R+1}|}\int_t^{t_0}\int_{B_{2R+1}(x_0)\cap\{u^{(m)}>1\}}\frac{u^{(m)m}-1}{m}dxds + CMR^2
\end{aligned} \tag{2.5}$$

where $C > 0$ is a constant. Since

$$\frac{z^m-1}{m} \le z\log z \le M\log M \quad \forall 1 \le z \le M.$$

The second term on the right hand side above is bounded above by

$$(t_0-t)(2R+1)^n M\log M. \tag{2.6}$$

By the mean value theorem there exists $t_3 \in (t, t_0)$ and $0 < \theta_1 < 1$ such that

$$
\begin{aligned}
(2.7)\qquad &\frac{(t_0 - t)}{m}\left\{1 - (1-m)u^{(m)m}(x_0, t_0)t_0^{-m/1-m}\left(\frac{t_0^{1/1-m} - t^{1/1-m}}{t_0 - t}\right)\right\}\\
&= \frac{(t_0 - t)}{m}\left\{1 - u^{(m)m}(x_0, t_0)(t_3/t_0)^{m/1-m}\right\}\\
&= (t_0 - t)(u^{(m)}(x_0, t_0)(t_3/t_0)^{1/1-m})^{\theta_1 m}\log\frac{(t_0/t_3)^{1/1-m}}{u^{(m)}(x_0, t_0)}\\
&\le TM\log\frac{(T/t_1)^{1/1-m_1}}{\delta} \quad \forall 0 < m \le m_1.
\end{aligned}
$$

Similarly there exists $t_4 \in (t, t_0)$ such that

$$
\begin{aligned}
(2.8)\qquad &\frac{(t_0 - t)}{m}\left\{1 - (1-m)u^{(m)m}(y, t)t_0^{-m/1-m}\left(\frac{t_0^{1/1-m} - t^{1/1-m}}{t_0 - t}\right)\right\}\\
&= \frac{(t_0 - t)}{m}\left\{1 - u^{(m)m}(y, t)(t_4/t)^{m/1-m}\right\}\\
&\ge \frac{(t_0 - t_2)}{m}\left\{1 - u^{(m)m}(y, t)(T/t_1)^{m/1-m}\right\}.
\end{aligned}
$$

By (2.5), (2.6), (2.7), (2.8), there exists a constant $c_1 > 0$ such that for all $0 < m \le m_1$ we have

$$
\begin{aligned}
(2.9)\qquad &\frac{(t_0 - t_2)}{m}\left\{1 - u^{(m)m}(y, t)(T/t_1)^{m/1-m}\right\} \le c_1\\
&\Rightarrow \quad u^{(m)}(y, t) \ge \left(1 - \frac{mc_1}{t_0 - t_2}\right)^{1/m}(t_1/T)^{1/1-m} \quad \forall 0 < m \le m_1.
\end{aligned}
$$

Since the right hand side of (2.9) tends to some positive constant independent of $y \in B_R$, $t_1 \le t \le t_2$, there exists constants $0 < m_0 \le m_1$ and $C > 0$ such that

$$
u^{(m)}(y, t) \ge C > 0 \quad \forall y \in B_R, t_1 \le t \le t_2, 0 < m \le m_0
$$

and the lemma follows. □

Theorem 2.3. *For $n = 1, 2$, if $u_0 \in L^1(R^n) \cap L^\infty(R^n)$ and $\varepsilon(k + |x|^2)^{-\alpha} \le u_0(x) \le M$ for some constants $\varepsilon > 0$, $0 < k < 1$, $1 < \alpha < 2$, $M \ge 1$, satisfying $\alpha > (1 - m_2)^{-1}$ and $0 < k < (4\alpha)^{-1/(\alpha(1-m_2)-1)}$ for some constant $0 < m_2 < 1/2$, then as $m \to 0$ the solution $u^{(m)}$ of (0.1) will converge uniformly on every compact subset of $R^n \times (0, T)$ to the unique maximal solution v of (0.2) where T is given by (0.3).*

Proof. We first consider the case $n = 2$. For any $0 < m \le m_2$, we have $(1-m)^{-1} < \alpha < 2(1-m)^{-1}$ and $\alpha(1-m) - 1 \ge \alpha(1-m_2) - 1$. Thus $0 < k < (4\alpha)^{-1/(\alpha(1-m_2)-1)} \le (4\alpha)^{-1/(\alpha(1-m)-1)}$ for all $0 < m \le m_2$. By Theorem 1.2 and Corollary 1.4,

$$(\varepsilon^{1-m} - (1-m)t)_+^{1/1-m}(k + |x|^2)^{-\alpha} \\ \le u^{(m)}(x,t) \le M \quad \forall x \in R^2, t > 0, 0 < m \le m_2.$$

