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In this work it is shown that, under appropriate hypotheses, the multiplicative Cousin problem on complex manifolds admits solutions that depend continuously on parameters.

1. Introduction.

The bulk of this paper is devoted to the study of continuous families of nonnegative divisors on a complex manifold. In particular our work leads to the solution, for a broad class of complex manifolds, of two problems proposed by Stoll [**St**, Problem A and Problem B, p. 155 and pp. 201-202].

If $\mathcal{M}$ is a complex-analytic manifold[1] of complex dimension $N \geq 1$, we denote by $\mathfrak{D}^+(\mathcal{M})$ the set of nonnegative divisors on $\mathcal{M}$. Moreover we denote by $\mathfrak{D}^+_P(\mathcal{M})$ the subset of $\mathfrak{D}^+(\mathcal{M})$ consisting of the principal divisors, i.e., the divisors of holomorphic functions. (It will also be convenient to use the notation that $\mathfrak{D}^+(\mathcal{M}; p)$ denotes the set of all nonnegative divisors with support not containing the point $p \in \mathcal{M}$, and that $\mathfrak{D}^+_P(\mathcal{M}; p)$ is the set of principal divisors in $\mathfrak{D}^+(\mathcal{M}; p)$.) An element $D \in \mathfrak{D}^+(\mathcal{M})$ can be understood as a formal sum $D = \sum m_j V_j$ with each m_j a nonnegative integer and with $\{V_j\}_{j=1,\dots}$ a locally finite family of irreducible complex hypersurfaces in $\mathcal{M}$. It is natural to identify this divisor with the *current of integration over* D, that is, the current defined by

$$D(\alpha) = \langle \alpha, D \rangle = \sum m_j \int_{V_j} \alpha,$$

for each $\mathcal{C}^\infty$ compactly supported $(N-1, N-1)$-form, α, on $\mathcal{M}$. Thus, $\mathfrak{D}^+(\mathcal{M})$ may be considered as a subset of $\mathcal{D}_{1,1}(\mathcal{M}) = \mathcal{D}'^{(1,1)}(\mathcal{M})$, the space of bihomogeneous currents on $\mathcal{M}$ of type $(1,1)$, which is dual to the space $\mathcal{D}^{(N-1,N-1)}(\mathcal{M})$ of compactly supported smooth forms of bidegree $(N-1, N-1)$ on $\mathcal{M}$.

From the point of view of functional analysis there are two natural topologies on $\mathcal{D}'^{(1,1)}(\mathcal{M})$: The *weak* topology* and the *strong topology*. It follows that $\mathfrak{D}^+(\mathcal{M})$ inherits from $\mathcal{D}'^{(1,1)}(\mathcal{M})$ two topologies: The relative topology induced by the weak* topology and the relative topology induced by

[1] In what follows, we assume all our manifolds to be countable at infinity. In the absence of explicit mention to the contrary, they are also assumed to be connected.

the strong topology. Stoll [**St**] introduced a third topology on the space $\mathfrak{D}^+(\mathcal{M})$, a topology particularly suited to the study of the "normal families of nonnegative divisors". In Section 3 of the paper we establish that on $\mathfrak{D}^+(\mathcal{M})$ these three topologies coincide; the unique topology they determine will be seen eventually to be separable and metrizable.

Stoll [**St**, Problem B, p. 155 and p. 202] posed the following problem: *Let $\mathcal{M}$ be a complex-analytic manifold. Suppose that every nonnegative divisor on $\mathcal{M}$ is a principal divisor. Let $\mathfrak{R}$ be a set of nonnegative divisors on $\mathcal{M}$. Does there exist a continuous map $h : \mathfrak{R} \to \mathcal{O}(\mathcal{M})$ such that* $\operatorname{Div} h(D) = D$ *for every $D \in \mathfrak{R}$?*

Stoll [**St**] solved this problem in certain cases involving domains in $\mathbb{C}^N$. In particular, he proved the following result [**St**, Theorems 1.9, 2.25 and 3.6].

Theorem 1.0. *Let Ω be a domain in $\mathbb{C}^N$ that contains the closed unit ball $\bar{\mathbb{B}}_N$. There is a continuous map $h : \mathfrak{D}^+(\Omega; 0) \to \mathcal{O}(\mathbb{B}_N)$ such that for each $D \in \mathfrak{D}^+(\Omega; 0)$,* Div *$h(D) = D|\mathbb{B}_N$.*

We shall exploit this result systematically in the sequel. The appendix to the paper gives a development of the theorem from a point of view somewhat different from that used by Stoll.

The main thrust of the present work is to obtain generalizations of this result of Stoll.

We shall establish in Theorem 6.4 that *if $\mathcal{M}$ is a complex manifold with $H^1(\mathcal{M}, \mathcal{O}) = 0$ and $H^1(\mathcal{M}, \mathbb{Z}) = 0$, then there exists a continuous map $\varsigma : \mathfrak{D}^+_P(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ with* $\operatorname{Div} \varsigma(D) = D$ *for all $D \in \mathfrak{D}^+_P(\mathcal{M})$*. There is a strong result in the converse direction, Theorem 6.6.

The proof will depend in an essential way on Theorem 1.0 quoted above and on certain topological methods.

We shall see in Section 7 that the conclusion of Theorem 6.4 can also be drawn if $\mathcal{M}$ is a domain in a Stein manifold that satisfies the geometric condition $H^1(\mathcal{M}, \mathbb{Z}) = 0$.

In Section 8 we show that there is an analogue of Theorem 6.4 in which the role of the space $\mathcal{O}(\mathcal{M})$ is played by the space $\mathcal{L}(\mathcal{M})$ of global sections of the holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}$.

The above results can be reformulated in terms of the solvability of the Second Cousin Problem for nonnegative divisors with continuous dependence on parameters as follows. Given a continuous family $\{D_x : x \in X\}$ in $\mathfrak{D}^+_P(\mathcal{M})$, parametrized by a topological space X, we ask: Does there exist a continuous function $F : X \times \mathcal{M} \to \mathbb{C}$ such that $F(x, \cdot) \in \mathcal{O}(\mathcal{M})$ and $\operatorname{Div} F(x, \cdot) = D_x$ for all $x \in X$?

Consider the continuous map $\mu : X \to \mathfrak{D}^+_P(\mathcal{M})$ defined by $\mu(x) = D_x$, for all $x \in X$. The existence of F amounts to the existence of a continuous map

$\tilde{\mu} : X \to \mathcal{O}(\mathcal{M})$ with $\operatorname{Div} \tilde{\mu}(x) = \mu(x)$ for all $x \in X$, for, if either of them exists, then we may define the other by $F(x, \cdot) = \tilde{\mu}(x)$, for all $x \in X$.

It is well to indicate with a simple example that for the kind of lifting problem we are concerned with, some geometric conditions must be imposed on the manifold in question. This shows up already in the plane.

Set $\Omega = \mathbb{C} \setminus \{0\}$, the punctured plane. Denote by γ the unit circle in $\mathbb{C}$. Define $\psi : [0,1] \to \mathcal{O}(\Omega) \setminus \{0\}$ by

$$\psi(t)(z) = tz + (1-t)\frac{1}{z}.$$

Define $\phi = \operatorname{Div} \circ \psi : [0,1] \to \mathfrak{D}^+(\Omega) = \mathfrak{D}^+_P(\Omega)$. This is continuous and satisfies $\phi(0) = \phi(1)$.

We ask: Does this map lift to a continuous map $\tilde{\phi} : [0,1] \to \mathcal{O}(\Omega)$ that satisfies $\tilde{\phi}(0) = \tilde{\phi}(1)$ and $\operatorname{Div} \circ \tilde{\phi} = \phi$?

Suppose such a $\tilde{\phi}$ to exist. Without loss of generality, $\tilde{\phi}(0) = \psi(0)$. (If not, replace $\tilde{\phi}$ by $\frac{\psi(0)}{\tilde{\phi}(0)}\tilde{\phi}$.)

Define $\chi(t) = \frac{1}{2\pi i}\int_\gamma d\log\left(\frac{\tilde{\phi}(t)}{\psi(t)}\right)$.

The function χ is an integer that depends continuously on t and so is constant. But compute:

$$\chi(0) = \frac{1}{2\pi i}\int_\gamma d\log\left(\frac{\tilde{\phi}(0)}{\psi(0)}\right) = 0.$$

Also,

$$\begin{aligned}\chi(1) &= \frac{1}{2\pi i}\int_\gamma d\left(\frac{\tilde{\phi}(1)}{\psi(1)}\right)\\ &= \frac{1}{2\pi i}\int_\gamma d\log\left(\frac{\psi(0)}{\psi(1)}\right)\\ &= \frac{1}{2\pi i}\int_\gamma d\log z^{-2} = -2.\end{aligned}$$

Contradiction.

This simple example shows that to obtain lifting theorems of the kind we are concerned with, it is necessary to impose *some* geometric condition on the manifolds under consideration.

Stoll [**St**, Problem A, p. 155 and pp. 201–202] posed also the following problem:

Let $\mathcal{M}$ be a complex-analytic manifold. Suppose that every nonnegative divisor on $\mathcal{M}$ is a principal divisor. Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be a normal family of nonnegative divisors on $\mathcal{M}$. Does there exist a normal family $\{h_\lambda\}_{\lambda\in\Lambda}$ of *holomorphic functions on $\mathcal{M}$ and a compact subset K of $\mathcal{M}$ such that, for each $\lambda \in \Lambda$, $\operatorname{Div} h_\lambda(D_\lambda) = D_\lambda$ and $h_\lambda(a_\lambda) = 1$ for some $a_\lambda \in K$?*

Stoll [**St**] solved this problem in the case that $\mathcal{M} = \mathbb{C}^N$.

In Section 9 we show that if there is a continuous map $\varsigma : \mathfrak{D}^+(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ with $\operatorname{Div}\circ\varsigma = id$, then the problem has a positive solution.

Remark. Our motivation for undertaking this work was the desire to apply some of the results obtained here to the study of certain hulls that generalize polynomially convex hulls. In addition to the well known polynomially convex hulls, another collection of hulls has been introduced by Basner [**Bas**]. Given a compact subset X of $\mathbb{C}^N$ there is a hull $h_q(X)$ defined for each integer q in the range $1 \le q \le N-1$. The hull $h_q(X)$ is defined in terms of polynomial mappings from $\mathbb{C}^N$ to $\mathbb{C}^q$. By using the results of the present paper, we show that the hull $h_q(X)$ can be described in terms of continuous families of polynomial maps from $\mathbb{C}^N$ to $\mathbb{C}^{q+1}$. This work will be published in a subsequent paper.

We shall need to use a selection theorem proved by Michael [**Mi**]: *If E and F are Fréchet spaces and if $T : E \to F$ is a surjective continuous linear transformation, then there is a continuous map $\varsigma : F \to E$ such that $T \circ \varsigma$ is the identity map on F.* In general the selection map ς cannot be chosen to be linear; it can always be chosen to be homogeneous. A perspicuous proof of the result is given in [**Ru1**].

Given an indexed family of sets $\{S_\alpha\}_{\alpha\in A}$, we understand $\{S_{\alpha\beta}\}_{\alpha,\beta\in A}$ to be the family of intersections $S_{\alpha\beta} = S_\alpha \cap S_\beta$. Similar notation will be used for triple, quadruple,..., intersections.

We shall also use the notation that $\mathbb{C}^*$ denotes the collection of nonzero complex numbers and that $\mathcal{C}(X)$ denotes the space of continuous $\mathbb{C}$-valued functions on the space X.

A referee has drawn our attention to the papers of McGrath [**MG**] and Siu [**Si**]. McGrath's thesis extends Stoll's results to the setting of polycylinders in $\mathbb{C}^N$. In Siu's paper further results are obtained that extend those of Stoll and that are closely related to the present work.

It should be remarked that the possibility of solving the First Cousin Problem with continuous dependence on real parameters is essentially due to Oka [**Ok**]; it was discussed in [**Na**].

2. The Topologies on $\mathfrak{D}^+(\mathcal{M})$.

The space $\mathfrak{D}^+(\mathcal{M})$ has three natural topologies, two with functional-analytic roots, the other based in function theory.

The first of these is the *weak* topology* in which a net $\{D_\alpha\}_{\alpha\in A}$ in $\mathfrak{D}^+(\mathcal{M})$ converges to $D_o \in \mathfrak{D}^+(\mathcal{M})$ if and only if for every compactly supported smooth form ϑ of bidegree $(N-1, N-1)$ on $\mathcal{M}$

$$\lim_{\alpha \in A} \int_{D_\alpha} \vartheta = \int_{D_o} \vartheta.$$

The second functional-analytic topology is the *strong topology* in which a net $\{D_\alpha\}_{\alpha \in A}$ in $\mathfrak{D}^+(\mathcal{M})$ converges to $D_o \in \mathfrak{D}^+(\mathcal{M})$ if and only if it converges in the weak* sense and if, moreover, the convergence is uniform on *bounded* sets in the space $\mathcal{D}^{N-1,N-1}(\mathcal{M})$.

Recall that if $\mathcal{D}(\mathcal{M})$ denotes the space of compactly supported functions on $\mathcal{M}$, then a subset $\mathcal{B}$ of $\mathcal{D}(\mathcal{M})$ is bounded if there is a fixed compact set K in $\mathcal{M}$ with $\operatorname{supp} f \subset K$ for all $f \in \mathcal{B}$. It is required, moreover that the set K be decomposed into a union of closed subsets $K_1, ..., K_r$, with each K_j contained in an open set U_j on which there are global smooth coordinates. For each j, there is a sequence of positive constants $\{k_\nu\}_{\nu=1,2,...}$ with the property that for each $f \in \mathcal{B}$ and each $j = 1, ..., r$, the derivatives of order less than ν of f with respect to the coordinates in U_j are bounded uniformly on K_j by k_ν. This gives rise to the notion of bounded set in the space of forms $\mathcal{D}^{N-1,N-1}(\mathcal{M})$.

The third topology we shall consider is that introduced in function-theoretic terms by Stoll [**St**]. This topology is defined by the condition that a net $\{D_\alpha\}_{\alpha \in A}$ in $\mathfrak{D}^+(\mathcal{M})$ converges to $D_o \in \mathfrak{D}^+(\mathcal{M})$ if and only if there is an open cover $\mathcal{V} = \{V_j\}_{j=1,...}$ of $\mathcal{M}$ such that for each $\alpha \in A$, there is an $f_{j\alpha} \in \mathcal{O}(V_j)$ such that $\operatorname{Div} f_{j\alpha} = D_\alpha | V_j$ and such that $f_{j\alpha}$ converges uniformly on compacta in V_j to $f_{j_o} \in \mathcal{O}(V_j)$, f_{j_o} a function with divisor $D_o | V_j$. Stoll verifies that this prescription does specify a topology on the space $\mathfrak{D}^+(\mathcal{M})$.

The principal goal of the present section is to show the equivalence of these three topologies. In this connection, certain results are immediate. It is plain that strong convergence implies convergence in the weak* sense. It is less evident, but essentially known, that convergence in the sense of Stoll's topology implies strong convergence. This is contained in a theorem of Andreotti and Norguet [**AN**] to the effect that *if $\mathcal{N}$ is a connected complex manifold, then the map that associates to a nonzero-function $f \in \mathcal{O}(\mathcal{N})$ its divisor, viewed as a current, is continuous when the space $\mathcal{O}(\mathcal{N})$ is endowed with the usual topology of uniform convergence on compacta and the space of currents is endowed with its strong topology, viewed as the dual of the topological vector space $\mathcal{D}^{N-1,N-1}(\mathcal{N})$ of compactly supported* $(N-1, N-1)$-*forms on $\mathcal{N}$.*

To complete the proof of the equivalence of the three topologies on $\mathfrak{D}^+(\mathcal{M})$, it suffices to show that on this space, convergence in the weak* sense implies convergence in the sense of the topology of Stoll.

This depends on Stoll's characterization of normal families in the space $\mathfrak{D}^+(\mathcal{M})$. Recall that, by definition, a subset of $\mathfrak{D}^+(\mathcal{M})$ is a normal family if it is a relatively compact subset, when the space of divisors is endowed with the topology introduced by Stoll.

For normal families, Stoll [**St**] gives the following characterization: *A subset $\mathfrak{R}$ of $\mathfrak{D}^+(\mathcal{M})$ is relatively compact with respect to to Stoll's topology if and only if $\mathfrak{R}$ is bounded on every compact subset $K \subset \mathcal{M}$ in the sense that, for a fixed Hermitian metric on $\mathcal{M}$ with associated fundamental form ω, there is a positive constant L_K such that for each $D \in \mathfrak{R}$, the area of D in K given by $D(\chi_K\omega^{N-1}) = \frac{1}{(N-1)!}\int_D \chi_K\omega^{N-1}$ is not more than L_K.* (Here χ_K denotes the characteristic function of the set K.)

Fix now a Hermitian metric on $\mathcal{M}$, and let ω denote its associated fundamental form. Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be a net in $\mathfrak{D}^+(\mathcal{M})$ that converges in the relative weak* topology to $D_o \in \mathfrak{D}^+(\mathcal{M})$. Fix a relatively compact open set U in $\mathcal{M}$, and let χ be a compactly supported $\mathcal{C}^\infty$ function on $\mathcal{M}$ that is indentically one on a neighborhood of $\bar{U}$ and is everywhere nonnegative.

