

*Pacific
Journal of
Mathematics*

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Volume 188 No. 2

April 1999

**A NEW CLASS OF “BOUNDARY REGULAR”
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We give a new criterion for the propagation up to the boundary of the analytic singularities of the solutions of microdifferential systems. The class of systems we are able to treat is larger than in D’Ancona-Tose-Zampieri, 1990; namely the condition of transversal ellipticity is here replaced by the non-microcharacteristicity only for the conormal to the boundary. The method also is far different. It is perhaps the most effective application of the theory of the second microlocalization at the boundary by Uchida-Zampieri, 1990.

The microlocal theory of boundary value problems originated from the works by Kataoka and Schapira in the early 80’s. In this frame the propagation of the singularities is now almost completely understood. Among other contributions we quote: Schapira, 1986, Kataoka, 1980, Schapira-Zampieri, 1987. This new contribution covers one of the few problems not yet explained at least in the case of transversal bicharacteristics.

Let M be a real analytic manifold, X a complexification of M , S a real analytic hypersurface of M , M^\pm the two open components of $M \setminus S$ (in a neighborhood of a point $x \in S$). Let $T^*X \xrightarrow{\pi} X$ be the cotangent bundle to X endowed with the canonical 2-form $\sigma = \sigma^{\mathbb{R}} + \sqrt{-1}\sigma^{\mathbb{I}}$, and $T_M^*X \xrightarrow{\pi_M} M$ the conormal bundle to M in X . (Sometimes, if no confusion may arise, we write π instead of π_M .) The latter is \mathbb{R} -Lagrangian (i.e. $\sigma^{\mathbb{R}}|_{TT_M^*X} = 0$) and \mathbb{I} -symplectic (i.e. $\sigma^{\mathbb{I}}|_{TT_M^*X}$ is non-degenerate). In particular if $H = H^{\mathbb{R}} + \sqrt{-1}H^{\mathbb{I}}$ is the Hamiltonian isomorphism, then we have three identifications:

$$\begin{aligned} H &: T^*T^*X \rightarrow TT^*X \\ H^{\mathbb{R}} &: T^*T^*X \rightarrow TT^*X, \\ H^{\mathbb{I}} &: T^*T_M^*X \rightarrow TT_M^*X. \end{aligned}$$

We shall deal with the sheaves of Sato’s microfunctions $\mathcal{C}_{M|X}$, $\mathcal{C}_{S|X}$ and the complexes of microfunctions at the boundary $\mathcal{C}_{M^\pm|X}$. Let V be a smooth involutive submanifold of \dot{T}_M^*X ($\stackrel{\text{def.}}{=} T_M^*X \setminus T_X^*X$), and \tilde{V} the \mathbb{R} -Lagrangian

submanifold obtained as the union of the complexifications of the bicharacteristic leaves of V . Assume there are real analytic functions r and s on T_M^*X such that

$$(1) \quad s|_V = 0, \quad r|_{S \times_M T_M^*X} = 0, \quad \text{and } \{s, r\} \equiv 1.$$

Let \tilde{W} be the union of the integral leaves of $H_{\Re r}^{\mathbb{R}}$ issued from $\tilde{V} \cap \{\Re r^{\mathbb{C}} = 0\}$; this also is an \mathbb{R} -Lagrangian submanifold. Let \mathcal{M} be a coherent \mathcal{E}_X -module (i.e. a microdifferential system) in a neighborhood of a point $p \in S \times_M V$ and denote by $\text{char}(\mathcal{M})$ the characteristic variety of \mathcal{M} . We note that, since $\mathcal{C}_{M \pm |X}|_{T_M^*X}$ are concentrated in degree 0, they are endowed in a natural way with a structure of \mathcal{E}_X -modules. Let $x = \pi(p)$, recall the identification $\pi_M^*(= {}^t\pi'_M) : T_x^*M \rightarrow T_p^*T_M^*X$, and take $\theta \in T_S^*M$.

Theorem 1. *Assume*

$$(2) \quad \pm H^{\mathbb{I}}(\pi_M^*\theta) \notin C_p(\text{char}(\mathcal{M}), \tilde{V}),$$

$$(3) \quad \pm H^{\mathbb{I}}(\pi_M^*\theta) \notin C_p(\text{char}(\mathcal{M}), \tilde{W}),$$

where $C(\cdot, \cdot)$ is the Withney normal cone (cf. [K-S 2]). Then

$$(4) \quad \Gamma_{\pi^{-1}(S)} \text{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M \pm |X})_p = 0.$$

Observe now that we have an identification $T_x^*M \hookrightarrow T_x^*X \hookrightarrow T_p^*T^*X$ where the first embedding is obtained by means of the complex structure of X and the second by means of π^* . Let $V^{\mathbb{C}}$ be the complexification of V in T^*X . As an application of Theorem 1 we get the boundary version of the microlocal Holmgren's Theorem by Bony [B]:

Example 1. Assume

$$\mathbb{C}H(\pi^*\theta) \cap C_p(\text{char}(\mathcal{M}), V^{\mathbb{C}}) = \{0\}.$$

(That is assume the embedding $S \hookrightarrow M$ be *non-microcharacteristic* for μ .) Then (4) follows. To see how this follows from Theorem 1, one just needs to remark that the assumption of Example 1 obviously implies (2) and (3).

