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A NEW CLASS OF "BOUNDARY REGULAR" MICRODIFFERENTIAL SYSTEMS

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We give a new criterion for the propagation up to the boundary of the analytic singularities of the solutions of microdifferential systems. The class of systems we are able to treat is larger than in D'Ancona-Tose-Zampieri, 1990; namely the condition of transversal ellipticity is here replaced by the non-microcharacteristicity only for the conormal to the boundary. The method also is far different. It is perhaps the most effective application of the theory of the second microlocalization at the boundary by Uchida-Zampieri, 1990.

The microlocal theory of boundary value problems originated from the works by Kataoka and Schapira in the early 80's. In this frame the propagation of the singularities is now almost completely understood. Among other contributions we quote: Schapira, 1986, Kataoka, 1980, Schapira-Zampieri, 1987. This new contribution covers one of the few problems not yet explained at least in the case of transversal bicharacteristics.

Let M be a real analytic manifold, X a complexification of M, S a real analytic hypersurface of M, $M^\pm$ the two open components of $M \setminus S$ (in a neighborhood of a point $x \in S$). Let $T^*X \overset{\pi}{\to} X$ be the cotangent bundle to X endowed with the canonical 2-form $\sigma = \sigma^{\mathbb{R}} + \sqrt{-1}\sigma^{\mathbb{I}}$, and $T^*_M X \overset{\pi_M}{\to} M$ the conormal bundle to M in X. (Sometimes, if no confusion may arise, we write π instead of π_M.) The latter is $\mathbb{R}$-Lagrangian (i.e. $\sigma^{\mathbb{R}}|_{TT^*_M X} = 0$) and $\mathbb{I}$-symplectic (i.e. $\sigma^{\mathbb{I}}|_{TT^*_M X}$ is non-degenerate). In particular if $H = H^{\mathbb{R}} + \sqrt{-1}H^{\mathbb{I}}$ is the Hamiltonian isomorphism, then we have three identifications:

$$H : T^*T^*X \to TT^*X$$
$$H^{\mathbb{R}} : T^*T^*X \to TT^*X,$$
$$H^{\mathbb{I}} : T^*T^*_M X \to TT^*_M X.$$

We shall deal with the sheaves of Sato's microfunctions $\mathcal{C}_{M|X}$, $\mathcal{C}_{S|X}$ and the complexes of microfunctions at the boundary $\mathcal{C}_{M^\pm|X}$. Let V be a smooth involutive submanifold of $\dot{T}^*_M X (:\overset{\text{def.}}{=} T^*_M X \setminus T^*_X X)$, and $\tilde{V}$ the $\mathbb{R}$-Lagrangian submanifold obtained as the union of the complexifications of the bicharacteristic leaves of V. Assume there are real analytic functions r and s on T^*_MX such that

$$s|_V = 0,\ r|_{S\times_M T^*_MX} = 0, \quad \text{and } \{s, r\} \equiv 1. \tag{1}$$

Let $\tilde{W}$ be the union of the integral leaves of $H^{\mathbb{R}}_{\Re er^{\mathbb{C}}}$ issued from $\tilde{V} \cap \{\Re er^{\mathbb{C}} = 0\}$; this also is an $\mathbb{R}$-Lagrangian submanifold. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module (i.e. a microdifferential system) in a neighborhood of a point $p \in S \times_M V$ and denote by $\mathrm{char}(\mathcal{M})$ the characteristic variety of $\mathcal{M}$. We note that, since $\mathcal{C}_{M^\pm|X}|_{T^*_MX}$ are concentrated in degree 0, they are endowed in a natural way with a structure of $\mathcal{E}_X$-modules. Let $x = \pi(p)$, recall the identification $\pi^*_M(= {}^t\pi'_M) : T^*_xM \to T^*_pT^*_MX$, and take $\theta \in T^*_SM$.

Theorem 1. *Assume*

$$\pm H^{\mathbb{I}}(\pi^*_M\theta) \notin C_p(\mathrm{char}(\mathcal{M}), \tilde{V}), \tag{2}$$

$$\pm H^{\mathbb{I}}(\pi^*_M\theta) \notin C_p(\mathrm{char}(\mathcal{M}), \tilde{W}), \tag{3}$$

where $C(\cdot,\cdot)$ is the Withney normal cone (cf. [**K-S 2**]*). Then*

$$\Gamma_{\pi^{-1}(S)}\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M^\pm|X})_p = 0. \tag{4}$$

Observe now that we have an identification $T^*_xM \hookrightarrow T^*_xX \hookrightarrow T^*_pT^*X$ where the first embedding is obtained by means of the complex structure of X and the second by means of π^*. Let $V^{\mathbb{C}}$ be the complexification of V in T^*X. As an application of Theorem 1 we get the boundary version of the microlocal Holmgren's Theorem by Bony [**B**]:

Example 1. Assume

$$\mathbb{C}H(\pi^*\theta) \cap C_p(\mathrm{char}(\mathcal{M}), V^{\mathbb{C}}) = \{0\}.$$

(That is assume the embedding $S \hookrightarrow M$ be *non-microcharacteristic* for μ.) Then (4) follows. To see how this follows from Theorem 1, one just needs to remark that the assumption of Example 1 obviously implies (2) and (3).