Since $(\varepsilon^{1-m} - (1-m)t)_+^{1/1-m} \ge \varepsilon/2$ for all $0 < t \le \varepsilon/2$ and $0 < m \le 1/2$,

$$M \ge u^{(m)}(x,t) \ge C_K > 0 \quad (x,t) \in K$$

for any compact subset K of $R^2 \times (0, \varepsilon/2)$. Let

$$T_0 = \max\left\{ s > 0 : \liminf_{m \to 0} u^{(m)}(x_0, s) > 0 \ \text{ for some } \ x_0 \in R^n \right\}.$$

Then $T_0 \ge \varepsilon/2 > 0$. For any $0 < t_1 < t_2 < T_0$, let $t_0 = (t_2 + T_0)/2$. Then there exists $x_0 \in R^n$ such that

$$\liminf_{m \to 0} u^{(m)}(x_0, t_0) > 0. \tag{2.10}$$

By Lemma 2.2 for any $R > 0$ there exists $0 < m_0 < m_2$ such that (2.4) holds. Then there exists constants $C_2 > 0$, $C_3 > 0$ independent of $0 < m < m_0$ such that

$$C_2 \le u^{(m)m-1}(x,t) \le C_3 \quad \forall x \in B_R, t_1 \le t \le t_2.$$

Hence (0.1) is uniformly parabolic for $\{u^{(m)}\}_{0<m<m_0}$ on any compact subset of $R^2 \times (0, T_0)$. By standard parabolic theory [**LSU**], $\{u^{(m)}\}_{0<m\le m_0}$ is uniformly equi-Holder continuous on any compact subset of $R^2 \times (0, T_0)$. Let $\{u^{(m_i)}\}$, $m_i \to 0$ as $i \to 0$, be a sequence of $u^{(m)}$. By the Ascoli Theorem and a diagonalization argument $u^{(m_i)}$ will have a subsequence $u^{(m_i')}$ converging uniformly on every compact subset K of $R^2 \times (0, T_0)$ to a continuous function v satisfying $M \ge v \ge C_K > 0$ on K for some constant $C_K > 0$. Without loss of generality we may assume $u^{(m_i)}$ converges uniformly on every compact subset K of $R^2 \times (0, T_0)$ to v as $i \to \infty$. Since $u^{(m_i)}$ satisfies (0.1), for any $\eta \in C_0^\infty(R^2 \times (0, T_0))$ we have

$$\int_{t_1'}^{t_2'} \int_{R^2} \left(u^{(m_i)}\eta_t + \frac{u^{(m_i)m_i} - 1}{m_i}\Delta\eta \right) dxds \\ = \int_{R^2} u^{(m_i)}\eta dx \Big|_{t_1'}^{t_2'} \quad \forall 0 < t_1' < t_2' < T_0. \tag{2.11}$$

Now by the mean value theorem, for each $(x,t)\in K$, there exists $\theta=\theta(x,t)$ such that

$$
\begin{aligned}
&\left|\frac{u^{(m_i)m_i}(x,t)-1}{m_i}-\log\,v\right|\\
&\le\left|\frac{u^{(m_i)m_i}(x,t)-1}{m_i}-\log\,u^{(m_i)}\right|+\left|\log\,u^{(m_i)}-\log\,v\right|\\
&\le\left|\frac{e^{m_i\log\,u^{(m_i)}}-1}{m_i}-\log\,u^{(m_i)}\right|+\left|\log\,u^{(m_i)}-\log\,v\right|\\
&=\left|u^{(m_i)\theta m_i}\log\,u^{(m_i)}-\log\,u^{(m_i)}\right|+\left|\log\,u^{(m_i)}-\log\,v\right|\\
&\le\left|u^{(m_i)\theta m_i}-1\right|\left|\log\,u^{(m_i)}\right|+\left|\log\,u^{(m_i)}-\log\,v\right|\\
&\le\max(M^{m_i}-1,|C_K^{m_i}-1|)\cdot\max(\log\,M,|\log\,C_K|)+\left|\log\,u^{(m_i)}-\log\,v\right|\\
&\to 0\quad\text{ as }\; i\to\infty
\end{aligned}
$$

uniformly on K. Hence letting $i\to\infty$ in (2.11), for any $\eta\in C_0^\infty(R^2\times(0,T_0))$ we have

$$\int_{t_1'}^{t_2'}\int_{R^2}\Big(v\eta_t+\log\,v\Delta\eta\Big)dxds=\int_{R^2}v\eta dx\Big|_{t_1'}^{t_2'}\quad\forall 0<t_1'<t_2'<T_0.$$

