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MAXIMALITY OF THE MICROSTATES FREE ENTROPY FOR R-DIAGONAL ELEMENTS

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A non-commutative non-self adjoint random variable z is called R-diagonal, if its $*$-distribution is invariant under multiplication by free unitaries: if a unitary w is $*$-free from z, then the $*$-distribution of z is the same as that of wz. Using Voiculescu's microstates definition of free entropy, we show that the R-diagonal elements are characterized as having the largest free entropy among all variables y with a fixed distribution of y^*y. More generally, let Z be a $d \times d$ matrix whose entries are non-commutative random variables X_{ij}, $1 \leq i, j \leq d$. Then the free entropy of the family $\{X_{ij}\}_{ij}$ of the entries of Z is maximal among all Z with a fixed distribution of Z^*Z, if and only if Z is R-diagonal and is $*$-free from the algebra of scalar $d \times d$ matrices. The results of this paper are analogous to the results of our paper [3], where we considered the same problems in the framework of the non-microstates definition of entropy.

1. Introduction.

Let (M, τ) be a tracial non-commutative W^*-probability space. A (non-self-adjoint) element $z \in M$ is called R-diagonal if its $*$-distribution is invariant under multiplication by free unitaries; i.e., if w is a unitary, $*$-free from z, the $*$-distributions of wz and z coincide. The concept of R-diagonality was introduced in [4], where it was shown to be equivalent to several conditions; we mention that if z^*z has a (possibly unbounded) inverse (in particular, if the distribution of z^*z is non-atomic), then z is R-diagonal if and only if in its polar decomposition $z = u(z^*z)^{1/2}$, u is $*$-free from $(z^*z)^{1/2}$ and satisfies $\tau(u^k) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$.

In our recent paper [3] R-diagonal elements appeared in connection with certain maximization problems in free entropy. Free entropy was introduced by Voiculescu in [8]; later, a different definition was given by him in [10]. The first definition involves approximating the given n-tuple of variables using finite-dimensional matrices (so-called microstates); the normalized limit of the logarithms of volumes of all such possible microstates is then the free entropy. On the other hand, Voiculescu's definition in [10] does not involve microstates, but uses free Fisher information measure and non-commutative Hilbert transform. At present it is not known whether the two definitions of free entropy always give the same quantity. Our approach in [3] used the second definition of Voiculescu.

In this paper we prove two theorems for the microstates free entropy, which are analogous to our results in [3] for the second (non-microstates) definition of entropy. One of our results can be interpreted as saying that R-diagonal elements z are characterized by the statement that the free entropy $\chi(z)$ is maximal among all possible $\chi(y)$, so that the distributions of y^*y and z^*z are the same.

When this paper was almost finished we received a preprint of Hiai and Petz [1], where the same kind of problems were considered.

If $Y_1, \dots, Y_n \in M$ (not necessarily self-adjoint), we denote by $\chi(Y_1, ..., Y_n)$ the free entropy of $Y_1, \dots, Y_n$ as defined by Voiculescu in [11]. We denote by $\chi^{\mathrm{sa}}(X_1, \dots, X_n)$ for $X_i \in M$ self-adjoint the free entropy of a self-adjoint n-tuple as defined in [8]; we give a brief review of these quantities below in §2.3. A unitary u in a non-commutative probability space (M, τ) is called a Haar unitary if $\tau(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Theorem 1. *Let $y \in M$, and let $u \in M$ be a Haar unitary which is $*$-free from $b = (y^*y)^{1/2}$. Let x be an element such that $\tau(x^{2k}) = \tau(b^{2k})$ and $\tau(x^{2k+1}) = 0$, for all $k \in \mathbb{N}$ (i.e., x is symmetric). Then:*

(a) $\chi(y) \leq \chi(ub)$.
(b) $\chi(ub) = \chi^{\mathrm{sa}}(b^2/2) + 3/4 + 1/2 \log 2\pi = 2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}x)$.
(c) *If $\chi(y) = \chi(ub) > -\infty$, then y is R-diagonal, i.e., in the polar decomposition $y = vb$ we have: v is a Haar unitary and is $*$-free from b.*

Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter; i.e., a homomorphism from the algebra $C(\mathbb{N})$ of all bounded (continuous) functions on $\mathbb{N}$ to $\mathbb{C}$, which is not given by the evaluation at a point in $\mathbb{N}$. For $d \in \mathbb{N}$ we write $d\omega$ for the free ultrafilter corresponding to the functional $f \mapsto \lim_{n\to\omega} f(dn)$. Given ω, one can construct (see [11] and see also a brief review below) free entropy quantities $\chi^{\mathrm{sa}\omega}$ and χ^{ω}, which have properties similar to those of χ^{sa} and χ; it is in fact not known whether these quantities are different. It is known that in the one-variable case, $\chi^{\mathrm{sa}}(X) = \chi^{\mathrm{sa}\omega}(X)$.

Theorem 2. *Let X_{ij}, $1 \leq i, j \leq d$ be a family of non-commutative random variables in a tracial non-commutative probability space $(M, \hat{\tau})$. Let $Z \in M \otimes M_d$ be given by*

$$Z = \sum_{i,j=1}^{d} X_{ij} \otimes e_{ij},$$

where e_{ij} are matrix units in the algebra of $d \times d$ matrices. We denote by τ the normalized trace on $M \otimes M_d$. Let ω be a free ultrafilter. Let X

be a self-adjoint variable with $\tau(X^{2n+1}) = 0$ *for all* $n \in \mathbb{N}$*, and such that* $\tau(X^{2n}) = \tau((Z^*Z)^n)$, $\forall n \in \mathbb{N}$*. Then we have:*

(a) $\chi^{\omega}(\{X_{ij}\}_{1\leq i,j\leq d}) \leq d^2\chi^{d\omega}(Z) - d^2\log d \leq 2d^2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) - d^2\log d$.