As the net converges in the weak* sense, there is a constant L large enough that for some $\lambda_o \in \Lambda$ if $\lambda > \lambda_o$, then

$$\int_{D_\lambda} \chi\omega^{N-1} < L.$$

By the characterization of normal families quoted above, it follows that the family $\{D_\lambda|U : \lambda > \lambda_o\}$ is a normal family. Accordingly, the net $\{D_\lambda|U\}_{\lambda>\lambda_o}$ has a cluster point, say $\tilde{D}$, in $\mathfrak{D}^+(U)$ with respect to Stoll's topology. As convergence in the sense of Stoll's topology implies convergence in the weak* sense, $\tilde{D}$ can only be the limit $D_o|U$. (Stoll proves that the topology he introduces satisfies the Hausdorff separation axiom.) This implies that the net $\{D_\lambda|U\}_{\lambda\in\Lambda}$ converges to $D_o|U$ in the sense of Stoll's topology.

As the open, relatively compact subsets of $\mathcal{M}$ constitute an open cover for $\mathcal{M}$, it follows, as we wished, that the initial net $\{D_\lambda\}_{\lambda\in\Lambda}$ converges to D_o in the sense of Stoll's topology.

We have now reached the desired conclusion that *the three naturally defined topologies on the space $\mathfrak{D}^+(\mathcal{M})$ of nonnegative divisors on $\mathcal{M}$ coincide.* In the sequel, we shall speak simply of *the* topology on $\mathfrak{D}^+(\mathcal{M})$.

The proof just given of the equivalence of the three topologies on the space $\mathfrak{D}^+(\mathcal{M})$ depends explicitly on the mechanism developed in [**St**]. It is of interest, and of some importance at a later point in this work, that there is a rather simple, direct proof of the equivalence of the weak* and strong topologies on this space. It runs as follows.

What is to be shown is that a net $\{D_\iota\}_{\iota\in I}$ converges to $D_o \in \mathfrak{D}^+(\mathcal{M})$ in the relative weak* topology if and only if it converges to D_o in the relative strong topology.

That strong convergence implies weak* convergence is evident.

Conversely, we consider a net $\{D_\iota\}_{\iota\in I}$ in $\mathfrak{D}^+(\mathcal{M})$ that converges in the weak* sense to D_o, and we show it to converge strongly.

To this end, consider a bounded set $\mathcal{B} \subset \mathfrak{D}^{N-1,N-1}(\mathcal{M})$. As $\mathcal{B}$ is bounded, there is a compact set $X \subset \mathcal{M}$ such that each $\alpha \in \mathcal{B}$ has support in X. If χ is a smooth function on $\mathcal{M}$, then the set $\chi\mathcal{B}$ defined by

$$\chi\mathcal{B} = \{\chi\alpha : \alpha \in \mathcal{B}\}$$

is a bounded set in $\mathfrak{D}^{N-1,N-1}(\mathcal{M})$. If $\chi_1 + \ldots + \chi_q = 1$, then

$$\mathcal{B} = \chi_1\mathcal{B} + \ldots + \chi_q\mathcal{B}.$$

If $\{D_\iota\}_{\iota\in I}$ converges uniformly on each $\chi_k\mathcal{B}$ to χD_o, then $\{D_\iota\}_{\iota\in I}$ converges uniformly on $\mathcal{B}$ to D_o. This remark, coupled with the existence of smooth partitions of unity on $\mathcal{M}$, permits us to localize our problem: We assume from here on that $\mathcal{M} = \mathbb{B}_N(2)$, the open ball of radius 2 centered in the origin in $\mathbb{C}^N$, and that $X = \overline{\mathbb{B}}_N$, the closed unit ball.

Since $\mathcal{B}$ is bounded, there exist positive constants k_r, $r = 0, 1, \ldots$, such that for each $\alpha \in \mathcal{B}$, if

$$\alpha = \sum_{|I|,|J|=N-1} \alpha_{IJ} dz^I \wedge d\bar{z}^J,$$

then

$$\left| \frac{\partial^{|P|+|Q|}\alpha_{IJ}}{\partial z^P \partial \bar{z}^Q}(z) \right| \leq k_{|P|+|Q|}$$

for all $z \in \mathbb{B}_N(2)$ and all multi-indices P, Q.

Denote by χ a real-valued nonnegative smooth function on $\mathbb{C}^N$ with $\chi = 1$ on a neighborhood of $\overline{\mathbb{B}}_N(1)$ and with $\chi(z) = 0$ if $|z| > \frac{3}{2}$. Let ω be the Kähler form $\frac{i}{2}\sum_{j=1}^N dz_j \wedge d\bar{z}_j$.

By hypothesis

$$\lim_\iota D_\iota(\chi\omega^p) = D_o(\chi\omega^p). \tag{1}$$

For each $\iota \in I$ write $D_\iota = \sum_j m_{\iota j} V_{\iota j}$ for suitable positive integers $m_{\iota j}$ and suitable irreducible complex hypersurfaces $V_{\iota j}$ in $\mathbb{B}_N(2)$. Then (1) implies the existence of an index $\iota_o \in I$ such that for a positive constant C_o and for all multi-indices I, J with $|I| = |J| = N - 1$

$$\left| \sum_j m_{\iota j} \int_{V_{\iota j}\cap\overline{\mathbb{B}}_N} dz^I \wedge \bar{z}^J \right| \leq C_o \tag{2}$$

provided $\iota_o \leq \iota$.

Let $\alpha = \sum_{|I|,|J|=N-1} \alpha_{IJ} dz^I \wedge d\bar{z}^J \in \mathcal{B}$ and compute: For $\iota_o \leq \iota$ we have

(3)

$$\begin{aligned}
&|D_\iota(\alpha) - D_o(\alpha)| \\
&= \left| \sum_j m_{\iota j} \int_{V_{\iota j}} \sum_{|I|=|J|=N-1} \alpha_{IJ} dz^I \wedge d\bar{z}^J \right. \\
&\qquad \left. - \sum_k m_{ok} \int_{V_{ok}} \sum_{|I|=|J|=N-1} \alpha_{IJ} dz^I \wedge d\bar{z}^J \right| \\
&\leq \sum_{|I|=|J|=N-1} \left| \sum_j m_{\iota j} \int_{V_{\iota j}} \alpha_{IJ} dz^I \wedge d\bar{z}^J - \sum_k m_{ok} \int_{V_{ok}} \alpha_{IJ} dz^I \wedge d\bar{z}^K \right|.
\end{aligned}$$

For each $\iota \in \mathcal{I}$ and for all multi-indices I and J, define a measure $\mu_{\iota IJ}$ on $\mathbb{B}_N(2)$ by

$$\int g d\mu_{\iota IJ} = \sum_j m_{\iota j} \int_{V_{\iota j} \cap \overline{\mathbb{B}}_N} g dz^I \wedge d\bar{z}^J,$$

for $g \in \mathcal{C}(\overline{\mathbb{B}}_N)$ and define μ_{oIJ}, associated with D_o in a similar way. By (2), $\|\mu_{\iota IJ}\|$ and $\|\mu_{oIJ}\|$ are bounded uniformly in $\iota, \iota_o < \iota$, say by the constant C_1. Let $\lambda = 8N$ and define $\psi_{\iota IJ}(\zeta)$ for $\zeta \in \mathbb{C}^N$ by

$$\psi_{\iota IJ}(\zeta) = \gamma \int |z - \zeta|^{2\lambda - 4N} \log |z - \zeta| d\mu_{\iota IJ}(z).$$

Define ψ_{oIJ} similarly. With a suitable choice of constant γ, these functions satisfy the equations

$$\Delta^\lambda \psi_{\iota IJ} = \mu_{\iota IJ}, \quad \text{and} \quad \Delta^\lambda \psi_{oIJ} = \mu_{oIJ}$$

in the sense of distributions. See [**GS**, p. 282]. (Here Δ denotes the Laplacian on $\mathbb{R}^{2N} = \mathbb{C}^N$.)

We may write

$$\begin{aligned}
&\left| \sum_j m_{\iota j} \int_{V_{\iota j}} \alpha_{IJ} dz^I \wedge d\bar{z}^I - \sum_k m_{ok} \int_{V_{ok}} \alpha_{IJ} dz^I \wedge d\bar{z}^J \right| \\
&= \left| \int_{\overline{\mathbb{B}}_N} \alpha_{IJ} d(\mu_{\iota IJ} - \mu_{oIJ}) \right| \\
&= \left| \int_{\overline{\mathbb{B}}_N} \Delta^\lambda \alpha_{IJ}(\zeta)(\psi_{\iota IJ}(\zeta) - \psi_{oIJ}(\zeta)) \omega^N(\zeta) \right| \\
&\leq M \int_{\overline{\mathbb{B}}_N} |\psi_{\iota IJ}(\zeta) - \psi_{oIJ}(\zeta)| \omega^N(\zeta)
\end{aligned}$$

for a suitable positive constant M.

Call the last integral $L_{\iota IJ}$.

The definition of the $\psi_{\iota IJ}$'s shows that they are uniformly bounded and that they satisfy a uniform Lipschitz condition on $\mathbb{B}_N(2)$. Moreover, the measures $\mu_{\iota IJ}$ converge in the weak* sense (in the space of measures) to μ_{oIJ}. Thus, $\psi_{\iota IJ}(\zeta) \to \psi_{oIJ}(\zeta)$ for each $\zeta \in \mathbb{B}_N(2)$.

From this it follows that $\lim_\iota L_{\iota IJ} = 0$: The numbers $L_{\iota IJ}$ are uniformly bounded, so the net $\{L_{\iota IJ}\}_\iota$ has cluster points. Let t_o be such a cluster point, and let $\{L_{\iota_\beta IJ}\}_{\beta\in B}$ be a subnet of such that $\lim_\beta L_{\iota_\beta IJ} = t_o$.

The net $\{\psi_{\iota_\beta IJ} - \psi_{oIJ}\}_{\beta\in B}$ is a net of continuous functions that are uniformly bounded and uniformly equicontinuous. Consequently, there is a subnet, which we may suppose to be $\{\psi_{\iota_\beta IJ} - \psi_{oIJ}\}_{\beta\in B}$ itself, that converges *uniformly* on $\overline{\mathbb{B}}_N$. As $\psi_{\iota IJ} \to \psi_{oIJ}$ pointwise, the limit can only be zero. As the convergence is uniform, we have

$$\lim_\beta L_{\iota_\beta IJ} = 0.$$

Consequently, $t_o = 0$. This means that $\lim_\iota L_{\iota IJ} = 0$.

The proposition is proved.

It is worth noting that this argument applies, *mutatis mutandis,* to yield the corresponding result for the space of holomorphic p-chains on M.

3. The Topology of $\mathfrak{D}^+(\mathcal{M})$.

We have now to treat certain aspects of $\mathfrak{D}^+(\mathcal{M})$ and $\mathfrak{D}^+_P(\mathcal{M})$ as topological spaces, the topology being that discussed in the preceding section.

Lemma 3.1. *If $p \in \mathcal{M}$, then the set of $D \in \mathfrak{D}^+(\mathcal{M})$ with $p \notin \operatorname{supp} D$ is open in $\mathfrak{D}^+(\mathcal{M})$.*

Proof. Fix a divisor $D_o \in \mathfrak{D}^+(\mathcal{M})$ with $p \notin \operatorname{supp} D_o$. Let ω be the fundamental form for some Hermitian metric on $\mathcal{M}$. Fix neighborhoods $U'' \Subset U' \Subset U$ of p with $U \cap \operatorname{supp} D_o = \emptyset$. Let χ be a nonnegative $\mathcal{C}^\infty$ function on $\mathcal{M}$ with χ identically one on U' and χ identically zero on $\mathcal{M} \setminus U$. There is a constant $C > 0$ small enough that if $D \in \mathfrak{D}^+(\mathcal{M})$ and $\operatorname{supp} D \cap U'' \neq \emptyset$, then $\int_{D_o} \chi\omega^{N-1} > C$. It follows that the set

$$\left\{D \in \mathfrak{D}^+(\mathcal{M}) : \int_D \chi\omega^{N-1} < C/2\right\}$$

is a (weak*) neighborhood V of D_o with the property that if $D \in V$, then $\operatorname{supp} D$ does not contain p. The lemma is proved.

Lemma 3.2. *Let p be a fixed point in the complex manifold $\mathcal{M}$. There exists a countable family $\{\varphi_j\}_{j\in\mathbb{N}}$ of maps from the unit ball $\mathbb{B}_N$ into $\mathcal{M}$ with the following properties:*

i. *Each* φ_j *carries* $\mathbb{B}_N$ *biholomorphically onto a domain* Ω_j *in* $\mathcal{M}$.
ii. *Each* φ_j *carries* $0 \in \mathbb{B}_N$ *to the point* p.
iii. *For suitable* $\rho_j \in (0,1)$, $\mathcal{M} = \cup\{\varphi(\rho_j\mathbb{B}_N) : j = 1, \dots\}$.

Proof. This depends on knowing that every pair of points in a complex manifold is contained in a biholomorphic copy of the ball.

Granted this, for each point $q \in \mathcal{M}$, fix a biholomorphic map ψ_q from $\mathbb{B}_N$ onto a domain in $\mathcal{M}$ that carries the origin to the point p and the range of which contains the point q. If $\rho_q \in (0,1)$ is large enough, then the point q will lie in the set $\psi_q(\rho_q\mathbb{B}_N)$.

Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of compact sets in $\mathcal{M}$ with union $\mathcal{M}$. A finite number of the sets $\psi_j(\rho_j\mathbb{B}_N)$ will cover a given K in the sequence, so the result follows.

To realize that it is possible to find a biholomorphic copy of the ball that contains a given pair of points in an arbitrary complex manifold, consider points p and q in $\mathcal{M}$. There is a real-analytic arc in $\mathcal{M}$ that contains both p and q. Then as a compact (real) line segment in $\mathbb{C}^N$ has a neighborhood basis that consists of biholomorphic copies of the ball, we see that the desired ball exists. The lemma is proved.

We next prove that the space $\mathfrak{D}^+(\mathcal{M};p)$ is metrizable.

Fix a point $p \in \mathcal{M}$. Let $\Omega_j, j \in \mathbb{N}$, be biholomorphic copies of $\mathbb{B}_N$ in $\mathcal{M}$ as introduced in the preceding lemma. Let Ω'_j be the corresponding concentric balls that cover $\mathcal{M}$.

There are restriction maps

$$r : \mathfrak{D}^+(\mathcal{M};p) \to \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega_j;p)$$

and

$$r' : \mathfrak{D}^+(\mathcal{M};p) \to \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega'_j;p)$$

given by

$$r(D) = \{D|\Omega_j : j \in \mathbb{N}\} \in \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega_j;p)$$

and

$$r'(D) = \{D|\Omega'_j : j \in \mathbb{N}\} \in \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega'_j;p).$$

In addition, there is a restriction map

$$\tilde{r} : \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega_j;p) \to \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega'_j;p)$$

defined in the evident way. These maps are all continuous when the product spaces are endowed with the product topologies. Plainly $\tilde{r} \circ r = r'$. Notice that the map r is a homeomorphism from $\mathfrak{D}^+(\mathcal{M};p)$ onto the closed subset

$$\{\{D_j : j \in \mathbb{N}\} \in \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega_j;p) : D_j|\Omega_j \cap \Omega_k = D_k|\Omega_j \cap \Omega_k\}$$

of $\Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega_j;p)$. Similarly, r' is a homeomorphism onto its range.

Consider now the map

$$\sigma_j : \mathfrak{D}^+(\Omega_j;p) \to \mathcal{O}(\Omega'_j)$$

given by Stoll that has the property that $\mathrm{Div} \circ \sigma_j(D) = D|\Omega'_j$. These maps taken together give a map

$$\sigma : \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega_j;p) \to \oplus_{j\in\mathbb{N}}\mathcal{O}(\Omega'_j)$$

given by

$$\sigma(\{D_j : j \in \mathbb{N}\}) = \{\sigma_j(D_j) : j \in \mathbb{N}\}.$$

Also let

$$\delta : \oplus_{j\in\mathbb{N}}\mathcal{O}(\Omega_j) \to \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega_j)$$

and

$$\delta' : \oplus_{j\in\mathbb{N}}\mathcal{O}(\Omega'_j) \to \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(\Omega'_j)$$

be the divisor maps

$$\delta(\{f_j : j \in \mathbb{N}\}) = \{\mathrm{Div} f_j : j \in \mathbb{N}\}$$

and with δ' defined in a similar way. These maps are continuous.

The map $\delta' \circ \sigma$ carries $r\mathfrak{D}^+(\mathcal{M};p)$ onto $r'\mathfrak{D}^+(\mathcal{M};p)$.

Finally, notice that if $D \in \mathfrak{D}^+(\mathcal{M};p)$, then $r'^{-1} \circ \delta' \circ \sigma \circ r(D)$ is simply D. This implies that $\sigma \circ r$ is a homeomorphism of $\mathfrak{D}^+(\mathcal{M};p)$ onto a subset of the metrizable space $\Pi_{j\in\mathbb{N}}\mathcal{O}(\Omega'_j)$.

