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LOCAL MAXIMAL MONOTONICITY,
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ARE ALL THE SAME FOR
CONTINUOUS LINEAR OPERATORS

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Heinz H. Bauschke and Jonathan M. Borwein

The concept of a monotone operator — which covers both linear positive semi-definite operators and subdifferentials of convex functions — is fundamental in various branches of mathematics. Over the last few decades, several stronger notions of monotonicity have been introduced: Gossez's maximal monotonicity of dense type, Fitzpatrick and Phelps's local maximal monotonicity, and Simons's monotonicity of type (NI). While these monotonicities are automatic for maximal monotone operators in reflexive Banach spaces and for subdifferentials of convex functions, their precise relationship is largely unknown.

Here, it is shown — within the beautiful framework of Convex Analysis — that for continuous linear monotone operators, *all these notions coincide and are equivalent to the monotonicity of the conjugate operator.* This condition is further analyzed and illustrated by examples.

1. Introduction.

Motivation.

A monotone operator is a (possibly set-valued) map from a Banach space to its dual satisfying a certain relation. In the simplest case, when the space is just the real line, this relation corresponds precisely to increasing (possibly set-valued) functions, hence the name.

Monotone operators appear in diverse areas such as Operator Theory, Numerical Analysis, Differentiability Theory of Convex Functions, and Partial Differential Equations, because the notion of a monotone operator is broad enough to cover two fundamental mathematical objects: *Linear positive semi-definite operators* and *subdifferentials of convex functions.* Although the former object gave rise to the field, it is the latter that has been receiving much of the recent attention. (For more on monotone operators, the reader is referred to the conference proceedings [11, 4, 31], the books

[**21**, **29**, **32**, **33**], the electronic lecture notes [**22**], and the historical account [**19**]; applications are discussed in [**6**, **10**, **20**].)

The urge to extract and study the quite strong monotonicity properties of subdifferentials of convex functions has led to the introduction of several new more powerful notions of monotonicity. While these notions are automatic for maximal monotone operators on reflexive Banach spaces, the situation in nonreflexive Banach spaces is far less well understood. Surprisingly, these notions of monotonicity were largely untested even for the most natural candidates: Continuous linear monotone operators. Thus:

The aim of this paper is to study the various notions of monotonicity for continuous linear monotone operators.

Using elegant and potent tools from Convex Analysis, we show that *these notions all coincide with the monotonicity of the conjugate operator.*

In contrast to the subdifferential case, this condition is *not* automatic and we present a new derivation of two classical counter-examples. Using Banach space theory, it can be shown that monotonicity of the conjugate operator is the rule — with the notable exception of spaces containing a complemented copy of ℓ_1.

Overview.

In Section 2, we introduce the various notions of monotonicity coined by Gossez, by Fitzpatrick and Phelps, and by Simons and then review their basic relationship. From Section 3 on, we focus on the case when the monotone operator is continuous and linear. The main result, whose proof depends crucially on Fenchel's Duality Theorem, is presented in Section 4. It allows us to give an affirmative answer to a question posed by Gossez more than two decades ago. In the last section, we derive and extend classical counter-examples by Gossez and by Fitzpatrick and Phelps *systematically* from a result that can be viewed as an "instruction manual" for constructing interesting continuous linear monotone operators whose skew parts have *nonmonotone* conjugates. We conclude by remarking that such strange operators occur only in a few classical Banach spaces like ℓ_1 and $L_1[0, 1]$ and that preliminary results on *regularizations* demonstrate the close relationship between the various monotonicities even in a nonlinear context.

Notation.

The notation we employ is standard. Throughout, we assume that

$$X \text{ is a real Banach space with norm } \|\cdot\| \text{ and dual } X^*.$$

The evaluation of a functional $x^* \in X^*$ at a point $x \in X$ is written as $\langle x^*, x \rangle$ or as $\langle x, x^* \rangle$. We often view X as a subspace in its bidual $X^{**} := (X^*)^*$. The *unit ball* $\{x \in X : \|x\| \leq 1\}$ is denoted B_X. If (x_α^*) is a net in some dual space, then we write $x_\alpha^* \overset{\mathrm{w}^*}{\to} x^*$ (resp. $x_\alpha^* \to x^*$) to indicate *convergence* in the weak* (resp. norm) topology with limit point x^*. If $x, y \in X$, then $[x, y]$

