FREDHOLM PROPERTIES OF TOEPLITZ OPERATORS ON DIRICHLET SPACES

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In this paper, the Fredholm properties of some Toeplitz operators on Dirichlet spaces be discussed, and the essential spectra of Toeplitz operators with symbols in C^1 or $H_1^\infty + C^1$ be computed.

1. Introduction.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $dA = \frac{1}{\pi}dxdy$ be the normalized area measure on $\mathbb{D}$. $L^{2,1}$ is the space of functions $u : \mathbb{D} \to \mathbb{C}$, for which the norm

$$\|u\|_{\frac{1}{2}} = \left[\int_{\mathbb{D}} \left(\left| \frac{\partial u}{\partial z}(z) \right|^2 + \left| \frac{\partial u}{\partial \bar{z}}(z) \right|^2 \right) dA \right]^{\frac{1}{2}} < \infty.$$

$L^{2,1}$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{\frac{1}{2}} = \left\langle \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right\rangle_{L^2(dA)} + \left\langle \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial \bar{z}} \right\rangle_{L^2(dA)}.$$

The Dirichlet space, $\mathcal{D}$, is the subspace of all analytic functions g in $L^{2,1}$ with $g(0) = 0$. Let P be the orthogonal projection from $L^{2,1}$ into $\mathcal{D}$. P is an integral operator represented by

$$P(u)(w) = \int_{\mathbb{D}} \frac{\partial u}{\partial z} \frac{\overline{\partial K(z,w)}}{\partial z} dA,$$

where $K(z,w) = \sum_{k=1}^{\infty} \frac{z^k \bar{w}^k}{k}$ is the reproducing kernel of $\mathcal{D}$ (see R. Rochberg and Z.J. Wu [**5**] and Wu [**7**]). Let G be a domain in $\mathbb{C}$, define

$$C^1(G) = \left\{ u \mid u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in C(G) \right\},$$

$$C^1(\bar{G}) = \left\{ u \mid u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in C(\bar{G}) \right\},$$

and

$$H_1^\infty(G) = \{ f \in H(G) \mid f' \in H^\infty(G) \},$$

where $C(G)$(or $C(\bar{G})$) and $H(G)$ are respectively the set of continuous functions on G(or $\bar{G}$) and the one of analytic functions on G.

In [**5**] and [**7**], R. Rochberg and Z.J. Wu define the Toeplitz operator with symbol μ as

$$T_\mu f = \int_{\mathbb{D}} f'(z) \overline{\frac{\partial K(z,w)}{\partial z}} d\mu \qquad (\forall f \in \mathcal{D}),$$

where μ is a measure on $\mathbb{D}$. If $d\mu = \varphi dA, \varphi \in L^\infty(\mathbb{D}, dA)$, one can define

$$T_\varphi f = \int_{\mathbb{D}} f'(z) \overline{\frac{\partial K(z,w)}{\partial z}} \varphi dA.$$

We see easily that $T_\varphi f \neq P(\varphi f)$ in general, in fact, $P(\varphi f)$ may be undefined for $\varphi \in L^\infty(\mathbb{D}, dA)$. However, it seems to be more natural that the Toeplitz operators be defined as the form $P(\varphi f)$. In this paper, we try to define the Toeplitz operators with some special symbols as following

Definition 1. Suppose $\varphi \in C^1(\mathbb{D})$, the operator

$$T_\varphi f(w) = P(\varphi f)(w) = \langle \varphi f, K \rangle_{\frac{1}{2}} = \int_{\mathbb{D}} \frac{\partial(\varphi f)}{\partial z} \overline{\frac{\partial K(z,w)}{\partial z}} dA(z)$$

is said to be the Toeplitz operator with symbol φ.

We will compute the spectra and essential spectra of these operators. For convenience, we use frequently the notation $\widetilde{T_\varphi}$ to denote the Toeplitz operator with symbol φ on Bergman space.

By the way, our results are also true for Toeplitz operators on weighted Dirichlet spaces $\mathcal{D}_\alpha$ ($\alpha < 1$). If $\alpha = \frac{1}{2}$, then $\mathcal{D}_{\frac{1}{2}} = \mathcal{D}$, the usual Dirichlet space (c.f. [**5**], [**7**]).

2. Toeplitz operators with symbols in $C^1(\bar{\mathbb{D}})$.

Throughout this paper, we use the symbol "$\langle .,. \rangle$" to represent the inner product in $L^2(\mathbb{D}, dA)$, and "$\langle .,. \rangle_{\frac{1}{2}}$" to that in $\mathcal{D}$.

Define the norm in $C^1(\bar{\mathbb{D}})$ as

$$\|\varphi\|_* = \max_{z \in \bar{\mathbb{D}}} \max \left\{ |\varphi|, \left| \frac{\partial \varphi}{\partial z} \right|, \left| \frac{\partial \varphi}{\partial \bar{z}} \right| \right\}, \qquad (\forall \varphi \in C^1(\bar{\mathbb{D}})).$$

It is well-known that $C^1(\bar{\mathbb{D}})$ is a norm-closed algebra relative to $\|.\|_*$.

Lemma 1. *For any* $\varphi \in C^1(\bar{\mathbb{D}})$, $H_\varphi f =: (I - P)(\varphi f)$ ($\forall f \in \mathcal{D}$) *is a compact operator from* $\mathcal{D}$ *to* $\mathcal{D}^\perp$.

Proof. For any $f \in \mathcal{D}, g \in \mathcal{D}^{\perp}$

$$\langle H_\varphi f, g \rangle_{\frac{1}{2}} = \langle \varphi f, g \rangle_{\frac{1}{2}} = \left\langle \frac{\partial \varphi}{\partial z} f + \varphi \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \right\rangle + \left\langle \frac{\partial \varphi}{\partial \bar{z}} f, \frac{\partial g}{\partial \bar{z}} \right\rangle$$

$$= \left\langle \frac{\partial \varphi}{\partial z} f, \frac{\partial g}{\partial z} \right\rangle + \left\langle \varphi \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \right\rangle + \left\langle \frac{\partial \varphi}{\partial \bar{z}} f, \frac{\partial g}{\partial \bar{z}} \right\rangle.$$

Note $g \perp \mathcal{D}$, so $\frac{\partial g}{\partial z} \perp L_a^2$ (L_a^2 is the classical Bergman space on $\mathbb{D}$). Let P_a is the orthogonal projection from $L^2(\mathbb{D}, dA)$ to L_a^2, then for any $f \in \mathcal{D}$ and $g \in \mathcal{D}^{\perp}$, we have

$$\left\langle \varphi f', \frac{\partial g}{\partial z} \right\rangle = \left\langle (I - P_a)(\varphi f'), \frac{\partial g}{\partial z} \right\rangle = \left\langle \widetilde{H}_\varphi f', \frac{\partial g}{\partial z} \right\rangle,$$

where $\widetilde{H}_\varphi$ is the Hankel operator from L_a^2 to $L_a^{2\perp}$ with symbol φ. Since $\varphi \in C^1(\bar{\mathbb{D}})$, $\widetilde{H}_\varphi$ is a compact operator. Now assume that $\{f_k\} \subset \mathcal{D}$ is a sequence which converges weakly 0 and satisfies $\|f_k\|_{\mathcal{D}} = 1$, we prove first that $\|f_k\|_{L^2} \to 0$ $(k \to \infty)$. In fact, without loss of generality, assume $f_k(z) = \sum_{n=1}^{\infty} a_n^{(k)} z^k$, then $a_n^{(k)} \to 0$ $(k \to \infty, \forall n)$ by $f_k \xrightarrow{w} 0$. Note $\|f_k\|_{\mathcal{D}}^2 = \|f_k'\|_{L^2}^2 = 1$, so $\sum_{n=1}^{\infty} |a_n^{(k)}|^2 n = 1$ $(\forall k)$, thus $\sum_{n=1}^{\infty} |a_n^{(k)}|^2 \leq 1$, furthermore, for any $\epsilon > 0$, there is a N_0 which is independent on k such that $\sum_{n \geq N_0+1} |a_n^{(k)}|^2 \frac{1}{n+1} < \frac{\epsilon}{2}$. Fix such a N_0, then

$$\|f_k\|_{L^2}^2 = \sum_{n=1}^{\infty} \left| a_n^{(k)} \right|^2 \frac{1}{n+1} \leq \sum_{n=1}^{N_0} \left| a_n^{(k)} \right|^2 \frac{1}{n+1} + \frac{\epsilon}{2}.$$