We have reached the following conclusion:

Proposition 3.3. *The space $\mathfrak{D}^+(\mathcal{M};p)$ is metrizable.*

Consequently, $\mathfrak{D}^+(\mathcal{M};p)$ and all its topological subspaces are paracompact.

4. Lifts of $\varphi : X \to \mathfrak{D}^+_P(\mathcal{M};p)$.

The main result of this section is the following intermediate proposition, which will serve as a stepping stone to more general results.

Proposition 4.1. *Let $\mathcal{M}$ be a complex manifold with $H^1(\mathcal{M},\mathcal{O}) = 0$. Let $p \in \mathcal{M}$. If X is a connected paracompact space with $H^1(X,\mathbb{Z}) = 0$, then given a continuous map $\psi : X \to \mathfrak{D}^+_P(\mathcal{M};p)$, there is a continuous map $\tilde{\psi} : X \to \mathcal{O}(\mathcal{M})$ with* $\mathrm{Div} \circ \tilde{\psi} = \psi$.

Proof. Again introduce a sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ of (biholomorphic copies of) balls in $\mathcal{M}$ and concentric balls Ω'_j , all centered at the point $p \in \mathcal{M}$ and with $\cup\Omega'_j = \mathcal{M}$. Let $\mathcal{U} = \{U_j\}_{j\in\mathbb{N}}$ be a locally finite refinement of the cover $\{\Omega'_j\}_{j\in\mathbb{N}}$ of $\mathcal{M}$ by connected domains U_j with the property that the intersections U_{jk} are all contractible. That such covers exist is shown in [**BT**, p. 42]. Denote by $\tau : \mathbb{N} \to \mathbb{N}$ a refining map so that for each j, $U_j \subset \Omega'_{\tau(j)}$.

Also, let $\sigma_j : \mathfrak{D}^+(\Omega_j;p) \to \mathcal{O}(\Omega'_j)$, be the map used above that satisfies $\mathrm{Div} \circ \sigma_j(D) = D|\Omega'_j$ for all $D \in \mathfrak{D}^+(\Omega_j;p)$. Define $r_j : \mathfrak{D}^+(\mathcal{M};p) \to$

$\mathfrak{D}^+(\Omega_j; p)$ to be the restriction map so that $r_j(D) = D|\Omega'_j$. The map r_j is continuous and satisfies

$$\mathrm{Div} \circ \sigma_j \circ r_j(D) = D|\Omega'_j$$

for all j.

For each $j = 1, \ldots,$ let $F_j : \mathfrak{D}^+_P(\mathcal{M}) \times \Omega'_j \to \mathbb{C}$ be defined by

$$F_j(D, z) = (\sigma_j \circ r_j(D))(z).$$

These functions have the following properties: i. $F_j \in \mathcal{C}(\mathfrak{D}^+_P(\mathcal{M}) \times \Omega'_j)$, ii. $F_j(D, \cdot) \in \mathcal{O}(\Omega'_j)$ for each j, and iii. $\mathrm{Div} F_j(D, \cdot)|\Omega'_{jk} = \mathrm{Div} F_k(D, \cdot)|\Omega'_{jk}$ for all j, k.

Define $f_{jk} : \mathfrak{D}^+_P(\mathcal{M}; p) \times U_{jk} \to \mathbb{C}$ by

$$f_{jk} = (F_{\tau(j)}|U_{jk})(F_{\tau(k)}|U_{jk})^{-1}.$$

This is a continuous zero-free function with the property that for all D, $f_{jk}(D, \cdot)$ is holomorphic on Ω'_{jk} for each choice of j, k.

With $\psi : X \to \mathfrak{D}^+_P(\mathcal{M}, p)$ as in the theorem, let the map $\psi_{jk} : X \times U_{jk} \to \mathbb{C}$ be given for each choice of j and k by

$$\psi_{jk}(x, z) = f_{jk}(\psi(x), z);$$

it is continuous and zero-free. Moreover, for each $x \in X$, the function $\psi_{jk}(x, \cdot)$ is holomorphic on U_{jk}.

By hypothesis, $H^1(X, \mathbb{Z}) = 0$, so the group $H^1(X \times U_{jk}, \mathbb{Z})$ also vanishes. Accordingly, the function ψ_{jk} has a continuous logarithm, to be denoted by λ_{jk}, on $X \times U_{jk}$. For each $x \in X$, $\lambda_{jk}(x, \cdot)$ is holomorphic on U_{jk}.

Define $c_{\alpha\beta\gamma}$ on $X \times U_{\alpha\beta\gamma}$ by

$$c_{\alpha\beta\gamma} = \lambda_{\beta\gamma} - \lambda_{\alpha\gamma} + \lambda_{\alpha\beta}.$$

This is some value of log 1. The data $\{c_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma\in\mathbb{N}}$ constitute a 2-cocycle for the covering $\mathcal{U}_X = \{X \times U_j\}_{j\in\mathbb{N}}$ of $X \times \mathcal{M}$ with values in the group $2\pi i\mathbb{Z}$. For fixed $x \in X$, this cocycle defines the Chern class of the set $\{\psi_{jk}(x, \cdot)\}_{j,k\in\mathbb{N}}$ of Cousin II data on $\mathcal{M}$. As the associated divisor is principal, this cocycle is a coboundary: There are integers $\{n_{\alpha\beta}\}_{\alpha,\beta\in\mathbb{N}}$ such that

$$c_{\alpha\beta\gamma} = 2\pi i(n_{\beta\gamma} - n_{\alpha\gamma} + n_{\alpha\beta}).$$

As X is connected, continuity implies that for each fixed $z \in \mathcal{M}$, $c_{\alpha\beta\gamma}$ is independent of x.

Given $j, k \in \mathbb{N}$, define a continuous map $\lambda'_{jk} : X \to \mathcal{O}(U_{jk})$ by

$$\lambda'_{jk} = \lambda_{jk} - n_{jk}.$$

For each $x \in X$, the family $\lambda'(x) = \{\lambda'_{jk}(x)\}_{jk\in\mathbb{N}}$ defines a 1-cocycle for the covering $\mathcal{U}$ with values in the sheaf $\mathcal{O} = \mathcal{O}_{\mathcal{M}}$, that is, an element of the space $Z^1(\mathcal{U}, \mathcal{O})$. Thus, we have a continuous map $\lambda' : X \to Z^1(\mathcal{U}, \mathcal{O})$, given by $x \mapsto \lambda'(x)$, for all $x \in X$.

For every nonnegative integer p the space $C^p(\mathcal{U}, \mathcal{O})$ of p-chains for the covering $\mathcal{U}$ with values in the sheaf $\mathcal{O}$ is the direct sum $\oplus_{\beta_o,\dots,\beta_p \in \mathbb{N}} \mathcal{O}(U_{\beta_o \dots \beta_p})$ of the Fréchet spaces $\mathcal{O}(U_{\beta_o \dots \beta_p})$. As this is a *countable* direct sum, $C^p(\mathcal{U}, \mathcal{O})$ is itself a Fréchet space.

The coboundary map $\delta^p : C^p(\mathcal{U}, \mathcal{O}) \to C^{p+1}(\mathcal{U}, \mathcal{O})$ is a continuous linear map, so its kernel $Z^p(\mathcal{U}, \mathcal{O})$ is a closed subspace of $C^p(\mathcal{U}, \mathcal{O})$, hence a Fréchet space.

By hypothesis, $H^1(\mathcal{M}, \mathcal{O})$ vanishes, so $H^1(\mathcal{U}, \mathcal{O}) = 0$ as well, whence the range of the coboundary map $\delta^0 : C^0(\mathcal{U}, \mathcal{O}) \to C^1(\mathcal{U}, \mathcal{O})$ is closed and so is a Fréchet space.

Accordingly, by the selection theorem of Michael quoted in the Introduction, there is a continuous map

$$\varsigma : Z^1(\mathcal{U}, \mathcal{O}) \to C^0(\mathcal{U}, \mathcal{O})$$

such that $\delta^0 \circ \varsigma$ is the identity. Consider then the continuous map $f = \varsigma \circ \lambda' : X \to C^0(\mathcal{U}, \mathcal{O})$. We have $f = \{f_j\}_{j \in \mathbb{N}}$, where each f_j is a continuous map from X into $\mathcal{O}(U_j)$, and the f_j's satisfy $f_j - f_k = \lambda'_{jk}$ on U_{jk}, for all $j, k \in \mathbb{N}$. It follows that $F_{\tau(j)} e^{-2\pi i f_j} = F_{\tau(k)} e^{-2\pi i f_k}$ on U_{jk}, for all $j, k \in \mathbb{N}$ and all $x \in X$.

Hence the map we seek is the map $\tilde{\psi} : X \to \mathcal{O}(\mathcal{M})$ given by

$$\tilde{\psi}(x)(z) = F_{\tau(k)}(x, z) e^{-2\pi i f_k(x)(z)}$$

for all $k \in \mathbb{N}$, $x \in X$ and $z \in U_k$.

We point out that the proof just given contains implicitly also the proof of the following fact:

Proposition 4.2. *Let $\mathcal{M}$ be a complex-analytic manifold of dimension $N \geq 1$. Let X be a topological space that is paracompact and connected and has the further property that the Čech cohomology group $H^1(X, \mathbb{Z}) = 0$. Let $\mu : X \to \mathfrak{D}^+(\mathcal{M})$ be a continuous map. Then all the divisors $\mu(x)$, $x \in X$ have the same Chern class.*

5. The Topology of $\mathfrak{D}^+_P$, Continued.

The following lemma will be used at several points below.

Lemma 5.1. *If E is an infinite dimensional locally convex Fréchet space, then for all $n = 0, 1, \dots$, the homotopy group $\pi_n(E \setminus \{0\})$ vanishes.*

Remark. This is known: $E \setminus \{0\}$ is known to be contractible. The reason we include Lemma 5.1, with a proof, is that it can be proved in a very elementary way by an argument for which we have no reference. It seems worth our while to include the argument for the convenience of the reader.

That $E \setminus \{0\}$ is contractible appears in the literature as follows. (See [**BP**].) By a theorem of Anderson and Kadec [**BP**, Theorem 5.2, p. 189], every infinite-dimensional separable Fréchet space is homeomorphic to $\mathbb{R}^{\mathbb{N}}$. On the other hand, by a theorem of Anderson, if A is a compact subset of $\mathbb{R}^{\mathbb{N}}$ (or more generally a countable union of compacta), then $\mathbb{R}^{\mathbb{N}} \setminus A$ is homeomorphic to the whole space $\mathbb{R}^{\mathbb{N}}$. This follows from [**BP**, Theorem 6.3, p. 166 and Corollary 6.2, p. 165]. It follows in particular that $E \setminus \{0\}$ is homeomorphic to E and so is contractible. A simpler approach than this, still granted the theorem of Anderson and Kadec, is to invoke a theorem of Klee [**Kl**], which is somewhat simpler than the more general result of Anderson.

Proof of the Lemma. The space $E \setminus \{0\}$ is connected, so let $n > 0$ from here on.

Denote by ρ a metric on E that has the property that the ball $\{y \in E : \rho(y, x) < c\}$ is convex for every choice of $x \in E$ and all $r > 0$.

Denote by Σ the simplex in R^{n+1} determined by the origin and the unit vectors $e_j = (0, \dots, 1, \dots, 0)$, 1 in the j^{th} place. The boundary, $b\Sigma$, of Σ is topologically the $n-$sphere.

Consider a continuous map $\varphi : b\Sigma \to E \setminus \{0\}$. As $\varphi(b\Sigma)$ is compact, there is a $\delta > 0$ small enough that $\rho(\varphi(x), 0) > \delta$ for all $x \in b\Sigma$.

For a sufficiently fine simplicial refinement of the given triangulation of $b\Sigma$, say with vertices $\{x_1, \dots, x_r\}$ and with simplexes $T_1, \dots, T_s$, the unique continuous map $\psi : \Sigma \to E \setminus \{0\}$ that agrees with φ at each x_j and that is real affine on each T has the property that both of the sets $\varphi(T_j)$ and $\psi(T_j)$ are contained in some ball with respect to the metric ρ that does not contain 0. Consequently, φ and ψ are homotopic as maps into $E \setminus \{0\}$.

The range of ψ is contained in a finite dimensional real linear subspace of E, which we can take to have real dimension $d > n + 1$. As every map of the $n-$sphere into $\mathbb{R}^d \setminus \{0\}$ is homotopic to a constant, it follows that the original map φ is homotopic in $E \setminus \{0\}$ to a constant.

Remark. Granted that all of the homotopy groups $\pi_n(E \setminus \{0\})$ vanish, it follows that the space $E \setminus \{0\}$ is contractible. See [**BP**, Theorem 6.3, p. 79 and Corollary 6.5, p. 76].

Lemma 5.2. *If $H^1(\mathcal{M}, \mathcal{O}) = 0$ and $H^1(\mathcal{M}, \mathbb{Z}) = 0$, then the cohomology groups $H^1(\mathfrak{D}^+_P(\mathcal{M}), \mathbb{Z})$ and $H^1(\mathfrak{D}^+_P(\mathcal{M}, p), \mathbb{Z})$ vanish.*

The cohomology in question is taken in the sense of Čech theory.

Proof. This lemma depends on the following simple fact: Because the group $H^1(\mathcal{M}, \mathbb{Z})$ vanishes, it follows that every zero-free continuous $\mathbb{C}-$valued function on $\mathcal{M}$ is of the form e^h for some continuous function h. As every continuous logarithm of a holomorphic function is itself holomorphic, it follows that the space $\mathcal{O}^*(\mathcal{M})$ of zero-free holomorphic functions on $\mathcal{M}$ is connected and, indeed, is arcwise connected: If $f = e^h$ with $h \in \mathcal{O}(\mathcal{M})$, then $t \mapsto e^{th}$, $t \in [0,1]$ is a curve in $\mathcal{O}^*(\mathcal{M})$ that connects f to 1. (Note that $\mathcal{O}^*(\mathcal{M})$ is not open in $\mathcal{O}(\mathcal{M})$.)

Consider first the group $H^1(\mathfrak{D}_P^+(\mathcal{M}), \mathbb{Z})$.

If $\mathcal{O}(\mathcal{M}) = \mathbb{C}$, then $\mathfrak{D}_P^+(\mathcal{M}) = \{0\}$, and the assertion of the lemma is true. Thus, assume $\mathcal{O}(\mathcal{M})$ to be infinite dimensional. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover for $\mathfrak{D}_P^+(\mathcal{M})$, and let $\{m_{\alpha\beta}\}_{\alpha\beta \in A}$ be a 1-cocycle for this cover with values in $\mathbb{Z}$ so that for each choice of $\alpha, \beta \in A$, $m_{\alpha\beta}$ is a continuous $\mathbb{Z}$-valued function on $U_{\alpha\beta}$.

Because the map Div is continuous, if $\tilde{U}_\alpha = \mathrm{Div}^{-1}(U_\alpha)$, then $\tilde{\mathcal{U}} = \{\tilde{U}_\alpha\}_{\alpha \in A}$ is an open cover for the space $\mathcal{O}(\mathcal{M}) \setminus \{0\}$. With $\tilde{m}_{\alpha\beta} = m_{\alpha\beta} \circ \mathrm{Div}$, the family $\{\tilde{m}_{\alpha\beta}\}_{\alpha,\beta \in A}$ is a 1-cocycle for the covering $\tilde{\mathcal{U}}$ with values in $\mathbb{Z}$.

As the space $\mathcal{O}(\mathcal{M}) \setminus \{0\}$ has vanishing first integral cohomology, it follows that the cohomology group $H^1(\tilde{\mathcal{U}}, \mathbb{Z})$ vanishes, for the canonical map from this group into $H^1(\mathcal{O}(\mathcal{M}) \setminus \{0\}, \mathbb{Z})$ is injective. (See the discussion of Leray's theorem given in [**Gu**].) Consequently, the 1-cocycle $\{\tilde{m}_{\alpha\beta}\}_{\alpha,\beta \in A}$ is a 1-coboundary: There are continuous, $\mathbb{Z}-$valued functions $\tilde{n}_\alpha$ on $\tilde{U}_\alpha$ with $\tilde{m}_{\alpha\beta} = \tilde{n}_\alpha | U_{\alpha\beta} - \tilde{n}_\beta | U_{\alpha\beta}$.

For any divisor $D \in \mathfrak{D}^+(\mathcal{M})$, the fiber $\mathrm{Div}^{-1}(D)$ is of the form $\{gh : g \in \mathcal{O}(\mathcal{M}), g \text{ zero} - \text{free}\}$ for any fixed element h of the fiber. That the set of zero-free functions holomorphic on $\mathcal{M}$ is connected implies that the fiber $\mathrm{Div}^{-1}(D)$ is connected.

It follows that the continuous, $\mathbb{Z}-$valued functions $\tilde{n}_\alpha$ are constant on the fibers of Div and so are of the form $\tilde{n}_\alpha = n_\alpha \circ \mathrm{Div}$ for suitable continuous $\mathbb{Z}-$valued functions n_α on U_α. Plainly $n_\alpha | U_{\alpha\beta} - n_\beta | U_{\alpha\beta} = m_{\alpha\beta}$. That is, the cocycle $\{m_{\alpha\beta}\}_{\alpha\beta} \in A$ is a coboundary. It follows that the group $H^1(\mathfrak{D}_P^+(\mathcal{M}), \mathbb{Z})$ is trivial as claimed.