(b) *If* Z *is* R*-diagonal and* $*$*-free from the algebra* $1 \otimes M_d$*, then* $\chi^{d\omega}(\{X_{ij}\}_{1\leq i,j\leq d}) = d^2\chi^{\omega}(Z) - d^2\log d = 2d^2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) - d^2\log d$.

(c) *If* $\chi^{\omega}(\{X_{ij}\}_{1\leq i,j\leq d}) = 2d^2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) - d^2\log d$ *and* $\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) \neq -\infty$*, then* Z *is* R*-diagonal and is* $*$*-free from the algebra* $1 \otimes M_d$*.*

The proof of the first theorem is quite different in nature than our proof in [**3**] (the microstates-free proof relied on the notion of free entropy with respect to a completely-positive map introduced in [**6**]). On the other hand, the proof of the second theorem is analogous to the one we gave in [**3**], and relies on the microstates analog [**5**] of the relative entropy [**10**] that we used in the microstates-free approach.

2. Maximality of microstates free entropy for R-diagonal pairs.

Let (M, τ) be a tracial W^*-probability space, and $b \in M$ be a fixed positive element. Let $u \in M$ be a Haar unitary which is $*$-free from b. Lastly, let $x \in M$ be such that for all $k \in \mathbb{N}$, $\tau(x^{2k+1}) = 0$ and $\tau(x^{2k}) = \tau(b^{2k})$. The main result of the section is:

Theorem 2.1. *Let* u*,* b *and* x *be as above. Assume that* $y \in M$ *satisfies* $(y^*y)^{1/2} = b$*. Then:*

(a) $\chi(y) \leq \chi(ub)$.

(b) $\chi(ub) = \chi^{\mathrm{sa}}(b^2/2) + 3/4 + 1/2\log 2\pi = 2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}x)$.

(c) *If* $\chi(y) = \chi(ub) > -\infty$*, then* y *is* R*-diagonal, i.e., in the polar decomposition* $y = vb$*, we have:* v *is a Haar unitary and is* $*$*-free from* b*.*

The same conclusions hold for χ^{ω} *in place of* χ*.*

Before starting the proof of the theorem, we fix some notation and definitions.

Notation 2.2. We use the following notation:

- $U(k)$ is the unitary group of $k \times k$ unitary matrices.
- M_k is the set of all $k \times k$ matrices; M_k^{sa} is the set of all self-adjoint matrices in M_k.
- $M_k^+ \subset M_k$ is the set of all positive $k \times k$ matrices.
- μ_k is the normalized Haar measure on $U(k)$; thus $\mu_k(U(k)) = 1$.
- λ_k is the measure on M_k, coming from its Euclidean structure $\langle a, b\rangle = \mathrm{Re}\,\mathrm{Tr}(ab^*)$, where Tr is the usual matrix trace, $\mathrm{Tr}(I) = k$; λ_k^{sa} is the Lebesgue measure on M_k^{sa} coming from its Euclidean structure $\langle a, b\rangle = \mathrm{Re}\,\mathrm{Tr}(ab^*)$.

- λ_k^+ is the measure on M_k^+ coming from its structure of a cone in the Euclidean space of $k \times k$ matrices.
- $P : U(k) \times M_k^+ \to M_k$ is given by $(v, p) \mapsto vp$.
- Ω_k is the canonical volume form on M_k giving rise to Lebesgue measure.
- $\Omega_k^u \wedge \Omega_k^+$ is the canonical volume form on $U(k) \times M_k^+$, giving rise to the product measure $\mu_k \times \lambda_k^+$.
- $\mathfrak{u}(k)$ is the Lie algebra of $U(k)$.
- C_k is the volume of $U(k)$ with respect to the bi-invariant volume form arising from the Euclidean structure on $\mathfrak{u}(k)$ coming from the Killing form $\langle a, b\rangle = \operatorname{Re}\operatorname{Tr}(ab)$.

2.3. Definitions of free entropy. Let $X_1, \ldots, X_n \in M$ be self-adjoint, and $Y_1, \ldots, Y_n \in M$ be not necessarily self-adjoint. Let $\epsilon > 0$, $R > 0$ be real numbers and $k > 0$, $m > 0$ be integers. Then define the sets (cf. [8, 11])

$$\Gamma_R^{\mathrm{sa}}(X_1, \ldots, X_n; m, k, \epsilon) = \Big\{ (x_1, \ldots, x_n) \in (M_k^{\mathrm{sa}})^n : \left| \frac{1}{k}\operatorname{Tr}(x_{i_1} \ldots x_{i_p}) - \tau(X_{i_1} \ldots X_{i_p}) \right| < \epsilon \text{ for all } p \le m, 1 \le i_j \le n, 1 \le j \le p \Big\};$$

$$\Gamma_R(Y_1, \ldots, Y_n; m, k, \epsilon) = \Big\{ (y_1, \ldots, y_n) \in (M_k)^n : \left| \frac{1}{k}\operatorname{Tr}(y_{i_1}^{g_1} \ldots y_{i_p}^{g_p}) - \tau(Y_{i_1}^{g_1} \ldots Y_{i_p}^{g_p}) \right| < \epsilon \text{ for all } p \le m, 1 \le i_j \le n, g_j \in \{*, \cdot\}, 1 \le j \le p \Big\}.$$