Since $a_n^{(k)} \to 0$ $(\forall n, k \to \infty)$, there is a K_0 such that for any $k > K_0$, $\sum_{n=1}^{N_0} |a_n^{(k)}|^2 \frac{1}{n+1} < \frac{\epsilon}{2}$, thus $\|f_k\|_{L^2}^2 < \frac{\epsilon}{2}$ $(k > K_0)$, hence $\|f_k\|_{L^2} \to 0$. Note

$$\|H_\varphi f_k\|_{L^{2,1}}^2 = \langle H_\varphi f_k, H_\varphi f_k \rangle_{\frac{1}{2}}$$

$$\leq \left\| \frac{\partial \varphi}{\partial z} f_k \right\|_{L^2} \|H_\varphi f_k\|_{L^{2,1}} + \left\| \widetilde{H}_\varphi f_k' \right\|_{L^2} \|H_\varphi f_k\|_{L^{2,1}}$$

$$+ \left\| \frac{\partial \varphi}{\partial \bar{z}} f_k \right\|_{L^2} \|H_\varphi f_k\|_{L^{2,1}}$$

$$\leq \|\varphi\|_* \|f_k\|_{L^2} \|H_\varphi f_k\|_{L^{2,1}} + \left\| \widetilde{H}_\varphi f_k' \right\|_{L^2} \|H_\varphi f_k\|_{L^{2,1}}$$

$$+ \|\varphi\|_* \|f_k\|_{L^2} \|H_\varphi f_k\|_{L^{2,1}},$$

we know easily that $\|H_\varphi f_k\|_{L^{2,1}} \to 0$. This shows that H_φ is compact. $\square$

Proposition 2. *For any $\varphi \in C^1(\bar{\mathbb{D}})$, T_φ is a bounded operator on $\mathcal{D}$.*

Proof. For any $f, g \in \mathcal{D}$,

$$\langle T_\varphi f, g \rangle_{\frac{1}{2}} = \langle \varphi f, g \rangle_{\frac{1}{2}} = \left\langle \frac{\partial(\varphi f)}{\partial z}, g' \right\rangle = \left\langle \frac{\partial \varphi}{\partial z} f, g' \right\rangle + \langle \varphi f', g' \rangle,$$

hence

$$\left| \langle T_\varphi f, g \rangle_{\frac{1}{2}} \right| \le \left\| \frac{\partial \varphi}{\partial z} f \right\|_{L^2} \|g'\|_{L^2} + \|\varphi f'\|_{L^2} \|g'\|_{L^2}$$

$$\le \|\varphi\|_*(\|f\|_{L^2}\|g\|_{\mathcal{D}} + \|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}).$$

It is not difficult to check $\|f\|_{L^2} \le \|f\|_{\mathcal{D}}$ for any $f \in \mathcal{D}$ (since $f(0) = 0$ for any $f \in \mathcal{D}$), hence $|\langle T_\varphi f, g \rangle_{\frac{1}{2}}| \le 2\|\varphi\|_*\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}$. This shows that $\|T_\varphi\| \le 2\|\varphi\|_*$. $\qquad\square$

Remark. In general, $\|T_\varphi\| \ne \|\varphi\|_*$ even $\varphi \in C^1(\mathbb{D}) \cap A(\mathbb{D})$ ($A(\mathbb{D})$ be the disk algebra), it may be that $\|\varphi\|_{C^1} < \|T_\varphi\| < 2\|\varphi\|_{C^1}$. For instance, if $\varphi(z) = z$, then $\|\varphi\|_* = \|\varphi\|_\infty = 1$, but $\|T_\varphi\| \ge \sqrt{2}$.

Proposition 3. *Let* $L^{\infty,1} = \{f \in L^{2,1} | f, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}} \in L^\infty(\mathbb{D}, dA)\}$, *then for any* $\varphi \in L^{\infty,1}$ *and* $\psi \in C^1(\bar{\mathbb{D}})$ $T_\varphi T_\psi - T_{\varphi\psi} \in \mathcal{K}(\mathcal{D})$ *(the compact operator algebra on* $\mathcal{D}$.)

Proof. For any $f, g \in \mathcal{D}$, as a direct computing, we see that

$$\langle (T_\varphi T_\psi - T_{\varphi\psi})f, g \rangle_{\frac{1}{2}} = \left\langle \frac{\partial \varphi}{\partial z} T_\psi f, \frac{\partial g}{\partial z} \right\rangle - \left\langle \frac{\partial \varphi}{\partial z}(\psi f), \frac{\partial g}{\partial z} \right\rangle$$

$$+ \left\langle \varphi \frac{\partial(-H_\psi f)}{\partial z}, \frac{\partial g}{\partial z} \right\rangle,$$

set $g = (T_\varphi T_\psi - T_{\varphi\psi})f$, we have

$$\|(T_\varphi T_\psi - T_{\varphi\psi})f\|_{\mathcal{D}}^2 \le \|\varphi\|_*[\|T_\psi f\|_{L^2} + \|\psi\|_*\|f\|_{L^2}$$
$$+ \|H_\psi f\|_{L^{2,1}}]\|(T_\varphi T_\psi - T_{\varphi\psi})f\|_{\mathcal{D}}.$$

Note for any sequence $\{f_k\} \subset \mathcal{D}$ which converges weakly 0 and satisfies $\|f_k\|_{\mathcal{D}} = 1$ (certainly, $T_\psi f_k \overset{w}{\to} 0$ in $\mathcal{D}$), $\|T_\psi f_k\|_{L^2} \to 0, \|f_k\|_{L^2} \to 0$, hence

$$\|(T_\varphi T_\psi - T_{\varphi\psi})f_k\|_{\mathcal{D}} \to 0$$

by H_ψ is compact. Futhermore, $T_\varphi T_\psi - T_{\varphi\psi}$ is compact. $\qquad\square$

Lemma 4. *For any* $\varphi \in L^{\infty,1}$, $T_\varphi^* - T_{\bar\varphi} \in \mathcal{K}(\mathcal{D})$.

Proof. Similar to the proof of Proposition 2, it is easy to see that T_φ is bounded for any $\varphi \in L^{\infty,1}$. Now assume $\varphi \in L^{\infty,1}$, then for any $f \cdot g \in \mathcal{D}$,

$\langle (T_\varphi^* - T_{\bar\varphi}) f.g \rangle_{\frac{1}{2}} = \langle \frac{\partial f}{\partial z}, \frac{\partial \varphi}{\partial z} g \rangle - \langle f, \frac{\partial \varphi}{\partial \bar z} \frac{\partial g}{\partial z} \rangle$. Thus

$$|\langle (T_\varphi^* - T_{\bar\varphi}) f, g \rangle_{\frac{1}{2}}| \le \left\| \frac{\partial f}{\partial z} \right\|_{L^2} \left\| \frac{\partial \varphi}{\partial z} g \right\|_{L^2} + \|f\|_{L^2} \left\| \frac{\partial \varphi}{\partial \bar z} \frac{\partial g}{\partial z} \right\|_{L^2}$$
$$\le \|\varphi\|_* \|g\|_{L^2} \|f\|_{\mathcal{D}} + \|\varphi\|_* \|f\|_{L^2} \|g\|_{\mathcal{D}},$$

where $\|\varphi\|_* = \mathrm{esssup}_{z \in \mathbb{D}} \max\{|\varphi(z)|, |\frac{\partial \varphi}{\partial z}(z)|, |\frac{\partial \varphi}{\partial \bar z}(z)|\}$. Hence for any $\{f_k\} \subset \mathcal{D}, \|f_k\| = 1, f_k \xrightarrow{w} 0$, we have

$$\|(T_\varphi^* - T_{\bar\varphi}) f_k\|_{\mathcal{D}}^2 \le \|\varphi\|_* \|(T_\varphi^* - T_{\bar\varphi}) f_k\|_{L^2} \|f_k\|_{\mathcal{D}} + \|\varphi\|_* \|f_k\|_{L^2} \|(T_\varphi^* - T_{\bar\varphi}) f_k\|_{\mathcal{D}}.$$

Note $(T_\varphi^* - T_{\bar\varphi}) f_k \xrightarrow{w} 0$, so $\|(T_\varphi^* - T_{\bar\varphi}) f_k\|_{L^2} \to 0$. Consequently, $\|(T_\varphi^* - T_{\bar\varphi}) f_k\|_{\mathcal{D}} \to 0$. This shows that $T_\varphi^* - T_{\bar\varphi}$ is compact. $\square$

Remark.   It is well-known that $T_\varphi^* = T_{\bar\varphi}$ for any $\varphi \in L^\infty(\mathbb{D}, dA)$ on L_a^2, but it is not difficult to check $T_\varphi^* \ne T_{\bar\varphi}$ in general. For example, if $\varphi(z) = \sum_{n=1}^\infty \frac{z^n}{n^2}$, then $\varphi \in H^\infty$, but $\|\varphi'\|_\infty = \infty$, one can check easily that T_φ is unbounded, however, $T_{\bar\varphi}$ is always bounded for every $\varphi \in H^\infty$, so is $T_{\bar\varphi}^*$, hence $T_{\bar\varphi}^* - T_\varphi$ is unbounded.

Proposition 5. If $\varphi \in C^1(\bar{\mathbb{D}})$, then T_φ is a compact operator on $\mathcal{D}$ if and only if $\varphi|_{\partial \mathbb{D}} \equiv 0$.