Example 2. We take coordinates $z \in X, x \in M, (z; \zeta) \in T^*X, (x; \sqrt{-1}\eta) \in T_M^*X, z = x + \sqrt{-1}y, \zeta = \xi + \sqrt{-1}\eta$, write $z = (z_1, z', z'')$, $\zeta = (\zeta_1, \zeta', \zeta'')$, and assume

$$S = \{x \in M : x_1 = 0\}, \quad V = \{\eta_1 = 0, \eta' = 0\}, \quad p = (0; \sqrt{-1}dx_n).$$

Then (2) is equivalent to

$$(5) \quad |\eta_1| \leq c[|\xi_1| + |\zeta'| + |\xi''| + |y''|] \quad \forall (z; \zeta) \in \text{char}\mathcal{M},$$

and (3) is equivalent to

$$(6) \quad |\eta_1| \leq c[|x_1| + |\zeta'| + |\xi''| + |y''|] \quad (z; \zeta) \in \text{char}\mathcal{M}.$$

Let us consider the case $\mathcal{M} = \frac{\mathcal{E}_X}{\mathcal{E}_X P}$ where $P = P(x, D)$ is a differential operator with principal symbol $\sigma(P) = \zeta_1^2 + a(z'', \zeta') + b(z'', \zeta', \zeta'')$ with a, b

real on T_M^*X homogeneous of order 2, and with $b|_{T_M^*X} \leq 0$. For S and V defined as above, (2) and (3) hold. In fact

$$\begin{cases} \Re\sigma(P) \leq \xi_1^2 - \eta_1^2 + |a(z'', \zeta')| \leq \xi_1^2 - \eta_1^2 + c|\zeta'|^2 & \text{if } \xi'' = 0, y'' = 0 \\ \Im\sigma(P) = 2\xi_1\eta_1 & \text{if } \xi' = 0, \xi'' = 0, y'' = 0. \end{cases}$$

Thus if $\sigma(P) = 0, \xi' = \xi'' = y'' = 0$ then either $\eta_1 = 0$ or $\xi_1 = 0$ whence $|\eta_1| \leq c|\eta'|$. By an easy variant of the local Bochner's tube theorem this implies

$$|\eta_1| \leq c[|\zeta'| + |\xi''| + |y''|] \quad \text{if } \sigma(P) = 0.$$

Thus for instance in \mathbb{R}^4 and with S defined by $x_1 = 0$, the operator

$$P_1 = D_1^2 \pm D_2^2 + D_3^2 + x_3^2 D_4^2,$$

verifies (4) at $p = (0; \pm\sqrt{-1}dx_4)$,

$$P_2 = D_1^2 \pm x_3^m D_2^2 + x_3^2 D_3^2 + (x_3^2 + x_4^2) D_4^2,$$

at any $p = (0; \sqrt{-1}\eta)$ with $\eta_1 = 0, \eta_2 = 0$, and finally

$$P_3 = D_1^2 + x_3^2 D_2^2 + x_3^2 D_3^2 + (x_2^2 + x_3^2 + x_4^2) D_4^2,$$

at any $p = (0; \sqrt{-1}\eta)$ with $\eta_1 = 0$. (The V 's we may use here are $V = \{\eta_1 = 0, \eta_2 = 0\}$ as for P_1, P_2 and $V = \{\eta_1 = 0\}$ as for P_3 respectively.) In particular the two traces on S of a real analytic solution u of $P_3u = 0$ on M^\pm are real analytic at 0.

Remark. In [D'A-T-Z, Corollary 1.2] one encounters the same statement as in Theorem 1 but with (2) replaced by:

$$(2\text{-bis}) \quad \dot{T}_V T_M^*X \cap C_p(\text{char}(\mathcal{M}), \tilde{V}) = \emptyset.$$

Note that (2-bis) implies (2) because $H^{\mathbb{I}}(\pi^*\theta)$ belongs to $\dot{T}_V T_M^*X$ due to (1). But the converse is false as for instance for the above symbol $\zeta_1^2 + a + b$ (with $b|_{T_M^*X} \leq 0$) which fulfills (2-bis) only when $a|_{T_M^*X} < 0$ (and (2) for any $a \gtrless 0$).