stands for the *line segment* $\{\lambda x + (1 - \lambda)y : \lambda \in [0,1]\}$. If U (resp. V) is a subset of X (resp. X^*), then $U^\perp$ (resp. $^\perp V$) stands for the *annihilator* $\{x^* \in X^* : \langle x^*, u \rangle = 0, \ \forall u \in U\}$ (resp. $\{x \in X : \langle x, v \rangle = 0, \ \forall v \in V\} = X \cap V^\perp$). If T is a continuous linear operator from X to some other Banach space, then the *conjugate* (or *adjoint, transpose*) is denoted T^*, the *restriction* of T to some subset U of X is written as $T|_U$, and $\ker T$ is the *kernel* (or *null space*) of T: $\ker T := \{x \in X : Tx = 0\}$. Suppose C is a subset of X. Then $\operatorname{span} C$ is the *span* of C (i.e. the set of all linear combinations of elements of C). Also, $\operatorname{cl} C$ (resp. $\operatorname{int} C$) stands for the *closure* (resp. *interior*) of C; here, the norm topology is the "default topology". If these operations are meant with respect to some other topology $\mathcal{T}$, then we indicate this by subscripts; for instance, $\operatorname{cl}_\mathcal{T} C$ would be the closure of C with respect to the topology $\mathcal{T}$.

Suppose Y, Z are sets and T is a *set-valued map* from Y to Z, *i.e.*, T is a map from Y to 2^Z. Then the *graph* of T is denoted $\operatorname{gra} T$; so $z \in Ty$ if and only if $(y, z) \in \operatorname{gra} T, \ \forall y \in Y, z \in Z$. The *domain* (resp. *range*) of T is given by $\operatorname{dom} T := \{y \in Y : Ty \neq \emptyset\}$ (resp. $\operatorname{ran} T := \{z \in Z : z \in Ty, \text{ for some } y \in Y\}$). The *inverse* of T, denoted T^{-1}, is the set-valued map from Z to Y defined by $y \in T^{-1}z$ if and only if $z \in Ty, \ \forall y \in Y, z \in Z$. If U is a subset of Y, then we write $T(U)$ for $\bigcup_{u \in U} Tu$.

Notation from Convex Analysis appears throughout the paper. For the reader's convenience, we review the definitions. The *indicator function* of a subset C of X, denoted ι_C, is given by

$$\iota_C : X \to \mathbb{R} \cup \{+\infty\} : x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

Suppose f is a convex lower semi-continuous function from X to $\mathbb{R} \cup \{+\infty\}$. Then the (*essential*) *domain* of f is the set $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. The conjugate of f, denoted f^*, is given by

$$f^* : X^* \to \mathbb{R} \cup \{+\infty\} : x^* \mapsto \sup_{x \in X} \big[\langle x^*, x \rangle - f(x) \big].$$

Note that f^* is defined on X^* and hence $f^{**} := (f^*)^*$ is defined on X^{**}. The *subdifferential* of f at $x \in \operatorname{dom} f$ is the set

$$\{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x), \quad \forall y \in X\}.$$

If $x \in \operatorname{int} \operatorname{dom} f$ and $\partial f(x)$ is singleton, then the element in $\partial f(x)$ coincides with the (*Gâteaux*) *gradient* $\nabla f(x)$.

Finally, the reals (resp. strictly positive integers $\{1, 2, 3, \dots\}$) are abbreviated $\mathbb{R}$ (resp. $\mathbb{N}$) and we used already $\forall$ (resp. $\exists$) as a short form for "for all" (resp. "there exists").

As general references on Functional Analysis, we recommend [**5**, **17**, **18**, **30**]; for more on Convex Analysis, see [**25**, **1**, **7**, **15**, **16**].

The single most important tool from Convex Analysis is the celebrated *Fenchel Duality Theorem*:

Fact 1.1. (Fenchel Duality; see, e.g., [**1**, Theorem 4.6.1].) Suppose A is a continuous linear operator from a Banach space X to a Banach space Y, f is a convex lower semi-continuous function on X and g is a convex lower semi-continuous function on Y. Consider the convex programs

(P) $$p := \inf_{x \in X} \big[f(x) + g(Ax) \big]$$

and

(D) $$d := - \inf_{y^* \in Y^*} \big[f^*(-A^*y^*) + g^*(y^*) \big].$$

Then $p \geq d$. If $A(\operatorname{dom} f) \cap \operatorname{int} \operatorname{dom} g \neq \emptyset$ and p is finite, then $p = d$ and d is attained.

2. General tools.

Recall that a set-valued map from X to X^* is a *monotone operator*, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \qquad \forall x, y \in X.$$

If T is monotone and $\operatorname{gra} T$ is a maximal subset in $X \times X^*$, then T is called *maximal monotone*. Zorn's Lemma guarantees the existence of maximal monotone extensions for any given monotone operator. Analogously, one can speak of (maximal) monotone operators from X^* to X or from X^{**} to X^* or of monotone operators whose graphs are maximal monotone with respect to some subsets and so forth.