Define next

$$\chi^{\mathrm{sa}}(X_1, \ldots, X_n; m, \epsilon) = \limsup_{k\to\infty} \left[\frac{1}{k^2}\log \lambda_k \Gamma_R^{\mathrm{sa}}(X_1, \ldots, X_n; m, k, \epsilon) + \frac{n}{2}\log k \right]$$

and similarly

$$\chi(Y_1, \ldots, Y_n; m, \epsilon) = \limsup_{k\to\infty} \left[\frac{1}{k^2}\log \lambda_k \Gamma_R(Y_1, \ldots, Y_n; m, k, \epsilon) + n\log k \right].$$

For ω a free ultrafilter on $\mathbb{N}$, the quantities $\chi^{\omega}(Y_1, \ldots, Y_n; m, \epsilon)$ and $\chi^{\mathrm{sa}\omega}(X_1, \ldots, X_n; m, \epsilon)$ are defined in exactly the same way, except that

$\limsup_{k\to\infty}$ is replaced by $\lim_{k\to\omega}$. Next, the free entropy is defined by

$$\chi^{\mathrm{sa}}(X_1,\ldots,X_n)=\sup_R\inf_{m,\epsilon}\chi^{\mathrm{sa}}(X_1,\ldots,X_n;m,\epsilon);$$

the quantities $\chi^{\mathrm{sa}\omega}$, χ, χ^{ω} are defined in exactly the same way, using in the place of $\chi^{\mathrm{sa}}(\cdots;m,\epsilon)$ the quantities $\chi^{\mathrm{sa}\omega}(\cdots;m,\epsilon)$, $\chi(\cdots;m,\epsilon)$, and $\chi^{\omega}(\cdots;m,\epsilon)$, respectively.

Definition 2.4. Let $(X_R(k,m,\epsilon),\mu^X_{R,k,m,\epsilon})$ and $(Y_R(k,m,\epsilon),\mu^Y_{R,k,m,\epsilon})$ be two sequences of measure spaces depending on $k,m\in\mathbb{N}$ and $R,\epsilon\in(0,+\infty)$. We shall say that X is asymptotically included in Y, if for all m, ϵ, R, there is k_0, $m'\geq m$, $\epsilon'\leq\epsilon$, $R'>R$, such that for all $k>k_0$, there is a map

$$\phi=\phi_{R',k,m',\epsilon'}:X_{R'}(k,m',\epsilon')\to Y_R(k,m,\epsilon),$$

which is measure preserving. We say that X and Y are asymptotically equal, if both X is asymptotically included in Y and Y is asymptotically included in X.

Remark 2.5. Note that if X is asymptotically included into Y, we obtain that

$$\begin{aligned}&\sup_R\inf_{m,\epsilon}\limsup_k\alpha_k\log\mu^X_{R,k,m,\epsilon}(X_R(k,m,\epsilon))+a_k\\&\leq\sup_R\inf_{m,\epsilon}\limsup_k\alpha_k\log\mu^Y_{R,k,m,\epsilon}(Y_R(k,m,\epsilon))+a_k,\end{aligned}$$

for all sequences a_k, α_k.

It is not hard to see that the sets

$$\Gamma_R(Y_1,\ldots,Y_n;k,m,\epsilon)$$

and

$$\Gamma^{\mathrm{sa}}_R(\mathrm{Re}(Y_1),\mathrm{Im}(Y_1),\ldots,\mathrm{Re}(Y_n),\mathrm{Im}(Y_n);k,m,\epsilon)$$

are asymptotically equal; the relevant maps ϕ send the n-tuple $(y_1,\ldots,y_n)$ of non-self-adjoint matrices to the $2n$-tuples of self-adjoint matrices $(\mathrm{Re}(y_1),\mathrm{Im}(y_1),\ldots,\mathrm{Re}(y_n),\mathrm{Im}(y_n))$. This implies (using the Remark 2.5) that

$$\chi(Y_1,\ldots,Y_n)=\chi^{\mathrm{sa}}(\mathrm{Re}(Y_1),\mathrm{Im}(Y_1),\ldots,\mathrm{Re}(Y_n),\mathrm{Im}(Y_n)).$$

We proceed to prove several lemmas that will be used in the proof of the main theorem.

Lemma 2.6. *Let $\Gamma\subset M_k^+$ and $U_k\subset U(k)$ be measurable sets. Let*

$$U_k\Gamma=\{vp:v\in U_k,p\in\Gamma\}\qquad\text{and}\qquad S(\Gamma)=\left\{\frac{p^2}{2}:p\in\Gamma\right\}.$$

Then

$$\lambda_k(U_k\Gamma)=C_k\,\mu_k(U_k)\lambda_k^+(S(\Gamma)).$$

In other words, the map $Q : (v, p) \mapsto v\sqrt{2p}$ from $U(k) \times M_k^+$, endowed with the measure $\mu_k \times C_k\lambda_k^+$, to M_k, endowed with the measure λ_k, is measure preserving.

Proof. Since invertible matrices are a set of comeasure zero in M_k, we see by existence of polar decomposition that $P : (v, p) \mapsto vp$ is invertible as a map of measure spaces. We start by computing the pull-back of Lebesgue measure on M_k to $U(k) \times M_k^+$. Note that since P is equivariant with respect to the actions of $U(k)$ by left multiplication, and Lebesgue measure is invariant under this action (since the Euclidean structure is), the resulting measure on $U(k) \times M_k^+$ is the product of Haar measure on $U(k)$ and some measure ν_k on M_k^+, hence $\lambda_k(U_k\Gamma) = \mu_k(U_k)\nu_k(\Gamma)$. It remains to identify ν_k.

We have the equation

$$d\mu_k(v)d\nu_k(p) = (P^*(\Omega_k) : \Omega_k^u \wedge \Omega_k^+)d\mu_k(v)d\lambda_k^+(p), \tag{1}$$

where $P^*(\Omega_k) : \Omega_k^u \wedge \Omega_k^+$ is the ratio of the two volume forms. Furthermore, in view of the mentioned invariance under an action of $U(k)$, it is sufficient to compute $(P^*(\Omega_k) : \Omega_k^u \wedge \Omega_k^+)$ in (1) at the point $(1, p) \in U(k) \times M_k^+$.