Proof. Assume $\varphi|_{\partial \mathbb{D}} \equiv 0$, then $\widetilde{T}_{|\varphi|^2}$ is a compact operator on $L_a^2(\mathbb{D}, dA)$, so for any $\{f_k\} \subset L_a^2, f_k \xrightarrow{w} 0, \|f_k\|_{L^2} = 1$, we have $\|\widetilde{T}_{|\varphi|} f_k\|_{L^2} \to 0 \ (k \to \infty)$. If T_φ is not compact on $\mathcal{D}$, then there is a sequence $\{F_k\} \subset \mathcal{D}, \|F_k\|_{\mathcal{D}} = 1$, $F_k \xrightarrow{w} 0$ such that $\|T_\varphi F_k\|_{\mathcal{D}} \nrightarrow 0$, thus $\|\varphi F_k\|_{L^{2,1}} \nrightarrow 0$, furthermore

$$\left\langle \frac{\partial(\varphi F_k)}{\partial z}, \frac{\partial(\varphi F_k)}{\partial z} \right\rangle + \left\langle \frac{\partial(\varphi F_k)}{\partial \bar z}, \frac{\partial(\varphi F_k)}{\partial \bar z} \right\rangle \nrightarrow 0.$$

Note

$$\frac{\partial(\varphi F_k)}{\partial z} = \frac{\partial \varphi}{\partial z} F_k + \varphi \frac{\partial F_k}{\partial z}, \ \frac{\partial(\varphi F_k)}{\partial \bar z} = \frac{\partial \varphi}{\partial \bar z} F_k,$$

thus

$$\left\langle \frac{\partial(\varphi F_k)}{\partial \bar z}, \frac{\partial(\varphi F_k)}{\partial \bar z} \right\rangle \le \|\varphi\|_*^2 \|F_k\|_{L^2}^2 \to 0,$$

and

$$\left\| \frac{\partial(\varphi F_k)}{\partial z} \right\|_{L^2}^2 = \left\langle \frac{\partial \varphi}{\partial z} F_k, \frac{\partial \varphi}{\partial z} F_k \right\rangle + \left\langle \frac{\partial \varphi}{\partial z} F_k, \varphi F_k' \right\rangle$$
$$+ \left\langle \varphi F_k', \frac{\partial \varphi}{\partial z} F_k \right\rangle + \langle \varphi F_k', \varphi F_k' \rangle.$$

Since $\|F_k'\|_{L^2} = \|F_k\|_{\mathcal{D}} = 1$, and $F_k' \xrightarrow{w} 0$ in L_a^2,

$$\langle \varphi F_k', \varphi F_k' \rangle = \langle |\varphi|^2 F_k', F_k' \rangle = \langle T_{|\varphi|^2} F_k', F_k' \rangle \to 0,$$

consequently, $\|\frac{\partial(\varphi F_k)}{\partial z}\|_{L^2}^2 \to 0$ by $|\langle \frac{\partial \varphi}{\partial z} F_k, \frac{\partial \varphi}{\partial z} F_k \rangle| \le \|\varphi\|_*^2 \|F_k\|_{L^2}^2$ and $|\langle \frac{\partial \varphi}{\partial z} F_k, \varphi F_k' \rangle| \le \|\varphi\|_*^2 \|F_k\|_{L^2} \|F_k'\|_{L^2}$, and $\|F_k\|_{L^2} \to 0$. This contradicts that $\|\varphi F_k\|_{L^{2,1}} \nrightarrow 0$. Hence T_φ must be a compact operator on $\mathcal{D}$.

Conversely, assume that T_φ is compact on $\mathcal{D}$. We need to prove that $\varphi|_{\partial \mathbb{D}} = 0$. Otherwise, $\widetilde{T}_{|\varphi|}$ is not compact on $L_a^2(\mathbb{D})$, thus there is a sequence $\{f_k\} \subset L_a^2$, $\|f_k\|_{L^2} = 1$, $f_k \xrightarrow{w} 0$ such that $\|\widetilde{T}_{|\varphi|} f_k\|_{L^2} \nrightarrow 0$, thus $\||\varphi| f_k\|_{L^2} \nrightarrow 0$, that is $\int_{\mathbb{D}} |\varphi|^2 |f_k|^2 dA \nrightarrow 0$. But T_φ is compact on $\mathcal{D}$, so for $F_k = \int_0^z f_k dw$, $\|T_\varphi F_k\|_{\mathcal{D}} \to 0$, further $\|T_\varphi^* T_\varphi F_k\|_{\mathcal{D}} \to 0$. Note $T_\varphi^* - T_{\overline{\varphi}} \in \mathcal{K}$, so $\|T_{|\varphi|^2} F_k\|_{\mathcal{D}} \to 0$ (since $F_k \xrightarrow{w} 0$ in $\mathcal{D}$ and $\|F_k\|_{\mathcal{D}} = 1$), thus $\langle T_{|\varphi|^2} F_k, F_k \rangle_{\frac{1}{2}} \to 0$, hence

$$\left\langle \frac{\partial(|\varphi|^2 F_k)}{\partial z}, F_k' \right\rangle \to 0,$$

that is

$$\left\langle \frac{\partial(|\varphi|^2)}{\partial z} F_k, F_k' \right\rangle + \langle |\varphi|^2 F_k', F_k' \rangle \to 0.$$

Note

$$\langle |\varphi|^2 F_k', F_k' \rangle = \langle |\varphi|^2 f_k, f_k \rangle = \int_{\mathbb{D}} |\varphi|^2 |f_k|^2 dA \nrightarrow 0,$$

and

$$\left| \left\langle \frac{\partial(|\varphi|^2)}{\partial z} F_k, F_k' \right\rangle \right| \le \||\varphi|^2\|_* \|F_k\|_{L^2} \|f_k\|_{L^2} \to 0,$$

hence $\langle T_{|\varphi|^2} F_k, F_k \rangle_{\frac{1}{2}} \nrightarrow 0$, this contradiction shows that $\varphi|_{\partial \mathbb{D}} \equiv 0$. $\qquad \square$

Remark. If $\varphi \notin C^1(\bar{\mathbb{D}})$, then T_φ may be non-compact even $\varphi \in C(\bar{\mathbb{D}})$ and $\varphi|_{\partial \mathbb{D}} = 0$. For instance, let $\varphi = \sqrt{1 - |z|^2}$, then it is not difficult to check T_φ is not a compact operator. In fact, T_φ is also a unbounded operator on $\mathcal{D}$.

Theorem 6. *Suppose* $\varphi \in C^1(\bar{\mathbb{D}})$, *then* $\sigma_e(T_\varphi) = \varphi(\partial \mathbb{D})$.

Proof. Without loss of generality, assume $0 \in \varphi(\partial \mathbb{D})$, thus there is a $\zeta \in \partial \mathbb{D}$ such that $\varphi(\zeta) = 0$. Write $f_k(z) = (\frac{1+z\bar{\zeta}}{2})^k$, then $\frac{f_k}{\|f_k\|_{\mathcal{D}}} \xrightarrow{w} 0$, and $\frac{\|T_\varphi f_k\|_{\mathcal{D}}}{\|f_k\|_{\mathcal{D}}} \le \frac{\|\varphi f_k\|_{L^{2,1}}}{\|f_k\|_{\mathcal{D}}}$. Clearly

$$\|\varphi f_k\|_{L^{2,1}}^2 = \left\langle \frac{\partial \varphi}{\partial z} f_k + \varphi f_k', \frac{\partial \varphi}{\partial z} f_k + \varphi f_k' \right\rangle + \left\langle \frac{\partial \varphi}{\partial \bar{z}} f_k, \frac{\partial \varphi}{\partial \bar{z}} f_k \right\rangle$$

$$= \left\langle \frac{\partial \varphi}{\partial z} f_k, \frac{\partial \varphi}{\partial z} f_k \right\rangle + \left\langle \varphi f_k', \frac{\partial \varphi}{\partial z} f_k \right\rangle + \left\langle \frac{\partial \varphi}{\partial z} f_k, \varphi f_k' \right\rangle$$

$$+ \langle \varphi f_k', \varphi f_k' \rangle + \left\langle \frac{\partial \varphi}{\partial \bar{z}} f_k, \frac{\partial \varphi}{\partial \bar{z}} f_k \right\rangle,$$

note

$$\frac{|\langle \frac{\partial \varphi}{\partial z} f_k, \frac{\partial \varphi}{\partial z} f_k \rangle + \langle \varphi f'_k, \frac{\partial \varphi}{\partial z} f_k \rangle + \langle \frac{\partial \varphi}{\partial z} f_k, \varphi f'_k \rangle + \langle \frac{\partial \varphi}{\partial \bar{z}} f_k, \frac{\partial \varphi}{\partial z} f_k \rangle|}{\|f_k\|_{\mathcal{D}}^2} \longrightarrow 0,$$

and

$$\frac{\langle \varphi f'_k, \varphi f'_k \rangle}{\|f_k\|_{\mathcal{D}}^2} = \frac{\int_{\mathbb{D}} |\varphi|^2 |f'_k|^2 dA}{\int_{\mathbb{D}} |f'_k|^2 dA}$$

$$= \frac{\int_{\mathbb{D}} |\varphi|^2 (\frac{1+z\bar{\zeta}}{2})^{2(k-1)} dA}{\int_{\mathbb{D}} (\frac{1+z\bar{\zeta}}{2})^{2(k-1)} dA}$$

$$= \frac{\int_{\mathbb{D}} |\varphi|^2 |f_{k-1}|^2 dA}{\int_{\mathbb{D}} |f_{k-1}|^2 dA} \to |\varphi(\zeta)|^2 = 0$$

(since f_k is a peak function at ζ). Thus $\frac{\|T_\varphi f_k\|_{\mathcal{D}}}{\|f_k\|_{\mathcal{D}}} \to 0$, this shows that T_φ can not be Fredholm, hence $0 \in \sigma_e(T_\varphi)$. That is $\varphi(\partial \mathbb{D}) \subset \sigma_e(T_\varphi)$.