By standard parabolic theory [**LSU**] $v\in C^\infty(R^2\times(0,T_0))$. Thus v is a solution of (0.3) in $R^2\times(0,T_0)$ with $\phi(v)=\log v$. We next claim that v has initial value u_0. To prove the claim we let $\eta\in C_0^\infty(R^2)$ be such that supp $\eta\subset B_R$ for some constant $R>2$ and fix a $0<t_2<T_0$. Then by the definition of T_0, if $t_0=(t_2+T_0)/2$, then there exists $x_0\in R^2$ such that (2.10) holds. By Lemma 2.2 there exists $\delta>0$ and $0<M_0<\min(m_1,m_2)$ such that

$$u^{(m)}(x,t_2)\ge\delta\quad\forall|x|\le R,0<m<M_0$$

where m_1 is as in Lemma 1.6. By Theorem 1.1, we have for any $0<m<M_0$,

$$
\begin{aligned}
u^{(m)}(x,t)&\ge\frac{u^{(m)}(x,t_2)}{t_2^{1/1-m}}t^{1/1-m}\\
&\ge\frac{\delta}{t_2^{1/1-m}}t^{1/1-m}\ge c_1t^{1/1-m}\quad\forall|x|\le R,0<t\le t_2
\end{aligned}
\tag{2.12}
$$

where $c_1=\delta/(1+T_0)^2$. Hence

$$
\begin{aligned}
&\left|\int_{R^2} u^{(m)}(x,t)\eta(x)dx-\int_{R^2} u_0\eta(x)dx\right| \\
&=\left|\int_0^t\int_{R^2} u_t^{(m)}\eta dxds\right| \\
&=\left|\int_0^t\int_{R^2}\Delta\left(\frac{u^{(m)m}-1}{m}\right)\eta dxds\right| \\
&=\left|\int_0^t\int_{B_R}\left(\frac{u^{(m)m}-1}{m}\right)\Delta\eta dxds\right| \\
&\le\|\Delta\eta\|_{L^\infty}\left(\int_0^t\int_{B_R\cap\{u^{(m)}\ge 1\}}\frac{u^{(m)m}-1}{m}dxds\right. \\
&\qquad\left.+\int_0^t\int_{B_R\cap\{u^{(m)}<1\}}\frac{1-u^{(m)m}}{m}dxds\right) \\
&\le\|\Delta\eta\|_{L^\infty}\left(\int_0^t\int_{B_R\cap\{u^{(m)}\ge 1\}}\frac{M^m-1}{m}dxds\right. \\
&\qquad\left.+\int_0^t\int_{B_R\cap\{u^{(m)}<1\}}\frac{1-(c_1s^{1/1-m})^m}{m}dxds\right) \\
&\le\|\Delta\eta\|_{L^\infty}|B_R|\left(tM\log\, M-\int_0^t(c_1s^{1/1-m})^{\theta m}\log\,(c_1s^{1/1-m})ds\right) \\
&\le C(t+t\log\, t)\quad\forall 0<t<t_2
\end{aligned}
$$

for some constants $C>0$ and $0<\theta<1$ by the mean value theorem. Letting $m=m_i\to 0$, we get

$$\left|\int_{R^2} v(x,t)\eta(x)dx-\int_{R^2} u_0\eta(x)dx\right|\le C(t+t\log\, t)\to 0 \text{ as } t\to 0.$$

Hence v have initial value u_0 and is thus a solution of (0.2) in $R^2\times(0,T_0)$. We next claim that v is the unique maximal solution of (0.2). To prove the claim we observe that by (2.12)

$$
\begin{aligned}
u^{(m)}(x,t)&\ge\frac{2^{2/1-m}\delta}{(1+T_0)^2}\left(\frac{2^m-1}{m}\right)^2\frac{m^2t^{1/1-m}}{|x|^{2/(1-m)}(|x|^m-1)^2} \\
&\ge\frac{Am^2t^{1/1-m}}{|x|^{2/(1-m)}(|x|^m-1)^2}\quad\forall|x|=2, 0<t\le t_2, 0<m\le M_0 \qquad (2.13)
\end{aligned}
$$