The following extensions have turned out to be useful when studying the nonreflexive case.

Definition 2.1. Suppose T is a set-valued map from X to X^*. Define set-valued maps T_1, T_0, $\overline{T}$ from X^{**} to X^* via their graphs as follows:

   (i) $(x^{**}, x^*) \in \operatorname{gra} T_1$, if there exists a bounded net (x_α, x_α^*) in $\operatorname{gra} T$ with $x_\alpha \overset{\text{w*}}{\to} x^{**}$ and $x_\alpha^* \to x^*$.
  (ii) $(x^{**}, x^*) \in \operatorname{gra} T_0$, if $\inf_{(y,y^*) \in \operatorname{gra} T} \langle y^* - x^*, y - x^{**} \rangle = 0$.
 (iii) $(x^{**}, x^*) \in \operatorname{gra} \overline{T}$, if $\inf_{(y,y^*) \in \operatorname{gra} T} \langle y^* - x^*, y - x^{**} \rangle \geq 0$.

Proposition 2.2. *Suppose T is a monotone operator from X to X^*. Then the following inclusions hold in $X^{**} \times X^*$:*

$$\operatorname{gra} T \subseteq \operatorname{gra} T_1 \subseteq \operatorname{gra} T_0 \subseteq \operatorname{gra} \overline{T} = \operatorname{gra} \overline{T_1} \cap (X^{**} \times X^*).$$

Proof. The inclusions $\operatorname{gra} T \subseteq \operatorname{gra} T_1$ and $\operatorname{gra} T_0 \subseteq \operatorname{gra} \overline{T} \supseteq \operatorname{gra} \overline{T_1} \cap (X^{**} \times X^*)$ are obvious (even without monotonicity). Fix an arbitrary $(x^{**}, x^*) \in \operatorname{gra} T_1$ and obtain a bounded net (x_α, x_α^*) in $\operatorname{gra} T$ with $x_\alpha \overset{\text{w*}}{\to} x^{**}$ and $x_\alpha^* \to x^*$. Then $\langle x_\alpha - y, x_\alpha^* - y^* \rangle \geq 0$, $\forall \alpha, (y, y^*) \in \operatorname{gra} T$; taking limits yields $\langle x^{**} -$

$y, x^* - y^* \rangle \geq 0$. On the other hand, $\langle x^{**} - x_\alpha, x^* - x_\alpha^* \rangle \to 0$; altogether, $0 = \inf_{(y,y^*)\in \operatorname{gra} T} \langle x^{**} - y, x^* - y^* \rangle$, i.e., $(x^{**}, x^*) \in \operatorname{gra} T_0$. Hence $\operatorname{gra} T_1 \subseteq \operatorname{gra} T_0$. Finally, pick $(z^{**}, z^*) \in \operatorname{gra} \overline{T}$. Then $0 \leq \inf_{(y,y^*)\in \operatorname{gra} T} \langle y^* - z^*, y - z^{**} \rangle \leq \lim_\alpha \langle x_\alpha^* - z^*, x_\alpha - z^{**} \rangle = \langle x^* - z^*, x^{**} - z^{**} \rangle$; so (z^{**}, z^*) is monotonically related to $\operatorname{gra} T_1$, hence $\operatorname{gra} \overline{T} \subseteq \operatorname{gra} \overline{T_1} \cap (X^{**} \times X^*)$. □

Definition 2.3. Suppose T is a monotone operator from X to X^*. Then:

(i) (Gossez [**14**]) T *is of dense type* or *of type* (D), if $T_1 = \overline{T}$.

(ii) (Simons [**28**, Definition 14]) T *is of range-dense type* or *of type* (WD), if for every $x^* \in \operatorname{ran} \overline{T}$, there exists a bounded net $(x_\alpha, x_\alpha^*) \in \operatorname{gra} T$ with $x_\alpha^* \to x^*$.

(iii) (Simons [**28**, Definition 10]) T *is of type* (NI), if $\inf_{(y,y^*)\in \operatorname{gra} T} \langle y^* - x^*, y - x^{**} \rangle \leq 0$, for all $(x^{**}, x^*) \in X^{**} \times X^*$. If this holds only on some subset of $X^{**} \times X^*$, then we say that T *is of type* (NI) *with respect to this subset*.

(iv) (Fitzpatrick and Phelps [**8**, Section 3]) T *is locally maximal monotone*, if $(\operatorname{gra} T^{-1}) \cap (V \times X)$ is maximal monotone in $V \times X$, for every convex open set V in X^* with $V \cap \operatorname{ran} T \neq \emptyset$.