Conversely, if $0 \notin \varphi(\partial \mathbb{D})$, then $|\varphi(\zeta)| > \epsilon_0 > 0$ for any $\zeta \in \partial \mathbb{D}$, thus $\widetilde{T}_\varphi$ is Fredholm on L_a^2, if T_φ is not Fredholm on $\mathcal{D}$, then there is a sequence $\{F_k\} \subset \mathcal{D}$ with $\|F_k\|_{\mathcal{D}} = 1, F_k \xrightarrow{w} 0$ such that $\|T_\varphi F_k\|_{\mathcal{D}} \to 0$, or $\|T_\varphi^* F_k\|_{\mathcal{D}} \to 0$. Since $\widetilde{T}_\varphi$ is Fredholm on L_a^2, there is an $S \in L(L_a^2)$ such that $[ST_\varphi] = [T_\varphi S] = [I]$ in the Calkin algebra $\frac{L(L_a^2)}{\mathcal{K}}$, thus $S^* T_\varphi^* = I + K, \ K \in \mathcal{K}(L_a^2)$, assume $\|T_\varphi^* F_k\|_{\mathcal{D}} \to 0$. (For the case of $\|T_\varphi F_k\|_{\mathcal{D}} \to 0$, we can complete the proof similarly.) Write $f_k = F'_k$, then $\|f_k\|_{L^2} = 1$, and $f_k \xrightarrow{w} 0$ in L_a^2, thus $\langle S^* T_\varphi^* f_k, f_k \rangle \to 1$, that is $\langle T_\varphi^* f_k, S f_k \rangle \to 1$.

On the other hand,

$$\left\langle T_\varphi^* F_k, \int_0^z (Sf_k)(w)dw \right\rangle_{\frac{1}{2}} = \left\langle F_k, \varphi \int_0^z (Sf_k)(w)dw \right\rangle_{\frac{1}{2}}$$

$$= \left\langle f_k, \frac{\partial \varphi}{\partial z} \int_0^z (Sf_k)(w)dw + \varphi Sf_k \right\rangle$$

$$= \left\langle f_k, \frac{\partial \varphi}{\partial z} \int_0^z Sf_k dw \right\rangle + \langle f_k, \varphi Sf_k \rangle.$$

Since $Sf_k \xrightarrow{w} 0$ in L_a^2, $\|\int_0^z Sf_k dw\|_{L_a^2} \to 0$, thus $\langle f_k, \frac{\partial \varphi}{\partial z} \int_0^z Sf_k(w)dw \rangle \to 0$. Note $\langle f_k, \varphi Sf_k \rangle = \langle T_\varphi^* f_k, Sf_k \rangle \to 1 \neq 0$, this shows that $\langle T_\varphi^* F_k, \int_0^z (Sf_k)(w)dw \rangle_{\frac{1}{2}} \to 1$, thus

$$\|T_\varphi^* F_k\|_{\mathcal{D}} \|S\|_{L(L_a^2)} \geq \left\langle T_\varphi^* F_k, \int_0^z Sf_k dw \right\rangle_{\frac{1}{2}} \to 1,$$

this contradicts that $\|T_\varphi^* F_k\|_{\mathcal{D}} \to 0$. Hence T_φ must be Fredholm, that is $0 \notin \sigma_e(T_\varphi)$. This follows the theorem. $\qquad \square$

Lemma 7. *For any $\varphi \in C^1(\bar{\mathbb{D}})$, $\|T_\varphi\|_e = \|[T_\varphi]\| = \|\varphi|_{\partial\mathbb{D}}\|_\infty$, where $\|T_\varphi\|_e$ denotes the essential norm of T_φ.*

Proof. Assume $\zeta \in \partial\mathbb{D}$ such that $|\varphi(\zeta)| = \|\varphi|_{\partial\mathbb{D}}\|_\infty$, set $f_\zeta(z) = \frac{1+z\bar\zeta}{2}$, then $f_k(z) = \frac{f_\zeta^k}{\|f_\zeta^k\|_\mathcal{D}} \xrightarrow{w} 0$, thus for any $K \in \mathcal{K}$, $\|Kf_k\| \to 0$. Note

$$\left| \langle T_\varphi f_k, f_k \rangle_{\frac{1}{2}} \right| = \left| \left\langle \frac{\partial\varphi}{\partial z} f_k, f_k' \right\rangle + \langle \varphi f_k', f_k' \rangle \right| \to |\varphi(\zeta)| = \|\varphi|_{\partial\mathbb{D}}\|_\infty$$

and $|\langle(T_\varphi + K)f_k, f_k\rangle_{\frac{1}{2}}| \le \|T_\varphi + K\|$. So $\|\varphi|_{\partial\mathbb{D}}\|_\infty \le \|T_\varphi + K\|$ for any $K \in \mathcal{K}$. Hence $\|\varphi|_{\partial\mathbb{D}}\|_\infty \le \|T_\varphi\|_e$.

Conversely, by Lemma 1.2 in C.K. Fong [**3**], there is an orthogonal sequence $\{f_k\} \subset \mathcal{D}$ such that $\|T_\varphi f_k\| \to \|T_\varphi\|_e$, so for any $\epsilon > 0$, there is a k_0 such that for any $k > k_0$, $\|T_\varphi f_k\| > \|T_\varphi\|_e - \epsilon$. Since $\varphi \in C^1(\bar{\mathbb{D}})$, there is a $0 < r < 1$ such that $|\varphi(z)| < \|\varphi|_{\partial\mathbb{D}}\|_\infty + \epsilon$ for any $|z| > r$. Note

$$\|T_\varphi f_k\|_\mathcal{D}^2 \le \|\varphi f_k\|_{L^{2,1}}^2 = \langle \varphi f_k, \varphi f_k \rangle_{\frac{1}{2}}$$

$$= \langle \varphi f_k', \varphi f_k' \rangle + \left\langle \frac{\partial\varphi}{\partial z} f_k, \varphi f_k' \right\rangle$$

$$+ \left\langle \varphi f_k', \frac{\partial\varphi}{\partial z} f_k \right\rangle + \left\langle \frac{\partial\varphi}{\partial z} f_k, \frac{\partial\varphi}{\partial z} f_k \right\rangle$$

$$+ \left\langle \frac{\partial\varphi}{\partial\bar z} f_k, \frac{\partial\varphi}{\partial\bar z} f_k \right\rangle,$$

clearly,

$$\left\langle \frac{\partial\varphi}{\partial z} f_k, \varphi f_k' \right\rangle + \left\langle \varphi f_k', \frac{\partial\varphi}{\partial z} f_k \right\rangle + \left\langle \frac{\partial\varphi}{\partial z} f_k, \frac{\partial\varphi}{\partial z} f_k \right\rangle + \left\langle \frac{\partial\varphi}{\partial\bar z} f_k, \frac{\partial\varphi}{\partial\bar z} f_k \right\rangle \to 0,$$

so there is a k_1 such that for any $k > k_1$, $\|T_\varphi f_k\|^2 \le |\langle \varphi f_k', \varphi f_k' \rangle| + \epsilon$. Since $f_k \xrightarrow{w} 0$, we know that $f_k \xrightarrow{\text{uniformly}} 0$ on any compact subset of $\mathbb{D}$, so for any $0 < t < 1$, $\int_{\{|z| \le t\}} |f_k|^2 dA \to 0$ $(k \to \infty)$, assume $f_k(z) = \sum_{i=1}^\infty a_n^{(k)} z^n$, thus

$$\sum_{i=1}^\infty a_n^{(k)} \frac{t^{2(n+1)}}{n+1} = \int_{\{|z| \le t\}} |f_k|^2 dA \to 0.$$