Proof of Theorem 1. We use the trick of the adjunction of an auxiliary variable due to M. Kashiwara. We put $\hat{M} = M \times \mathbb{R}, \hat{S} = S \times \mathbb{R}, \hat{X} = X \times \mathbb{C}, \hat{M}^\pm = M^\pm \times \mathbb{R}$, denote by t (resp. τ) the new variable in \mathbb{R} , (resp. \mathbb{C}), denote by $j : X \hookrightarrow \hat{X}$ the embedding, and pick up $\hat{p} \in p \times (\{0\} \times_{\mathbb{R}} \dot{T}_{\mathbb{R}}^*\mathbb{C})$. Then from the exact sequence:

$$0 \rightarrow \mathcal{O}_{\hat{X}} \xrightarrow{\tau} \mathcal{O}_{\hat{X}} \rightarrow j_*\mathcal{O}_X \rightarrow 0,$$

we get, by applying the functor $\mu\text{hom}(\mathbb{Z}_{\hat{M}^\pm}, \cdot)$, a new exact sequence

$$(7) \quad 0 \rightarrow (\mathcal{C}_{M^\pm|X})_p \xrightarrow{\otimes \delta_t} (\mathcal{C}_{\hat{M}^\pm|\hat{X}})_{\hat{p}} \xrightarrow{t} (\mathcal{C}_{\hat{M}^\pm|\hat{X}})_{\hat{p}}.$$

Therefore by the injectivity of the morphism $\otimes \delta_t$ on the left of (7) we can treat our problem at $\hat{p} \in \hat{V} = V \times \hat{T}_{\mathbb{R}}^* \mathbb{C}$, or else assume from the beginning V regular (involutive) i.e. suppose that the 1-form does not vanish on TV . We can then find complex symplectic homogeneous coordinates $(z, \zeta) = (x + \sqrt{-1}y; \xi + \sqrt{-1}\eta) \in T^*X$, $(x; \sqrt{-1}\eta) \in T_M^*X$ such that $r = x_1$, $s = \eta_1$, $V = \{(x; \sqrt{-1}\eta) \in T_M^*X; \eta_1 = \eta' = 0\}$. We put

$$\begin{aligned} X &= \mathbb{C} \times X' \times X'', & M &= \mathbb{R} \times M' \times M'', \\ S &= \{0\} \times M' \times M'', & M^\pm &= \mathbb{R}^\pm \times M' \times M'', \\ (8) \quad M_1 &= \mathbb{R} \times X' \times M'', & S_1 &= \{0\} \times X' \times M'' & M_1^\pm &= \mathbb{R}^\pm \times X' \times M''. \end{aligned}$$

We identify $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$, $z_1 \mapsto (x_1, y_1)$, and complexify (x_1, y_1) to $(x_1^{\mathbb{C}}, y_1^{\mathbb{C}}) \in \mathbb{C}^2$. We set

$$\begin{aligned} \tilde{X} &= \mathbb{C}^2 \times X' \times X'', & \tilde{M} &= \mathbb{R}^2 \times X' \times M'', \\ \tilde{S} &= (\{0\} \times \mathbb{R}) \times X' \times M'', & \tilde{M}^\pm &= (\mathbb{R}^\pm \times \mathbb{R}) \times X' \times M''. \end{aligned}$$

Note that (identifying $\tilde{M}, \tilde{S}, \tilde{M}^\pm$ to subsets of X), we have

$$V = M \times_{\tilde{M}} T_M^*X, \quad \tilde{V} = T_{\tilde{M}}^*X, \quad \tilde{W} = T_{\tilde{S}}^*X.$$

We shall deal with the sheaves (resp. complexes of sheaves) of usual (resp. “boundary”) microfunctions $\mathcal{C}_{S|X}, \mathcal{C}_{M|X}, \mathcal{C}_{\tilde{S}|\tilde{X}}, \mathcal{C}_{\tilde{M}|\tilde{X}}$ (resp. $\mathcal{C}_{M^\pm|X}, \mathcal{C}_{\tilde{M}^\pm|\tilde{X}}$).

Let $T^*X \xleftarrow{tj'} X \times_{\tilde{X}} T^*\tilde{X} \xrightarrow{j^\pi} T^*\tilde{X}$, be the mappings canonically associated to the embedding $j : X \hookrightarrow \tilde{X}$. Let \mathcal{M} be a coherent \mathcal{E}_X -module (i.e. a microdifferential system) on X , and $\mathcal{O}_{\tilde{\mathbb{C}}}$ (the module associated to) the Cauchy-Riemann equation $\bar{\partial}_{z_1}$. The proof of Theorem 1 will require several steps.