(v) T *is unique*, if all maximal monotone extensions of T in $X^{**} \times X^*$ coincide.

A monotone operator which is either maximal monotone and of dense type or locally maximal monotone is certainly maximal monotone; the converse is true in reflexive spaces:

Fact 2.4. (See, e.g., Phelps's [**22**, Example 3.2.(b) and Proposition 4.4].) Suppose X is reflexive and T is a monotone operator from X to X^*. Then TFAE: (i) T is maximal monotone; (ii) T is maximal monotone and of dense type; (iii) T is locally maximal monotone.

It is known and very useful that subdifferentials of convex functions are "everything": Maximal monotone, of dense type, and locally maximal monotone.

Fact 2.5. Suppose f is convex lower semi-continuous proper function on X. Then:

(i) (Rockafellar [**26**]) ∂f is maximal monotone.

(ii) (Gossez [**12**, Théorème 3.1]) ∂f is of dense type and $(\partial f)_1 = (\partial f^*)^{-1}$.

(iii) (Simons [**27**]) ∂f is locally maximal monotone.

In general, the following is known to be true.

Fact 2.6. (Simons's [**28**, Lemma 15 and Theorem 19].) For any monotone operator T from X to X^*, the following implications hold:

$$\text{dense type} \implies \text{range-dense type} \implies \text{type (NI)} \implies \text{unique}.$$

Moreover, TFAE:

(i) T is unique.
(ii) $\overline{T}$ is the unique maximal monotone extension of T in $X^{**} \times X^*$.
(iii) $\overline{T}$ is maximal monotone.
(iv) $\overline{T}$ is monotone.

It is sometimes more handy to work with the following reformulations of the various monotonicities.

Proposition 2.7. *Suppose T is a monotone operator from X to X^*. Then:*

(i) *T is of dense type if and only if T_1 is maximal monotone.*
(ii) *T is of range-dense type if and only if $\operatorname{ran} T_1 = \operatorname{ran} \overline{T}$.*
(iii) *T is of type (NI) if and only if $T_0 = \overline{T}$.*
(iv) *(Phelps's [**22**, Proposition 4.3]) T is locally maximal monotone if and only if for every weak* closed convex bounded subset C of X^* with $\operatorname{ran} T \cap \operatorname{int} C \neq \emptyset$, and for every $x_0 \in X$, $x_0^* \in (\operatorname{int} C) \setminus T x_0$, there exist $(z, z^*) \in \operatorname{gra} T \cap (X \times C)$ with $\langle z^* - x_0^*, z - x_0 \rangle < 0$.*

Proof. (i): "$\Rightarrow$": T is of dense type $\Leftrightarrow T_1 = \overline{T}$. Now T_1 is monotone (because T is), hence so is $\overline{T}$. By Fact 2.6, $\overline{T} = T_1$ is maximal monotone. "$\Leftarrow$": Pick $(x^{**}, x^*) \in \operatorname{gra} \overline{T}$. Then (by Proposition 2.2) $(x^{**}, x^*) \in \operatorname{gra} \overline{T_1}$, i.e., this point is monotonically related to $\operatorname{gra} T_1$. Now T_1 is maximal monotone, hence $(x^{**}, x^*) \in \operatorname{gra} T_1$.

(ii): "$\Rightarrow$": Pick $x^* \in \operatorname{ran} \overline{T}$. By assumption, there exists a bounded net (x_α, x_α^*) in $\operatorname{gra} T$ such that $x_\alpha^* \to x^*$. Without loss, we can assume that $x_\alpha \overset{\text{w*}}{\to} x^{**}$. Then $(x^{**}, x^*) \in \operatorname{gra} T_1$ and in particular $x^* \in \operatorname{ran} T_1$. "$\Leftarrow$" is even simpler.

(iii): Let us abbreviate $\inf_{(y,y^*) \in \operatorname{gra} T} \langle x^{**} - y, x^* - y^* \rangle$ by I. "$\Rightarrow$": If $(x^{**}, x^*) \in \operatorname{gra} \overline{T}$, then $I \geq 0$. Now T is of type (NI), hence $I \leq 0$. Thus $I = 0$. "$\Leftarrow$": Fix $(x^{**}, x^*) \in X^{**} \times X^*$. If $(x^{**}, x^*) \notin \operatorname{gra} \overline{T}$, then $I < 0$. Otherwise, $(x^{**}, x^*) \in \overline{T} = T_0$ and hence $I = 0$. $\square$

3. Linear tools.

For a continuous linear operator T from X to X^*, the extension T_1 has the following explicit description:

Fact 3.1. (Gossez's [**13**, End of Section 2].) *Suppose T is a continuous linear operator from X to X^*. Then:*

$$\operatorname{gra} T_1 = \operatorname{cl}_{\text{weak*} \times \|\cdot\|} \operatorname{gra} T = (\operatorname{gra} T^{**}) \cap (X^{**} \times X^*).$$

Recall that a continous linear operator T from X to X^* is *weakly compact*, if $\operatorname{ran} T^{**}|_{X^{**} \setminus X} \subseteq X^*$; equivalently, if $\operatorname{cl} T(B_X)$ is weakly compact or if T^* is weakly compact. Fact 3.1 yields immediately:

Corollary 3.2. *Suppose T is a continuous linear operator from X to X^*. Then T is weakly compact if and only if $T_1 = T^{**}$.*

Further recall that if T is a continuous linear operator from X to X^* with $\langle Tx, x \rangle \geq 0$, $\forall x \in X$, then T is called *positive* or *positive semi-definite*. The following result is part of the folklore.

Proposition 3.3. *Suppose T is a continuous linear operator from X to X^*. Then* TFAE: (i) T *is positive;* (ii) T *is monotone;* (iii) T *is maximal monotone.*

Proof. By linearity of T, (i) and (ii) are equivalent; also, (iii) implies (ii). For "(ii)$\Rightarrow$(iii)" see, e.g., [**22**, Proof of Example 1.5.(b)]. $\square$

Monotonicity of type (NI) relates to monotonicity of the conjugate operator:

Proposition 3.4. *Suppose T is a continuous linear operator from X to X^*. Then T is monotone and of type* (NI) *with respect to* $\mathrm{gra}\,(-T^*)$ *if and only if T^* is monotone.*

Proof. Clearly, if T^* is monotone, then so is T. So suppose T is monotone. Fix $x^{**} \in X^{**}$ and $x \in X$. Then $\langle Tx + T^*x^{**}, x - x^{**} \rangle = \langle Tx, x \rangle - \langle Tx, x^{**} \rangle + \langle T^*x^{**}, x \rangle - \langle T^*x^{**}, x^{**} \rangle \geq -\langle T^*x^{**}, x^{**} \rangle$. Hence $-\langle T^*x^{**}, x^{**} \rangle \leq \inf_{x \in X} \langle Tx + T^*x^{**}, x - x^{**} \rangle \leq \langle T0 + T^*x^{**}, 0 - x^{**} \rangle = -\langle T^*x^{**}, x^{**} \rangle$ and thus:

$$\inf_{(y,y^*) \in \mathrm{gra}\,T} \langle y^* - (-T^*x^{**}), y - x^{**} \rangle = \inf_{x \in X} \langle Tx + T^*x^{**}, x - x^{**} \rangle$$
$$= -\langle T^*x^{**}, x^{**} \rangle.$$

The result follows. $\square$

Recall also that a continuous linear operator from X to X^* is *symmetric* (resp. *skew*), if $T^*|_X = T$ (resp. $T^*|_X = -T$); equivalently, if $\langle Tx, y \rangle = \langle x, Ty \rangle$ (resp. $\langle Tx, y \rangle = -\langle x, Ty \rangle$), $\forall x, y \in X$.

Our study of continuous linear monotone operators relies also on the following easy-to-prove yet immensely useful decomposition principle.

Proposition 3.5. *Suppose T is a continuous linear operator from X to X^*. Then T can be written as the sum of two continuous linear operators, $T = P + S$, where P is symmetric and S is skew. This decomposition is unique; in fact:*

$$Px = \frac{1}{2}Tx + \frac{1}{2}T^*x \quad and \quad Sx = \frac{1}{2}Tx - \frac{1}{2}T^*x, \qquad \forall x \in X.$$

We refer to P (resp. S) as the symmetric part (*resp.* skew part) *of T.*

Symmetric operators.

Theorem 3.6. *Suppose T is a continuous linear operator from X to X^* with symmetric part P. Let $q(x) := \frac{1}{2}\langle x, Tx\rangle, \forall x \in X$. Then:*

$$q \text{ is convex} \Leftrightarrow T \text{ is monotone} \Leftrightarrow P \text{ is monotone.}$$

Assume in addition that T is monotone. Then:

   (i) $\nabla q = P$ *and* $q^* \circ P = q$.
   (ii) $P_1 = P_0 = \overline{P} = P^* = P^{**}$. *Hence: P is maximal monotone of dense type, weakly compact, and locally maximal monotone; P^* is monotone and symmetric.*
   (iii) *For every $x^{**} \in X^{**}$, there exists a bounded net (x_α) in X such that $x_\alpha \overset{\mathrm{w*}}{\to} x^{**}$ and $Px_\alpha \to P^*x^{**} = P^{**}x^{**}$.*
   (iv) $q^*(P^*x^{**}) = \frac{1}{2}\langle x^{**}, P^{**}x^{**}\rangle = q^{**}(x^{**})$, *for every $x^{**} \in X^{**}$ and $\nabla q^{**} = P^{**} = P^*$.*