Example 2. We take coordinates $z \in X$, $x \in M$, $(z;\zeta) \in T^*X$, $(x; \sqrt{-1}\eta) \in T^*_MX$, $z = x + \sqrt{-1}y$, $\zeta = \xi + \sqrt{-1}\eta$, write $z = (z_1, z', z'')$, $\zeta = (\zeta_1, \zeta', \zeta'')$, and assume

$$S = \{x \in M : x_1 = 0\},\ V = \{\eta_1 = 0, \eta' = 0\},\ p = (0; \sqrt{-1}\mathrm{d}x_n).$$

Then (2) is equivalent to

$$|\eta_1| \le c[|\xi_1| + |\zeta'| + |\xi''| + |y''|] \quad \forall (z;\zeta) \in \mathrm{char}\mathcal{M}, \tag{5}$$

and (3) is equivalent to

$$|\eta_1| \le c[|x_1| + |\zeta'| + |\xi''| + |y''|] \quad (z;\zeta) \in \mathrm{char}\mathcal{M}. \tag{6}$$

Let us consider the case $\mathcal{M} = \frac{\mathcal{E}_X}{\mathcal{E}_X P}$ where $P = P(x, D)$ is a differential operator with principal symbol $\sigma(P) = \zeta_1^2 + a(z'', \zeta') + b(z'', \zeta', \zeta'')$ with a, b

real on $T^*_M X$ homogeneous of order 2, and with $b|_{T^*_M X} \le 0$. For S and V defined a above, (2) and (3) hold. In fact

$$\begin{cases} \Re\mathrm{e}\sigma(P) \le \xi_1^2 - \eta_1^2 + |a(z'', \zeta')| \le \xi_1^2 - \eta_1^2 + c|\zeta'|^2 & \text{if } \xi'' = 0,\ y'' = 0 \\ \Im\mathrm{m}\sigma(P) = 2\xi_1\eta_1 & \text{if } \xi' = 0,\ \xi'' = 0,\ y'' = 0. \end{cases}$$

Thus if $\sigma(P) = 0$, $\xi' = \xi'' = y'' = 0$ then either $\eta_1 = 0$ or $\xi_1 = 0$ whence $|\eta_1| \le c|\eta'|$. By an easy variant of the local Bochner's tube theorem this implies

$$|\eta_1| \le c[|\zeta'| + |\xi''| + |y''|] \quad \text{if } \sigma(P) = 0.$$

Thus for instance in $\mathbb{R}^4$ and with S defined by $x_1 = 0$, the operator

$$P_1 = D_1^2 \pm D_2^2 + D_3^2 + x_3^2 D_4^2,$$

verifies (4) at $p = (0; \pm\sqrt{-1}\mathrm{d}x_4)$,

$$P_2 = D_1^2 \pm x_3^m D_2^2 + x_3^2 D_3^2 + (x_3^2 + x_4^2) D_4^2,$$

at any $p = (0; \sqrt{-1}\eta)$ with $\eta_1 = 0$, $\eta_2 = 0$, and finally

$$P_3 = D_1^2 + x_3^2 D_2^2 + x_3^2 D_3^2 + (x_2^2 + x_3^2 + x_4^2) D_4^2,$$

at any $p = (0; \sqrt{-1}\eta)$ with $\eta_1 = 0$. (The V's we may use here are $V = \{\eta_1 = 0, \eta_2 = 0\}$ as for P_1, P_2 and $V = \{\eta_1 = 0\}$ as for P_3 respectively.) In particular the two traces on S of a real analytic solution u of $P_3 u = 0$ on $M^\pm$ are real analytic at 0.

Remark. In [**D'A-T-Z**, Corollary 1.2] one enconters the same statement as in Theorem 1 but with (2) replaced by:

(2-bis) $$\dot{T}_V T^*_M X \cap C_p(\mathrm{char}(\mathcal{M}), \tilde{V}) = \emptyset.$$

Note that (2-bis) implies (2) because $\mathrm{H}^{\mathbb{I}}(\pi^*\theta)$ belongs to $\dot{T}_V T^*_M X$ due to (1). But the converse is false as for instance for the above symbol $\zeta_1^2 + a + b$ (with $b|_{T^*_M X} \le 0$) which fulfills (2-bis) only when $a|_{T^*_M X} < 0$ (and (2) for any $a \gtrless 0$).

Proof of Theorem 1. We use the trick of the adjunction of an auxillary variable due to M. Kashiwara. We put $\hat{M} = M \times \mathbb{R}$, $\hat{S} = S \times \mathbb{R}$, $\hat{X} = X \times \mathbb{C}$, $\hat{M}^\pm = M^\pm \times \mathbb{R}$, denote by t (resp. τ) the new variable in $\mathbb{R}$, (resp. $\mathbb{C}$), denote by $j : X \hookrightarrow \hat{X}$ the embedding, and pick up $\hat{p} \in p \times \left(\{0\} \times_{\mathbb{R}} \dot{T}^*_{\mathbb{R}}\mathbb{C}\right)$. Then from the exact sequence:

$$0 \to \mathcal{O}_{\hat{X}} \xrightarrow{\tau} \mathcal{O}_{\hat{X}} \to j_* \mathcal{O}_X \to 0,$$

we get, by applying the functor $\mu\mathrm{hom}(\mathbb{Z}_{\hat{M}^\pm}, \cdot)$, a new exact sequence