Note that the tangent space $T_{1,p}(U(k) \times M_k^+)$ is isomorphic to the direct sum $\mathfrak{u}(k) \times M_k^{\mathrm{sa}}$, where $\mathfrak{u}(k) = iM_k^{\mathrm{sa}}$ is the Lie algebra of $U(k)$. Identify $T_{(1,p)}(U(k) \times M_k^+) = iM_k^{\mathrm{sa}} \oplus M_k^{\mathrm{sa}}$ with $M_k = T_p(M_k)$. Then the inner product given by the trace $\langle a, b\rangle = \operatorname{Re}\operatorname{Tr}(ab^*)$ defines on $T_{1,p}$ a Euclidean structure, for which the subspaces M_k^{sa} and iM_k^{sa} are perpendicular. Since the restriction of this inner product to $\mathfrak{u}(k)$ is the Killing form on this Lie algebra, and the restriction to $T_pM_k^+$ is the inner product we chose before on this space, Ω_k (which via the above identification is a volume form on $U(k) \times M_k^+$) has the form $C_k\Omega_k^u \wedge \Omega_k^+$. Further, C_k is the ratio of the volume form on $U(k)$ arising from the Euclidean structure on $\mathfrak{u}(k)$ coming from the Killing form and the volume form corresponding to the normalized Haar measure. Hence C_k is just the volume of $U(k)$ with respect to the volume form arising from the Euclidean structure on $\mathfrak{u}(k)$ coming from the Killing form.

Thus from (1) we get that

$$d\nu_k(p)d\mu_k(v) = C_k d\mu_k(v)\det(DP)(p)d\lambda_k^+(p).$$

It remains to compute DP. We note that P is the identity map restricted to M_k^+. Choose a basis in which p is diagonal with eigenvalues $l_1, \ldots, l_k$, and let $e_{ij} \in M_k$ be the matrix all of whose entries are zero, except that the i, j-th entry is 1. Consider the orthonormal basis $\xi_{\alpha\beta}$ for iM_k^{sa}, given by:

$$\xi_{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{2}}(e_{\alpha\beta} - e_{\beta\alpha}) & \text{if } \alpha < \beta \\ ie_{\alpha\alpha} & \text{if } \alpha = \beta \\ i\frac{1}{\sqrt{2}}(e_{\alpha\beta} + e_{\beta\alpha}) & \text{if } \alpha > \beta. \end{cases}$$

Then

$$DP(\xi_{\alpha\beta})p = \xi_{\alpha\beta}p = \frac{1}{2}(l_\alpha + l_\beta)\xi_{\alpha\beta} + \frac{1}{2}(l_\alpha - l_\beta)\eta_{\alpha\beta}, \qquad \eta_{\alpha\beta} \in M_k^{\mathrm{sa}}.$$

It follows that

$$\det(DP)(p) = \frac{1}{2^{k^2}} \prod_{\alpha,\beta=1}^{k} (l_\alpha + l_\beta).$$

Hence we record the final answer:

$$d\nu_k(p) = C_k 2^{-k^2} \prod_{\alpha,\beta=1}^{k} (l_\alpha + l_\beta) d\lambda_k^+(p)$$

where l_i are the eigenvalues of p.

Consider the map $S : p \mapsto \frac{p^2}{2}$ from M_k^+ to itself. This map is a.e. invertible; moreover, its Jacobian $\det(DS)$ at p is given by $\det(\frac{1}{2}(1 \otimes p + p \otimes 1))$, where $1 \otimes p$ and $p \otimes 1$ are viewed as elements of $M_k \otimes M_k \cong M_{k^2}$ (see e.g. [8]). To compute this determinant, let ζ_i, $i = 1, \dots, k$ be orthonormal eigenvectors of p, such that $p\zeta_i = l_i\zeta_i$. Then $\zeta_i \otimes \zeta_j$ is an orthonormal basis for $\mathbb{C}^{k^2}$, on which $M_{k^2} = M_k \otimes M_k$ acts naturally. Moreover, $\frac{1}{2}(1 \otimes p + p \otimes 1)(\zeta_i \otimes \zeta_j) = \frac{1}{2}(l_i + l_j)\zeta_i \otimes \zeta_j$. So the determinant is $2^{-k^2} \prod_{\alpha,\beta=1}^{k}(l_\alpha + l_\beta)$. Hence the push-forward of ν_k by S is given by

$$d(S_*\nu_k)(p) = C_k 2^{-k^2} \prod_{\alpha,\beta=1}^{k} (l_\alpha + l_\beta) d\lambda_k^+(p) \cdot \det(DS)^{-1}(p) = C_k d\lambda_k^+(p).$$

Thus we have

$$S_*\nu_k = C_k \lambda_k^+,$$

which is our assertion. □

We have the following standard lemma (see [8]).

Lemma 2.7. *Let p be a positive element in M. Then the sequences of sets $\Gamma_R^{\mathrm{sa}}(p, m, k, \epsilon)$ and $\Gamma_R^{\mathrm{sa}}(p, m, k, \epsilon) \cap M_k^+$, each taken with the measure λ_k, are asymptotically equal.*

Lemma 2.8. $\lim_k \frac{1}{k^2} \log(C_k) + \frac{1}{2} \log k = \frac{3}{4} + \frac{1}{2} \log 2\pi$.

In this exact form this lemma can be found, for example, in [2] (the reader is cautioned that the cited paper uses a slightly different normalization of the Killing form, different from ours by a factor).