Proof. Since q is continuous, it suffices to check midpoint convexity; fixing two arbitrary points $x, y \in X$, we have $q(\frac{1}{2}x + \frac{1}{2}y) \le \frac{1}{2}q(x) + \frac{1}{2}q(y) \Leftrightarrow 0 \le \langle x - y, T(x - y)\rangle \Leftrightarrow 0 \le \langle x - y, P(x - y)\rangle$. The displayed equivalences follow.

(i): q is continuous, convex, and finite on X; hence q is subdifferentiable everywhere. So fix an arbitrary $x_0 \in X$ and pick $x^* \in \partial q(x_0)$. Then $t\langle x^*, h\rangle \le q(x_0 + th) - q(x_0), \forall h \in X, t > 0$; this simplifies to $\langle x^*, h\rangle \le \langle \frac{1}{2}Tx_0 + \frac{1}{2}T^*x_0, h\rangle + \frac{1}{2}t\langle h, Th\rangle$. Letting t tend to 0 yields $x^* = \frac{1}{2}Tx_0 + \frac{1}{2}T^*x_0 = Px_0$. Now

$$q^*(Px_0) = \sup_{x \in X}\langle Px_0, x\rangle - q(x) = -\inf_{x \in X}\{q(x) + \langle -Px_0, x\rangle\};$$

this last infimum can be viewed as a little optimization problem which is easy to solve: Indeed, after taking gradients, we learn that the set of minimizers equals $x_0 + \ker P$. It follows that $q^*(Px_0) = q(x_0)$.

(ii): By (i), P is the subdifferential of q. Consequently (Fact 2.5), P is maximal monotone, of dense type, and locally maximal monotone. In particular (Fact 2.6 and Proposition 2.7.(iii)), $P_1 = P_0 = \overline{P}$ and P is of type (NI). It follows that on the one hand, P^* is a *maximal monotone extension of P* (Proposition 3.4 and Proposition 3.3). On the other hand, $\overline{P}$ is *the unique maximal monotone extension of P in $X^{**} \times X^*$* (Fact 2.6). Altogether, $\overline{P} = P^*$. Now $P^* = P^{**}$, because $P_1 = P^*$ and $\mathrm{gra}\, P_1 \subseteq \mathrm{gra}\, P^{**}$ (Fact 3.1). Thus the weak compactness of P follows from Corollary 3.2.

(iii): By (ii), $P_1 = P^* = P^{**}$.

(iv): Fix $x^{**} \in X^{**}$ and define $g(x) := \langle -P^*x^{**}, x\rangle + \frac{1}{2}\langle x^{**}, P^*x^{**}\rangle$, $\forall x \in X$. Then, by (ii), $(x^{**}, P^*x^{**}) \in \mathrm{gra}\, P_0$ and hence

$$0 = \frac{1}{2}\inf_{x \in X}\langle Px - P^*x^{**}, x - x^{**}\rangle = \inf_{x \in X} q(x) + g(x).$$

The conjugate of g is given by $g^*(x^*) = -\frac{1}{2}\langle x^{**}, P^*x^{**}\rangle + \iota_{\{-P^*x^{**}\}}(x^*)$, $\forall x^* \in X^*$. Fact 1.1 yields

$$0 = -\inf_{x^* \in X^*}\{q^*(x^*) + g^*(-x^*)\} = \frac{1}{2}\langle x^{**}, P^*x^{**}\rangle - q^*(P^*x^{**}),$$

which is the first equality. To prove the second equality, we first note that the first equality implies

$$q^{**}(x^{**}) \geq \langle x^{**}, P^*x^{**}\rangle - q^*(P^*x^{**}) = \frac{1}{2}\langle x^{**}, P^*x^{**}\rangle.$$

On the other hand, by (iii), there is a bounded net (x_α) in X such that $x_\alpha \overset{\text{w*}}{\to} x^{**}$ and $Px_\alpha \to P^*x^{**}$. Then for every $x^* \in X^*$, we estimate

$$q^*(x^*) \geq \lim_\alpha \langle x^*, x_\alpha\rangle - \frac{1}{2}\langle x_\alpha, Px_\alpha\rangle = \langle x^{**}, x^*\rangle - \frac{1}{2}\langle x^{**}, P^*x^{**}\rangle.$$

This in turn implies $\frac{1}{2}\langle x^{**}, P^*x^{**}\rangle \geq \sup_{x^* \in X^*}\langle x^{**}, x^*\rangle - q^*(x^*) = q^{**}(x^{**})$, which yields the second equality. Applying (i) to P^{**}, which is monotone and symmetric, finally yields $\nabla q^{**} = P^{**}$. □

Skew operators.