(7) $$0 \to (\mathcal{C}_{M^\pm|X})_p \xrightarrow{\otimes\delta_t} (\mathcal{C}_{\hat{M}^\pm|\hat{X}})_{\hat{p}} \xrightarrow{t} (\mathcal{C}_{\hat{M}^\pm|\hat{X}})_{\hat{p}}.$$

Therefore by the injectivity of the morphism $\otimes\delta_t$ on the left of (7) we can treat our problem at $\hat{p} \in \hat{V} = V \times \dot{T}^*_{\mathbb{R}}\mathbb{C}$, or else assume from the beginning V regular (involutive) i.e. suppose that the 1–form does not vanish on TV. We can then find complex symplectic homogeneous coordinates $(z, \zeta) = (x + \sqrt{-1}y; \xi + \sqrt{-1}\eta) \in T^*X$, $(x; \sqrt{-1}\eta) \in T^*_M X$ such that $r = x_1,\ s = \eta_1,\ V = \{(x; \sqrt{-1}\eta) \in T^*_M X; \eta_1 = \eta' = 0\}$. We put

$$X = \mathbb{C} \times X' \times X'', \quad M = \mathbb{R} \times M' \times M'',$$
$$S = \{0\} \times M' \times M'', \quad M^\pm = \mathbb{R}^\pm \times M' \times M'',$$
$$(8) \quad M_1 = \mathbb{R} \times X' \times M'', \quad S_1 = \{0\} \times X' \times M'' \quad M_1^\pm = \mathbb{R}^\pm \times X' \times M''.$$

We identify $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2, z_1 \mapsto (x_1, y_1)$, and complexify (x_1, y_1) to $(x_1^{\mathbb{C}}, y_1^{\mathbb{C}}) \in \mathbb{C}^2$. We set

$$\tilde{X} = \mathbb{C}^2 \times X' \times X'', \quad \tilde{M} = \mathbb{R}^2 \times X' \times M'',$$
$$\tilde{S} = (\{0\} \times \mathbb{R}) \times X' \times M'', \quad \tilde{M}^\pm = (\mathbb{R}^\pm \times \mathbb{R}) \times X' \times M''.$$

Note that (identifying $\tilde{M}$, $\tilde{S}$, $\tilde{M}^\pm$ to subsets of X), we have

$$V = M \times_{\tilde{M}} T^*_{\tilde{M}} X, \quad \tilde{V} = T^*_{\tilde{M}} X, \quad \tilde{W} = T^*_{\tilde{S}} X.$$

We shall deal with the sheaves (resp. complexes of sheaves) of usual (resp. "boundary") microfunctions $\mathcal{C}_{S|X}, \mathcal{C}_{M|X}, \mathcal{C}_{\tilde{S}|\tilde{X}}, \mathcal{C}_{\tilde{M}|\tilde{X}}$ (resp. $\mathcal{C}_{M^\pm|X}, \mathcal{C}_{\tilde{M}^\pm|\tilde{X}}$). Let $T^*X \overset{{}^t j'}{\leftarrow} X \times_{\tilde{X}} T^*\tilde{X} \overset{j_\pi}{\hookrightarrow} T^*\tilde{X}$, be the mappings canonically associated to the embedding $j : X \hookrightarrow \tilde{X}$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module (i.e. a microdifferential system) on X, and $\mathcal{O}_{\bar{\mathbb{C}}}$ (the module associated to) the Cauchy-Riemann equation $\bar{\partial}_{z_1}$. The proof of Theorem 1 will require several steps.

Proposition 2. (2) *and* (3) *imply that the natural morphisms*

$$(9) \qquad \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1|X}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\pi^{-1}(M_1)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{C}_{\tilde{M}|\tilde{X}})[+1]$$
$$\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{S_1|X}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\pi^{-1}(S_1)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{C}_{\tilde{S}|\tilde{X}})[+1],$$

are isomorphisms.

(Remark that ${}^t j'$ is injective over $j_\pi^{-1}(\mathrm{char}(\mathcal{O}_{\bar{\mathbb{C}}}))$. For this reason we neglect the functor $\mathrm{R}^t j'_* j_\pi^{-1}$ in the terms on the right side of the above isomorphisms. We shall often act similarly in the following.)

Proof. We consider the commuting diagrams:

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} \\ \uparrow & & \uparrow \\ M_1 & \hookrightarrow & \tilde{M}, \end{array} \qquad \begin{array}{ccc} X & \hookrightarrow & \tilde{X} \\ \uparrow & & \uparrow \\ S_1 & \hookrightarrow & \tilde{S}. \end{array}$$