In its essence, the argument to prove that $H^1(\mathfrak{D}_P^+(\mathcal{M}, p), \mathbb{Z}) = 0$ is parallel to the preceding argument, but some preliminaries are necessary.

If the set $\mathcal{F}_p$ is defined by $\mathcal{F}_p = \{f \in \mathcal{O}(\mathcal{M}) : f(p) \notin \mathbb{C} \setminus (-\infty, 0]\}$, then $\mathrm{Div}\mathcal{F}_p = \mathfrak{D}_P^+(\mathcal{M}; p)$. This is so, for if $D \in \mathfrak{D}_P^+(\mathcal{M}; p)$, then $D = \mathrm{Div} f$ for an $f \in \mathcal{O}(\mathcal{M})$ with $f(p) \neq 0$, which yields that $D = \mathrm{Div}(f/f(p))$. The function $f/f(p)$ lies in $\mathcal{F}_p$. Conversely, it is evident that $\mathrm{Div}\mathcal{F}_p \subset \mathfrak{D}_P^+(\mathcal{M}; p)$.

The set $\mathcal{F}_p$ is open in $\mathcal{O}(\mathcal{M})$. It is also contractible. To see the latter point, define $H : [0,2] \times \mathcal{F}_p \to \mathcal{F}_p$ by

$$H(t,f)(z) = \begin{cases} f(p) e^{(1-t)\int_p^z \frac{df}{f}} & \text{when } t \in [0,1],\ f \in \mathcal{F}_p,\ z \in \mathcal{M} \\ \\ (2-t) f(p) + t - 1 & \text{when } t \in [1,2],\ f \in \mathcal{F}_p,\ z \in \mathcal{M}. \end{cases}$$

That the integral is well defined depends on the hypothesis that $H^1(\mathcal{M}, \mathbb{Z}) = 0$. The map H is continuous, and its range is contained in $\mathcal{F}_p$, $H(0,\cdot)$ is the identity on $\mathcal{F}_p$, and, finally, $H(2,\cdot)$ is the function identically 1.

As $\mathcal{F}_p$ is contractible, it follows that $H^1(\mathcal{F}_p, \mathbb{Z})$ vanishes.

If $D \in \mathfrak{D}^+_P(\mathcal{M}; p)$, then the fiber $(\mathrm{Div}^{-1} D) \cap \mathcal{F}_p$ be described as follows. Fix an $f_o \in \mathcal{O}(\mathcal{M})$ with $\mathrm{Div} f_o = D$. Without loss of generality, f_o can be chosen to take the value 1 at p. The condition that $\mathrm{Div} g = D$ is then the condition that for some $h \in \mathcal{O}^*(\mathcal{M})$, $g = hf_o$ The function g lies in $\mathcal{F}$ if and only if $h(p) \notin (-\infty, 0]$. It follows that the fiber $\mathrm{Div}^{-1}(D) \cap \mathcal{F}_p$ is (arcwise) connected: Fix $h \in \mathcal{O}^*(\mathcal{M})$ with $h(p) \notin (-\infty, 0]$. If $\lambda : [0, 1] \to \mathcal{O}^*(\mathcal{M})$ is defined by

$$\lambda(t)(z) = f_o(z) e^{(1-t)\int_p^z \frac{dh}{h} + (1-t)L(h(p))}$$

where L denotes the branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$ that vanishes at the point 1, then λ is a curve in $\mathcal{O}^*(\mathcal{M})$ that connects g to the function f_o. Note that as $\lambda(t)(p) = f_o(p)e^{(1-t)L(h(p))}$ for all t, the range of λ is contained in the set $\mathcal{F}_p$. Thus, as claimed, the fiber is connected.

The rest of the argument can proceed as in the case of $\mathfrak{D}^+_P(\mathcal{M})$: If $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of $\mathfrak{D}^+_P(\mathcal{M}; p)$, then the sets $\tilde{U}_\alpha = \mathcal{F}_p \cap \mathrm{Div}^{-1}(U_\alpha)$ constitute an open cover for $\mathcal{F}_p$. An integral 1-cocycle on $\mathfrak{D}^+_P(\mathcal{M}; p)$ induces a corresponding integral 1-cocycle on $\mathcal{F}_p$, which is a coboundary, for $H^1(\mathcal{F}_p, \mathbb{Z}) = 0$. The result follows as before.

Corollary 5.3. *If $H^1(\mathcal{M}, \mathcal{O})$ and $H^1(\mathcal{M}, \mathbb{Z})$ vanish, and if $p \in \mathcal{M}$, then there is a continuous map $\varsigma_p : \mathfrak{D}^+_P(\mathcal{M}; p) \to \mathcal{O}(\mathcal{M})$ with $\mathrm{Div} \circ \varsigma_p(D) = D$ for all $D \in \mathfrak{D}^+_P(\mathcal{M}; p)$.*

Proof. This is a consequence of the preceding lemma and Proposition 4.1.

Corollary 5.4. *If $H^1(\mathcal{M}, \mathcal{O})$ and $H^1(\mathcal{M}, \mathbb{Z})$ vanish, then the map $\mathrm{Div} : \mathcal{O}(\mathcal{M}) \setminus \{0\} \to \mathfrak{D}^+_P(\mathcal{M})$ is open.*

Proof. As usual, the case that $\mathcal{O}(\mathcal{M})$ consists only of constants requires separate comment, and is trivial. Therefore suppose that there is a nonconstant holomorphic function f on $\mathcal{M}$. Let $D_o \in \mathfrak{D}^+_P(\mathcal{M})$ be given. Fix $p \in \mathcal{M} \setminus \mathrm{supp} D_o$. There is an associated section $\varsigma_p : \mathfrak{D}^+_P(\mathcal{M}; p) \to \mathcal{O}(\mathcal{M})$.

Let $f_o \in \mathcal{O}(\mathcal{M})$ satisfy $\mathrm{Div}\, f_o = D_o$. We show that if W_o is a neighborhood of f_o in $\mathcal{O}(\mathcal{M}) \setminus \{0\}$, then $\mathrm{Div}\, W_o$ contains a neighborhood of D_o in $\mathfrak{D}^+_P(\mathcal{M})$. To do this, define $\sigma : \mathfrak{D}^+_P(\mathcal{M}; p) \to \mathcal{O}(\mathcal{M})$ by $\sigma(D) = \frac{f_o}{\varsigma_p(D_o)} \varsigma_p(D)$. This map is continuous, and it satisfies $\sigma(D_o) = f_o$. By continuity there is a neighborhood $\tilde{W}_o$ of D_o such that $\sigma(D) \in W_o$ if $D \in \tilde{W}_o$. This implies that the map Div carries W_o onto a set that contains the neighborhood $\tilde{W}_o$ of D_o.

Thus, as claimed, Div is an open map.

Corollary 5.5. *If $H^1(\mathcal{M}, \mathcal{O})$ and $H^1(\mathcal{M}, \mathbb{Z})$ vanish, then the topology on the space $\mathfrak{D}^+_P(\mathcal{M})$ is the quotient topology induced by the map $\mathrm{Div} : \mathcal{O}(\mathcal{M}) \setminus \{0\} \to \mathfrak{D}^+_P(\mathcal{M})$.*

This follows from the continuity and openness of the map Div. (See [**Ke**, p. 95].)

Corollary 5.6. *If $H^1(\mathcal{M}, \mathcal{O})$ and $H^1(\mathcal{M}, \mathbb{Z})$ vanish then the space $\mathfrak{D}^+_P(\mathcal{M})$ is metrizable.*

Remark. Granted the corollary, it is easy to see that *for every complex manifold $\mathcal{M}$, the space $\mathfrak{D}^+(\mathcal{M})$ of nonnegative divisors is metrizable.* Fix $\mathcal{M}$ and let $\{B_j\}_{j\in\mathbb{N}}$ be a countable collection of domains in $\mathcal{M}$ each of which is biholomorphic to a ball and the union of which covers $\mathcal{M}$. Define a map $\rho : \mathfrak{D}^+(\mathcal{M}) \to \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(B_j)$ by $\rho D = \{D|B_j : j \in \mathbb{N}\}$. The map ρ is a homeomorphism from $\mathfrak{D}^+(\mathcal{M})$ onto the closed subset $\{\{D_j : j \in \mathbb{N}\} \in \Pi_{j\in\mathbb{N}}\mathfrak{D}^+(B_j) : D_j|B_{jk} = D_k|B_{jk} \text{ forall j, k} \in \mathbb{N}\}$ of $\Pi_{j\in\mathbb{N}}\mathfrak{D}^+(B_j)$. As the product is countable and as each $\mathfrak{D}^+(B_j) = \mathfrak{D}^+_P(B_j)$ is metrizable, the metrizability of $\mathfrak{D}^+(\mathcal{M})$ follows.

Proof of the corollary. Begin by recalling that the topology of a topological space Y is *regular* if given a point $y \in Y$ and a neighborhood U of y, there is a neighborhood V of y with $\bar{V} \subset U$.

The space $\mathcal{O}(\mathcal{M}) \setminus \{0\}$ is metrizable and separable, so its topology has a countable basis. As the map Div is open, it follows that the topology on $\mathfrak{D}^+_P(\mathcal{O})$ has a countable base.

Moreover, the topology of $\mathfrak{D}^+_P(\mathcal{M})$ is regular. For this, work with the relative weak* topology. If $D_o \in \mathfrak{D}^+_P(\mathcal{M})$, and if W is an open set containing D_o, then there is a collection $\{\alpha_1, \dots, \alpha_r\}$ of compactly supported smooth $(N-1, N-1)$-forms on $\mathcal{M}$ such that W contains the weak* neighborhood

$$W_1 = \left\{D \in \mathfrak{D}^+_P(\mathcal{M}) : \left|\int_D \alpha_j - \int_{D_o} \alpha_j\right| < 1 \text{ for all } j = 1, ..., r\right\}$$

of D_o But then the closed neighborhood

$$\bar{W}_{\frac{1}{2}} = \left\{D \in \mathfrak{D}^+_P(\mathcal{M}) : \left|\int_D \alpha_j - \int_{D_o} \alpha_j\right| \le \frac{1}{2} \text{ for all } j = 1, ..., r\right\}$$

is contained in W_1. Thus, the topology of $\mathfrak{D}^+_P(\mathcal{M})$ is regular as claimed.

The classical metrization theorem of Uryson and of Tikhonov [**Ke**, p. 125] implies that $\mathfrak{D}^+_P(\mathcal{M})$ is metrizable.

Recall now the notion of an ANR [**BP**]. A topological space Y is an ANR (*absolute neighborhood retract*) if it is metrizable and if whenever it is embedded topologically as a closed subset of a metric space, X, there is a neighborhood of X that retracts onto Y. We shall need three general facts about ANR's. One is a theorem of Hanner [**BP**, p. 69]: *A paracompact space that is locally an ANR is an ANR.* There is also the fact that *a retract of an ANR is an ANR* [**BP**, p. 68]. Finally, *an open set in a locally convex metrizable topological vector space is an ANR* [**BP**, p. 69].

The following corollary is now evident:

Corollary 5.7. *If $H^1(\mathcal{M}, \mathcal{O})$ and $H^1(\mathcal{M}, \mathbb{Z})$ vanish, then the space $\mathfrak{D}^+_P(\mathcal{M})$ is an ANR as is the space $\mathfrak{D}^+(\mathcal{M}; p)$.*

Proof. If $\mathcal{O}(\mathcal{M})$ consists of the constants the result is clear.

If $\mathcal{O}(\mathcal{M})$ contains a nonconstant function, the case of $\mathfrak{D}^+_P(\mathcal{M})$ is a consequence of case of $\mathfrak{D}^+_P(\mathcal{M}; p)$, for as noted above, a space that is locally an ANR is itself an ANR.

To treat the case of $\mathfrak{D}^+_P(\mathcal{M}; p)$, notice that as the group $H^1(\mathfrak{D}^+_P(\mathcal{M}; p), \mathbb{Z})$ vanishes, Proposition 4.1 applies to yield a map $\sigma : \mathfrak{D}^+_P(\mathcal{M}; p) \to \mathcal{O}(\mathcal{M})$ with $\mathrm{Div} \circ \sigma(D) = D$ for every $D \in \mathfrak{D}^+_P(\mathcal{M}; p)$. Consequently, the map $\sigma \circ \mathrm{Div} : \mathrm{Div}^{-1}(\mathfrak{D}^+(\mathcal{M}; p)) \to \sigma(\mathfrak{D}^+(\mathcal{M}; p))$ is a retraction of the open subset $\mathrm{Div}^{-1}(\mathfrak{D}^+(\mathcal{M}; p))$ of $\mathcal{O}(\mathcal{M})$ onto the range of σ. It follows that $\mathfrak{D}^+(\mathcal{M}; p)$ is an ANR as desired.

Corollary 5.8. *Let $\mathcal{M}$ be a complex manifold with $H^1(\mathcal{M}, \mathcal{O}) = 0$ and $H^1(\mathcal{M}, \mathbb{Z}) = 0$, and let $p \in \mathcal{M}$. For each $n = 0, 1, 2, \dots,$ the Čech and the singular cohomology groups of $\mathfrak{D}^+_P(\mathcal{M})$ of dimension n with integral coefficients are isomorphic as are the corresponding groups of $\mathfrak{D}^+_P(\mathcal{M}; p)$.*

Proof. It is known [**Bo**, p. 107], [**Ma**] that for topological spaces of type ANR the Čech and singular cohomology groups agree.

6. Liftings Over $\mathfrak{D}^+_P$.

If X is a topological space and $\mathcal{M}$ is a complex manifold, we denote by $\mathcal{C}_{X;\mathcal{O}(\mathcal{M})}$ and by $\mathcal{C}_{X;\mathcal{O}^*(\mathcal{M})}$ the sheaves of germs of continuous $\mathcal{O}(\mathcal{M})$−valued functions and of continuous $\mathcal{O}^*(\mathcal{M})$−valued functions on X, respectively. These are sheaves of abelian groups.

Lemma 6.1. *Let $\mathcal{M}$ be a complex-analytic manifold of dimension $N \geq 1$ with $H^1(\mathcal{M}, \mathbb{Z}) = 0$, and let X be a paracompact topological space.*

1) *Every continuous map $f : X \to \mathcal{O}^*(\mathcal{M})$ has a continuous logarithm if and only if the Čech cohomology group $H^1(X, \mathbb{Z}) = 0$ vanishes.*
2) *For $q \geq 1$ the Čech cohomology groups $H^q(X, \mathcal{C}_{X;\mathcal{O}^*(\mathcal{M})})$ and $H^{q+1}(X, \mathbb{Z})$ are isomorphic.*

Proof. Define $E : \mathcal{C}_{X;\mathcal{O}(\mathcal{M})} \to \mathcal{C}_{X;\mathcal{O}^*(\mathcal{M})}$ by $Eg = e^{2\pi i g}$. This is a homomorphism of sheaves of abelian groups.

The map E is surjective, that is, if $x_o \in X$ and $f_o : V \to \mathcal{O}^*(\mathcal{M})$ is a continuous map, V some neighborhood of x_o in X, then there is a neighborhood $W \subset V$ of x_o on which there is defined a continuous map $g : W \to \mathcal{O}(\mathcal{M})$ that satisfies $Eg(x) = f_o(x)$ for all $x \in W$. This is seen as follows.

Fix $\zeta \in \mathcal{M}$ and define a function $\epsilon : V \to \mathbb{C}^*$ by $\epsilon(x) = f_o(x)(\zeta)$, for all $x \in V$. (A word on notation may be in order here: For each $x \in V$, $f_o(x)$ is an element of $\mathcal{O}^*(\mathcal{M})$ and so has a value at the point $\zeta \in \mathcal{M}$, viz., $f_o(x)(\zeta)$.)

The evaluation function ϵ is continuous, so there is a neighborhood W of x_o, $W \subset V$, on which ϵ has a continuous logarithm: There is a continuous function $\lambda : W \to \mathbb{C}$ with $e^{\lambda(x)} = \epsilon(x)$ for all $x \in W$.

Define $\tilde{g} : W \to \mathcal{O}(\mathcal{M})$ by

$$\tilde{g}(x)(z) = \int_\zeta^z \frac{df_o(x)}{f_o(x)}$$

for all $x \in W$ and for all $z \in \mathcal{M}$. By hypothesis $H^1(\mathcal{M}, \mathbb{Z}) = 0$, so the integral does not depend on the choice of the path of integration from ζ to z. Hence $\tilde{g}$ is a well-defined, $\mathcal{O}(\mathcal{M})-$valued function on W; it depends continuously on $x \in W$.

The map $g : W \to \mathcal{O}(\mathcal{M})$ given by

$$g(x) = \frac{1}{2\pi i}(\tilde{g}(x) + \lambda(x)), \quad \text{for all } \; x \in W,$$

is continuous and satisfies $Eg = f_o$. Thus, E is surjective.