Without loss of generality, assume $t > r$, thus

$$\int_{\{|z| \le r\}} |f_k'|^2 dA = \sum_{i=1}^\infty \left| a_n^{(k)} \right|^2 nr^{2n} = \sum_{i=1}^\infty \left| a_n^{(k)} \right|^2 \frac{t^{2(n+1)}}{n+1} n(n+1)t^{-2} \left(\frac{r}{t} \right)^{2n},$$

since $(\frac{r}{t})^{2n}n(n+1)t^{-2} \to 0$ $(n \to \infty)$, it is clearly that $\int_{\{|z| \leq r\}} |f'_k|^2 dA \to 0$, so there is a k_2 such that for any $k > k_2$, $\int_{\{|z| \leq r\}} |f'_k|^2 dA < \epsilon$, consequently,

$$
\begin{aligned}
|\langle \varphi f'_k, \varphi f'_k \rangle| &= \int_{\mathbb{D}} |\varphi|^2 |f'_k|^2 dA \\
&= \int_{\{|z| \leq r\}} |\varphi|^2 |f'_k|^2 dA + \int_{\{|z| > r\}} |\varphi|^2 |f'_k|^2 dA \\
&\leq \|\varphi\|^2_\infty \int_{\{|z| \leq r\}} |f'_k|^2 dA + (\|\varphi|_{\partial\mathbb{D}}\|_\infty + \epsilon)^2 \int_{\{|z| > r\}} |f'_k|^2 dA \\
&\leq \|\varphi\|^2_\infty \int_{\{|z| \leq r\}} |f'_k|^2 dA + (\|\varphi|_{\partial\mathbb{D}}\|_\infty + \epsilon)^2 \\
&< \|\varphi\|^2_\infty \epsilon + (\|\varphi|_{\partial\mathbb{D}}\|_\infty + \epsilon)^2.
\end{aligned}
$$

Furthermore, for any $k > \max\{k_0, k_1, k_2\}$

$$
\begin{aligned}
\|T_\varphi f_k\|^2 &\leq \|\varphi\|^2_\infty \int_{\{|z| \leq r\}} |f'_k|^2 dA + (\|\varphi|_{\partial\mathbb{D}}\|_\infty + \epsilon)^2 + \epsilon \\
&\leq (\|\varphi\|^2_\infty + 1)\epsilon + (\|\varphi|_{\partial\mathbb{D}}\|_\infty + \epsilon)^2.
\end{aligned}
$$

Hence $\lim_{k\to\infty} \|T_\varphi f_k\| \leq \|\varphi|_{\partial\mathbb{D}}\|_\infty$ by the arbitrariness of ϵ. This shows that $\|T_\varphi\|_e \leq \|\varphi|_{\partial\mathbb{D}}\|_\infty$. We are done. $\qquad\square$

Theorem 8. *Let $\mathcal{I} = \mathcal{I}(C^1)$ be the C^*-algebra generated by $\{T_\varphi | \varphi \in C^1(\bar{\mathbb{D}})\}$, then the commutator ideal $\mathcal{C}(C^1)$ of $\mathcal{I}$ equals $\mathcal{K}(\mathcal{D})$, and $\frac{\mathcal{I}}{\mathcal{K}} \cong C(\partial\mathbb{D})$. Consequently, following short sequence*

$$(*) \qquad\qquad 0 \to \mathcal{K} \to \mathcal{I} \to C(\partial\mathbb{D}) \to 0$$

is exact.

Proof. By Proposition 3, we know that $\mathcal{C}(C^1) \subset \mathcal{K}$, since $\mathcal{K}$ is minimal, $\mathcal{C}(C^1) = \mathcal{K}$. By Lemma 4, $[T^*_\varphi] = [T_{\bar\varphi}]$ in $\frac{\mathcal{I}}{\mathcal{K}}$. Define $\xi : \{[T_\varphi] | \varphi \in C^1(\bar{\mathbb{D}})\} \to C(\partial\mathbb{D})$ as $\xi([T_\varphi]) = \varphi|_{\partial\mathbb{D}}$, it is easy to see that ξ is well-defined, and one-to-one by Proposition 5. By Proposition 3, Lemma 4 and Lemma 7, we see that ξ is an isometric *-homomorphism. Hence ξ can be extended to $\frac{\mathcal{I}}{\mathcal{K}}$, in fact, for any $[T] \in \frac{\mathcal{I}}{\mathcal{K}}$, there is a sequence $[T_{\varphi_k}]$ such that $\|[T_{\varphi_k}] - [T]\| \to 0$, thus $\|(\varphi_k - \varphi_j)|_{\partial\mathbb{D}}\|_\infty = \|[T_{\varphi_k - \varphi_j}]\| = \|[T_{\varphi_k}] - [T_{\varphi_j}]\| \to 0$. Hence there is a $\varphi \in C(\partial\mathbb{D})$ such that $\|\varphi_k|_{\partial\mathbb{D}} - \varphi\|_\infty \to 0$. Let $\xi([T]) = \varphi$, then ξ is well-defined on $\frac{\mathcal{I}}{\mathcal{K}}$, and $\|\xi([T])\|_\infty = \|\varphi\|_\infty = \lim_{k\to\infty} \|\varphi_k|_{\partial\mathbb{D}}\|_\infty = \lim_{k\to\infty} \|[T_{\varphi_k}]\| = \|[T]\|$, so ξ is an isometry from $\frac{\mathcal{I}}{\mathcal{K}}$ into $C(\partial\mathbb{D})$. For any $\varphi \in C(\partial\mathbb{D})$, there is a polynomial sequence $\{p_k\} \subset C^1(\bar{\mathbb{D}})$ such that $\|p_k|_{\partial\mathbb{D}} - \varphi\|_\infty \to 0$, thus $\|(p_k - p_j)|_{\partial\mathbb{D}}\|_\infty \to 0$. Furthermore, $\|[T_{p_k}] - [T_{p_j}]\| \to 0$, so there is a $[T] \in \frac{\mathcal{I}}{\mathcal{K}}$ such that $\|[T_{p_k}] - [T]\| \to 0$, hence $\xi([T]) = \varphi$, i.e. ξ is a surjection onto $C(\partial\mathbb{D})$. This shows that ξ is a *-isomorphism between $\frac{\mathcal{I}}{\mathcal{K}}$ and $C(\partial\mathbb{D})$. The proof is thus completed. $\qquad\square$

3. Toeplitz operators with symbols in $H_1^\infty + C^1(\bar{\mathbb{D}})$.

It is well-known that $H^\infty + C(\partial\mathbb{D})$ is a norm-closed algebra (c.f. R.G. Douglas [**2**]), Rudin [**6**] proved that $H^\infty(\partial B_n) + C(\partial B_n)$ is also a norm-closed algebra, where B_n is the unit ball of $\mathbb{C}^n$. The Toeplitz operators with symbols in $H^\infty + C$ on Hardy or Bergman spaces have many important properties, their essential spectra and Fredholm index can be completely determined by their symbols. In this section, we prove that $H_1^\infty + C^1(\bar{\mathbb{D}})$ is a norm-closed space relative to a suitable norm, and obtain the representation of essential spectra of the Toeplitz operators with symbols in $H_1^\infty + C^1(\bar{\mathbb{D}})$.

For any $\varphi \in H_1^\infty + C^1(\bar{\mathbb{D}})$, define

$$\|\varphi\|_* = \sup_{z \in \mathbb{D}} \max \left\{ |\varphi|, \left|\frac{\partial\varphi}{\partial z}\right|, \left|\frac{\partial\varphi}{\partial\bar{z}}\right| \right\},$$

it is clearly that $\| \,.\, \|_*$ is a norm on $H_1^\infty + C^1(\bar{\mathbb{D}})$.

Theorem 9. $H_1^\infty + C^1(\bar{\mathbb{D}})$ *is a closed space relative to the norm* $\| \,.\, \|_*$.