Lemma 2.9. *Let $y \in (M, \tau)$ be a (not necessarily self-adjoint) random variable. Then*

$$\chi(y) \leq \chi^{\mathrm{sa}}\left(\frac{y^*y}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi.$$

Proof. Denote by $S : M_k \to M_k^+$ the map

$$y \mapsto \frac{y^* y}{2}.$$

Note that

$$S(\Gamma_R(y; m, k, \epsilon)) \subset \Gamma_{R^2}^{\mathrm{sa}}\left(\frac{y^* y}{2}; m/2, k, \epsilon\right),$$

hence the former is asymptotically included in the latter. Note that

$$\Gamma_R(y; m, k, \epsilon) \subset U(k)\Gamma_R(y; m, k, \epsilon).$$

We therefore get

$$\begin{aligned}
\lambda_k(\Gamma_R(y; m, k, \epsilon)) &\leq \lambda_k(U(k)\Gamma_R(y; m, k, \epsilon)) \\
&\leq \lambda_k(U(k)\{(a^* a)^{1/2} : a \in \Gamma_R(y; m, k, \epsilon)\}) \\
&\leq C_k \lambda_k(S(\Gamma_R(y; m, k, \epsilon))) \\
&\leq C_k \lambda_k\left(\Gamma_{R^2}^{\mathrm{sa}}\left(\frac{y^* y}{2}; \frac{m}{2}, k, \epsilon\right)\right).
\end{aligned}$$

Taking the logarithm and passing to the limits gives the result. □

Lemma 2.10. *Let $u, b \in (M, \tau)$ be such that u is a Haar unitary $*$-free from the positive element b. Let $z = ub$. Given $\delta > 0$, there exists k_0, such that for all $k > k_0$, there is a subset $X_k \subset U(k) \times \left(\Gamma_R^{\mathrm{sa}}(\frac{z^* z}{2}; m, k, \epsilon) \cap M_k^+\right)$,*

$$\log \frac{\mu_k \times \lambda_k^+(X_k)}{\mu_k \times \lambda_k^+(U_k \times \Gamma_R^{\mathrm{sa}}(\frac{z^* z}{2}; m, k, \epsilon))} \geq -\delta,$$

such the map

$$Q : (v, p) \mapsto v\sqrt{2p} \tag{2}$$

is an asymptotic inclusion of X_k, endowed with the measure $\mu_k \times C_k \lambda_k^+$, into $\Gamma_R(z; m, k, \epsilon)$, endowed with the measure λ_k.

Proof. Note that by Lemma 2.6, the map defined in Equation (2) is measure preserving.

Let $R > 0$, $\epsilon > 0$ and $\delta > 0$ be fixed. For $x \in M_k^+$, let $U_k(x, \epsilon) \subset U(k)$ be the maximal set of unitaries, for which $U_k(x, \epsilon) \cdot x \in \Gamma_R(wx; m, k, \epsilon)$, where w is a Haar unitary $*$-free from x (in other words, "elements of $U_k(x, \epsilon)$ and x are $*$-free to order m, ϵ"). Note that $U_k(x, \epsilon)$ is open. By Corollary 2.12 of [**11**], there exists k_0, such that for all $k > k_0$, and any $x \in M_k^+$, $\|x\| < R$, $\log \mu_k(U_k(x, \epsilon/2)) > -\delta$. Let

$$\hat{X}_k = \bigcup_{x \in \Gamma_R^{\mathrm{sa}}(\frac{z^* z}{2}; m, k, \epsilon) \cap M_k^+} U_k(x, \epsilon) \times \{x\}.$$

Since whenever $x \in \Gamma_R^{\mathrm{sa}}\left(\frac{z^* z}{2}; m, k, \epsilon\right) \cap M_k^+$, $U_k(x) \cdot \sqrt{2x} \subset \Gamma_R(z; m, k, \epsilon)$, $Q(\hat{X}_k)$ lies in $\Gamma_R(z; m, k, \epsilon)$.

We claim that there exists a measurable subset $X_k \subset \hat{X}_k$ of measure at least $\exp(-\delta)$ times that of $\Gamma_R^{\mathrm{sa}}(\frac{z^*z}{2}; m, k, \epsilon)$. First, it is sufficient to show (because of Lemma 2.7) that the measure of X_k is at least $\exp(-\delta)$ times that of $\Gamma_R^{\mathrm{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+$. Next, let $x \in \Gamma_R^{\mathrm{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+$, and let $V(x)$ be an open neighborhood of x for the norm topology. Then if $V(x)$ is sufficiently small, for all $x' \in V$, $U_k(x, \epsilon/2) \subset U_k(x', \epsilon)$. Hence

$$O(x) = U_k(x, \epsilon/2) \times V(x) \subset \hat{X}_k.$$

Moreover, the volume of $O(x)$ is at least $\exp(-\delta)$ times the volume of $V(x)$. Let

$$X_k = \bigcup_x O(x).$$

Then X_k is open, and its volume is at least $\exp(-\delta)$ times that of $\Gamma_R^{\mathrm{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+$. □

Proof of 2.1(a) *and* 2.1(b) *in Theorem* 2.1. Assume that x, u and b are as in the statement of Theorem 2.1(b) and let $z = ub$; note that z is R-diagonal. By Lemma 2.10 and Lemma 2.8, we have that

$$\chi^{\mathrm{sa}}\left(\frac{z^*z}{2}\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi \leq \chi(z).$$

Since, by Lemma 2.9, we always have the other inequality, we obtain

$$\chi(z) = \chi^{\mathrm{sa}}\left(\frac{z^*z}{2}\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi. \tag{3}$$

This can be expressed in terms of the free entropy of the symmetric variable x as follows (by using the explicit formula for χ^{sa} of one variable given by Voiculescu in [8]):

$$\begin{aligned}
\chi(z) &= \chi^{\mathrm{sa}}\left(\frac{z^*z}{2}\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi \\
&= 2\left(\frac{3}{4} + \frac{1}{2}\log 2\pi\right) + \iint \log|s-t| d\mu_{\frac{z^*z}{2}}(s) d\mu_{\frac{z^*z}{2}}(t) \\
&= 2\left(\frac{3}{4} + \frac{1}{2}\log 2\pi\right) + 2\iint \log|s-t| d\mu_{2^{-1/2}x}(s) d\mu_{2^{-1/2}x}(t) \\
&= 2\chi^{\mathrm{sa}}(2^{-1/2}x).
\end{aligned}$$

This proves 2.1(b).