Proposition 2. (2) and (3) imply that the natural morphisms

$$\begin{aligned} (9) \quad \mathrm{RHom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1|X}) &\xrightarrow{\sim} \mathrm{R}\Gamma_{\pi^{-1}(M_1)} \mathrm{RHom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{\mathbb{C}}}, \mathcal{C}_{\tilde{M}|\tilde{X}})[+1] \\ \mathrm{RHom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{S_1|X}) &\xrightarrow{\sim} \mathrm{R}\Gamma_{\pi^{-1}(S_1)} \mathrm{RHom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{\mathbb{C}}}, \mathcal{C}_{\tilde{S}|\tilde{X}})[+1], \end{aligned}$$

are isomorphisms.

(Remark that tj' is injective over $j_\pi^{-1}(\mathrm{char}(\mathcal{O}_{\tilde{\mathbb{C}}}))$. For this reason we neglect the functor $\mathrm{R}^t j'_* j_\pi^{-1}$ in the terms on the right side of the above isomorphisms. We shall often act similarly in the following.)

Proof. We consider the commuting diagrams:

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} & & X & \hookrightarrow & \tilde{X} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ M_1 & \hookrightarrow & \tilde{M}, & & S_1 & \hookrightarrow & \tilde{S}. \end{array}$$

According to [K-S 1, Th. 2.3.1], what we need to prove is that the embedding $M_1 \hookrightarrow \tilde{M}$ (resp. $S_1 \hookrightarrow \tilde{S}$) is *microhyperbolic* for the system $\mathcal{O}_{\tilde{C}} \otimes \mathcal{M}$. (As for the additional condition (2.3.1) of loc. cit., this is always satisfied in a suitable neighborhood U of p (and with \mathcal{M} still being the induced system of $\mathcal{O}_{\tilde{C}} \otimes \mathcal{M}|_U$ on ${}^t f' f_\pi^{-1}(U)$.) *Microhyperbolicity* means that in the identification:

$$(10) \quad T_x^* \tilde{M} \hookrightarrow T_x^* \tilde{M} \oplus (T_{\tilde{M}}^* \tilde{X})_x \simeq T_x^* \tilde{X} \xrightarrow{\pi^*} T_p^* T^* \tilde{X},$$

(which follows from the fact that $\mathbb{R}^2 \times M''$ is totally real in $\mathbb{C}^2 \times X''$), we have

$$(11) \quad \mathbb{H}^{\mathbb{R}} \left(\pi^* \left(T_{M_1}^* \tilde{M} \right)_x \right) \cap C_p \left(\text{char} \left(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}} \right), T_{\tilde{M}}^* \tilde{X} \right) = \{0\},$$

and

$$(12) \quad \mathbb{H}^{\mathbb{R}} \left(\pi^* \left(T_{S_1}^* \tilde{S} \right)_x \right) \cap C_p \left(\text{char} \left(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}} \right), T_{\tilde{S}}^* \tilde{X} \right) = \{0\}$$

respectively. Let $(x_1^{\mathbb{C}}, y_1^{\mathbb{C}}; \xi_1^{\mathbb{C}}, \eta_1^{\mathbb{C}})$ be coordinates in $T^* \mathbb{C}_{(x_1^{\mathbb{C}}, y_1^{\mathbb{C}})}^2$; then (11) and (12) are equivalent, for $\sigma(P)(x_1^{\mathbb{C}}, \xi_1^{\mathbb{C}}, z', \xi', z'', \xi'') = 0$ and $\xi_1^{\mathbb{C}} + \sqrt{-1} \eta_1^{\mathbb{C}} = 0$, to:

$$(13) \quad |\Re \eta_1^{\mathbb{C}}| \leq c[|\Re \xi_1^{\mathbb{C}}| + |\Im x_1^{\mathbb{C}}| + |\zeta'| + |\xi''| + |y''|],$$

and

$$(14) \quad |\Re \eta_1^{\mathbb{C}}| \leq c[|x_1^{\mathbb{C}}| + |\zeta'| + |\xi''| + |y''|]$$

respectively. But by the substitution $\Re \eta_1^{\mathbb{C}} = -\Im x_1^{\mathbb{C}}$, (13) and (14) are immediate consequences of (2) and (3) respectively. \square

Proposition 3. *Assume (2) and (3). Then the natural morphism*

$$(15) \quad \mathcal{R}Hom_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1^\pm|X}) \xrightarrow{\sim} R\Gamma_{\pi^{-1}(M_1)} \mathcal{R}Hom_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}}, \mathcal{C}_{\tilde{M}^\pm|\tilde{X}})$$

is an isomorphism.