Proof. Our proof is similar to that of Rudin [**6**]. Assume $\varphi \in cl(H_1^\infty + C^1(\bar{\mathbb{D}}))$, to prove $\varphi \in H_1^\infty + C^1(\bar{\mathbb{D}})$, we first prove that for any $\psi \in H_1^\infty + C^1(\bar{\mathbb{D}})$, there are $\psi_1 \in H_1^\infty$, and $\psi_2 \in C^1(\bar{\mathbb{D}})$ such that $\psi = \psi_1 + \psi_2$ and $\|\psi_1\|_* \leq 3\|\psi\|_*, \|\psi_2\|_* \leq 2\|\psi\|_*$. In fact, if $\psi = \tilde{\psi}_1 + \tilde{\psi}_2 \in H_1^\infty + C^1(\bar{\mathbb{D}})$, $\tilde{\psi}_1 \in H_1^\infty$, $\tilde{\psi}_2 \in C^1(\bar{\mathbb{D}})$, then write $\tilde{\psi}_2^r(z) = \tilde{\psi}_2(rz)$, clearly, $\|\tilde{\psi}_2^r - \tilde{\psi}_2\|_\infty \to 0$ $(r \to 1^-)$, and $\frac{\partial\tilde{\psi}_2^r}{\partial z}(z) = r\frac{\partial\tilde{\psi}_2}{\partial z}(rz)$, $\frac{\partial\tilde{\psi}_2^r}{\partial\bar{z}}(z) = r\frac{\partial\tilde{\psi}_2}{\partial\bar{z}}(rz)$, so $\|\tilde{\psi}_2^r - \tilde{\psi}_2\|_* \to 0$. Fix a r_0 such that $\|\tilde{\psi}_2^{r_0} - \tilde{\psi}_2\|_* \leq \|\psi\|_*$. Set

$$\psi_2 = \tilde{\psi}_2 - \tilde{\psi}_2^{r_0} + \psi^{r_0}, \ \psi_1 = \tilde{\psi}_1 - \tilde{\psi}_1^{r_0},$$

then $\psi = \psi_1 + \psi_2$, and $\|\psi_1\|_* = \|\psi - \psi_2\|_* \leq 3\|\psi\|_*$, $\|\psi_2\|_* \leq 2\|\psi\|_*$. Since $\varphi \in cl(H_1^\infty + C^1(\bar{\mathbb{D}}))$, there are $\varphi_i \in H_1^\infty + C^1(\bar{\mathbb{D}})$, such that $\|\varphi_i\|_* \leq 2^{-i}$ $(i \geq 2)$, and $\varphi = \sum_{i=1}^\infty \varphi_i$ (in fact, there is a sequence $\{p_k\} \subset H_1^\infty + C^1(\bar{\mathbb{D}})$ such that $\|p_k - \varphi\|_* \to 0$, thus there is a subsequence $\{p_k\}$ such that $\|p_{k_{i+1}} - p_{k_i}\|_* < \frac{1}{2^i}$, write $\varphi_1 = p_{k_1}, \varphi_i = p_{k_{i+1}} - p_{k_i}$ $(i \geq 2)$, then $\sum_{i=1}^\infty \varphi_i = \varphi$ and $\|\varphi_i\|_* < \frac{1}{2^i}$ $(i \geq 2)$). For each φ_i, there is a $\varphi_i^{(1)} \in H_1^\infty$, $\varphi_i^{(2)} \in C^1(\bar{\mathbb{D}})$ such that $\|\varphi_i^{(1)}\|_* \leq 3\|\varphi_i\|_*$, $\|\varphi_i^{(2)}\|_* \leq 2\|\varphi_i\|_*$, $\varphi_i = \varphi_i^{(1)} + \varphi_i^{(2)}$, thus $\varphi^{(1)} = \sum_{i=1}^\infty \varphi_i^{(1)} \in H_1^\infty$ (since H_1^∞ is closed relative to $\|.\|_*$), $\varphi^{(2)} = \sum_{i=1}^\infty \varphi_i^{(2)} \in C^1(\bar{\mathbb{D}})$, and $\varphi = \varphi^{(1)} + \varphi^{(2)}$. That is $\varphi \in H_1^\infty + C^1(\bar{\mathbb{D}})$. □

Proposition 10. *If* $\varphi \in H_1^\infty + C^1(\bar{\mathbb{D}})$ *satisfies* $\varphi|_{\partial\mathbb{D}} = 0$, *then* T_φ *is a compact operator on* $\mathcal{D}$, *where* $\varphi|_{\partial\mathbb{D}}$ *denotes the radial boundary values of* φ.

Proof. Suppose $\varphi = \varphi_1 + \varphi_2, \varphi_1 \in H_1^\infty, \varphi_2 \in C^1(\bar{\mathbb{D}})$, since $\varphi|_{\partial\mathbb{D}} = 0$, $\varphi_1|_{\partial\mathbb{D}} = -\varphi_2|_{\partial\mathbb{D}}$, thus $\varphi_1|_{\partial\mathbb{D}} \in C(\partial\mathbb{D})$, further $\varphi_1(z) = P[\varphi_1|_{\partial\mathbb{D}}] \in H_1^\infty \cap A(\mathbb{D})$ (where $P[\varphi|_{\partial\mathbb{D}}]$ denotes the Poisson integral of $\varphi|_{\partial\mathbb{D}}$), consequently, $\varphi \in C(\bar{\mathbb{D}}) \cap (H_1^\infty + C^1(\bar{\mathbb{D}}))$.

Suppose $\{f_k\} \subset \mathcal{D}$, $\|f_k\|_{\mathcal{D}} = 1$, $f_k \xrightarrow{w} 0$, then

$$
\begin{aligned}
\|T_\varphi f_k\|_{\mathcal{D}}^2 &\leq \|\varphi f_k\|_{L^{2,1}}^2 = \langle \varphi f_k, \varphi f_k \rangle_{\frac{1}{2}} \\
&= \left\langle \frac{\partial \varphi}{\partial z} f_k + f_k' \varphi, \frac{\partial \varphi}{\partial z} f_k + f_k' \varphi \right\rangle + \left\langle \frac{\partial \varphi}{\partial \bar{z}} f_k, \frac{\partial \varphi}{\partial \bar{z}} f_k \right\rangle \\
&= \left\| \frac{\partial \varphi}{\partial z} f_k \right\|_{L^2}^2 + 2\mathrm{Re} \int_{\mathbb{D}} f_k' \varphi \overline{\frac{\partial \varphi}{\partial z} f_k} dA + \|\varphi f_k'\|_{L^2}^2 + \left\| \frac{\partial \varphi}{\partial \bar{z}} f_k \right\|_{L^2}^2 \\
&\leq \|\varphi\|_*^2 \|f_k\|_{L^2}^2 + 2\|\varphi\|_*^2 \|f_k\|_{\mathcal{D}} \|f_k\|_{L^2} \\
&\quad + \|\varphi f_k'\|_{L^2}^2 + \|\varphi\|_*^2 \|f_k\|_{L^2}^2.
\end{aligned}
$$

Since $\|f_k\|_{\mathcal{D}} = 1$ and $f_k \xrightarrow{w} 0$ in $\mathcal{D}$, $\|f_k\|_{L^2} \to 0$, so

$$
\|\varphi\|_*^2 \|f_k\|_{L^2}^2 + 2\|\varphi\|_*^2 \|f_k\|_{\mathcal{D}} \|f_k\|_{L^2} + \|\varphi\|_*^2 \|f_k\|_{L^2}^2 \to 0.
$$

Note $\varphi \in C(\bar{\mathbb{D}})$, and $\varphi|_{\partial \mathbb{D}} = 0$, hence for any $\epsilon > 0$, there is a $0 < \delta < 1$ such that $|\varphi(z)| < \epsilon$ for any $|z| > \delta$, thus

$$
\begin{aligned}
\int_{\mathbb{D}} |\varphi|^2 |f_k'|^2 dA &\leq \epsilon^2 \int_{\{|z| > \delta\}} |f_k'|^2 dA + \int_{\{|z| \leq \delta\}} |\varphi|^2 |f_k'|^2 dA \\
&\leq \epsilon^2 + \|\varphi\|_*^2 \int_{\{|z| \leq \delta\}} |f_k'|^2 dA.
\end{aligned}
$$

By $f_k \xrightarrow{w} 0$, we know that $\int_{\{|z| \leq \delta\}} |f_k'|^2 dA \to 0$, hence $\int_{\mathbb{D}} |\varphi|^2 |f_k'|^2 dA \to 0$ $(k \to \infty)$, furthermore $\|T_\varphi f_k\|_{\mathcal{D}}^2 \to 0$, this shows that T_φ is compact on $\mathcal{D}$. $\qquad \square$

Proposition 11. *If $\varphi \in H_1^\infty(\mathbb{D})$, then on Dirichlet space $\mathcal{D}$, $\sigma(T_\varphi) = \overline{\varphi(\mathbb{D})}$.*

Proof. Note for any $f \in \mathcal{D}$,

$$
\langle f, T_\varphi^* K \rangle_{\frac{1}{2}} = \langle T_\varphi f, K \rangle_{\frac{1}{2}} = \langle \varphi f, K \rangle_{\frac{1}{2}} = \varphi(z) f(z),
$$

so $T_\varphi^* K(z, w) = \overline{\varphi(z)} K(z, w)$, this shows that $\overline{\varphi(\mathbb{D})} \subset \sigma(T_\varphi)$.