According to [**K-S 1**, Th. 2.3.1], what we need to prove is that the embedding $M_1 \hookrightarrow \tilde{M}$ (resp. $S_1 \hookrightarrow \tilde{S}$) is *microhyperbolic* for the system $\mathcal{O}_{\bar{\mathbb{C}}} \otimes \mathcal{M}$. (As for the additional condition (2.3.1) of loc. cit., this is always satisfied in a suitable neighborhood U of p (and with $\mathcal{M}$ still being the induced system of $\mathcal{O}_{\bar{\mathbb{C}}} \otimes \mathcal{M}|_U$ on ${}^t f' f_\pi^{-1}(U)$).) *Microhyperbolicity* means that in the identification:

$$T_x^*\tilde{M} \hookrightarrow T_x^*\tilde{M} \oplus (T_{\tilde{M}}^*\tilde{X})_x \simeq T_x^*\tilde{X} \underset{\pi^*}{\hookrightarrow} T_p^*T^*\tilde{X}, \tag{10}$$

(which follows from the fact that $\mathbb{R}^2 \times M''$ is totally real in $\mathbb{C}^2 \times X''$), we have

$$\mathrm{H}^{\mathbb{R}}\left(\pi^*\left(T_{M_1}^*\tilde{M}\right)_x\right) \cap C_p\left(\mathrm{char}\left(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}_{z_1}}\right), T_{\tilde{M}}^*\tilde{X}\right) = \{0\}, \tag{11}$$

and

$$\mathrm{H}^{\mathbb{R}}\left(\pi^*\left(T_{S_1}^*\tilde{S}\right)_x\right) \cap C_p\left(\mathrm{char}\left(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}_{z_1}}\right), T_{\tilde{S}}^*\tilde{X}\right) = \{0\} \tag{12}$$

respectively. Let $(x_1^{\mathbb{C}}, y_1^{\mathbb{C}}; \xi_1^{\mathbb{C}}, \eta_1^{\mathbb{C}})$ be coordinates in $T^*\mathbb{C}^2_{(x_1^{\mathbb{C}}, y_1^{\mathbb{C}})}$; then (11) and (12) are equivalent, for $\sigma(P)(x_1^{\mathbb{C}}, \xi_1^{\mathbb{C}}, z', \xi', z'', \xi'') = 0$ and $\xi_1^{\mathbb{C}} + \sqrt{-1}\eta_1^{\mathbb{C}} = 0$, to:

$$|\Re\mathrm{e}\eta_1^{\mathbb{C}}| \leq c[|\Re\mathrm{e}\xi_1^{\mathbb{C}}| + |\Im\mathrm{m}x_1^{\mathbb{C}}| + |\zeta'| + |\xi''| + |y''|], \tag{13}$$

and

$$|\Re\mathrm{e}\eta_1^{\mathbb{C}}| \leq c[|x_1^{\mathbb{C}}| + |\zeta'| + |\xi''| + |y''|] \tag{14}$$

respectively. But by the substitution $\Re\mathrm{e}\eta_1^{\mathbb{C}} = -\Im\mathrm{m}\xi_1^{\mathbb{C}}$, (13) and (14) are immediate consequences of (2) and (3) respectively. □

Proposition 3. *Assume* (2) *and* (3). *Then the natural morphism*

$$\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1^{\pm}|X}) \xrightarrow{\sim} R\Gamma_{\pi^{-1}(M_1)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{C}_{\tilde{M}^{\pm}|\tilde{X}}) \tag{15}$$

is an isomorphism.

Proof. (Again we neglect here the functor $\mathrm{R}^t j'_* j_\pi^{-1}$ in the right of (15).) The morphism $\mathcal{C}_{A|X} \to \mathrm{R}^t j'_* j_\pi^{-1}\mathrm{R}\Gamma_{\pi^{-1}(A)}\mathcal{C}_{\tilde{A}|\tilde{X}}$ ($A = M_1^{\pm}$, M_1, S_1) induces the vertical arrows in the following commuting diagram in the category $D^b(T^*X)$:

(16)

$$\begin{array}{ccc} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{S_1|X}) & \to & \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1|X}) \ldots \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\pi^{-1}(S_1)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{C}_{\tilde{S}|\tilde{X}}) & \to & \mathrm{R}\Gamma_{\pi^{-1}(M_1)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{C}_{\tilde{M}|\tilde{X}}) \ldots \end{array}$$

$$\begin{array}{c} \cdots \to \bigoplus_{\pm} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M_1^{\pm}|X}) \\ \downarrow \\ \ldots \bigoplus_{\pm} \mathrm{R}\Gamma_{\pi^{-1}(M_1)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{C}_{\tilde{M}^{\pm}|\tilde{X}}). \end{array}$$

By Proposition 3 the two first vertical arrows are isomorphisms. Hence the third is an isomorphism too. □

For V_1 defined in $T^*_M X$ by $\eta' = 0$, let us recall the complex by [**U-Z**] of 2-hyperfunctions at the boundary along V_1:

$$\mathcal{B}^{2,V_1}_{M^\pm|X} = \mathrm{R}\Gamma_{\pi^{-1}(M)}(\mathcal{C}_{M^\pm_1|X})[d], \tag{17}$$

$(d = \operatorname{codim} V_1)$. We put

$\tilde{M}_2 = \mathbb{R}^2 \times M' \times M''$, $\tilde{S}_2 = (\{0\} \times \mathbb{R}) \times M' \times M''$, $\tilde{M}^\pm_2 = (\mathbb{R}^\pm \times \mathbb{R}) \times M' \times M''$,

and

$\tilde{M}_3 = (\mathbb{R} \times \mathbb{C}) \times X' \times M''$, $\tilde{S}_3 = (\{0\} \times \mathbb{C}) \times X' \times M''$, $\tilde{M}^\pm_3 = (\mathbb{R}^\pm \times \mathbb{C}) \times X' \times M''$.