The kernel of the sheaf homomorphism E is the sheaf $\mathbb{Z}$ of germs of continuous integer-valued functions on X, so we infer that, under the hypothesis $H^1(\mathcal{M}, \mathbb{Z}) = 0$, there is an exact sheaf sequence

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{C}_{X;\mathcal{O}(\mathcal{M})} \xrightarrow{E} \mathcal{C}_{X;\mathcal{O}^*(\mathcal{M})} \to 0.$$

This gives rise to the associated exact cohomology sequence, which contains the segments

$$\Gamma(X, \mathcal{C}_{X;\mathcal{O}(\mathcal{M})}) \to \Gamma(X, \mathcal{C}_{X;\mathcal{O}^*(\mathcal{M})}) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{C}_{X;\mathcal{O}(\mathcal{M})})$$

and

$$H^q(X, \mathcal{C}_{X;\mathcal{O}(\mathcal{M})}) \to H^q(X, \mathcal{C}_{X;\mathcal{O}^*(\mathcal{M})}) \to H^{q+1}(X, \mathbb{Z}) \to H^{q+1}(X, \mathcal{C}_{X;\mathcal{O}(\mathcal{M})}).$$

Moreover the cohomology group $H^q(X, \mathcal{C}_{X;\mathcal{O}(\mathcal{M})})$ is zero for $q \geq 1$, since the sheaf $\mathcal{C}_{X;\mathcal{O}(\mathcal{M})}$ of germs of continuous $\mathcal{O}(\mathcal{M})-$valued functions on the paracompact space X is a *fine* sheaf, as it is because it is a sheaf of modules over the fine sheaf $\mathcal{C}_X$ of complex-valued continuous functions on X. Then the two statements of the lemma follow at once.

Lemma 6.2. *Let $\mathcal{M}$ be a complex-analytic manifold of dimension $N \geq 1$, such that $H^1(\mathcal{M}, \mathcal{O})$ and $H^1(\mathcal{M}, \mathbb{Z})$ vanish. Let X be a topological space that is paracompact and connected and has the further property that the Čech cohomology groups $H^1(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ vanish. Let $\mu : X \to \mathfrak{D}_P^+(\mathcal{M})$ be a continuous map. Then there exists a continuous map $\tilde{\mu} : X \to \mathcal{O}(\mathcal{M})$ such that* $\operatorname{Div} \tilde{\mu}(x) = \mu(x)$ *for all $x \in X$.*

Proof. The case that $\mathcal{O}(\mathcal{M}) = \mathbb{C}$ is clear, so we assume from here on that there is a nonconstant holomorphic function on $\mathcal{M}$.

To begin with, the Lemma is correct locally in X. Let $x \in X$. The point $\mu(x)$ lies in an open set $\mathfrak{D}_P^+(\mathcal{M}; p)$ for some choice of $p \in \mathcal{M}$. By

Corollary 5.3, there is a section $\varsigma_p : \mathfrak{D}^+_P(\mathcal{M};p) \to \mathcal{O}(\mathcal{M})$ of the map Div. By continuity, there is an open set V_x in X that contains x and such that $\mu(V_x) \subset \mathfrak{D}^+_P(\mathcal{M};p)$. As $\mathrm{Div} \circ \varsigma_p \circ \mu(x) = \mu(x)$, the Lemma is seen to be correct locally on X.

Let $\mathcal{V} = \{V_j\}_{j\in\mathbb{N}}$ be a locally finite open covering of X with the property that for each j there is a map $\mu_j : V_j \to \mathcal{O}(\mathcal{M})$ that satisfies $\mathrm{Div} \circ \mu_j = \mu$ in V_j. If $\mu_{jk} : V_{jk} \to \mathcal{O}(\mathcal{M})$ is defined by $\mu_{jk}(x)(z) = \frac{\mu_j(x)(z)}{\mu_k(x)(z)}$, then μ_{jk} is continuous, and $\{\mu_{jk}\}_{j,k\in\mathbb{N}}$ defines an element of the group of cocycles $\mathbb{Z}^1(\mathcal{V}, \mathcal{C}_{X;\mathcal{O}^*(\mathcal{M})})$. By hypothesis, $H^2(X,\mathbb{Z}) = 0$, so by Lemma 6.1, this cocycle is a coboundary: There are maps $\nu_j : V_j \to \mathcal{O}^*(\mathcal{M})$ with $\frac{\mu_j}{\mu_l} = \frac{\nu_j}{\nu_k}$ on V_{jk}. This implies that on V_{jk} the functions $\mu_j\nu_j^{-1}$ and $\mu_k\nu_k^{-1}$ agree, whence we can obtain a well defined map $\tilde{\mu} : X \to \mathcal{O}(\mathcal{M})$ by requiring that on V_j, $\tilde{\mu} = \mu_j\nu_j^{-1}$. This map $\tilde{\mu}$ satisfies $\mathrm{Div} \circ \tilde{\mu} = \mu$, so the Lemma is proved.

Lemma 6.3. *Let $\mathcal{M}$ be a complex manifold with $H^1(\mathcal{M},\mathcal{O}) = 0$ and $H^1(\mathcal{M}, Z) = 0$. For every nonnegative integer n the homotopy group $\pi_n(\mathfrak{D}^+_P(\mathcal{M}))$ is zero.*

Proof. The lemma is trivial if there are no nonconstant holomorphic functions on $\mathcal{M}$, in which case $\mathfrak{D}^+_P(\mathcal{M})$ contains only $0-$the zero divisor.

Thus, assume that $\mathcal{O}(\mathcal{M})$ contains a nonconstant function.

That $\pi_0(\mathfrak{D}^+_P(\mathcal{M})) = 0$ means simply that the space $\mathfrak{D}^+_P(\mathcal{M})$ is arcwise connected, which it is.

In order to prove that $\pi_1(\mathfrak{D}^+_P(\mathcal{M})) = 0$, we begin by recalling that, as $H^1(\mathcal{M},\mathbb{Z}) = 0$, the space $\mathcal{O}^*(\mathcal{M})$ is arcwise connected.

We have to prove that each continuous map $\gamma : [0,1] \to \mathfrak{D}^+_P(\mathcal{M})$ with $\gamma(0) = \gamma(1)$ is homotopic to the constant map $t \mapsto \gamma(0)$. To do this, first lift γ to a closed curve μ in $\mathcal{O}(\mathcal{M})$. That is, μ is to be a closed curve that satisfies $\mathrm{Div} \circ \mu = \gamma$. Lemma 6.2 or, alternatively, Proposition 4.1, implies that there is a continuous map $\gamma^* : [0,1] \to \mathcal{O}(\mathcal{M})$ that satisfies $\mathrm{Div} \circ \gamma^* = \gamma$. In general, γ^* will not be a closed curve. However, the hypothesis that $H^1(\mathcal{M},\mathbb{Z}) = 0$ lets us modify γ^* to obtain a closed lifting, μ, of γ: The function $\frac{\gamma^*(0)}{\gamma^*(1)}$ is holomorphic and zero-free on $\mathcal{M}$ and so has a holomorphic logarithm, say λ. For μ take the function given by $\mu(t) = e^{t\lambda}\gamma^*(t)$. As $\mathrm{Div} \circ \mu = \gamma$, the range of μ is contained in the space $\mathcal{O}(\mathcal{M}) \setminus \{0\}$, which, by Lemma 5.1, has vanishing fundamental group, so the map μ is homotopic to the constant map $t \mapsto \mu(0)$: There is a map $\tilde{H} : [0,1] \times [0,1] \to \mathcal{O}(\mathcal{M}) \setminus \{0\}$ with $\tilde{H}(0,t) = \mu(t)$, $\tilde{H}(1,t) = \mu(0)$ for all $t \in [0,1]$ and $\tilde{H}(0,s) = H(1,s)$ for all s. Then the map $H = \mathrm{Div} \circ \tilde{H} : [0,1] \times [0,1] \to \mathfrak{D}^+_P(\mathcal{M})$ is a homotopy in $\mathfrak{D}^+_P(\mathcal{M})$ connecting γ with the constant map $t \mapsto \gamma(0)$.

Hence $\mathfrak{D}^+_P(\mathcal{M})$ is simply connected.

Next, assume that $n \geq 2$. There is a distinction between the case $n = 2$ and the case $n \geq 3$.

Consider first the case $n \geq 3$. We are to show that each continuous map $\phi : S^n \to \mathfrak{D}_P^+(\mathcal{M})$ is homotopic to a constant map. As the cohomology groups $H^1(S^n, \mathbb{Z})$ and $H^2(S^n, \mathbb{Z})$ vanish, Lemma 6.2 yields a continuous lift $\tilde{\varphi} : S^n \to \mathcal{O}(\mathcal{M})$ with $\operatorname{Div} \tilde{\varphi}(x) = \varphi(x)$ for all $x \in S^n$. The range of $\tilde{\varphi}$ is contained in the space $\mathcal{O}(\mathcal{M}) \setminus \{0\}$, which, by Lemma 5.1, has vanishing n^{th} homotopy group, so the map $\tilde{\varphi}$ is homotopic to a constant map: There is a map $\tilde{H} : [0,1] \times S^n \to \mathcal{O}(\mathcal{M}) \setminus \{0\}$ with $\tilde{H}(0,x) = \tilde{\varphi}(x)$ and $\tilde{H}(1,x) = f_o$ for all $x \in S^n$, for some fixed $f_o \in \mathcal{O}(\mathcal{M}) \setminus \{0\}$ that is independent of x. Then the map $H = \operatorname{Div} \circ \tilde{H} : [0,1] \times S^n \to \mathfrak{D}_P^+(\mathcal{M})$ is a homotopy in $\mathfrak{D}_P^+(\mathcal{M})$ connecting φ with a constant map.

The case $n = 2$ requires something more.

Let $\varphi : S^2 \to \mathfrak{D}_P^+(\mathcal{M})$ be a continuous map with S^2 realized in the usual way as

$$S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

Denote by Σ^+ the subset $S^2 \setminus \{(-1,0,0)\}$ of S^2 and by Σ^- the subset $S^2 \setminus \{(1,0,0)\}$. Lemma 6.2 (considered for $X = \Sigma^+, \Sigma^-$ and $\mu = \varphi|\Sigma^+, \varphi|\Sigma^-$, respectively) yields continuous maps $\varphi^+ : \Sigma^+ \to \mathcal{O}(\mathcal{M})$ and $\varphi^- : \Sigma^- \to \mathcal{O}(\mathcal{M})$ with $\operatorname{Div} \circ \varphi^+ = \varphi$ on Σ^+ and $\operatorname{Div} \circ \varphi^- = \varphi$ on Σ^-. After multiplying φ^- by a function in $\mathcal{O}^*(\mathcal{M})$, we can suppose that $\varphi^+(0,0,1) = \varphi^-(0,0,1)$.

Set $\Sigma_{+-} = \Sigma^+ \cap \Sigma^-$.

Let z_o be a fixed point of $\mathcal{M}$, and define $h : \Sigma_{+-} \to \mathbb{C}^*$ by

$$h(x) = \frac{\varphi^+(x)(z_o)}{\varphi^-(x)(z_o)}, \quad \text{for all } \; x \in \Sigma_{+-}.$$

Define $f : \Sigma_{+-} \to \mathcal{O}^*(\mathcal{M})$ by

$$f(x) = \frac{1}{h(x)} \frac{\varphi^+(x)}{\varphi^-(x)} \quad \text{for all } \; x \in \Sigma_{+-}.$$

This is a continuous $\mathcal{O}^*(\mathcal{M})$-valued function with the properties that $f(x)(z_o) = 1$ for all $x \in \Sigma_{+-}$ and $f(0,0,1) \equiv 1$ on $\mathcal{M}$.

Define then $g : \Sigma_{+-} \to \mathcal{O}(\mathcal{M})$ by

$$g(x)(z) = \int_{z_o}^{z} \frac{df(x)}{f(x)} \quad \text{for all } \; x \in \Sigma_{+-}.$$

The integral is independent of the choice of the path of integration from z_o to $z \in \mathcal{M}$, for, if γ and γ' are two such integration paths, the map $\Sigma_{+-} \to \mathbb{C}$ given by $x \mapsto \int_{\gamma - \gamma'} \frac{df}{f(x)}$ for all $x \in \Sigma_{+-}$, being a continuous $2\pi i\mathbb{Z}-$valued map which is zero at the point $(0,0,1)$, is identically zero on Σ_{+-}. Consequently g is a well-defined continuous map from Σ_{+-} into $\mathcal{O}(\mathcal{M})$, and it satisfies $e^{g(x)(z)} = f(x)(z)$, for all $(x,z) \in \Sigma_{+-} \times \mathcal{M}$.

Since $H^1(S^2, \mathcal{C}_{S^2;\mathcal{O}(\mathcal{M})}) = 0$, there are continuous functions $g^\pm : \Sigma^\pm \to \mathcal{O}(\mathcal{M})$ with $g^+ - g^- = g$ on Σ_{+-}. Define $f^+ = e^{g^+}$, $f^- = e^{g^-}$ on Σ^+ and Σ^-, respectively. We have $\frac{f^+}{f^-} = f$ on Σ_{+-}.

Define $\tilde{\varphi}^+ = \frac{1}{f^+}\varphi^+$ on Σ^+ and $\tilde{\varphi}^- = \frac{1}{f^-}\varphi^-$ on Σ^-. Then $\tilde{\varphi}^+$ and $\tilde{\varphi}^-$ are continuous $\mathcal{O}(\mathcal{M})-$valued functions on Σ^+ and Σ^-, respectively. Moreover, $\tilde{\varphi}^+ = \tilde{\varphi}^-$ on Σ_{+-}. Accordingly, if we set $\tilde{\varphi} = \tilde{\varphi}^+$ on Σ^+ and $\tilde{\varphi} = \tilde{\varphi}^-$ on Σ^-, then $\tilde{\varphi}$ is a well-defined map from S^2 to $\mathcal{O}(\mathcal{M})$ with $\mathrm{Div} \circ \tilde{\varphi} = \varphi$.

As before, this implies that φ is homotopic to a constant, and the proof for the case $n = 2$ is complete.

The lemma is proved.

Now we are ready to prove our main theorems concerning $\mathfrak{D}_P^+(\mathcal{M})$.

Theorem 6.4. *If $\mathcal{M}$ is a complex-analytic manifold of dimension $N \geq 1$ such that $H^1(\mathcal{M}, \mathbb{Z}) = 0$ and $H^1(\mathcal{M}, \mathcal{O}) = 0$, then there exists a continuous map $\varsigma : \mathfrak{D}_P^+(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ with the property that $\mathrm{Div}\, \varsigma(D) = D$ for all $D \in \mathfrak{D}_P^+(\mathcal{M})$.*

Proof. By Corollaries 5.6 and 5.7, $\mathfrak{D}_P^+$ is a metrizable ANR and so is also paracompact. Also, all the homotopy groups of $\mathfrak{D}_P^+(\mathcal{M})$ vanish. By a theorem of Hurewicz [**Hu**, p. 57], all the singular homology groups $H_n^s(\mathfrak{D}_P^+(\mathcal{M}))$ vanish, whence, by the universal coefficients theorem, the singular cohomology groups vanish. As the space is an ANR, this implies that the Čech cohomology groups vanish. The theorem now follows from Lemma 6.2.

Corollary 6.5. *If $\mathcal{M}$ is a complex-analytic manifold of dimension $N \geq 1$ such that $H^1(\mathcal{M}, \mathbb{Z}) = 0$ and $H^1(\mathcal{M}, \mathcal{O}) = 0$, then given an arbitrary topological space X and a continuous map $\psi \to \mathfrak{D}_P^+(\mathcal{M})$, there is a continuous map $\tilde{\psi} : X \to \mathcal{O}(\mathcal{M})$ with $\mathrm{Div} \circ \tilde{\psi} = \psi$.*

There is a result in the direction converse to Theorem 6.4 that requires fewer hypotheses on $\mathcal{M}$.

Theorem 6.6. *Let $\mathcal{M}$ be a complex-analytic manifold of dimension $N \geq 1$. If there exists a continuous map $\varsigma : \mathfrak{D}_P^+(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ with the property that $\mathrm{Div}\, h(D) = D$ for all $D \in \mathfrak{D}_P^+(\mathcal{M})$, then the space $\mathfrak{D}_P^+(\mathcal{M})$ is contractible, and every zero-free holomorphic function on $\mathcal{M}$ has a holomorphic logarithm.*

Remark. The condition that every-zero free holomorphic function on $\mathcal{M}$ have a logarithm is not, in general, equivalent to the condition that $H^1(\mathcal{M}, \mathbb{Z}) = 0$, i.e., to the condition that every zero-free *continuous* function have a continuous logarithm, though the equivalence is correct for Stein manifolds and, indeed, on all complex manifolds $\mathcal{M}$ with $H^1(\mathcal{M}, \mathcal{O}) = 0$. This follows by considering the cohomology sequence associated with the exact sheaf sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$.

Proof. The existence of a continuous map $\varsigma : \mathfrak{D}_P^+(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ such that $\operatorname{Div} \varsigma(D) = D$ for all $D \in \mathfrak{D}_P^+(\mathcal{M})$ implies that $\mathfrak{D}_P^+(\mathcal{M})$ is an ANR, following the proof of Corollary 5.7.