Conversely, if $0 \notin \overline{\varphi(\mathbb{D})}$, then there is a $\delta > 0$ such that $|\varphi(z)| > \delta > 0$ $(\forall z \in \mathbb{D})$, thus $\varphi^{-1} \in H^\infty$. Note

$$
\left| \frac{\partial(\varphi^{-1})}{\partial z} \right| = \left| -\frac{\varphi'}{\varphi^2} \right| \leq \frac{1}{\delta^2} |\varphi'| \leq \frac{1}{\delta^2} \|\varphi\|_* < \infty,
$$

we have $\varphi^{-1} \in H_1^\infty$, so $T_{\varphi^{-1}} T_\varphi = T_\varphi T_{\varphi^{-1}} = I$, that is T_φ is invertible, hence $\sigma(T_\varphi) \subset \overline{\varphi(\mathbb{D})}$. The proof is thus complete. $\qquad \square$

Theorem 12. *If $\varphi = \varphi_1 + \varphi_2 \in H_1^\infty + C^1(\bar{\mathbb{D}})$, where $\varphi_1 \in H_1^\infty$, $\varphi_2 \in C^1(\bar{\mathbb{D}})$, then on Dirichlet space $\mathcal{D}$,*

$$
\sigma_e(T_\varphi) = \overline{\cap_{0 < \delta < 1} \{\varphi(z) | |z| > \delta\}} = \varphi(\partial \mathbb{D}).
$$

Proof. Without loss of generality, assume $0 \in \overline{\cap_{0<\delta<1}\{\varphi(z)||z| > \delta\}}$, then there is a sequence $\{z_k\} \subset \mathbb{D}$ with $z_n \to \zeta \in \partial\mathbb{D}$, such that $\varphi(z_n) \to 0$, thus $\tilde{T}_\varphi$ is not Fredholm on L_a^2 (c.f. G.F. Cao [**1**] or MacDonald [**4**]), hence there is a sequence $\{f_k\} \subset L_a^2$ with $\|f_k\|_{L^2} = 1, f_k \overset{w}{\to} 0$ such that $\|\tilde{T}_\varphi f_k\|_{L^2} \to 0$ or $\|\tilde{T}_\varphi^* f_k\|_{L^2} = \|\tilde{T}_{\bar{\varphi}} f_k\|_{L^2} \to 0$. If T_φ is Fredholm on $\mathcal{D}$, then there is a bounded operator S on $\mathcal{D}$ such that $ST_\varphi - I, T_\varphi S - I \in \mathcal{K}(\mathcal{D})$, further $S^*T_\varphi^* - I, T_\varphi^* S^* - I \in \mathcal{K}(\mathcal{D})$. Write $F_k(z) = \int_0^z f_k(w)dw$, then $F_k \in \mathcal{D}$ and $\|F_k\|_{\mathcal{D}} = \|f_k\|_{L^2} = 1$, so $\|F_k\|_{L^2} \to 0$. In addition, it is clearly that $F_k \overset{w}{\to} 0$ in $\mathcal{D}$. Thus

$$
\begin{aligned}
\lim_{k\to\infty} \langle ST_\varphi F_k, F_k \rangle_{\frac{1}{2}} &= \lim_{k\to\infty} \langle T_\varphi S F_k, F_k \rangle_{\frac{1}{2}} \\
&= \lim_{k\to\infty} \langle T_\varphi^* S^* F_k, F_k \rangle_{\frac{1}{2}} \\
&= \lim_{k\to\infty} \langle S^* T_\varphi^* F_k, F_k \rangle_{\frac{1}{2}} = 1.
\end{aligned}
$$

Hence, without loss of generality, we can assume that $\|\tilde{T}_\varphi^* f_k\|_{L^2} \to 0$ (similarly for the case of $\|\tilde{T}_\varphi f_k\|_{L^2} \to 0$). Set $G_k = SF_k$, then $\|G_k\|_{\mathcal{D}} \le \|S\|\|F_k\|_{\mathcal{D}} = \|S\|$, and $G_k \overset{w}{\to} 0$, so $\|G_k\|_{L^2} \to 0$, and

$$
\begin{aligned}
\left| \langle T_\varphi S F_k, F_k \rangle_{\frac{1}{2}} \right| &= \left| \langle \varphi(SF_k)', f_k \rangle + \left\langle \frac{\partial\varphi}{\partial z} SF_k, f_k \right\rangle \right| \\
&\le \left| \langle (SF_k)', \tilde{T}_\varphi^* f_k \rangle \right| + \|\varphi\|_* \|SF_k\|_{L^2} \|f_k\|_{L^2} \\
&\le \|(SF_k)'\|_{L^2} \left\| \tilde{T}_\varphi^* f_k \right\|_{L^2} + \|\varphi\|_* \|G_k\|_{L^2} \|f_k\|_{L^2} \\
&= \|G_k\|_{\mathcal{D}} \left\| \tilde{T}_\varphi^* f_k \right\|_{L^2} + \|\varphi\|_* \|G_k\|_{L^2} \|f_k\|_{L^2} \to 0.
\end{aligned}
$$

This contradicts that $\langle T_\varphi S F_k, F_k \rangle_{\frac{1}{2}} \to 1$. hence T_φ must be non-Fredholm. That is $0 \in \sigma_e(T_\varphi)$, consequently $\overline{\cap_{0<\delta<1}\{\varphi(z)||z| > \delta\}} \subset \sigma_e(T_\varphi)$.

Conversely, assume $0 \notin \overline{\cap_{0<\delta<1}\{\varphi(z)||z| > \delta\}}$, thus there are $\epsilon, \delta > 0$ such that $|\varphi(z)| > \epsilon$ for any $|z| > \delta$, we prove that T_φ is Fredholm on $\mathcal{D}$. Otherwise, there is a sequence $\{F_k\} \subset \mathcal{D}$ with $\|F_k\|_{\mathcal{D}} = 1, F_k \overset{w}{\to} 0$, such that $\|T_\varphi F_k\|_{\mathcal{D}} \to 0$ or $\|T_\varphi^* F_k\|_{\mathcal{D}} \to 0$. Similar to above proof, we can assume $\|T_\varphi^* F_k\|_{\mathcal{D}} \to 0$ (if $\|T_\varphi F_k\|_{\mathcal{D}} \to 0$, the proof will be simpler). It is well-known that $\overset{\to}{} T_\varphi$ is Fredholm on L_a^2 if $|\varphi(z)| > \epsilon$ for any $|z| > \delta$, hence there is a $S \in L(L_a^2)$ such that $S\tilde{T}_\varphi - I, \tilde{T}_\varphi S - I \in \mathcal{K}(L_a^2)$, further, $S^*\tilde{T}_\varphi^* - I, \tilde{T}_\varphi^* S^* - I \in \mathcal{K}(L_a^2)$. Set $f_k = F_k', g_k = Sf_k$, then $f_k \overset{w}{\to} 0, g_k \overset{w}{\to} 0$ in L_a^2 and $\|f_k\|_{L_a^2} = \|F_k\|_{\mathcal{D}} = 1, \|g_k\|_{L_a^2} \le \|S\|\|f_k\|_{L_a^2} = \|S\|$, so $\langle S^*\tilde{T}_\varphi^* f_k, f_k \rangle \to$

1, that is $\langle \tilde{T}_\varphi^* f_k, S f_k \rangle \to 1$. Note

$$\left\langle T_\varphi^* F_k, \int_0^z S f_k(w) dw \right\rangle_{\frac{1}{2}} = \left\langle F_k, \varphi \int_0^z S f_k dw \right\rangle_{\frac{1}{2}}$$

$$= \left\langle f_k, \left(\frac{\partial \varphi}{\partial z} \int_0^z S f_k dw + \varphi S f_k \right) \right\rangle_{L^2}$$

$$= \left\langle f_k, \frac{\partial \varphi}{\partial z} \int_0^z g_k dw \right\rangle + \langle f_k, \varphi g_k \rangle$$

and $g_k \overset{w}{\to} 0$, so $\| \int_0^z g_k dw \|_{L^2} \to 0$, and

$$\left| \left\langle f_k, \frac{\partial \varphi}{\partial z} \int_0^z g_k dw \right\rangle \right| \leq \| f_k \|_{L_a^2} \| \varphi \|_* \left\| \int_0^z g_k dw \right\|_{L^2} \to 0,$$

but $\langle f_k, \varphi g_k \rangle = \langle \tilde{T}_\varphi^* f_k, S f_k \rangle \to 1 \neq 0$, hence $\langle T_\varphi^* F_k, \int_0^z g_k dw \rangle_{\frac{1}{2}} \to 1 \neq 0$, this contradicts that $\| T_\varphi^* F_k \|_{\mathcal{D}} \to 0$. It shows that T_φ must be Fredholm on $\mathcal{D}$. That is $0 \notin \sigma_e(T_\varphi)$. Note the functions with derivatives in H^∞ are continuous in the closed unit disk, hence $H_1^\infty + C^1(\bar{\mathbb{D}}) \subset C(\bar{\mathbb{D}})$, consequently, $\overline{\cap_{0<\delta<1} \{\varphi(z) | |z| > \delta\}} = \varphi(\partial \mathbb{D})$. We are done. $\qquad \square$

Remark. If $\varphi \in H^\infty$, then $T_{\bar\varphi}$ is always bounded on $\mathcal{D}$, in fact, for any $f, g \in \mathcal{D}$,

$$\left| \langle T_{\bar\varphi} f, g \rangle_{\frac{1}{2}} \right| = \left| \langle \bar\varphi f, g \rangle_{\frac{1}{2}} \right| = |\langle \bar\varphi f', g' \rangle|$$

$$= \left| \left\langle \tilde{T}_{\bar\varphi} f', g' \right\rangle \right| \leq \left\| \tilde{T}_{\bar\varphi} \right\| \| f' \|_{L^2} \| g' \|_{L^2} \leq \| \varphi \|_\infty \| f \|_{\mathcal{D}} \| g \|_{\mathcal{D}},$$

hence $\| T_{\bar\varphi} f \| \leq \| \varphi \|_\infty \| f \|_{\mathcal{D}}$. However, for $\varphi \in H^\infty$, T_φ may be unbounded on $\mathcal{D}$. Let

$$L_1^{\infty,1} = \left\{ f \in L^{2,1} \Big| f, \frac{\partial f}{\partial z} \in L^\infty(\mathbb{D}, dA) \right\},$$

then $L_1^{\infty,1} \cup \overline{H^\infty}$ is perhaps the most suitable symbol space of Toeplitz operators which are bounded. Also, we can prove that if $\varphi \in L_1^{\infty,1}$ has a compact support set, then T_φ is compact on $\mathcal{D}$.