where

$$A=\min\left((2/3)^4,\frac{(\log\, 2)^2\delta}{16^8(1+T_0)^2}\right)$$

since $(2^m-1)/m = 2^{\theta m}\log 2 \ge \log 2$ for some constant $0<\theta<1$. By Lemma 1.6, (2.12), and the maximum principle (Lemma 3.4 of [**HP**]), for any $0<m\le M_0$ we have

$$u^{(m)}(x,t) \ge Am^2t^{1/1-m}|x|^{-2/(1-m)}(|x|^m-1)^{-2} \quad \forall 0<t\le t_2, |x|\ge 2 \tag{2.14}$$

$$\Rightarrow \quad v(x,t)=\lim_{i\to\infty}u^{(m_i)}(x,t)\ge \frac{At}{(|x|\log|x|)^2} \quad \forall 0<t\le t_2, |x|\ge 2. \tag{2.15}$$

Then by Theorem 1.8 v is the unique maximal solution of (0.2) in $R^2\times(0,t_2)$ for all $0<t_2<T_0$. Hence v is the unique maximal solution of (0.2) in $R^2\times(0,T_0)$ and $u^{(m)}$ converges uniformly to v on any compact subset of $R^2\times(0,T_0)$ as $m\to 0$. Suppose $T_0<T$ where $T=\int_{R^2}u_0dx/4\pi$. By Theorem 1.5 v can be extended to the unique maximal solution of (0.2) in $R^2\times(0,T)$. Since $v>0$ in $R^2\times(0,T)$ and $v\in C^\infty(R^2\times(0,T))$, there exists a constant $\delta_1>0$ such that

$$v(x,t)\ge\delta_1>0 \quad \forall|x|\le 2, T_0/2\le t\le (T_0+T)/2.$$

Let

$$A_1=\min\left((2/3)^4, \frac{(\log 2)^2(\delta_1/2)}{16^8(1+T_0)^2}\right)$$

and choose $t_3>0$ such that $T_0/(1+A_1)<t_3<T_0$. Since $u^{(m)}\to v$ as $m\to 0$ uniformly on $B_2\times\{t_3\}$. There exists $0<M_1\le m_1$ such that

$$u^{(m)}(x,t_3)\ge\delta_1/2 \quad \forall|x|\le 2, 0<m\le M_1.$$

By the same argument as the proof of (2.14) we get

$$u^{(m)}(x,t)\ge\frac{A_1m^2t^{1/1-m}}{|x|^{2/(1-m)}(|x|^m-1)^2} \quad \forall|x|\ge 2, 0<t\le t_3, 0<m\le M_1.$$

Let $\alpha_m=(1-m)^{-1}+1+m$, $k_1=(2A_1(1+T_0)^2/\delta_1)^{1/\alpha_m}$, and let w be as in Lemma 1.3 with $T_1=(1+A_1)t_3$ and $k=k_1$. Then $1/(1-m)<\alpha_m<2/(1-m)<4$ for $0<m<1/2$ and

$$\alpha_m(1-m)-1=(1+m)(1-m)=1-m^2>\frac{1}{2} \quad \forall 0<m<1/2$$
$$\Rightarrow \quad \frac{1}{\alpha_m(1-m)-1}<2<\frac{8}{\alpha_m}$$
$$\Rightarrow \quad 0<k_1=(2A_1(1+T_0)^2/\delta_1)^{1/\alpha_m}\le 16^{-8/\alpha_m}\le(4\alpha_m)^{-1/\alpha_m(1-m)-1}.$$

Since by the mean value theorem there exists $0<\theta<1$ such that

$$\frac{x^m-1}{m}=|x|^{\theta m}\log|x|\le|x|^m\log|x|\le|x|^{1+m} \quad \forall|x|\ge 2$$