Also, the existence of ς implies that every continuous map μ from any topological space X into $\mathfrak{D}_P^+(\mathcal{M})$ lifts to a continuous map $\tilde{\mu} : X \to \mathcal{O}(\mathcal{M})$ such that $\operatorname{Div} \circ \tilde{\mu} = \mu$.

Then all the homotopy groups of $\mathfrak{D}_P^+(\mathcal{M})$ vanish, following the part of the proof of Lemma 6.3 concerning the case $n \geq 3$. It is known [**Hu**, Th. 8.2, p. 218] that an ANR with vanishing homotopy groups is contractible.

It remains to prove that every zero-free holomorphic function on $\mathcal{M}$ has a logarithm.

Each continuous map $\alpha : S^1 \to \mathfrak{D}_P^+(\mathcal{M})$ lifts to a continuous map $\tilde{\alpha} : S^1 \to \mathcal{O}(\mathcal{M}) \setminus \{0\}$ such that $\operatorname{Div} \circ \tilde{\alpha} = \alpha$.

Given a function $g \in \mathcal{O}^*(\mathcal{M})$, choose a continuous map $H : [0,1] \to \mathcal{O}(\mathcal{M}) \setminus \{0\}$ with $H(0) = 1$ and $H(1) = g$. (Note: $H(0)$ is the function identically one.) Define $\varphi : [0,1] \to \mathfrak{D}_P^+(\mathcal{M})$ by $\varphi(t) = \operatorname{Div} H(t)$. This is a continuous path which is *closed*, for $\operatorname{Div} H(0) = \operatorname{Div} H(1) = 0$−the zero divisor.

This closed path φ lifts to a closed path $\Phi : [0,1] \to \mathcal{O}(\mathcal{M}) \setminus \{0\}$ with $\operatorname{Div} \Phi(t) = \varphi(t)$ for all $t \in [0,1]$. (That Φ is a closed path means that $\Phi(0) = \Phi(1)$.) There is a continuous map $\sigma : [0,1] \to \mathcal{O}^*(\mathcal{M})$ such that $H(t) = \sigma(t)\Phi(t)$ for all t. Then

$$g = \frac{H(1)}{H(0)} = \frac{\sigma(1)\Phi(1)}{\sigma(0)\Phi(0)} = \frac{\sigma(1)}{\sigma(0)}.$$

As σ is continuous and always zero-free, if c is a piecewise smooth closed path in $\mathcal{M}$, then $\frac{1}{2\pi i}\int_c \frac{d\sigma(t)}{\sigma(t)}$ is a continuous integer and so is constant. Thus,

$$\frac{1}{2\pi i}\int_c \frac{d\sigma(0)}{\sigma(0)} = \frac{1}{2\pi i}\int_c \frac{d\sigma(1)}{\sigma(1)}.$$

This implies that $\frac{1}{2\pi i}\int_c \frac{dg}{g} = 0$. As this is correct for all c, the function g has a logarithm.

The theorem is proved.

We give an additional result in the direction of Theorem 6.4 in which the geometric condition on $\mathcal{M}$ is replaced by a geometric condition on the space of parameters.

Theorem 6.7. *Let $\mathcal{M}$ be a complex manifold with $H^1(\mathcal{M}, \mathcal{O}) = 0$, and let X be a topological space with $H^1(X, \mathbb{Z}) = 0$. If $\mu : X \to \mathfrak{D}_P^+(\mathcal{M})$ is a continuous map, then there is a continuous map $\tilde{\mu} : X \to \mathcal{O}(\mathcal{M})$ with* $\operatorname{Div} \circ \tilde{\mu} = \mu$.

Proof. If $\Omega \subset \mathcal{M}$ is a domain biholomorphically equivalent to a ball, then as there is a continuous map $\varsigma_\Omega : \mathfrak{D}_P^+(\Omega) \to \mathcal{O}(\Omega)$ with $\operatorname{Div} \circ \varsigma_\Omega(D) = D$ for all $D \in \mathfrak{D}_P^+(\Omega)$, we can define $\tilde{\mu}_\Omega : X \to \mathcal{O}(\Omega)$ by $\tilde{\mu}_\Omega(x) = \varsigma_\Omega(\mu(x)|\Omega)$ to get a map with $\mathrm{Div} \circ \tilde{\mu}_\Omega(x) = \mu(x)|\Omega$ for all $x \in X$.

Consequently, there is a locally finite cover $\{U_j\}_{j\in\mathbb{N}}$ such that the intersections $U_{j_1\cdots j_r}$ are all contractible and such that for all $j \in \mathbb{N}$, there is a continuous function $F_j : X \times U_j \to \mathbb{C}$ with $F_j(x, \cdot) \in \mathcal{O}(U_j)$ for all $x \in X$ and with $\mathrm{Div}\, F_j(x, \cdot) = \mu(x)|U_j$ for all $x \in X$ and all $j \in \mathbb{N}$.

Define $\psi_{jk} : X \times U_{jk} \to \mathbb{C}$ by $\psi_{jk}(x, z) = \frac{F_j(x,z)}{F_k(x,z)}$. From here, the proof simply follows the last nine paragraphs of the proof of Proposition 4.1.

Corollary 6.8. *If $\mathcal{M}$ be a complex-analytic manifold such that $H^1(\mathcal{M}, \mathcal{O}) = 0$, then the following four conditions are equivalent:*

1) $H^1(\mathcal{M}, \mathbb{Z}) = 0$*;*
2) *The space $\mathfrak{D}_P^+(\mathcal{M})$ is simply connected;*
3) *The space $\mathfrak{D}_P^+(\mathcal{M})$ is contractible;*
4) *There exists a continuous map $h : \mathfrak{D}_P^+(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ with the property that $\mathrm{Div}\, h(D) = D$ for all $D \in \mathfrak{D}_P^+(\mathcal{M})$.*

Proof. That the condition 1) implies 4) is the content of Theorem 6.4. Theorem 6.6 yields that 4) implies 3). Plainly, 3) implies 2). What remains to be seen is that condition 2) implies condition 1).

To this end, we show that if $\mathfrak{D}_P^+(\mathcal{M})$ is simply connected, then each zero-free holomorphic function on $\mathcal{M}$ is an exponential. Accordingly, let $g \in \mathcal{O}(\mathcal{M})$ be zero free. Let $H : [0, 1] \to \mathcal{O}(\mathcal{M})$ be a continuous function with $H(0) = 1$, $H(1) = g$. The map $\varphi = \mathrm{Div} \circ H : [0, 1] \to \mathfrak{D}_P^+$ is a closed curve in $\mathfrak{D}_P^+$. As $\mathfrak{D}_P^+$ is contractible there is a continuous map $m : [0, 1] \times [0, 1] \to \mathfrak{D}_P^+(\mathcal{M})$ with $m(1, s) = D_1$ for all $s \in [0, 1]$, D_1 some fixed divisor, with $m(0, s) = \varphi(s)$ for all $s \in [0, 1]$ and with $m(t, 0) = m(t, 1)$ for all $t \in [0, 1]$. The preceding theorem provides a lift of the map m to a map $\tilde{m} : [0, 1] \times [0, 1] \to \mathcal{O}(\mathcal{M})$ with $\mathrm{Div} \circ \tilde{m} = m$. This implies that the map φ lifts to a *closed* map $\Phi : [0, 1] \to \mathcal{O}(\mathcal{M})$ with $\mathrm{Div} \circ \Phi = \varphi$. We are now in the situation encountered at the end of Theorem 6.6. The argument there shows that g is an exponential. It follows that $H^1(\mathcal{M}, \mathbb{Z}) = 0$.

The Corollary is proved.

Another corollary of Theorem 6.7 is the following:

Corollary 6.9. *If $\mathcal{M}$ is a complex manifold with $H^1(\mathcal{M}, \mathcal{O}) = 0$, then the homotopy groups $\pi_n(\mathfrak{D}_P^+(\mathcal{M}))$ vanish for $n = 0, 2, 3, \ldots$.*

7. Domains in Stein Manifolds.

Theorem 6.4 is established under two hypotheses: the analytic hypothesis that $H^1(\mathcal{M}, \mathcal{O})$ vanish and the geometric hypothesis that $H^1(\mathcal{M}, \mathbb{Z})$ vanish. Theorem 6.6 establishes the necessity of the geometric hypothesis. We shall show that for manifolds that are domains in Stein manifolds, the analytic hypothesis can be abandoned.

Let $\mathcal{M}$ be a domain in a Stein manifold. A theorem of Rossi [**Ro**] implies that $\mathcal{M}$ has an envelope of holomorphy, $\tilde{\mathcal{M}}$, which is itself a Stein manifold.

For each $g \in \mathcal{O}(\mathcal{M})$, let $\tilde{g} \in \mathcal{O}(\tilde{\mathcal{M}})$ denote the extension of g to $\tilde{\mathcal{M}}$. The map $g \mapsto \tilde{g}$ effects a topological isomorphism between the Fréchet spaces $\mathcal{O}(\mathcal{M})$ and $\mathcal{O}(\tilde{\mathcal{M}})$, which carries $\mathcal{O}^*(\mathcal{M})$ onto $\mathcal{O}^*(\tilde{\mathcal{M}})$.

We will need below the remark that if $H^1(\mathcal{M}, \mathbb{Z}) = 0$, then $H^1(\tilde{\mathcal{M}}, \mathbb{Z}) = 0$, too. This is evident: The vanishing of $H^1(\mathcal{M}, \mathbb{Z})$ implies that each zero-free $f \in \mathcal{O}(\mathcal{M})$ is of the form e^g, $g \in \mathcal{O}(\mathcal{M})$ whence each zero-free $\tilde{f} \in \mathcal{O}(\tilde{\mathcal{M}})$ is of form $e^{\tilde{g}}$, $\tilde{g} \in \mathcal{O}(\tilde{\mathcal{M}})$. As $\tilde{\mathcal{M}}$ is a Stein manifold, the latter condition implies that $H^1(\tilde{\mathcal{M}}, \mathbb{Z}) = 0$.

Lemma 7.1. *The map* $\psi : \mathfrak{D}^+_P(\mathcal{M}) \to \mathfrak{D}^+_P(\tilde{\mathcal{M}})$ *defined by by* $\psi(\operatorname{Div} g) = \operatorname{Div} \tilde{g}$, *is a homeomorphism.*

Proof. The map ψ is injective. If $f, g \in \mathcal{O}(\mathcal{M}) \setminus \{0\}$ and if $\operatorname{Div} \tilde{f} = \operatorname{Div} \tilde{g}$, then $\tilde{f}/\tilde{g} \in \mathcal{O}^*(\tilde{\mathcal{M}})$, whence $f/g \in \mathcal{O}^*(\mathcal{M})$.

The map ψ is plainly surjective.

The inverse $\psi^{-1} : \mathfrak{D}^+_P(\tilde{\mathcal{M}}) \to \mathfrak{D}^+_P(\mathcal{M})$ is continuous: If, for $\tilde{g}_o, \tilde{g}_n \in \mathcal{O}(\tilde{\mathcal{M}}) \setminus \{0\}$, $n = 1, 2, \dots$, $\operatorname{Div} \tilde{g}_n \to \operatorname{Div} \tilde{g}_o$, then, with $g_o = \tilde{g}_o | \mathcal{M}$ and $g_n = \tilde{g}_n | \mathcal{M}$, $\operatorname{Div} g_n \to \operatorname{Div} g_o$.

It is less evident that ψ itself is continuous.

To prove the continuity of ψ, let g_n, $n = 1, 2, \dots$ and g_o be elements of $\mathcal{O}(\mathcal{M}) \setminus \{0\}$ such that $\operatorname{Div} g_n \to \operatorname{Div} g_o$. Set $D_n = \operatorname{Div} g_n$ and $D_o = \operatorname{Div} g_o$. As $D_n \to D_o$, the set $\mathcal{F} = \{D_1, D_2, \dots\}$ is a normal family of nonnegative divisors on $\mathcal{M}$.

Put $\tilde{\mathcal{F}} = \{\tilde{D}_1, \tilde{D}_2, \dots\}$, where $\tilde{D}_n = \operatorname{Div} \tilde{g}_n$. The set $\tilde{\mathcal{F}}$ is a set of nonnegative divisors in the Stein manifold $\tilde{\mathcal{M}}$. According to results of Oka and Fujita - see [**Ba**] and the references it contains - the *domain of normality* of the family $\tilde{\mathcal{F}}$ is a domain of holomorphy in $\tilde{\mathcal{M}}$, say $\mathcal{M}_1$. (The domain of normality of a family of divisors is the largest domain on which the family is a normal family.) As $\mathcal{F}$ is a normal family on $\mathcal{M}$, we have $\mathcal{M} \subset \mathcal{M}_1 \subset \tilde{\mathcal{M}}$. This implies that $\mathcal{M}_1 = \tilde{\mathcal{M}}$, for $\tilde{\mathcal{M}}$ is the envelope of holomorphy of $\mathcal{M}$. That is, the family $\tilde{\mathcal{F}}$ is a normal family in $\tilde{\mathcal{M}}$. Consequently, the sequence $\{\tilde{D}_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{\tilde{D}_{n_j}\}_{j=1}^{\infty}$. Let $\tilde{D}$ be the limit of this sequence. The divisor $\tilde{D}$ on $\tilde{\mathcal{M}}$ is a divisor that on $\mathcal{M}$ agrees with D_o. This implies that $\tilde{D} = \psi(D_o)$. We thus have that the sequence $\{\tilde{D}_n\}_{n=1}^{\infty}$ converges to $\psi(D_o)$. Accordingly, the map ψ is continuous.

The lemma is proved.

Theorem 7.2. *If* $\mathcal{M}$ *is a domain in a Stein manifold* $\mathcal{N}$, *and if* $H^1(\mathcal{M}, \mathbb{Z}) = 0$, *then there is a continuous map* $\varsigma : \mathfrak{D}^+_P(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ *with* $\operatorname{Div} \circ \varsigma(D) = D$ *for all* $D \in \mathfrak{D}^+_P(\mathcal{M})$.

Proof. Denote by $\tilde{\mathcal{M}}$ the envelope of holomorphy of $\mathcal{M}$. It is a Stein manifold and so satisfies $H^1(\tilde{\mathcal{M}}, \mathcal{O}) = 0$. We shall regard $\mathcal{M}$ as being a subset of $\tilde{\mathcal{M}}$. Let $\psi : \mathfrak{D}^+_P(\mathcal{M}) \to \mathfrak{D}^+_P(\tilde{\mathcal{M}})$ be the homeomorphism of Lemma 7.1. Theorem 6.4 applies to the manifold $\tilde{\mathcal{M}}$, so there is a map $\sigma : \mathfrak{D}^+_P(\tilde{\mathcal{M}})$ that satisfies $\operatorname{Div} \circ \sigma(D) = D$ for all $D \in \mathfrak{D}^+_P(\tilde{\mathcal{M}})$. The desired map ς is given by $\varsigma(D) = \sigma(\psi(D))|\mathcal{M}$.

The theorem is proved.

8. Line Bundles.

Some of the preceding results have analogues for holomorphic line bundles. Given a complex manifold $\mathcal{M}$ and a holomorphic line bundle $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ over $\mathcal{M}$, denote by $\mathcal{L}(\mathcal{M})$ the space of sections of $\mathcal{L}$ over $\mathcal{M}$. Given a holomorphic section, s, of $\mathcal{L}$, there is an associated divisor Div $s \in \mathfrak{D}^+(\mathcal{M})$. If $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ denotes the space of all these divisors, the map $\operatorname{Div} : \mathcal{L}(\mathcal{M}) \setminus \{0\} \to \mathfrak{D}^+(\mathcal{M})$ defined in this way is continuous as a consequence of the theorem of Andreotti and Norguet used above since the bundle $\mathcal{L}$ is locally trivial. As $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ is a subset of the space $\mathfrak{D}^+(\mathcal{M})$, the natural topology on it is metrizable,

In this section we shall be concerned mainly with line bundles $\mathcal{L}$ for which the space $\mathcal{L}(\mathcal{M})$ is infinite dimensional over $\mathbb{C}$.

Lemma 8.1. *If $\mathcal{M}$ is a complex manifold for which $H^1(\mathcal{M}, \mathbb{Z})$ vanishes, then the group $H^1(\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}), \mathbb{Z})$ vanishes.*

Proof. The point is that for a given divisor $D \in \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$, the fiber $\operatorname{Div}^{-1}(D)$ consists of all multiples of some fixed section $s \in \mathcal{L}(\mathcal{M})$ by zero-free holomorphic functions on $\mathcal{M}$. As we have seen earlier, the hypothesis that $H^1(\mathcal{M}, \mathbb{Z})$ vanish implies the connectedness of the space $\mathcal{O}^*(\mathcal{M})$. Granted this, the proof can proceed exactly along the lines of the first part of the proof of Lemma 5.2.