4. An index formula of Toeplitz operators.

The classical index formula shows that for $\varphi \in C(\partial \mathbb{D})$, if $|\varphi| > \epsilon > 0$, then T_φ is a Fredholm operator on Hardy space $H^2(\partial \mathbb{D})$, and $\text{Ind} T_\varphi = -\text{wind} \varphi$. If $\varphi \in C(\bar{\mathbb{D}})$ with $|(\varphi|_{\partial \mathbb{D}})(\zeta)| > \epsilon > 0$, then $\text{Ind} T_\varphi = -\text{wind} \varphi|_{\partial \mathbb{D}}$ on Bergman space $L_a^2(\mathbb{D})$. The proof of these index formulas is relative to the topology homotopy of symbol functions, it can not be directly extended to the case of Dirichlet space since we do not know whether there is a C^1-function H_t which is continuous with respect to $t \in [0, 1]$ such that $H_0 = \varphi, H_1 = \psi$ if $\varphi, \psi \in C^1(\bar{\mathbb{D}})$ and $\text{wind} \varphi|_{\partial \mathbb{D}} = \text{wind} \psi|_{\partial \mathbb{D}}$. In this section, we use the

short exact sequence $(*)$ to prove an analogy of the above index formulas for C^1-symbols.

Theorem 13. *Suppose $\varphi \in C^1(\bar{\mathbb{D}})$ such that $T_{\bar{\varphi}}$ is Fredholm on $\mathcal{D}$, then*

$$\mathrm{Ind} T_{\varphi} = -\mathrm{wind}\varphi|_{\partial\mathbb{D}}.$$

Proof. By Theorem 8, if $\varphi, \psi \in C^1(\bar{\mathbb{D}})$ such that T_{φ} and T_{ψ} are Fredholm on $\mathcal{D}$ and $\mathrm{wind}\varphi|_{\partial\mathbb{D}} = \mathrm{wind}\psi|_{\partial\mathbb{D}}$, then there is a $H_t \in C([0,1] \times \partial\mathbb{D})$ such that $H_t \in GC(\partial\mathbb{D})$ (the set of invertible elements in $C(\partial\mathbb{D})$) for each $t \in [0,1]$ and $H_0 = \varphi|_{\partial\mathbb{D}}, H_1 = \psi|_{\partial\mathbb{D}}$. Note ξ is an isometry isomorphism, so $\xi^{-1}(H_t)$ is continuous with respect to t. On the other hand, H_t^{-1} is also continuous on $[0,1] \times \partial\mathbb{D}$, and $\xi^{-1}(H_t)\xi^{-1}(H_t^{-1}) = \xi^{-1}(H_t^{-1})\xi^{-1}(H_t) = \xi^{-1}(H_t H_t^{-1}) = [I]$. Hence, $\xi^{-1}(H_t)$ is invertible in $\frac{\mathcal{I}}{\mathcal{K}}$. Furthermore, it is easy to see that $\mathrm{Ind} T_{\varphi} = \mathrm{Ind} T_{\psi}$.

Now suppose $\varphi \in C^1(\bar{\mathbb{D}})$ such that T_{φ} is Fredholm on $\mathcal{D}$ and $\mathrm{wind}\varphi|_{\partial\mathbb{D}} = k$, note T_z is Fredholm on $\mathcal{D}$ with $\mathrm{Ind} T_z = -1 = -\mathrm{wind}z|_{\partial\mathbb{D}}$, let

$$\tilde{z}^k = \begin{cases} z^k, & \text{if } k > 0, \\ \bar{z}^{-k}, & \text{otherwise}; \end{cases}$$

we see that $\mathrm{wind}\varphi|_{\partial\mathbb{D}} = \mathrm{wind}\tilde{z}^k|_{\partial\mathbb{D}}$, thus $\mathrm{Ind} T_{\varphi} = \mathrm{Ind} T_{\tilde{z}^k}$ by above proof, consequently, $\mathrm{Ind} T_{\varphi} = -k = -\mathrm{wind}\varphi|_{\partial\mathbb{D}}$. The proof is thus complete. $\square$

Proposition 14. *Suppose $\varphi \in H^{\infty}$ such that $T_{\bar{\varphi}}$ is Fredholm on $\mathcal{D}$, then*

$$\mathrm{Ind} T_{\bar{\varphi}} = -\lim_{r \to 1^-} \mathrm{wind}\bar{\varphi}_r|_{\partial\mathbb{D}},$$

where $\varphi_r(z) = \varphi(rz)$.

Proof. If $f \in \mathrm{Ker} T_{\bar{\varphi}}$, then for any $g \in \mathbb{D}$,

$$0 = \langle T_{\bar{\varphi}}f, g \rangle_{\frac{1}{2}} = \langle \bar{\varphi}f, g \rangle_{\frac{1}{2}} = \langle \bar{\varphi}f', g' \rangle = \left\langle \tilde{T}_{\bar{\varphi}}f', g' \right\rangle,$$

since $\{g'|g \in \mathcal{D}\} = L_a^2$, $\tilde{T}_{\bar{\varphi}}f' = 0$, hence $\{f'|f \in \mathrm{Ker} T_{\bar{\varphi}}\} \subset \mathrm{Ker}\tilde{T}_{\bar{\varphi}}$. Consequently, assume $\tilde{T}_{\bar{\varphi}}f = 0, f \in L_a^2$, set $F(z) = \int_0^z f dw$, then $F \in \mathcal{D}$, and for any $G \in \mathcal{D}$,

$$\langle T_{\bar{\varphi}}F, G \rangle_{\frac{1}{2}} = \langle \bar{\varphi}F', G' \rangle = \left\langle \tilde{T}_{\bar{\varphi}}f, G' \right\rangle = 0,$$

so $T_{\bar{\varphi}}F = 0$, further $\{\int_0^z f(w)dw | f \in \mathrm{Ker}\tilde{T}_{\bar{\varphi}}\} \subset \mathrm{Ker} T_{\bar{\varphi}}$. This shows that

$$\dim \mathrm{Ker} T_{\bar{\varphi}} = \dim \mathrm{Ker}\tilde{T}_{\bar{\varphi}}.$$

Now assume $f \in \mathcal{D}$ such that $T_{\bar{\varphi}}^* f = 0$, then for any $g \in \mathcal{D}$,

$$0 = \left\langle T_{\bar{\varphi}}^* f, g \right\rangle_{\frac{1}{2}} = \langle f, \bar{\varphi}g \rangle_{\frac{1}{2}}$$
$$= \langle f', \bar{\varphi}g' \rangle = \langle \varphi f' g' \rangle = \left\langle \tilde{T}_{\varphi}f', g' \right\rangle,$$

hence $\tilde{T}_\varphi f' = 0$, that is $\{f'|f \in \mathrm{Ker}T^*_{\bar{\varphi}}\} \subset \mathrm{Ker}\tilde{T}_\varphi$.

Similarly to above proof, we have also $\{\int_0^z f(w)dw|f \in \mathrm{Ker}\tilde{T}_\varphi\} \subset \mathrm{Ker}T^*_{\bar{\varphi}}$. Hence dim $\mathrm{Ker}T^*_{\bar{\varphi}} = \dim \mathrm{Ker}\tilde{T}_\varphi$, consequently,

$$\mathrm{Ind}T_{\bar{\varphi}} = -\mathrm{Ind}\tilde{T}_\varphi = \lim_{r \to 1^-} \mathrm{wind}\varphi_r|_{\partial\mathbb{D}} = -\lim_{r \to 1^-} \mathrm{wind}\bar{\varphi}_r|_{\partial\mathbb{D}}.$$

$\square$

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