Combining the above with Lemma 2.9 we get 2.1(a):

$$\begin{aligned}\chi(y) &\le \chi^{\mathrm{sa}}\left(\frac{y^*y}{2}\right)+\frac{3}{4}+\frac{1}{2}\log 2\pi\\ &= \chi^{\mathrm{sa}}\left(\frac{z^*z}{2}\right)+\frac{3}{4}+\frac{1}{2}\log 2\pi\\ &= \chi(z).\end{aligned}$$

□

Proposition 2.11 (A change of variables formula for polar decomposition)**.** *Let $y_1,\dots,y_n$ be elements of a W^*-probability space (M,τ), and let $y_i = v_i(y_i^*y_i)^{1/2}$ be their polar decompositions. Assume that $f_i : [0,+\infty) \to [0,+\infty)$ are C^1-diffeomorphisms, and let $z_i = v_i[2f(y_i^*y_i/2)]^{1/2}$. Then*

$$\chi(z_1,\dots,z_n) = \chi(y_1,\dots,y_n) + \sum_{j=1}^{n}\iint \log\left|\frac{f(s)-f(t)}{s-t}\right| d\mu_i(s)d\mu_i(t), \tag{4}$$

where μ_i is the distribution of $y_i^*y_i/2$ for $i=1,\dots,n$. The same statement holds for χ^ω in the place of χ.

Proof. If for some i the distribution of $y_i^*y_i$ contains atoms, then so does the distribution of $z_i^*z_i$. In this case we have

$$\chi(y_1,\dots,y_n) \le \sum_j \chi(y_j) = -\infty,$$

since by Lemma 2.9, $\chi(y_i) \le \chi^{\mathrm{sa}}(y_i^*y_i/2) + \mathrm{const} = -\infty$. Similarly, $\chi(z_1,\dots, z_n) = -\infty$, and there is nothing to prove. Hence we may assume that the distributions of $y_i^*y_i$, and thus the distributions of $z_i^*z_i$, are non-atomic for all i; in particular, that v_i are unitaries.

We may also assume that f_i for $i \ne 1$ are the identity diffeomorphisms; moreover, by replacing f_i with f_i^{-1}, we only need to prove that the left-hand side of the statement of Equation (4) is greater than or equal to the right hand side. We write $f = f_1$.

Consider the mappings

$$T : M_k \ni x \mapsto v[2f(x^*x/2)]^{1/2} \in M_k,$$

where $x = v(x^*x)^{1/2}$ is the polar decomposition of x, and

$$\hat{T} : M_k^n \ni (x_1,\dots,x_n) \mapsto (T(x_1),x_2,\dots,x_n) \in M_k^n.$$

Note that the set $\hat{T}(\Gamma_R(y_1,\dots,y_n;m,k,\epsilon))$, taken with the measure $\lambda_k \times \cdots \times \lambda_k$ is asymptotically included into the set $\Gamma_R(z_1,\dots,z_n;m,k,\epsilon)$, taken with the same measure. Moreover, the infimum of the Jacobian of $\hat{T}$ on the set $\Gamma_R(y_1,\dots,y_n;m,k,\epsilon)$ is not less than the infimum of the Jacobian of T on the set $\Gamma_R(y_1;m,k,\epsilon)$. View T as a map from $U(k)\times M_k^+$ to itself, using the identification of measure spaces $U(k) \times M_k^+ \cong M_k$, $(v,p) \mapsto vp$. Then T acts trivially on the unitary component. Recall that the measure on M_k^+, arising from the identification of M_k with $U(k) \times M_k^+$, is the push-forward of Lebesgue measure on M_k^+ to M_k^+ by the map $p \mapsto p^2/2$. Hence the infimum of the Jacobian of T is equal to the infimum of the Jacobian of the map $p \mapsto [2f(p^2/2)]^{1/2}$ viewed as a map from M_k^+ endowed with Lebesgue measure to itself, on the set $\Gamma_R(y_1^* y_1/2; m, k, \epsilon)$. The rest of the computation is exactly as in the proof of Proposition 3.1 of [**9**]. □

Remark 2.12. Let $B \subset M$ be a subalgebra of M. The proof of the proposition above also works if we replace $\chi(\cdot)$ with the relative entropy $\chi(\cdot|B)$ introduced in [**5**]; we leave the details to the reader.

Proof of 2.1(c) *of Theorem* 2.1. Assume that $\chi(y) = \chi(ub) > -\infty$. Because of part 2.1(b), we conclude that $\chi(b) > -\infty$; in particular, the distribution of b is non-atomic (see [**8**]). Since $(y^*y)^{1/2} = b$, this implies that in the polar decomposition of $y = v(y^*y)^{1/2}$, v is a unitary.