Theorem 3.7. *Suppose S is a continuous linear skew operator from X to X^*. If $(x^{**}, x^*) \in X^{**} \times X^*$, then:*
  (i) *$x^* \in S_1 x^{**} \Leftrightarrow x^* = S^{**}x^{**} = -S^*x^{**}$.*
  (ii) *$x^* \in S_0 x^{**} \Leftrightarrow x^* = -S^*x^{**}$ and $\langle S^*x^{**}, x^{**}\rangle = 0$.*
  (iii) *$x^* \in \overline{S}x^{**} \Leftrightarrow x^* = -S^*x^{**}$ and $\langle S^*x^{**}, x^{**}\rangle \leq 0$.*
Hence: S^* is skew if and only if S is weakly compact.
   If S^* is monotone and $x^{**} \in X^{**}$ with $\langle S^*x^{**}, x^{**}\rangle = 0$, then $S^{**}x^{**} = -S^*x^{**}$.

Proof. First "If" part: Fix an arbitrary $y \in X$. Then, using the skewness of S, $\langle x^{**} - y, x^* - Sy\rangle = \langle x^{**}, x^*\rangle - \langle y, S^*x^{**} + x^*\rangle$. Hence

$$\inf_{(y,y^*) \in \text{gra } S}\langle x^{**} - y, x^* - y^*\rangle = \begin{cases} \langle x^{**}, x^*\rangle, & \text{if } x^* = -S^*x^{**}; \\ -\infty, & \text{otherwise.} \end{cases}$$

(ii) and (iii) follow readily. For (i), observe that: $x^* \in S_1 x^{**} \Leftrightarrow (x^{**}, x^*) \in \text{gra } S_1 \cap \text{gra } S_0$ (Proposition 2.2) $\Leftrightarrow x^* = S^{**}x^{**} = -S^*x^{**} \in X^*$ ((ii) and Fact 3.1) $\Leftrightarrow x^* = S^{**}x^{**} = -S^*x^{**}$ (ran $S^* \subseteq X^*$).
   "Hence" part: S^* skew $\Leftrightarrow S^{**} = -S^* \Leftrightarrow S_1 = S^{**}$ (use (i)) $\Leftrightarrow S$ weakly compact (Corollary 3.2).
   Second "If" part: Fix an arbitrary $y^{**} \in X^{**}$ and $\lambda > 0$. Thus:

$$0 = \langle S^*x^{**}, x^{**}\rangle$$
$$= \langle S^*(x^{**} + \lambda y^{**}), x^{**} + \lambda y^{**}\rangle - \lambda\langle S^*(x^{**} + \lambda y^{**}), y^{**}\rangle - \lambda\langle S^*y^{**}, x^{**}\rangle$$
$$\geq -\lambda\langle S^*x^{**}, y^{**}\rangle - \lambda\langle S^*y^{**}, x^{**}\rangle - \lambda^2\langle S^*y^{**}, y^{**}\rangle.$$

Now divide by λ and then let λ tend to 0 to conclude $\langle S^*x^{**}, y^{**}\rangle \geq -\langle S^*y^{**}, x^{**}\rangle$, $\forall y^{**} \in X^{**}$. The result follows. $\qquad\square$

4. Characterizations.

We are now ready for the main result.

Theorem 4.1. *Suppose T is a continuous linear operator from X to X^* with symmetric part P and skew part S. Then* TFAE:

  (i) *T is monotone and of dense type.*
 (ii) *T is monotone and of range-dense type.*
(iii) *T is monotone and of type (NI).*
 (iv) *T is locally maximal monotone.*
  (v) *T^* is monotone.*
 (vi) *P and S^* are monotone.*
(vii) *P is monotone and S is of dense type.*
(viii) *P is monotone and S is of range-dense type.*
 (ix) *P is monotone and S is of type (NI).*
  (x) *P is monotone and S is locally maximal monotone.*

Proof. Throughout, let $q(x) := \frac{1}{2}\langle Tx, x\rangle = \frac{1}{2}\langle Px, x\rangle$, $\forall x \in X$.

"(i)$\Rightarrow$(ii)$\Rightarrow$(iii)": Fact 2.6.

"(iii)$\Rightarrow$(v)": Proposition 3.4.