Along with V_1 we also consider in $T^*_{\tilde{M}_2}\tilde{X}$, $V_2 = \{\eta' = 0\}$, $V_3 = \{\Im\mathrm{m}\eta^{\mathbb{C}}_1 = \eta' = 0\}$. We define similarly to (17):

$$\begin{aligned} \mathcal{B}^{2,V_2}_{\tilde{M}^\pm_2|\tilde{X}} &= \mathrm{R}\Gamma_{\pi^{-1}(\tilde{M}_2)}(\mathcal{C}_{\tilde{M}^\pm|\tilde{X}})[d], \\ \mathcal{B}^{2,V_3}_{\tilde{M}^\pm_2|\tilde{X}} &= \mathrm{R}\Gamma_{\pi^{-1}(\tilde{M}_2)}(\mathcal{C}_{\tilde{M}^\pm_3|\tilde{X}})[d+1]. \end{aligned} \tag{18}$$

According to [**U-Z**, Th. 2.6], $\mathcal{B}^{2,V_1}_{M^\pm|X}|_{V_1}$ and $\mathcal{B}^{2,V_i}_{\tilde{M}^\pm_2|\tilde{X}}|_{V_i}$, $i = 2, 3$ are all concentrated in degree 0, (whence they are naturally endowed with a structure of $\mathcal{E}_X$ or $\mathcal{E}_{\tilde{X}}$–modules). We also recall the complexes of usual 2–hyperfunctions by Kashiwara ([**K**]):

$$\mathcal{B}^{2,V_1}_{M|X},\ \mathcal{B}^{2,V_i}_{\tilde{M}|\tilde{X}}\,(i = 2, 3),\ \mathcal{B}^{2,V_1}_{S|X},\ \mathcal{B}^{2,V_i}_{\tilde{S}_2|\tilde{X}}\,(i = 2, 3), \tag{19}$$

defined similarly to (17), (18). It is classical that they are all concentrated in degree 0. We apply $\mathrm{R}\Gamma_{\pi^{-1}(M)}(\cdot)[d]$ to (9), (15) and get

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{B}^{2,V_1}_{M|X}) &\xrightarrow{\sim} \mathrm{R}\Gamma_{\pi^{-1}(M)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{B}^{2,V_2}_{\tilde{M}_2|\tilde{X}}) \\ \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{B}^{2,V_1}_{S|X}) &\xrightarrow{\sim} \mathrm{R}\Gamma_{\pi^{-1}(S)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{B}^{2,V_2}_{\tilde{S}_2|\tilde{X}}) \\ \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{B}^{2,V_1}_{M^\pm|X}) &\xrightarrow{\sim} \mathrm{R}\Gamma_{\pi^{-1}(M)}\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{B}^{2,V_2}_{\tilde{M}^\pm_2|\tilde{X}}). \end{aligned} \tag{20}$$

The natural (restriction) morphism $\mathbb{Z}_{M_1} \to \mathbb{Z}_M$, resp. $\mathbb{Z}_{\tilde{M}_3} \to \mathbb{Z}_{\tilde{M}}$, induces a morphism

$$\mathcal{C}_{M|X}|_{V_1} \to \mathcal{B}^{2,V_1}_{M|X}, \quad \text{resp. } \mathcal{B}^{2,V_2}_{\tilde{M}_2|\tilde{X}}\Big|_{V_3} \to \mathcal{B}^{2,V_3}_{\tilde{M}_2|\tilde{X}}. \tag{21}$$

It is classical (cf. [**K**]) that the first is injective. We show now:

Proposition 4. *The morphism*

$$\mathcal{B}^{2,V_2}_{\tilde{M}_2|\tilde{X}}\Big|_{V_3} \to \mathcal{B}^{2,V_3}_{\tilde{M}_2|\tilde{X}} \tag{22}$$

is injective.

Proof. Fix $p = (x_o; \sqrt{-1}\eta'' \mathrm{d}x'') \in V_3$ and let Z_2, resp. Z_3, describe the family of closed convex subsets of $\mathbb{R}^{n+1}_{(\Im\mathrm{m} x_1^{\mathbb{C}}, \Im\mathrm{m} y_1^{\mathbb{C}}, y', y'')}$ such that

$$\begin{cases} Z_2 \subset \{y| < y'', \eta'' > \geq \epsilon(|y''| + |\Im\mathrm{m} x_1^{\mathbb{C}}| + |\Im\mathrm{m} y_1^{\mathbb{C}}|)\}, \\ Z_2 \cap \{y'' = 0,\ \Im\mathrm{m} x_1^{\mathbb{C}} = 0,\ \Im\mathrm{m} y_1^{\mathbb{C}} = 0\} \subset \{0\}, \end{cases}$$

resp.

$$\begin{cases} Z_3 \subset \{y| < y'', \eta'' > \geq \epsilon(|\Im\mathrm{m} x_1^{\mathbb{C}}| + |y''|)\}, \\ Z_3 \cap \{\Im\mathrm{m} x_1^{\mathbb{C}} = 0,\ y'' = 0\} \subset \{0\}. \end{cases}$$