Theorem 8.2. *Let $\mathcal{M}$ be a complex manifold for which the groups $H^1(\mathcal{M}, \mathbb{Z})$ and $H^1(\mathcal{M}, \mathcal{O})$ both vanish. If $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ is a holomorphic line bundle for which the space $\mathcal{L}(\mathcal{M})$ is infinite dimensional, then there is a continuous map $\varsigma : \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \to \mathcal{L}(\mathcal{M})$ with* $\operatorname{Div} \circ \varsigma(D) = D$ *for all $D \in \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$.*

Note. The hypotheses of this theorem involve the cohomology of $\mathcal{M}$ with values in the sheaf $\mathcal{O}_{\mathcal{M}}$, not with values in the sheaf $\mathcal{L}_{\mathcal{M}}$ of germs of sections of the line bundle $\mathcal{L}$ as might be expected.

Proof. Fix an open cover $\mathcal{U} = \{U_j\}_{j \in \mathbb{N}}$ of the manifold $\mathcal{M}$ by contractible Stein open sets U_j for which the intersections U_{jk} are also contractible.

For each j, let $r_j : \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \to \mathfrak{D}^+(U_j)$ be the restriction map given by $r_j(D) = D|U_j$. The hypotheses imply that the the bundle $\mathcal{L}$ is trivial over U_j; let $s_j \in \mathcal{L}(U_j)$ be a zero-free section. By Theorem 6.4, there is a continuous map $\sigma_j : \mathfrak{D}^+(U_j) \to \mathcal{O}(U_j)$ with $\mathrm{Div} \circ \sigma_j(D) = D$ for every $D \in \mathfrak{D}^+(U_j)$. Define then $\varsigma_j : \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \to \mathcal{L}(U_j)$ by

$$\varsigma_j = (\sigma_j \circ r_j)s_j.$$

Thus, $\mathrm{Div} \circ \varsigma_j(D) = D|U_j$ for all $D \in \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$.

Define $g_{jk}(D,z) = \frac{\varsigma_j(D)}{\varsigma_k(D)}(z)$ so that g_{jk} is a continuous, zero-free function on $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \times U_{jk}$ with the property that for each D, $g_{jk}(D,\cdot) \in \mathcal{O}(U_{jk})$.

If there are zero-free continuous functions g_j on $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \times U_j$ with $g_j(D,\cdot) \in \mathcal{O}(U_j)$ and $\frac{g_j}{g_k} = g_{jk}$ on $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \times U_{jk}$, then the the map $\varsigma : \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \to \mathcal{L}(\mathcal{M})$ defined to be $g_j^{-1}\varsigma_j$ on U_j is a well defined continuous map from $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ to $\mathcal{L}$ with the property we seek.

For each D, the functions $g_{jk}(D,\cdot)$ define a cohomology class in the group $H^1(\mathcal{M},\mathcal{O}^*)$. Attached to this cohomology class is the Chern class $c(D) \in H^2(\mathcal{M},\mathbb{Z})$. The exact cohomology sequence associated with the sheaf sequence $0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{M}} \to \mathcal{O}^*_{\mathcal{M}} \to 0$ shows that, under the hypothesis that $H^1(\mathcal{M},\mathcal{O})$ vanish, the cocycle $\{g_{jk}(D,\cdot)\}_{j,k\in\mathbb{N}}$ is trivial if and only if its Chern class is.

The Chern classes in question may be determined as follows: Since $H^1(\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}),\mathbb{Z})$ vanishes, the continuous zero-free functions g_{jk} have continuous logarithms, say λ_{jk} on U_{jk}. The Chern class $c(D)$ is the cohomology class in $H^2(\mathcal{M},\mathbb{Z})$ determined by the integral 2-cocycle $\{c_{jkl}(D)\}_{j,k,l\in\mathbb{N}}$ given by

$$c_{jkl}(D) = \frac{1}{2\pi i}\{\lambda_{jk} + \lambda_{kl} + \lambda_{lj}\}.$$

This is a continuous integer-valued function, so as $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ is connected, it is independent of D.

Fix a divisor $D_o \in \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$. By definition, there is a section $s_o \in \mathcal{L}(\mathcal{M})$ with Div $s_o = D_o$. If we define $\kappa_k = \frac{\varsigma_j(D_o)}{s_o}$, then $\kappa_j \in \mathcal{O}^*(U_j)$, and $\frac{\kappa_j}{\kappa_k} = g_{jk}(D_o,\cdot)$. Thus, the cohomology class $\{c_{jkl}\}_{j,k,l\in\mathbb{N}}$ is trivial: There are integers $\{m_{jk}\}_{jk\in\mathbb{N}}$ with $m_{kl} - m_{jl} + m_{jk} = c_{jkl}$ on U_{jkl}.

Then the family $\{\log g_{jk}(D,\cdot) - m_{jk}\}_{j,k\in\mathbb{Z}}$ is a continuous family of 1-cocycles with values in $\mathcal{O}_{\mathcal{M}}$ for the covering $\mathcal{U}$. Again invoke Michael's selection theorem provides continuous functions g_j on $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M}) \times U_j$ with $g_j(D,z)$ depending holomorphically on $z \in U_j$ and with $g_{jk} = g_j - g_k$.

It follows that if $\varsigma(D)(z) = \frac{\varsigma_j(D)(z)}{g_j(D,z)}$, then ς is a well defined map from $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ to $\mathcal{L}(\mathcal{M})$ with $\mathrm{Div}\varsigma(D) = D$ for all $D \in \mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$.

The Theorem is proved.

In contrast with the case of holomorphic functions-the case of the trivial bundle, there is no apparent extension of this result to the case of arbitrary domains in a Stein manifold. Whereas if D is a domain in a Stein manifold, the sheaf $\mathcal{O}_D$ of germs of holomorphic functions on D extends naturally to the sheaf $\mathcal{O}_{\tilde{D}}$ of germs of holomorphic functions on the envelope of holomorphy, $\tilde{D}$, of D, in general a locally free sheaf of rank 1 or, equivalently, a holomorphic line bundle, on D will not extend to a locally free sheaf on $\tilde{D}$.

As a particular instance in which the theorem applies, the following may be cited. Let Σ be a simply connected Stein manifold of dimension n. If one has a continuous family $\{D_x\}_{x\in X}$ of divisors each of which is known to be the divisor of a holomorphic n-form on Σ, then there is a corresponding continuous family $\{\Omega_x\}_{x\in X}$ of holomorphic n-forms on Σ with $\operatorname{Div}\Omega_x = D_x$.

As in the earlier-considered case of holomorphic functions, Theorem 8.2 yields some topological information about the space $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$.

Corollary 8.3. *If $\mathcal{M}$ is a complex manifold for which the groups $H^1(\mathcal{M},\mathbb{Z})$ and $H^1(\mathcal{M},\mathcal{O})$ vanish and if $\mathcal{L}\xrightarrow{\pi}\mathcal{M}$ is a holomorphic line bundle with $\mathcal{L}(\mathcal{M})$ infinite dimensional, then the space of divisors $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ is a contractible ANR. In particular, all of its homotopy groups vanish.*

We conclude this section with some comments on the situation that obtains when the space of sections $\mathcal{L}(\mathcal{M})$ is finite dimensional. This can occur in either of two ways. It may be that $\mathcal{L}(\mathcal{M})=\{0\}$. In this case $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ is empty by definition.

Alternatively, $\dim_{\mathbb{C}}\mathcal{L}(\mathcal{M})$ may be a positive integer, say d. Then necessarily $\mathcal{O}(\mathcal{M})$ consists only of the constants: If s is a global section of $\mathcal{L}$ other than zero section, then $f\mapsto fs$ is a linear isomorphism of $\mathcal{O}(\mathcal{M})$ into $\mathcal{L}(\mathcal{M})$. If the latter space is finite dimensional, then necessarily $\mathcal{O}(\mathcal{M})$ reduces to the constants. (Recall here the standing convention that we are dealing *only* with connected manifolds.)

Thus, for $D\in\mathfrak{D}^+(\mathcal{L})$ the fiber $\pi^{-1}(D)$ is $\mathbb{C}^*$. Consequently, $\mathfrak{D}^+_{\mathcal{L}}(\mathcal{M})$ is topologically equivalent to the complex projective space $\mathbb{P}^{d-1}(\mathbb{C})$.

9. Lifting Normal Families.

In this section we remark that, in essence, Stoll's Theorem B has a solution whenever his Problem A has a general solution.

Theorem 9.1. *Let $\mathcal{M}$ be a complex-analytic manifold of dimension $N\geq 1$. Assume that there exists a continuous map $h:\mathfrak{D}^+_P(\mathcal{M})\to\mathcal{O}(\mathcal{M})$ with the property that $\operatorname{Div}h(D)=D$ for all $D\in\mathfrak{D}^+_P(\mathcal{M})$. If $\mathfrak{N}\subset\mathfrak{D}^+_P(\mathcal{M})$ is a compact set of nonnegative principal divisors on $\mathcal{M}$, there exists a compact set $K\subset\mathcal{M}$ and a compact set $\mathcal{K}$ of holomorphic functions on $\mathcal{M}$ with $\operatorname{Div}\mathcal{K}=\mathfrak{N}$ and such that, for each $f\in\mathcal{K}$, there is a point $z_f\in K$ with $f(z_f)=1$.*

Proof. First, fix a compact set $X_o\subset\mathcal{M}$, and let $W_o\supset X_o$ be an open set with compact closure. Denote by $\mathfrak{N}_o$ the set of all divisors in $\mathfrak{N}$ the support of which is disjoint from W_o. This is plainly a closed subset of $\mathfrak{N}$, so it is a compact set of divisors. Let $h : \mathfrak{D}_P^+(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$ be a continuous map as in the statement of the theorem. As the map is continuous, the set $h(\mathfrak{N}_0)$ is a compact subset of $\mathcal{O}(\mathcal{M})$, which we shall denote by $\mathcal{K}_o$.

Let z_o be a fixed point of the set K, and let ε_o denote the functional of evaluation at the point z_o, so that, for $f \in \mathcal{O}(\mathcal{M})$, $\varepsilon_o(f) = f(z_o)$. This is a continuous linear functional, and on the compact set $\mathfrak{N}_o$ it omits the value zero. Accordingly, there are positive numbers r_o and R_o such that $r_o < |\epsilon_o(f)| < R_o$ for every $f \in \mathcal{K}_o$. Define $\tilde{h}_o : \mathfrak{N} \to \mathcal{O}(\mathcal{M})$ by

$$\tilde{h}_o(D) = \frac{h(D)}{h(D)(z_o)}.$$

Then for all $D \in \mathfrak{N}_o$, we have $\tilde{h}_o(D)(z_o) = 1$. Moreover, the set $\tilde{\mathcal{K}}_o = \{\tilde{h}_o(D) : D \in \mathfrak{N}_o\}$ is a compact set in $\mathcal{O}(\mathcal{M})$.

To continue, denote by $\mathfrak{N}_1$ the subset of $\mathfrak{N}$ that consists of all the divisors in $\mathfrak{N}$ the supports of which meet $\overline{W}_o$. This is a compact set; it is not disjoint from $\mathfrak{N}_o$. Let $\{z_j\}_{j=1,2,...}$ be a countable dense set in $\mathcal{M}$, and for each j, let $\{V_{j,k}\}_{k=1,2,...}$ be a countable neighborhood basis for z_j that consists of relatively compact open sets. For each $j, k = 1, 2, \ldots,$ let

$$\mathfrak{N}_{j,k} = \left\{D \in \mathfrak{N}_1 : (\operatorname{supp} D) \cap \bar{V}_{j,k} = \emptyset\right\}.$$

As $\bar{V}_{j,k}$ is compact, the set $\mathfrak{N}_{j,k}$ is open in $\mathfrak{N}_1$. We have that

$$\bigcup_{j,k=1,2,...} \mathfrak{N}_{j,k} = \mathfrak{N}_1,$$

so by compactness a finite number of the $\mathfrak{N}_{j,k}$ cover $\mathfrak{N}_1$. Choose such a finite set, say $\{\mathfrak{N}_{j_\nu,k_\nu}\}_{\nu=1,...,q}$. Let z_ν be the z_j associated with $\mathfrak{N}_{j_\nu,k_\nu}$. Each of the sets $\mathfrak{K}_\nu = \overline{\mathfrak{N}}_{j_\nu,k_\nu}, \nu = 1, \ldots, q$, is compact. Consequently, each of the sets $\mathcal{K}_\nu = h(\mathfrak{K}_\nu)$ is a compact subset of $\mathcal{O}(\mathcal{M})$. For every ν, let $\tilde{\mathcal{K}}_\nu = \left\{\frac{f}{f(z_\nu)} : f \in \mathcal{K}_\nu\right\}$. The set $\tilde{\mathcal{K}}_\nu$ is compact, $\operatorname{Div} \tilde{\mathcal{K}}_\nu = \mathfrak{K}_\nu$, and if $f \in \tilde{\mathcal{K}}_\nu$, then $f(z_\nu) = 1$.

The compact set K we seek in $\mathcal{M}$ is the set $\{z_o, z_1, \ldots, z_q\}$, and the set $\mathcal{K}$ is the set $\cup_{\nu=1,...,q} \tilde{\mathcal{K}}_\nu$.

The proof of the Theorem is completed.

Appendix. Proof of Theorem 1.0.

Our object here is to give a proof of Theorem 1.0 along lines somewhat different from those occurring in Stoll's proof. In fact, we shall prove a formally different theorem:

Theorem 10.1. *If Ω is a domain in $\mathbb{C}^N$ that contains the closed unit ball $\bar{\mathbb{B}}_N$, then there is a map $h : \mathfrak{D}^+(\Omega; 0) \to \mathcal{O}(\mathbb{B}_N)$ that satisfies* $\operatorname{Div} \circ h(D) = D|\mathbb{B}_N$ *for all $D \in \mathfrak{D}^+(\Omega; 0)$ and that is continuous when the space $\mathfrak{D}^+(\Omega; 0)$ is endowed with the relative weak* topology and the space $\mathcal{O}(\mathbb{B}_N)$ is endowed with its usual topology of uniform convergence on compacta.*

This result is formally different from Theorem 1.0 in that here the space of divisors is taken to have the relative weak* topology whereas in Theorem 1.0, it is understood to be endowed with the topology introduced by Stoll. As convergence in Stoll's sense implies convergence in the weak* sense, as we noted in Section 5, the theorem just stated implies Theorem 1.0.

In the proof indicated below, it will be useful to know that on the space $\mathfrak{D}^+(\Omega)$ the relative weak* topology is identical with the relative strong topology. In Section 5 we proved this, but the first proof given there, that is, the proof of the equivalence of the relative weak* topology, the relative strong topology and Stoll's topology, depends on the work in Stoll's paper [**St**], which draws on Theorem 1.0. It was to avoid this circularity that we gave in Section 5 an independent proof of the equivalence of the relative weak* and the relative strong topologies on $\mathfrak{D}^+(\mathcal{M})$.

The proof we give for this follows well known lines; the whole point is to get solutions that vary continuously.

To begin the proof of Theorem 10.1, fix Ω, and fix $R_o > 0$ small enough that $\Omega \supset R_o\bar{\mathbb{B}}_N$. Define $\psi : (\mathbb{C}^N \setminus \{0\}) \times [0,1] \to \mathbb{C}^N$ to be the real-analytic map given by

$$\psi(z,t) = (1-t)z + tR_o\frac{z}{|z|}.$$

The partial map $\psi(\cdot,0)$ is the identity, and $\psi(\cdot,1)$ is the radial retraction of $\mathbb{C}^N \setminus \{0\}$ onto $R_o\mathbb{S}^{2N-1}$.

For an irreducible complex hypersurface, V, in Ω that does not pass through the origin, define a current $T_V \in \mathcal{D}_1(R_o\mathbb{B}_N)$ by the condition that if $\beta \in \mathcal{D}^{2N-1}(R_o\mathbb{B}_N)$, then

$$T_V(\beta) = \int_{V\times[0,1]} \psi^*\beta. \tag{4}$$

The integral is well defined, for since the support of β is a compact subset of $R_o\mathbb{B}_N$, the support of $\psi^*\beta$ is a compact subset of $(V \times [0,1)$. The variety V has locally finite volume (of dimension $2N-2$) in Ω, so the real variety $V\times\mathbb{R}$ has locally finite volume (of dimension $2N-1$) in $\Omega\times\mathbb{R}$. This implies that if $\gamma \in \mathcal{D}^{2N-1}(\Omega\times\mathbb{R})$, then $\int_{V\times\mathbb{R}}\gamma$ is defined and that the map $\gamma \mapsto \int_{V\times\mathbb{R}}\gamma$ is an element, $[V\times\mathbb{R}]$, of $\mathcal{D}_{2N-1}(\mathbb{C}^N\times\mathbb{R})$. It also implies that if γ is *any* $(2N-1)-$form on $\mathbb{C}^N\times\mathbb{R}$ with locally bounded, measurable coefficients such that $\operatorname{supp}\gamma\cap(V\times\mathbb{R})$ is compact, then the integral $\int_{V\times\mathbb{R}}\gamma$ exists. In particular $T_V(\beta)$ is defined when $\beta \in \mathcal{D}^{2N-1}(R_o\mathbb{B}_N)$.