and
(2.16)
$$T_1 = (1 + A_1)t_3 \ \Rightarrow \ ((1 - m)(T_1 - t_3)_+)^{1/1-m} \le (A_1 t_3)^{1/1-m} \le A_1 t_3^{1/1-m}$$
hence for all $|x| \ge 2$, $0 < m < M_1$, we have
$$\begin{aligned}\frac{A_1 m^2 t_3^{1/1-m}}{|x|^{2/(1-m)}(|x|^m - 1)^2} &\ge \frac{A_1 t_3^{1/1-m}}{|x|^{2+2m+2/(1-m)}} \ge \frac{((1-m)(T_1 - t_3)_+)^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} \\ \Rightarrow \quad u^{(m)}(x, t_3) &\ge \frac{((1-m)(T_1 - t_3)_+)^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} = w(x, t_3) \\ &\qquad\qquad\qquad \forall |x| \ge 2, 0 < m \le M_1.\end{aligned}$$
Since $k_1^{\alpha_m} = 2A_1(1 + T_0)^2/\delta_1$, by (2.16) for any $|x| \le 2$, $0 < m < M_1$, we have
$$\begin{aligned}u^{(m)}(x, t_3) \ge \frac{\delta_1}{2} = \frac{A_1(1 + T_0)^2}{k_1^{\alpha_m}} &\ge \frac{A_1 t_3^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} \\ &\ge \frac{((1-m)(T_1 - t_3)_+)^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} = w(x, t_3).\end{aligned}$$
Hence
$$u^{(m)}(x, t_3) \ge w(x, t_3) \quad \forall x \in R^2.$$
By the maximum principle,
$$u^{(m)}(x, t) \ge w(x, t) \quad \forall x \in R^2, t_3 \le t < T_1, 0 < m \le M_1.$$
Since $T_1 = (1 + A_1)t_3 > T_0$, we have $(T_0 + T_1)/2 > T_0$ and
$$u^{(m)}(x, (T_0 + T_1)/2) \ge w(x, (T_0 + T_1)/2) > 0 \quad \forall x \in R^2.$$
This contradicts the maximality of T_0. Hence $T_0 = T$ where T is given by (0.3) and the theorem for the case $n = 2$ follows. For $n = 1$ by an argument similar to the case $n = 2$ but with the subsolution u_2 of Lemma 1.7 in place of u_1 of Lemma 1.6 in the argument we get that for $n = 1$ as $m \to 0$ the solution $u^{(m)}$ also converges uniformly on every compact subset of $R \times (0, \infty)$ to the maximal solution v of (0.2) in $R \times (0, \infty)$ and the theorem follows. □

Theorem 2.4. *If $n = 1$, 2, and $u_0 \in L^1(R^n) \cap L^\infty(R^n)$, then as $m \to 0$, the solution $u^{(m)}$ of (0.1) will converge uniformly on every compact subset of $R^n \times (0, T)$ to the unique maximal solution v of (0.2) where T is given by (0.3).*

Proof. Choose $1 < \alpha < 2$, $0 < m_2 < 1/2$, $0 < k < 1$, satisfying the conditions $\alpha > (1 - m_2)^{-1}$ and $0 < k < (4\alpha)^{-1/(\alpha(1-m_2)-1)}$ and let $u_j^{(m)}$ be the solution of (0.1) with initial value

$$u_{0j}(x) = u_0(x) + \frac{1}{j}(k + |x|^2)^{-\alpha} \quad j = 1, 2, \dots$$

Then by Theorem 2.3 for each $j = 1, 2, \cdots, u_j^{(m)}$ will converge uniformly on every compact subset of $R^n \times (0, T_j)$ to the unique maximal solution v_j of (0.2) with initial value u_{0j} as $m \to 0$ where

$$T_j = \begin{cases} \infty & \text{if } n = 1 \\ \frac{1}{4\pi}\int_{R^2} u_{0,j}(x)dx & \text{if } n = 2. \end{cases}$$

Suppose the theorem is false. Then there exists $\varepsilon > 0$, $R > 0$, $0 < t_1 < t_2 < T$ and a sequence $u^{(m_i)}$, $m_i \to 0$ as $i \to \infty$, of $u^{(m)}$, such that

$$\|u^{(m_i)} - v\|_{L^\infty(B_R \times (t_1,t_2))} \ge \varepsilon \quad \forall i = 1, 2, \dots \tag{2.17}$$

where v is the unique maximal solution of (0.2) with initial value u_0. Since $\|u^{(m)}\|_{L^\infty} \le \|u_0\|_{L^\infty}$, $u^{(m_i)}$ will have a subsequence $u^{(m_i')}$ converging weakly in $L^\infty(R^n \times (0, T))$ and a.e. $(x, t) \in R^n \times (0, T)$ to some function $\widetilde{v}$ as $i \to \infty$. By the uniqueness of maximal solution and comparsion principle for maximal solution of (0.2) (Lemma 4.2 of [**H**] or Lemma 4.1 of [**DP**] for $n = 2$ and [**ERV**] for $n = 1$) we have $v_1 \ge v_2 \ge \cdots \ge v > 0$ in $R^n \times (0, T)$ Hence v_j are uniformly bounded below by some positive constant on any compact subset of $R^n \times (0, T)$. Thus (0.4) with $\phi(u) = \log u$ is uniformly parabolic for v_j on any compact subset of $R^n \times (0, T)$. Hence v_j are equi-Holder continuous on any compact subset of $R^n \times (0, T)$ and v_j will converge uniformly on any compact subset of $R^n \times (0, T)$ to some function $\overline{v} \ge v$ as $j \to \infty$.