Proof. (Again we neglect here the functor $R^t j'_* j_\pi^{-1}$ in the right of (15).) The morphism $\mathcal{C}_{A|X} \rightarrow R^t j'_* j_\pi^{-1} R\Gamma_{\pi^{-1}(A)} \mathcal{C}_{\tilde{A}|\tilde{X}}$ ($A = M_1^\pm, M_1, S_1$) induces the vertical arrows in the following commuting diagram in the category $D^b(T^*X)$:

$$(16) \quad \begin{array}{ccc} \mathcal{R}Hom_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{S_1|X}) & \rightarrow & \mathcal{R}Hom_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1|X}) \dots \\ \downarrow & & \downarrow \\ R\Gamma_{\pi^{-1}(S_1)} \mathcal{R}Hom_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}}, \mathcal{C}_{\tilde{S}|\tilde{X}}) & \rightarrow & R\Gamma_{\pi^{-1}(M_1)} \mathcal{R}Hom_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}}, \mathcal{C}_{\tilde{M}|\tilde{X}}) \dots \\ & & \dots \rightarrow \bigoplus_{\pm} \mathcal{R}Hom_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1^\pm|X}) \\ & & \downarrow \\ & & \dots \bigoplus_{\pm} R\Gamma_{\pi^{-1}(M_1)} \mathcal{R}Hom_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}}, \mathcal{C}_{\tilde{M}^\pm|\tilde{X}}). \end{array}$$

By Proposition 3 the two first vertical arrows are isomorphisms. Hence the third is an isomorphism too. \square

For V_1 defined in T_M^*X by $\eta' = 0$, let us recall the complex by $[\mathbf{U-Z}]$ of 2-hyperfunctions at the boundary along V_1 :

$$(17) \quad \mathcal{B}_{M^\pm|X}^{2,V_1} = \mathrm{R}\Gamma_{\pi^{-1}(M)}(\mathcal{C}_{M^\pm|X})[d],$$

($d = \mathrm{codim} V_1$). We put

$$\tilde{M}_2 = \mathbb{R}^2 \times M' \times M'', \quad \tilde{S}_2 = (\{0\} \times \mathbb{R}) \times M' \times M'', \quad \tilde{M}_2^\pm = (\mathbb{R}^\pm \times \mathbb{R}) \times M' \times M'',$$

and

$$\tilde{M}_3 = (\mathbb{R} \times \mathbb{C}) \times X' \times M'', \quad \tilde{S}_3 = (\{0\} \times \mathbb{C}) \times X' \times M'', \quad \tilde{M}_3^\pm = (\mathbb{R}^\pm \times \mathbb{C}) \times X' \times M''.$$

Along with V_1 we also consider in $T_{\tilde{M}_2}^* \tilde{X}$, $V_2 = \{\eta' = 0\}$, $V_3 = \{\Im m \eta_1^{\mathbb{C}} = \eta' = 0\}$. We define similarly to (17):

$$(18) \quad \begin{aligned} \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_2} &= \mathrm{R}\Gamma_{\pi^{-1}(\tilde{M}_2)}(\mathcal{C}_{\tilde{M}_2^\pm|\tilde{X}})[d], \\ \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_3} &= \mathrm{R}\Gamma_{\pi^{-1}(\tilde{M}_2)}(\mathcal{C}_{\tilde{M}_3^\pm|\tilde{X}})[d + 1]. \end{aligned}$$

According to $[\mathbf{U-Z}, \text{Th. 2.6}]$, $\mathcal{B}_{M^\pm|X}^{2,V_1}|_{V_1}$ and $\mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_i}|_{V_i}$, $i = 2, 3$ are all concentrated in degree 0, (whence they are naturally endowed with a structure of \mathcal{E}_X or $\mathcal{E}_{\tilde{X}}$ -modules). We also recall the complexes of usual 2-hyperfunctions by Kashiwara ($[\mathbf{K}]$):

$$(19) \quad \mathcal{B}_{M|X}^{2,V_1}, \mathcal{B}_{\tilde{M}|\tilde{X}}^{2,V_i} \ (i = 2, 3), \mathcal{B}_{S|X}^{2,V_1}, \mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_i} \ (i = 2, 3),$$

defined similarly to (17), (18). It is classical that they are all concentrated in degree 0. We apply $\mathrm{R}\Gamma_{\pi^{-1}(M)}(\cdot)[d]$ to (9), (15) and get

$$(20) \quad \begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{B}_{M|X}^{2,V_1}) &\simeq \mathrm{R}\Gamma_{\pi^{-1}(M)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\mathbb{C}}, \mathcal{B}_{\tilde{M}_2|\tilde{X}}^{2,V_2}) \\ \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{B}_{S|X}^{2,V_1}) &\simeq \mathrm{R}\Gamma_{\pi^{-1}(S)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\mathbb{C}}, \mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_2}) \\ \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{B}_{M^\pm|X}^{2,V_1}) &\simeq \mathrm{R}\Gamma_{\pi^{-1}(M)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\mathbb{C}}, \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_2}). \end{aligned}$$