Thus the arrow in (22) can be represented, between the stalks at p, by:

$$\varinjlim_{B, Z_2} \mathrm{H}^n_{\tilde{M}_2 + \sqrt{-1} Z_2}(B, \mathcal{O}_X) \to \varinjlim_{B, Z_3} \mathrm{H}^n_{\tilde{M}_2 + \sqrt{-1} Z_3}(B, \mathcal{O}_X),$$

for B describing a fundamental system of neighborhoods of $x_o = \pi(p)$. Now for any B (convex) and for any Z_2, Z_3, there exist $Z_2' \supset Z_2$ such that $Z_3 \setminus Z_2' \subset\subset B$. If then $K_2 = \bar{B} \cap (\tilde{M}_2 + \sqrt{-1} Z_2')$, $K_3 = \bar{B} \cap (\tilde{M}_2 + \sqrt{-1} Z_3)$, we have $K_3 \setminus K_2 = (\tilde{M}_2 + \sqrt{-1}(Z_3 \setminus Z_2')) \cap B$ and therefore

$$\mathrm{H}^{n-1}_{\tilde{M}_2 + \sqrt{-1}(Z_3 \setminus Z_2')}(B, \mathcal{O}_X) = \mathrm{H}^{n-1}_{K_3 \setminus K_2}(B, \mathcal{O}_X) = 0,$$

by a celebrated theorem due to M. Kashiwara. □

Note that the first morphism in (21) is a particular case of the second. Hence Proposition 5 provides also a proof of the injectivity of the former.

The natural morphisms $\mathbb{Z}_{M_1^{\pm}} \to \mathbb{Z}_{M^{\pm}}$, resp. $\mathbb{Z}_{\tilde{M}_3^{\pm}} \to \mathbb{Z}_{\tilde{M}^{\pm}}$, in turn induce morphisms:

$$\mathcal{C}_{M^{\pm}|X}|_{V_1} \to \mathcal{B}^{2,V_1}_{M^{\pm}|X} \quad \text{resp. } \mathcal{B}^{2,V_2}_{\tilde{M}_2^{\pm}|\tilde{X}}\Big|_{V_3} \to \mathcal{B}^{2,V_3}_{\tilde{M}\pm_2|\tilde{X}}. \tag{23}$$

Neither of them is injective ([U-Z, Remark 2.7]). Nevertheless they can be injective when restricted to solutions of non-characteristic systems. Let $V_4 = {}^t j'^{-1}(V) \cap T^*_{\tilde{M}} \tilde{X}$ (i.e. V_4 is the submanifold of $T^*_{\tilde{M}_2} \tilde{X}$ defined by $\Im\mathrm{m}\xi_1^{\mathbb{C}} = \Im\mathrm{m}\eta_1^{\mathbb{C}} = \eta' = 0$); note that $V_4 = V_2 \cap \mathrm{char}(\mathcal{O}_{\bar{\mathbb{C}}_{z_1}}) = V_3 \cap \mathrm{char}(\mathcal{O}_{\bar{\mathbb{C}}_{z_1}})$. We have:

Proposition 5. *Let $S^{\mathbb{C}} \hookrightarrow X$ be non-characteristic for $\mathcal{M}$, and consider the sequence of morphisms:*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M^{\pm}|X})|_V &\to \mathrm{Hom}_{\mathcal{E}_X}\left(\mathcal{M}, \mathcal{B}^{2,V_1}_{M^{\pm}|X}\right)\Big|_V \\ &\xrightarrow{\sim} \Gamma_{\pi^{-1}(M)} \mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}_{z_1}}, \mathcal{B}^{2,V_2}_{\tilde{M}_2^{\pm}|\tilde{X}}\right)\Big|_{V_4} \\ &\hookrightarrow \mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}_{z_1}}, \mathcal{B}^{2,V_2}_{\tilde{M}_2^{\pm}|\tilde{X}}\right)\Big|_{V_4} \\ &\to \mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}_{z_1}} \mathcal{B}^{2,V_3}_{\tilde{M}_2^{\pm}|\tilde{X}}\right)\Big|_{V_4}, \end{aligned} \tag{24}$$

with the first and the fourth arrow induced by (23), *the second by* (20), *and the third being the natural identification. Then the composition of the morphisms in* (23) *is injective.*

Remark 6. In particular the first morphism in (24) is injective. Our proof will show that this is in fact injective on the whole V_1 (not only on V) according to [**U-Z**, Th. 2.8]. However the full generalization of this statement (in analogy with Proposition 4), i.e. the injectivity of the last morphism in (24) is not clear to us because of the lack of a 2-microlocal version of the *watermelon-cut* Theorem (cf. [**S**]).