Arguing as in Lemma 4.2 of [**9**], we may assume that there exists a family f_i of C^1 diffeomorphisms on $[0, +\infty)$, and a continuous function $f : [0, +\infty) \to [0, +\infty)$, such that $f(\frac{y^*y}{2})$ is the square of a $(0, 1)$-semicircular random variable, $\|f_j(y^*y) - f(y^*y)\| \to 0$ as $j \to \infty$, $W^*(y^*y) = W^*(f(y^*y))$, and $\lim_j \chi^{\rm sa}(f_j(y^*y)) = \chi^{\rm sa}(f(y^*y))$. Let $y = v(y^*y)^{1/2}$ be the polar decomposition of y; let $z = v[2f(y^*y/2)]^{1/2}$, and similarly $z_j = v[2f_j(y^*y/2)]^{1/2}$. Then by Proposition 2.11 and the explicit formula for the free entropy of one variable given by Voiculescu (Proposition 4.5 in [**8**]), we get for all j,

$$\chi(z_j) = \chi(y) + \chi^{\rm sa}\left(f_j\left(\frac{y^*y}{2}\right)\right) - \chi^{\rm sa}\left(\frac{y^*y}{2}\right).$$

Applying Proposition 2.6 of [**8**], we get that

$$\begin{aligned} \chi(z) &\geq \limsup_j \chi(z_j) \\ &= \limsup_j \left[\chi(y) + \chi^{\rm sa}\left(f_j\left(\frac{y^*y}{2}\right)\right) - \chi^{\rm sa}\left(\frac{y^*y}{2}\right)\right] \\ &= \chi(y) + \chi^{\rm sa}\left(f\left(\frac{y^*y}{2}\right)\right) - \chi^{\rm sa}\left(\frac{y^*y}{2}\right). \end{aligned}$$

Since $\chi(y) = \chi(ub)$ by assumption, and $\chi(ub) = \chi^{\rm sa}(\frac{y^*y}{2}) + \frac{3}{4} + \frac{1}{2}\log 2\pi$ by Theorem 2.1(b) we get that

$$\chi(z) \geq \chi^{\rm sa}\left(f\left(\frac{y^*y}{2}\right)\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi.$$

By assumption, the distribution of $(z^*z)^{1/2}$ is quarter-circular (i.e., it is the absolute value of a $(0, 2)$-semicircular). Let c be a circular variable (i.e., its real and imaginary parts are free $(0,1)$-semicircular variables). Then, since c is R-diagonal (see [4]), we have by 2.1(b), that

$$\begin{aligned}\chi(c) &= \chi^{\mathrm{sa}}\left(\frac{c^*c}{2}\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi \\ &= \chi^{\mathrm{sa}}\left(f\left(\frac{y^*y}{2}\right)\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi,\end{aligned}$$

since c^*c has the same distribution as $z^*z = 2f(y^*y/2)$. Hence $\chi(z) \geq \chi(c)$.

On the other hand, c is R-diagonal, with the same distribution of the positive part as z, so by 2.1(a), we have $\chi(z) \leq \chi(c)$. So $\chi(z) = \chi(c)$.

We claim that z is circular. This will prove the proposition, since then the polar and positive parts of z are $*$-free (see [7] or [4]), and thus the polar and positive parts of y are $*$-free, since the polar part of y is the same as the polar part of z, and the positive part of y is some function of the positive part of z.

Now, for the claim that z is circular, let γ be a complex number of modulus one; then $\chi(\gamma z) = \chi(z)$. Let

$$X_\gamma = \frac{1}{2}(\gamma z + \overline{\gamma} z^*), \quad Y_\gamma = \frac{1}{2i}(\gamma z - \overline{\gamma} z^*).$$

Then

$$\tau(X_\gamma^2) = \frac{1}{4}\left[2\tau(zz^*) + \gamma^2\tau(z^2) + \overline{\gamma^2}\cdot\overline{\tau(z^2)}\right].$$

Similarly,

$$\tau(Y_\gamma^2) = \frac{1}{4}\left[2\tau(zz^*) - \gamma^2\tau(z^2) - \overline{\gamma^2}\cdot\overline{\tau(z^2)}\right].$$

We choose γ such that $\gamma^2\tau(z^2)$ is purely imaginary. Since $\tau(z^*z) = 2$, we have then $\tau(X_\gamma^2) = \tau(Y_\gamma^2) = 1$. But $\chi(z) = \chi(c) = \chi^{\mathrm{sa}}(x_1, x_2)$, where x_i are free $(0,1)$ semicircular variables. Hence we have

$$\chi(z) = \chi^{\mathrm{sa}}(X_\gamma, Y_\gamma) = \chi(\gamma z) = \chi^{\mathrm{sa}}(x_1, x_2),$$

where X_γ and Y_γ are some self-adjoint random variables of covariance 1. But then by Voiculescu's Proposition 2.4 of [9], X_γ and Y_γ are both semicircular and free, so that γz is circular, so z is circular. □

3. Maximization of free entropy for matrices.

Theorem 3.1. *Let X_{ij}, $1 \leq i,j \leq d$ be non-commutative random variables in a tracial non-commutative probability space $(M, \hat{\tau})$. Let $Z \in M \otimes M_d$ be given by*

$$Z = \sum_{i,j=1}^{d} X_{ij} \otimes e_{ij},$$

where e_{ij} are matrix units in the algebra of $d \times d$ matrices. We denote by τ the normalized trace on $M \otimes M_d$. Let ω be a free ultrafilter. Let X

be a self-adjoint variable with $\tau(X^{2n+1}) = 0$ *for all* $n \in \mathbb{N}$*, and such that* $\tau(X^{2n}) = \tau((Z^*Z)^n)$, $\forall n \in \mathbb{N}$*. Then we have:*

(a) $\chi^{d\omega}(\{X_{ij}\}_{1\leq i,j\leq d}) \leq d^2\chi^{\omega}(Z) - d^2\log d \leq 2d^2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) - d^2\log d$.