The natural (restriction) morphism $\mathbb{Z}_{M_1} \rightarrow \mathbb{Z}_M$, resp. $\mathbb{Z}_{\tilde{M}_3} \rightarrow \mathbb{Z}_{\tilde{M}}$, induces a morphism

$$(21) \quad \mathcal{C}_{M|X}|_{V_1} \rightarrow \mathcal{B}_{M|X}^{2,V_1}, \quad \text{resp.} \quad \mathcal{B}_{\tilde{M}_2|\tilde{X}}^{2,V_2}|_{V_3} \rightarrow \mathcal{B}_{\tilde{M}_2|\tilde{X}}^{2,V_3}.$$

It is classical (cf. $[\mathbf{K}]$) that the first is injective. We show now:

Proposition 4. *The morphism*

$$(22) \quad \mathcal{B}_{\tilde{M}_2|\tilde{X}}^{2,V_2}|_{V_3} \rightarrow \mathcal{B}_{\tilde{M}_2|\tilde{X}}^{2,V_3}$$

is injective.

Proof. Fix $p = (x_o; \sqrt{-1}\eta'' dx'') \in V_3$ and let Z_2 , resp. Z_3 , describe the family of closed convex subsets of $\mathbb{R}^{n+1}_{(\mathfrak{S}mx_1^C, \mathfrak{S}my_1^C, y', y'')}$ such that

$$\begin{cases} Z_2 \subset \{y | \langle y'', \eta'' \rangle \geq \epsilon(|y''| + |\mathfrak{S}mx_1^C| + |\mathfrak{S}my_1^C|)\}, \\ Z_2 \cap \{y'' = 0, \mathfrak{S}mx_1^C = 0, \mathfrak{S}my_1^C = 0\} \subset \{0\}, \end{cases}$$

resp.

$$\begin{cases} Z_3 \subset \{y | \langle y'', \eta'' \rangle \geq \epsilon(|\mathfrak{S}mx_1^C| + |y''|)\}, \\ Z_3 \cap \{\mathfrak{S}mx_1^C = 0, y'' = 0\} \subset \{0\}. \end{cases}$$

Thus the arrow in (22) can be represented, between the stalks at p , by:

$$\lim_{\substack{\rightarrow \\ B, Z_2}} H^n_{\tilde{M}_2 + \sqrt{-1}Z_2}(B, \mathcal{O}_X) \rightarrow \lim_{\substack{\rightarrow \\ B, Z_3}} H^n_{\tilde{M}_2 + \sqrt{-1}Z_3}(B, \mathcal{O}_X),$$

for B describing a fundamental system of neighborhoods of $x_o = \pi(p)$. Now for any B (convex) and for any Z_2, Z_3 , there exist $Z'_2 \supset Z_2$ such that $Z_3 \setminus Z'_2 \subset\subset B$. If then $K_2 = \tilde{B} \cap (\tilde{M}_2 + \sqrt{-1}Z'_2)$, $K_3 = \tilde{B} \cap (\tilde{M}_2 + \sqrt{-1}Z_3)$, we have $K_3 \setminus K_2 = (\tilde{M}_2 + \sqrt{-1}(Z_3 \setminus Z'_2)) \cap B$ and therefore

$$H^{n-1}_{\tilde{M}_2 + \sqrt{-1}(Z_3 \setminus Z'_2)}(B, \mathcal{O}_X) = H^{n-1}_{K_3 \setminus K_2}(B, \mathcal{O}_X) = 0,$$

by a celebrated theorem due to M. Kashiwara. □

Note that the first morphism in (21) is a particular case of the second. Hence Proposition 5 provides also a proof of the injectivity of the former.

The natural morphisms $\mathbb{Z}_{M_1^\pm} \rightarrow \mathbb{Z}_{M^\pm}$, resp. $\mathbb{Z}_{\tilde{M}_3^\pm} \rightarrow \mathbb{Z}_{\tilde{M}^\pm}$, in turn induce morphisms:

$$(23) \quad \mathcal{C}_{M^\pm|X}|_{V_1} \rightarrow \mathcal{B}^{2, V_1}_{M^\pm|X} \quad \text{resp.} \quad \mathcal{B}^{2, V_2}_{\tilde{M}_2^\pm|\tilde{X}}|_{V_3} \rightarrow \mathcal{B}^{2, V_3}_{M^\pm|\tilde{X}}.$$

Neither of them is injective ([U-Z, Remark 2.7]). Nevertheless they can be injective when restricted to solutions of non-characteristic systems. Let $V_4 = {}^t j'^{-1}(V) \cap T^*_{\tilde{M}} \tilde{X}$ (i.e. V_4 is the submanifold of $T^*_{\tilde{M}} \tilde{X}$ defined by $\mathfrak{S}m\xi_1^C = \mathfrak{S}m\eta_1^C = \eta' = 0$); note that $V_4 = V_2 \cap \text{char}(\mathcal{O}_{\tilde{C}_{z_1}}) = V_3 \cap \text{char}(\mathcal{O}_{\tilde{C}_{z_1}})$. We have:

Proposition 5. *Let $S^C \hookrightarrow X$ be non-characteristic for \mathcal{M} , and consider the sequence of morphisms:*

$$\begin{aligned} (24) \quad \text{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M^\pm|X})|_V &\rightarrow \text{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{B}^{2, V_1}_{M^\pm|X})|_V \\ &\xrightarrow{\sim} \Gamma_{\pi^{-1}(M)} \text{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}}, \mathcal{B}^{2, V_2}_{\tilde{M}_2^\pm|\tilde{X}})|_{V_4} \\ &\hookrightarrow \text{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}}, \mathcal{B}^{2, V_2}_{\tilde{M}_2^\pm|\tilde{X}})|_{V_4} \\ &\rightarrow \text{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}}, \mathcal{B}^{2, V_3}_{\tilde{M}_2^\pm|\tilde{X}})|_{V_4}, \end{aligned}$$

with the first and the fourth arrow induced by (23), the second by (20), and the third being the natural identification. Then the composition of the morphisms in (23) is injective.

Remark 6. In particular the first morphism in (24) is injective. Our proof will show that this is in fact injective on the whole V_1 (not only on V) according to [U-Z, Th. 2.8]. However the full generalization of this statement (in analogy with Proposition 4), i.e. the injectivity of the last morphism in (24) is not clear to us because of the lack of a 2-microlocal version of the *watermelon-cut* Theorem (cf. [S]).

Proof. We consider

$$(25) \quad \begin{array}{ccc} \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_2} \Big|_{\tilde{S}_2 \times_{\tilde{M}_3} T_{\tilde{M}_3}^* \tilde{X}} & \rightarrow & \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_3} \Big|_{\tilde{S}_2 \times_{\tilde{M}_3} T_{\tilde{M}_3}^* \tilde{X}} \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\tilde{F}^\pm} \mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_2} \Big|_{\tilde{S}_2 \times_{\tilde{M}_3} T_{\tilde{M}_3}^* \tilde{X}}[1] & \rightarrow & \mathrm{R}\Gamma_{\tilde{F}^\pm} (\mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_3}) \Big|_{\tilde{S}_2 \times_{\tilde{M}_3} T_{\tilde{M}_3}^* \tilde{X}}[+1], \end{array}$$

where $\tilde{F}^\pm = (\tilde{S}_2 \times_{\tilde{M}_2} T_{\tilde{M}_2}^* \tilde{X}) \pm \mathbb{R}^+\theta$ with θ the exterior conormal to \tilde{M}^+ in \tilde{M} . Remark that the vertical arrows of (25) are induced by the natural morphisms $\mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_i} \rightarrow \mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_i}[1]$ which factorize through $\mathrm{R}\Gamma_{\tilde{F}^\pm} \mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_i}[1]$ (due to $\mathrm{supp}(\mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2,V_i}) \cap \mathrm{supp}(\mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_i}) \subset \tilde{F}^\pm$). If we apply $\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}}, \cdot)$ to (25) and take the 0-th cohomology, the arrow on the bottom becomes injective. In fact let $\tilde{Y} = \tilde{S}_2^{\mathbb{C}}$ be the complexification of S_2 (i.e. $\tilde{Y} = \mathbb{C}_{y_1^{\mathbb{C}}} \times X' \times X''$), denote by $k : \tilde{Y} \rightarrow \tilde{X}$ the natural embedding, and let $V'_i = {}^t k' k_\pi^{-1}(V_i)$. By the aid of division formulas for $\mathcal{B}_{\tilde{S}_2|\tilde{X}}^{2,V_i}$, the above injectivity is reduced to the injectivity of

$$\mathcal{B}_{\tilde{S}_2|\tilde{Y}}^{2,V'_2} \Big|_{V'_3} \hookrightarrow \mathcal{B}_{\tilde{S}_2|\tilde{Y}}^{2,V'_3}.$$

But this is, under different notations, the same statement as in Proposition 4. We consider now:

$$(26) \quad \begin{array}{ccc} \mathcal{C}_{M^\pm|X} \Big|_{S \times_{M_1} T_{M_1}^* X} & \rightarrow & \mathcal{B}_{M^\pm|X}^{2,V_1} \Big|_{S \times_{M_1} T_{M_1}^* X} \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{F^\pm} (\mathcal{C}_{S|X}) \Big|_{S \times_{M_1} T_{M_1}^* X}[1] & \rightarrow & \mathrm{R}\Gamma_{F^\pm} (\mathcal{B}_{S|X}^{2,V_1}) \Big|_{S \times_{M_1} T_{M_1}^* X}[1], \end{array}$$

with $F^\pm = S \times_M T_M^* X \pm \mathbb{R}^+\theta$. The arrow in the bottom is injective, over solutions of \mathcal{M} , for the same argument as for (25). Concerning the first

vertical arrow, this is represented at each point $p \in S \times_M T_M^* X$ by

$$(\mathcal{C}_{M^\pm|X})_p \simeq \left(\frac{\mathcal{C}_{\bar{M}^\pm|X}}{\mathcal{C}_{S|X}} \right)_p \hookrightarrow \mathcal{H}_{F^\pm}^1(\mathcal{C}_{S|X})_p,$$

(where $\mathcal{C}_{\bar{M}^\pm|X}$ are the Kataoka's microfunctions along the closed half-spaces \bar{M}^\pm) whose injectivity is immediately proved by the aid of a Legendre transformation (cf. [Kat] and [S]).