Proof. We consider

$$\begin{array}{ccc} \mathcal{B}^{2,V_2}_{\tilde{M}_2^{\pm}|\tilde{X}}\Big|_{\tilde{S}_2\times_{\tilde{M}_3}T^*_{\tilde{M}_3}\tilde{X}} & \to & \mathcal{B}^{2,V_3}_{\tilde{M}_2^{\pm}|\tilde{X}}\Big|_{\tilde{S}_2\times_{\tilde{M}_3}T^*_{\tilde{M}_3}\tilde{X}} \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\tilde{F}^{\pm}}\mathcal{B}^{2,V_2}_{\tilde{S}_2|\tilde{X}}\Big|_{\tilde{S}_2\times_{\tilde{M}_3}T^*_{\tilde{M}_3}\tilde{X}}[1] & \to & \mathrm{R}\Gamma_{\tilde{F}^{\pm}}(\mathcal{B}^{2,V_3}_{\tilde{S}_2|\tilde{X}})\Big|_{\tilde{S}_2\times_{\tilde{M}_3}T^*_{\tilde{M}_3}\tilde{X}}[+1], \end{array} \tag{25}$$

where $\tilde{F}^{\pm} = (\tilde{S}_2 \times_{\tilde{M}_2} T^*_{\tilde{M}_2}\tilde{X}) \pm \mathbb{R}^+\theta$ with θ the exterior conormal to $\tilde{M}^+$ in $\tilde{M}$. Remark that the vertical arrows of (25) are induced by the natural morphisms $\mathcal{B}^{2,V_i}_{\tilde{M}_2^{\pm}|\tilde{X}} \to \mathcal{B}^{2,V_i}_{\tilde{S}_2|\tilde{X}}[1]$ which factorize through $\mathrm{R}\Gamma_{\tilde{F}^{\pm}}\mathcal{B}^{2,V_i}_{\tilde{S}_2|\tilde{X}}[1]$ (due to $\mathrm{supp}(\mathcal{B}^{2,V_i}_{\tilde{M}_2^{\pm}|\tilde{X}}) \cap \mathrm{supp}(\mathcal{B}^{2,V_i}_{\tilde{S}_2|\tilde{X}}) \subset \tilde{F}^{\pm}$). If we apply $\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}_{z_1}}, \cdot)$ to (25) and take the 0-th cohomology, the arrow on the bottom becomes injective. In fact let $\tilde{Y} = \tilde{S}_2^{\mathbb{C}}$ be the complexification of S_2 (i.e. $\tilde{Y} = \mathbb{C}_{y_1^{\mathbb{C}}} \times X' \times X''$), denote by $k : \tilde{Y} \to \tilde{X}$ the natural embedding, and let $V_i' = {}^t k' k_\pi^{-1}(V_i)$. By the aid of division formulas for $\mathcal{B}^{2,V_i}_{\tilde{S}_2|\tilde{X}}$, the above injectivity is reduced to the injectivity of

$$\mathcal{B}^{2,V_2'}_{\tilde{S}_2|\tilde{Y}}\Big|_{V_3'} \hookrightarrow \mathcal{B}^{2,V_3'}_{\tilde{S}_2|\tilde{Y}}.$$

But this is, under different notations, the same statement as in Proposition 4. We consider now:

$$\begin{array}{ccc} \mathcal{C}_{M^{\pm}|X}|_{S\times_{M_1}T^*_{M_1}X} & \to & \mathcal{B}^{2,V_1}_{M^{\pm}|X}\Big|_{S\times_{M_1}T^*_{M_1}X} \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{F^{\pm}}(\mathcal{C}_{S|X})|_{S\times_{M_1}T^*_{M_1}X}[1] & \to & \mathrm{R}\Gamma_{F^{\pm}}(\mathcal{B}^{2,V_1}_{S|X})\Big|_{S\times_{M_1}T^*_{M_1}X}[1], \end{array} \tag{26}$$

with $F^{\pm} = S \times_M T^*_M X \pm \mathbb{R}^+\theta$. The arrow in the bottom is injective, over solutions of $\mathcal{M}$, for the same argument as for (25). Concerning the first vertical arrow, this is represented at each point $p \in S \times_M T^*_M X$ by

$$(\mathcal{C}_{M^\pm|X})_p \simeq \left(\frac{\mathcal{C}_{\bar{M}^\pm|X}}{C_{S|X}}\right)_p \hookrightarrow \mathcal{H}^1_{F^\pm}(\mathcal{C}_{S|X})_p,$$

(where $\mathcal{C}_{\bar{M}^\pm|X}$ are the Kataoka's microfunctions along the closed half-spaces $\bar{M}^\pm$) whose injectivity is immediately proved by the aid of a Legendre transformation (cf. [**Kat**] and [**S**]).

We are ready to conclude. We apply $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \cdot)$ to (26) and $\mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \otimes \mathcal{O}_{\bar{\mathbb{C}}_{z_1}}, \cdot)$ to (25) respectively (and neglect $\mathrm{R}^t j'_* j_\pi^{-1}$). We glue the diagrams so obtained by means of the second and third of (20) and by the natural morphism $\mathrm{R}\Gamma_{\pi^{-1}(M)}(\cdot) \to \cdot$. We thus obtain a long diagram with the first vertical and all the bottom horizontal arrows injective over the 0-th cohomology. The composition of the upper horizontal arrows (which is precisely the sequence of morphisms in (24)) is therefore also injective. □