By Theorem 1.2 and Fatou's lemma, for any $0 < t < T$ we have

$$\begin{aligned}
&\int_{R^n} \left|u_j^{(m)}(x,t) - u^{(m)}(x,t)\right| dx \le \int_{R^n} |u_{0j} - u_0| dx \\
&\le \frac{1}{j}\int_{R^n} (k + |x|^2)^{-\alpha} dx \le \frac{C}{j} \\
&\Rightarrow \quad \int_0^T \int_{R^n} \left|u_j^{(m)}(x,t) - u^{(m)}(x,t)\right| dxdt \le \frac{CT}{j} \\
&\Rightarrow \quad \int_0^T \int_{R^n} |v_j(x,t) - \widetilde{v}(x,t)| dxdt \le \frac{CT}{j} \qquad \text{as } m = m_i' \to 0 \\
&\Rightarrow \quad \int_0^T \int_{R^n} |\overline{v}(x,t) - \widetilde{v}(x,t)| dxdt = 0 \qquad \text{as } j \to \infty \\
&\Rightarrow \quad \overline{v}(x,t) = \widetilde{v}(x,t) \qquad\qquad \text{a.e. } (x,t) \in R^n \times (0,T).
\end{aligned}$$

Since $\overline{v} \ge v > 0$ in $R^n \times (0,T)$, $\widetilde{v}(x,t) > 0$ for a.e. $(x,t) \in R^n \times (0,T)$. Thus there exists a set $E \subset R^n \times (0,T)$ of measure zero such that

$$\lim_{i\to\infty} u^{(m_i')}(x,t) = \widetilde{v}(x,t) = \overline{v}(x,t) \ge v(x,t) > 0 \quad \forall (x,t) \in (R^n \times (0,T))\backslash E.$$

Hence for any $R' > 0$, $0 < t_1' < t_2' < T$, there exist $x_0 \in R^n$, $t_2' < t_0 < T$, such that

$$\lim_{i\to\infty} u^{(m_i)}(x_0,t_0) > 0.$$

By Lemma 2.2 there exists a constant $C > 0$ such that

$$\|u_0\|_{L^\infty} \ge u^{(m_i')}(x,t) \ge C > 0 \quad \forall x \in B_{R'}, t_1' \le t \le t_2', i = 1,2,\dots.$$

Then there exists constants $C_1 > 0$, $C_2 > 0$, independent of m_i' such that

$$C_1 \le u^{(m_i')m_i'-1}(x,t) \le C_2 \quad \forall x \in B_{R'}, t_1' \le t \le t_2', i = 1,2,\dots.$$

Hence (1.2) is uniformly parabolic for $u^{(m_i')}$. By standard parabolic theory [**LSU**], $\{u^{(m_i')}\}$ is uniformly equi-Holder continuous on any compact subset of $R^2 \times (0,T)$. By the Ascoli Theorem and a diagonalization argument similar to the proof of Theorem 2.3 $u^{(m_i')}$ will have a subsequence converging uniformly on every compact subset of $R^2 \times (0,T)$ to the maximal solution v of (0.2). This contradicts (2.17). Hence the theorem must be true and $u^{(m)}$ converges uniformly on every compact subset of $R^2 \times (0,T)$ to the maximal solution v of (0.2) as $m \to 0$ and we are done. □

By the proof of Theorems 2.3 and 2.4 we have the following corollary:

Corollary 2.5. *If v is the maximal solution of (0.2) in $R^n \times (0,T)$, $n = 1$, 2, where T is given by (0.3), then for any $0 < T_0 < T$ there exists a constant $C > 0$ such that*

$$v(x,t) \ge \begin{cases} \frac{Ct}{|x|^2} & \forall |x| \ge 2, 0 < t \le T_0 \quad \text{if } n = 1 \\ \frac{Ct}{(|x|\log|x|)^2} & \forall |x| \ge 2, 0 < t \le T_0 \quad \text{if } n = 2. \end{cases}$$

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