The current T_V satisfies

$$dT_V = -[V]|R_o\mathbb{B}_N \tag{5}$$

where, as usual, $[V]$ denotes the current of integration over the variety V. This fact is simply Stokes's theorem: By definition, $dT_V(\alpha) = T_V(d\alpha)$, which gives

$$T_V(d\alpha) = \int_{V\times[0,1]} \psi^* d\alpha = \int_{V\times[0,1]} d\psi^*\alpha,$$

and, by Stokes's theorem, this is

$$\int_{b(V\times[0,1])} \psi^*\alpha = \int_{b(V\times[0,1])} \psi^* d\alpha = -\int_V \alpha.$$

If $D = \sum_j m_j V_j$ is a nonnegative divisor on Ω with $V_1, V_2, \dots$ distinct, irreducible complex hypersurfaces in Ω none of which pass through the origin, define

$$(6) \qquad T_D = \sum_j m_j T_{[V_j]}.$$

The family $V_j, j = 1, 2, \dots$, is locally finite in Ω, so the sum (6) is finite for each choice of D. We have

$$dT_D = -\sum_j m_j [V_j] | R_o\mathbb{B}_N.$$

Lemma 10.2. *The map $D \mapsto T_D$ is continuous from $\mathfrak{D}^+(\Omega; 0)$ to $\mathcal{D}_1(R_o\mathbb{B}_N)$ when the two spaces are given their respective weak* topologies.*

Proof. Given a net $\{D_\iota\}_{\iota\in I}$ in $\mathfrak{D}^+(\Omega; 0)$ that converges to $D_o \in \mathfrak{D}^+(\Omega; 0)$ in the sense that for each $\beta \in \mathcal{D}^{2N-2}(\Omega)$, $\lim_{\iota\in I} D_\iota(\beta) = D_o(\beta)$, we are to prove that for each $\beta \in \mathcal{D}^{2N-1}(R_o\mathbb{B}_N)$, $\lim T_{D_\iota}(\beta) = T_{D_o}(\beta)$, i.e., that if $D_\iota = \sum_j m_{\iota j} V_{\iota j}$ with $\{V_{\iota j}\}$ for fixed ι a locally finite family of irreducible complex hypersurfaces in Ω and if $D_o = \sum_j m_{oj} V_{oj}$ is the corresponding decomposition of D_o, then

$$(7) \qquad \lim_\iota \sum_j m_{\iota j} \int_{V_{\iota j}\times[0,1]} \psi^*\beta = \sum_j m_{oj} \int_{V_{oj}\times[0,1]} \psi^*\beta$$

for each $\beta \in \mathcal{D}^{2N-1}(R_o\mathbb{B}_N)$.

To this end, note first that there is $\delta_o > 0$ sufficiently small that for $\iota \in I$ sufficiently large, $\operatorname{supp} D_\iota \cap \delta_o\mathbb{B}_N = \emptyset$.

As $D_\iota \to D_o$ in the weak* sense, the convergence also takes place in the sense of the strong topology, i.e., uniformly on bounded sets in $\mathcal{D}^{2N-1}(\Omega)$. (Recall the discussion at the end of Section 3.)

With $\beta \in \mathcal{D}^{2N-1}(\Omega)$, write $\beta = \sum_{|J|+|K|=2N-1} b_{JK}\, dz^J \wedge d\bar{z}^K$ for a suitable choice of functions $b_{JK} \in \mathcal{D}^0(R_o\mathbb{B}_N)$. The support of β is contained in the ball $(R_o - \delta_1)\mathbb{B}_N$ for some $\delta_1 > 0$. As $\psi(z,t) = (1-t)z + tR_o(\frac{z}{|z|}) \geq |z|$, it follows that for every $t \in [0,1]$ and for all J, K $\operatorname{supp} b_{JK}(\psi(z,t))$, qua function of z, is contained in the ball $(R_o - \delta_1)\mathbb{B}_N$. Moreover, if $\epsilon_o > 0$ is fixed, then the derivatives with respect to z of order no more than k of the functions $b_{JK}(z,t)$ are bounded uniformly in t (and in J and K) on the spherical region $\epsilon_o < |z| < R_o - \delta_1$. That is to say, the set $\{b_{JK}(\cdot,t)\}_{t\in[0,1]}$ is a bounded set in $\mathcal{D}^0(R_o\mathbb{B}_N \setminus \epsilon_o\mathbb{B}_N)$.

Write $\psi^*\beta = B' + B'' \wedge dt$ where $B' \in D^{2N-1}(R_o\mathbb{B}_N)$ does not contain the factor dt and where $B'' \in \mathcal{D}^{2N-2}(R_o\mathbb{B}_N)$ is also free of the factor dt.

Then

$$T_{D_\iota}(\beta) = \sum_j m_{\iota j} \int_0^1 \left(\int_{V_{\iota j}} B'' \right) dt.$$

The last paragraph implies that the family $B'' = \sum_{J,K} b''_{JK}(z,t)dz^J \wedge d\bar{z}^K$ of forms of degree $2N-2$ in z and $\bar{z}$ indexed by the parameter $t \in [0,1]$ is a bounded family in $\mathcal{D}^{2N-2}(R_o\mathbb{B}_N)$. Thus the convergence of $\sum_j m_{\iota j} \int_{V_{\iota j}} B''$ to $\sum_j m_{oj} \int_{V_{oj}} B''$ is uniform in t. Accordingly, we can integrate with respect to $t, t \in [0,1]$ to find that, as desired, $T_{D_\iota}(\beta) \to T_{D_o}(\beta)$.

The lemma is proved.

As noted above, if $D \in \mathfrak{D}^+(\Omega;0)$, then the current $-T_D$ satisfies $d(-T_D) = [D]|(R_o\mathbb{B}_N)$. Set $-T_D = S_{0,1} + S_{1,0}$ with $S_{0,1} \in \mathcal{D}_{0,1}(R_o\mathbb{B}_N)$ and $S_{1,0} \in \mathcal{D}_{1,0}(R_o\mathbb{B}_N)$. As $d = \partial + \bar\partial$ and as $[D]$ is of bidegree $(1,1)$, it follows that the current $S_{0,1}$ satisfies the equation $\bar\partial S_{0,1} = 0$ on $R_o\mathbb{B}_N$. As a $\bar\partial-$closed current, it is $\bar\partial$-exact. We want to see that it is possible to choose a $\bar\partial-$primitive for $S_{0,1}$ that depends continuously on T_D and so continuously on D. In fact, this analysis will be carried out not on the full ball $R_o\mathbb{B}_N$ but rather on the smaller ball $\mathbb{B}_N$.

We shall need the observation that T_D is a current with measure coefficients, whence the same is true of S_{01}.

Recall the well known explicit solution for $\bar\partial$ on the ball that is given in detail, e.g., in [**Ru2**]. Let $\eta : \mathbb{B}_N \times \bar{\mathbb{B}}_N \to [0,1]$ be a function of class $\mathcal{C}^\infty$ that satisfies i) $\eta = 1$ near the diagonal $\Delta = \{(z,z) : z \in \mathbb{B}_N\}$ and ii) $\eta = 0$ on a neighborhood of $\mathbb{B}_N \times b\mathbb{B}_N$. The map $s : \mathbb{B}_N \times \bar{\mathbb{B}}_N \to \mathbb{C}^N$ is defined by

$$s(z,\zeta) = \eta(z,\zeta)(\zeta - z) + [1 - \eta(z,\zeta)](\bar\zeta - \bar z).$$

In terms of s the kernel K_s is defined by

$$K_s(z,\zeta) = \langle \zeta - z, s(z,\zeta)\rangle^{-N} \omega'(\bar{s}(z,\zeta)) \wedge \omega(\zeta).$$

In this definition, the following notation is used. $\omega(\zeta)$ denotes the holomorphic $N-$form $d\zeta_1 \wedge \cdots \wedge d\zeta_N$. If the vector $s(z,\zeta)$ has coordinates $(s_1,\ldots,s_N)$, then $\omega'(\bar{s}(z,\zeta))$ is the $(0,N-1)-$form given by

$$\omega'(\bar{s}(z,\zeta)) = \sum_{j=1}^N (-1)^{j-1} s_j \bar\partial_\zeta s_1 \wedge \cdots [j] \cdots \wedge \bar\partial_\zeta s_N.$$

Finally, $\langle \zeta - z, s(z,\zeta)\rangle = \sum_{j=1}^N (\zeta_j - z_j)s_j$. An application of the kernel K_s is that with it, one can solve $\bar\partial$ as follows. (See [**Ru2**, p. 351]). If

$\alpha = \sum_{j=1}^{N} a_j d\bar{\zeta}_j$ is a $\bar{\partial}-$closed form of bidegree $(0,1)$ on $\bar{\mathbb{B}}_N$ with coefficients of class $\mathcal{C}^1$ then the function u_α given for $z \in \mathbb{B}_N$ by

$$u_\alpha(z) = \frac{1}{Nc_N} \int_{\mathbb{B}_N} K_s(z,\zeta) \wedge \alpha(\zeta) \tag{8}$$

is of class $\mathcal{C}^1$ on $\mathbb{B}_N$ and satisfies there $\bar{\partial} u_\alpha = \alpha$.

In this statement, c_N denotes the constant $\frac{1}{N!}(-1)^{N(N-1)/2}(2\pi i)^N$.

Write

$$K_s(z,\zeta) = \langle \zeta - z, s(\zeta,z)\rangle^{-N} \sum_{j=1}^{N} \Theta_j(z,\zeta)\omega_{[j]}(\bar{\zeta}) \wedge \omega(\zeta)$$

where by $\omega_{[j]}(\bar{\zeta})$ we understand the form obtained from $\omega(\bar{\zeta})$ by deleting the factor $d\bar{\zeta}_j$. Thus,

$$K_s(z,\zeta) \wedge \alpha = \langle \zeta - z, s(\zeta,z)\rangle^{-N} \sum_{j=1}^{N} (-1)^j \Theta_j(z,\zeta) a_j(\zeta) \omega(\bar{\zeta}) \wedge \omega(\zeta).$$

In this, Θ_j is a combination of the functions s_j and their first derivatives with respect to $\bar{\zeta}$.

As remarked above, the current $S_{0,1}$ has measure coefficients, say

$$S_{0,1} = \sum_{j=1}^{N} \mu_j d\bar{\zeta}_j.$$

Consider then the function

$$U_D(z) = \frac{1}{nc'_n} \sum_{j=1}^{N} \int \frac{(-1)^j \Theta_j(z,\zeta)}{\langle \zeta - z, s(\zeta,z)\rangle^N} d\mu_j(\zeta). \tag{9}$$

Here the constant c'_N is the constant chosen so that $\frac{1}{c_N}\omega(\bar{\zeta}) \wedge \omega(\zeta) = \frac{1}{c'_N} d\mathcal{L}$ where we understand by $d\mathcal{L}$ Lebesgue measure on $\mathbb{C}^N$. Notice that the integration in (8) is supported in $\cup \operatorname{supp} \mu_j$. The function U_D satisfies $\bar{\partial} u_D = S_{0,1}$.

The last assertion requires a preliminary remark about regularity. The denominator $\langle \zeta - z, s(\zeta,z)\rangle$ has positive real part away from the diagonal, and near the diagonal, it agrees with the $|\zeta - z|^{2N}$. It follows that the term $\frac{\Theta_j(z,\zeta)}{\langle \zeta - z, s(\zeta,z)\rangle^N}$ is majorized by $\frac{\text{constant}}{|\zeta - z|^{2N-1}}$. As the convolution of a measure with compact support and a locally integrable function is locally integrable, it follows that the function U_D is locally integrable on $\mathbb{B}_N$. Thus, the equation $\bar{\partial} U_D = S_{0,1}$ is at least meaningful in the sense of distributions.

To prove that this equation is *correct*, we argue as follows. We are to see that if $\beta = \sum_{j=1}^{N} \beta_j \omega(\zeta) \wedge \omega_{[j]}(\bar{\zeta})$ is a smooth form on $\mathbb{B}_N$ with compact support, then

$$\int_{\mathbb{B}_N} U_D \bar{\partial}\beta = S_{0,1}(\beta) = \sum_{j=1}^{N} \int \beta_j(z) d\mu_j(z).$$

Introduce a smooth approximate identity $\{\chi_\epsilon\}_{\epsilon>0}$ with $\chi_\epsilon(z) = \epsilon^{-2N}\chi(\frac{z}{\epsilon^{2N}})$ for a nonnegative compactly supported smooth even function χ on $\mathbb{C}^n$ with integral one. Introduce also the convolution $S^\epsilon = \chi_\epsilon * S_{0,1}$.

This is a smooth form: $S^\epsilon = \sum_{j=1}^{N} \chi_\epsilon * \mu_j d\bar{\zeta}_j$, and it is $\bar{\partial}-$closed, for $S_{0,1}$ is $\bar{\partial}-$closed. Let u_{S^ϵ} be the solution of $\bar{\partial}u = S^\epsilon$ given by (8), so that, by definition,

$$u_{S^\epsilon}(z) = \frac{1}{Nc'_N} \int_{|\zeta|<1} \int \sum_{j=1}^{N} \frac{(-1)^j \Theta_j(z,\zeta)\chi_\epsilon(\zeta-\xi)}{\langle \zeta - z, s(\zeta,z)\rangle^N} d\mu_j(\xi) d\mathcal{L}(z).$$

For a smooth, compactly supported form β on $\mathbb{B}_N$

$$\begin{aligned}\int u_{S^\epsilon}\bar{\partial}\beta &= \int \bar{\partial}u_{S^\epsilon} \wedge \beta \\ &= \int S^\epsilon \wedge \beta \to S_{0,1}(\beta) = \sum_{j=1}^{N} \int \beta_j d\mu_j.\end{aligned}$$

Also, if b is the compactly supported smooth function on $\mathbb{B}_N$ such that

$$\int u_{S^\epsilon}\bar{\partial}\beta = \int \bar{\partial}u_{S^\epsilon}(z) b(z) d\mathcal{L}(z),$$

then

$$\begin{aligned}&\int u_{S^\epsilon}\bar{\partial}\beta \\ &= \int \frac{1}{Nc'_N} \sum_{j=1}^{N} \int \int \frac{(-1)^j \Theta_j(z,\zeta)\chi_\epsilon(\zeta-\xi)b(z)}{\langle \zeta - z, s(\zeta,z)\rangle^N} d\mu_j(\xi) d\mathcal{L}(\zeta) d\mathcal{L}(z) \\ &= \int \frac{1}{Nc'_N} \sum_{j=1}^{N} \int \int \frac{(-1)^j \Theta_j(z,\zeta)\chi_\epsilon(\zeta-\xi)b(z)}{\langle \zeta - z, s(\zeta,z)\rangle^N} d\mathcal{L}(\zeta) d\mathcal{L}(z) d\mu_j(\xi).\end{aligned}$$

As $\epsilon \to 0$, this tends to

$$\begin{aligned}&\frac{1}{Nc'_N} \sum_{j=1}^{N} \int \int \frac{(-1)^j \Theta_j(z,\xi)b(z)}{\langle (\xi - z, s(\xi,z))\rangle^N} d\mathcal{L}(z) d\mu_j(\xi) \\ &= \int U_D(z) b(z) d\mathcal{L}(z).\end{aligned}$$

This gives the desired equation

$$\int U_D \bar{\partial}\beta = S_{0,1}(\beta).$$

The integral (9) defines $U_D \in L^1_{loc}(\mathbb{B}_N)$, and U_D satisfies $\bar{\partial} U_D = S_{0,1}$, whence $\partial\bar{\partial} u_D = D$ on $\mathbb{B}_N$. The divisor $D|\mathbb{B}_N$ is principal: $D|\mathbb{B}_N = \text{Div} f$ for some $f \in \mathcal{O}(\mathbb{B}_N)$, so $D|\mathbb{B}_N = \partial\bar{\partial} \log|f|$ also. Thus the function $w_D = U_D - log|f|$ is pluriharmonic on $\mathbb{B}_N$. This implies that U_D is pluriharmonic on $\mathbb{B}_N \setminus \text{supp} D$. It follows further that $U_D = \log|fe^g|$ for some $g \in \mathcal{O}(\mathbb{B}_N)$, and thus, if we denote by V_D the pluriharmonic conjugate of U_D that vanishes at the origin, then $e^{U_D + iV_D} = fe^g$ is holomorphic and has divisor $D|\mathbb{B}_N$.

Finally, we must verify that the map $D \mapsto e^{U_D + iV_D}$ is continuous with respect to the weak* (or strong) topology on the space of divisors and the topology of uniform convergence on compacta on the space of holomorphic functions on $\mathbb{B}_N$. Granted that we are specifying that V_D is the pluriharmonic conjugate of U_D that vanishes at the origin, the function V_D depends continuously on U_D. Thus, it is enough to see that as $D_\iota \to D_o$, $U_{D_\iota}(z) \to U_{D_o}(z)$ uniformly on compacta in $\mathbb{B}_N \setminus \text{supp}\, D_o$. For this, notice that if K is a compact subset of $\mathbb{B}_N \setminus \text{supp}\, D_o$, then there is an open set $W \subset \mathbb{B}_N$ with $\text{supp}\, D_\iota \cap W = \emptyset$ for all $\iota > \iota_o$. Since the convergence of D_ι to D_o is in the strong sense, it follows that the corresponding functions U_{D_ι} converge to U_{D_o} uniformly on K, and we are done.

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