We are ready to conclude. We apply $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \cdot)$ to (26) and $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}}, \cdot)$ to (25) respectively (and neglect $\mathrm{R}^t j_* j_\pi^{-1}$). We glue the diagrams so obtained by means of the second and third of (20) and by the natural morphism $\mathrm{R}\Gamma_{\pi^{-1}(M)}(\cdot) \rightarrow \cdot$. We thus obtain a long diagram with the first vertical and all the bottom horizontal arrows injective over the 0-th cohomology. The composition of the upper horizontal arrows (which is precisely the sequence of morphisms in (24)) is therefore also injective. \square

End of proof of Theorem 1. Let $\pm\theta$ be the exterior conormals to \tilde{M}^\pm in \tilde{M} identified to vectors $\mathrm{H}^{\mathbb{R}}(\pm\pi^*\theta)$ of $T_p T^* \tilde{X}$ (cf. (10)). Then clearly

$$(27) \quad \mathrm{H}^{\mathbb{R}}(\pm\pi^*(\theta)) \notin C(\mathrm{char}(\mathcal{O}_{\tilde{C}}), \tilde{V}_3).$$

Let $\mathrm{SS}(\mathbb{Z}_{\tilde{M}_3^\pm})$ denote the microsupport of $\mathbb{Z}_{\tilde{M}_3^\pm}$ in the sense of [K-S 2]. One easily checks that

$$\mathrm{SS}(\mathbb{Z}_{\tilde{M}_3^\pm}) = \left(\overline{\tilde{M}_3^\pm} \times T_{\tilde{M}_3}^* \tilde{X} \right) \pm \mathbb{R}^+\theta.$$

It is also easy to see that (27) implies

$$-\mathrm{H}^{\mathbb{R}}(\pm\pi^*\theta) \notin C\left(\mathrm{char}(\mathcal{O}_{\tilde{C}}), \mathrm{SS}(\mathbb{Z}_{\tilde{M}_3^\pm})\right).$$

It follows, merely by definition of SS:

$$\mathrm{R}\Gamma_{\pi^{-1}(\tilde{S}_3)} \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{\tilde{X}}}(\mathcal{O}_{\tilde{C}}, \mathcal{C}_{\tilde{M}_3^\pm|\tilde{X}}) = 0,$$

and thus, by applying $\mathrm{R}\Gamma_{\pi^{-1}(\tilde{M}_2)}(\cdot)[d+1]$:

$$(28) \quad \mathrm{R}\Gamma_{\pi^{-1}(\tilde{S}_2)} \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{O}_{\tilde{C}}, \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2, V_3}\right) = 0.$$

In conclusion we have

$$\begin{aligned} \Gamma_{\pi^{-1}(S)} \mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M^\pm|X})|_V &\hookrightarrow \Gamma_{\pi^{-1}(S)} \mathrm{Hom}_{\mathcal{E}_X}\left(\mathcal{M}, \mathcal{B}_{M^\pm|X}^{2, V_1}\right)|_V \\ &\xrightarrow{\sim} \Gamma_{\pi^{-1}(S)} \mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}}, \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2, V_2}\right)|_{V_4} \\ &\rightarrow \Gamma_{\pi^{-1}(S)} \mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{M} \otimes \mathcal{O}_{\tilde{C}_{z_1}}, \mathcal{B}_{\tilde{M}_2^\pm|\tilde{X}}^{2, V_3}\right)|_{V_4} \\ &= 0, \end{aligned}$$

(where the first “ \hookrightarrow ” follows from Remark 6, the second “ $\xrightarrow{\sim}$ ” from (20), the third “ \rightarrow ” from (23), and the last “ $=$ ” from (28) respectively). On the other hand the composition of “ \hookrightarrow ”, “ $\xrightarrow{\sim}$ ” and “ \rightarrow ” is injective by Proposition 5; hence $\Gamma_{\pi^{-1}(S)}\text{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M^\pm|X})|_V = 0$. The proof is complete. \square

Acknowledgements. The author is grateful to an anonymous referee for pointing out the problem in the second part of Remark 6 which lead to Proposition 5.

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Received April 22, 1997 and revised May 27, 1998.

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