(b) *If* Z *is* R*-diagonal and* $*$*-free from the algebra* $1 \otimes M_d$*, then* $\chi^{d\omega}(\{X_{ij}\}_{1\leq i,j\leq d}) = d^2\chi^{\omega}(Z) - d^2\log d = 2d^2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) - d^2\log d$.

(c) *If* $\chi^{d\omega}(\{X_{ij}\}_{1\leq i,j\leq d}) = 2d^2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) - d^2\log d$*, and* $\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) \neq -\infty$*, then* Z *is* R*-diagonal and is* $*$*-free from the algebra* $1 \otimes M_d$*.*

Proof. Let $B = 1 \otimes M_d$. We have by [**5**] that

$$\chi^{d\omega}(\{X_{ij}\}) = d^2\chi^{\omega}(Z|B) - d^2\log d \leq d^2\chi^{\omega}(Z) - d^2\log d.$$

(We have the summand $-d^2\log d$ rather than $-\frac{d^2}{2}\log d$ appearing above because we are dealing with χ, not χ^{sa}.) Moreover, $d^2\chi^{\omega}(Z) \leq 2d^2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X)$ by Theorem 2.1, hence 3.1(a).

If Z is $*$-free from B, then, by [**5**], we have $\chi^{\omega}(Z|B) = \chi^{\omega}(Z)$. Moreover, if Z is R-diagonal, we have, by Theorem 2.1, that $\chi^{\omega}(Z) = 2\chi^{\mathrm{sa}}(X/\sqrt{2})$, which proves 3.1(b).

Assuming the conditions in 3.1(c) are satisfied, we get that $\chi^{\omega}(Z) = 2\chi^{\mathrm{sa}}(2^{-\frac{1}{2}}X) > -\infty$, so Z is R-diagonal by Theorem 2.1(c), i.e, Z has polar decomposition $Z = v(Z^*Z)^{1/2}$, where v is a Haar unitary, which is $*$-free from Z^*Z. Note also that we are given that $\chi^{\omega}(Z|B) = \chi^{\omega}(Z)$. We may assume, as in the proof of statement 2.1(c) of Theorem 2.1 that there exists a family f_i of C^1 diffeomorphisms on $[0,+\infty)$, and a continuous function $f : [0,+\infty) \to [0,+\infty)$, such that $f(\frac{Z^*Z}{2})$ is the square of a $(0,1)$-semicircular random variable, $\|f_j(Z^*Z) - f(Z^*Z)\| \to 0$ as $j \to \infty$, $W^*(Z^*Z) = W^*(f(Z^*Z))$, and $\lim_j \chi^{\mathrm{sa}}(f_j(Z^*Z)) = \chi^{\mathrm{sa}}(f(Z^*Z))$. Given the polar decomposition $Z = v(Z^*Z)^{1/2}$, let $z = v[2f(Z^*Z/2)]^{1/2}$, and similarly $z_j = v[2f_j(Z^*Z/2)]^{1/2}$. Notice that z is circular; moreover, since $W^*(Z^*Z) = W^*(f(Z^*Z)) = W^*(z^*z)$, we have that $Z \in W^*(z)$. Hence it will suffice to prove that z is $*$-free from B, as then also Z is $*$-free from B.

By Remark 2.12 and the explicit formula for the free entropy of one variable given by Voiculescu (Proposition 4.5 in [**8**]), we get for all j,

$$\chi^{\omega}(z_j|B) = \chi^{\omega}(Z|B) + \chi^{\mathrm{sa}}\left(f_j\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\mathrm{sa}}\left(\frac{Z^*Z}{2}\right).$$

We get

$$\begin{aligned}\chi^{\omega}(z|B) &\geq \limsup_j \chi^{\omega}(z_j|B)\\ &= \limsup_j \left[\chi^{\omega}(Z|B) + \chi^{\mathrm{sa}}\left(f_j\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\mathrm{sa}}\left(\frac{Z^*Z}{2}\right)\right]\\ &= \chi^{\omega}(Z|B) + \chi^{\mathrm{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\mathrm{sa}}\left(\frac{Z^*Z}{2}\right).\end{aligned}$$

By assumption, we have that $\chi^\omega(Z|B) = \chi^\omega(Z)$; moreover, by R-diagonality of Z we get by Theorem 2.1(b) that $\chi(Z) = \chi^{\mathrm{sa}}(Z^*Z/2)+3/4+(1/2)\log 2\pi$. Therefore, we get that

$$\begin{aligned}
\chi^\omega(z|B) &\geq \chi^{\mathrm{sa}}\left(\frac{Z^*Z}{2}\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi \\
&\quad + \chi^{\mathrm{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\mathrm{sa}}\left(\frac{Z^*Z}{2}\right) \\
&= \frac{3}{4} + \frac{1}{2}\log 2\pi + \chi^{\mathrm{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right).
\end{aligned}$$

But z is circular, in particular R-diagonal; moreover, $z^*z/2 = f(Z^*Z/2)$. So from the formula in 2.1(b), we get that

$$\chi^\omega(z) = \frac{3}{4} + \frac{1}{2}\log 2\pi + \chi^{\mathrm{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right).$$

Thus $\chi^\omega(z|B) \geq \chi^\omega(z)$. Since $\chi^\omega(z|B) \leq \chi^\omega(z)$ in general, we get that $\chi^\omega(z|B) = \chi^\omega(z)$.

Now let S_1, S_2 be the real and imaginary parts of z. Then we have that $\chi^{\mathrm{sa}}(S_1, S_2|B) = \chi^{\mathrm{sa}}(S_1, S_2)$. Since S_1 and S_2 are two free semicircular variables, it follows by Theorem 4.5 from [**5**] that $W^*(S_1, S_2)$ is free from B. Hence z is $*$-free from B; hence Z is $*$-free from B. □

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