End of proof of Theorem 1. Let $\pm\theta$ be the exterior conormals to $\tilde{M}^\pm$ in $\tilde{M}$ identified to vectors $\mathrm{H}^{\mathbb{R}}(\pm\pi^*\theta)$ of $T_p T^* \tilde{X}$ (cf. (10)). Then clearly

$$\mathrm{H}^{\mathbb{R}}(\pm\pi^*(\theta)) \notin C(\mathrm{char}(\mathcal{O}_{\bar{\mathbb{C}}}), \tilde{V}_3). \tag{27}$$

Let $\mathrm{SS}(\mathbb{Z}_{\tilde{M}_3^\pm})$ denote the microsupport of $\mathbb{Z}_{\tilde{M}_3^\pm}$ in the sense of [**K-S 2**]. One easily checks that

$$\mathrm{SS}(\mathbb{Z}_{\tilde{M}_3^\pm}) = \left(\overline{\tilde{M}_3^\pm} \times T^*_{\tilde{M}_3} \tilde{X}\right) \pm \mathbb{R}^+\theta.$$

It is also easy to see that (27) implies

$$-\mathrm{H}^{\mathbb{R}}(\pm\pi^*\theta) \notin C\left(\mathrm{char}(\mathcal{O}_{\bar{\mathbb{C}}}), \mathrm{SS}(\mathbb{Z}_{\tilde{M}_3^\pm})\right).$$

It follows, merely by definition of SS:

$$\mathrm{R}\Gamma_{\pi^{-1}(\tilde{S}_3)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}(\mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{C}_{\tilde{M}_3^\pm|\tilde{X}}) = 0,$$

and thus, by applying $\mathrm{R}\Gamma_{\pi^{-1}(\tilde{M}_2)}(\cdot)[d+1]$:

$$\mathrm{R}\Gamma_{\pi^{-1}(\tilde{S}_2)} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{O}_{\bar{\mathbb{C}}}, \mathcal{B}^{2,V_3}_{\tilde{M}_2^\pm|\tilde{X}}\right) = 0. \tag{28}$$

In conclusion we have

$$\begin{aligned}\Gamma_{\pi^{-1}(S)}\mathrm{Hom}_{\mathcal{E}_X}\left(\mathcal{M}, \mathcal{C}_{M^\pm|X}\right)|_V &\hookrightarrow \Gamma_{\pi^{-1}(S)}\mathrm{Hom}_{\mathcal{E}_X}\left(\mathcal{M}, \mathcal{B}^{2,V_1}_{M^\pm|X}\right)\Big|_V \\ &\xrightarrow{\sim} \Gamma_{\pi^{-1}(S)}\mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{M}\otimes\mathcal{O}_{\bar{\mathbb{C}}_{z_1}}, \mathcal{B}^{2,V_2}_{\tilde{M}_2^\pm|\tilde{X}}\right)\Big|_{V_4} \\ &\to \Gamma_{\pi^{-1}(S)}\mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}\left(\mathcal{M}\otimes\mathcal{O}_{\bar{\mathbb{C}}_{z_1}}, \mathcal{B}^{2,V_3}_{\tilde{M}_2^\pm|\tilde{X}}\right)\Big|_{V_4} \\ &= 0,\end{aligned}$$

(where the first "$\hookrightarrow$" follows from Remark 6, the second "$\xrightarrow{\sim}$" from (20), the third "$\to$" from (23), and the last "=" from (28) respectively). On the other hand the composition of "$\hookrightarrow$", "$\xrightarrow{\sim}$" and "$\to$" is injective by Proposition 5; hence $\Gamma_{\pi^{-1}(S)}\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M^\pm|X})|_V = 0$. The proof is complete. □

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References

[B] J.M. Bony, *Extension du Théorème de Holmgren*, Sém. Goulaouic-Schwartz, (1975-76), Exp. 17.

[D'A-T-Z] P. D'Ancona, N. Tose and G. Zampieri, *Propagation of singularities up to the boundary along leaves*, Comm. in Partial Differential Equations, **15**(**4**) (1990), 453-460.

[K] M. Kashiwara, *Talks in Nice*, (1972).

[K-S 1] M. Kashiwara and P. Schapira, *Microhyperbolic systems*, Acta Math., **142** (1979), 1-55.

[K-S 2] ———, *Microlocal study of sheaves*, Astérisque, **128** (1985).

[Kat] K. Kataoka, *Microlocal theory of boundary value problems* I, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **35** (1980), 355-399; II, ibid., **28** (1981), 31-56.

[S-K-K] M. Sato, M. Kashiwara and T. Kawai, *Hyperfunctions and pseudodifferential equations*, Springer Lecture Notes in Math., **287** (1973), 265-529.

[S] P. Schapira, *Front d'onde analytique au bord* I *and* II, C.R. Acad. Sci. Paris, **302**(**10**) (1986), 383-386; Sém. E.D.P. École Polyt., Exp., **13** (1986).

[S-Z] P. Schapira and G. Zampieri, *Regularity at the boundary for systems of microdifferential equations*, Pitman Res. Notes in Math., **158** (1987), 186-201.

[Sj] J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque, **95** (1982).

[U-Z] M. Uchida and G. Zampieri, *Second microlocalization at the boundary and microhyperbolicity*, Publications of the Research Institute for Mathematical Sciences, Kyoto University, **26** (1990), 205-219.

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