"(v)$\Rightarrow$(vi)": T and P are monotone, because T^* is. Fix an arbitrary $x^{**} \in X^{**}$. By Theorem 3.6.(iii), obtain a bounded net (x_α) in X such that $x_\alpha \overset{\text{w*}}{\to} x^{**}$ and $Px_\alpha \to P^*x^{**}$. Now

$$0 \leq \langle T^*(x^{**} - x_\alpha), x^{**} - x_\alpha\rangle = \langle T^*x^{**}, x^{**} - x_\alpha\rangle - \langle T^*x_\alpha, x^{**} - x_\alpha\rangle$$
$$= \langle T^*x^{**}, x^{**} - x_\alpha\rangle - \langle Px_\alpha - Sx_\alpha, x^{**} - x_\alpha\rangle$$
$$= \langle T^*x^{**} - Px_\alpha, x^{**} - x_\alpha\rangle + \langle x_\alpha, S^*x^{**}\rangle \to \langle x^{**}, S^*x^{**}\rangle;$$

consequently, S^* is monotone and (vi) holds.

"(vi)$\Rightarrow$(i)": We start by noting that if (x^{**}, x^*) belongs to $X^{**} \times X^*$, then

$$\frac{1}{2}\inf_{x \in X}\langle Tx - x^*, x - x^{**}\rangle = \frac{1}{2}\langle x^{**}, x^*\rangle - q^*\left(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**}\right).$$

Now fix an arbitrary $(x^{**}, x^*) \in \operatorname{gra}\overline{T}$. Then, on the one hand, $q^*(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**}) \leq \frac{1}{2}\langle x^{**}, x^*\rangle$. On the other hand, Theorem 3.6.(iii) yields a bounded net (x_α) in X such that $x_\alpha \overset{\text{w*}}{\to} x^{**}$ and $Px_\alpha \to P^*x^{**}$. Using the monotonicity of S^*, we conclude altogether

$$\frac{1}{2}\langle x^{**}, x^*\rangle \geq q^*\left(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**}\right) \geq \lim_\alpha\left\langle \frac{1}{2}x^* + \frac{1}{2}T^*x^{**}, x_\alpha\right\rangle - \frac{1}{2}\langle x_\alpha, Px_\alpha\rangle$$
$$= \frac{1}{2}\langle x^*, x^{**}\rangle + \frac{1}{2}\langle S^*x^{**}, x^{**}\rangle \geq \frac{1}{2}\langle x^{**}, x^*\rangle.$$

Hence $\langle S^* x^{**}, x^{**} \rangle = 0$ and $q^*(\frac{1}{2}x^* + \frac{1}{2}T^* x^{**}) = \frac{1}{2}\langle x^{**}, x^* \rangle$. This has two important consequences: firstly, by Theorem 3.7,

$$S^{**} x^{**} = -S^* x^{**}.$$

Secondly, using Theorem 3.6.(iv), $\langle \frac{1}{2}x^* + \frac{1}{2}T^* x^{**}, x^{**} \rangle = \frac{1}{2}\langle x^{**}, x^* \rangle + \frac{1}{2}\langle x^{**}, P^* x^{**} \rangle = q^*(\frac{1}{2}x^* + \frac{1}{2}T^* x^{**}) + q^{**}(x^{**})$; thus: $x^{**} \in \partial q^*(\frac{1}{2}x^* + \frac{1}{2}T^* x^{**}) \Rightarrow \frac{1}{2}x^* + \frac{1}{2}T^* x^{**} \in \partial q^{**}(x^{**}) = \{P^* x^{**}\} \Leftrightarrow$

$$x^* = P^* x^{**} - S^* x^{**}.$$

Altogether, $x^* = P^* x^{**} - S^* x^{**} = P^{**} x^{**} + S^{**} x^{**} = T^{**} x^{**} \in X^*$, so that (Fact 3.1) $(x^{**}, x^*) \in \operatorname{gra} T_1$, as desired.

"(iv)$\Rightarrow$(v)": T is maximal monotone (use $V = X^*$ in Definition 2.3.(iv)), hence so is P and the function q is convex (Theorem 3.6). Fix an arbitrary $x_0^{**} \in X^{**}$. We aim for $\langle T^* x_0^{**}, x_0^{**} \rangle \geq 0$ and can thus assume WLOG that $x_0^* := T^* x_0^{**} \neq 0$. Select $x_1 \in X$ with $\langle x_0^*, x_1 \rangle < 0$ and let $x_1^* := T x_1$. Let $x_0 := 0$, fix an arbitrary $\epsilon > 0$, and define

$$C_\epsilon := [x_0^*, x_1^*] + \epsilon B_{X^*}.$$

Then C_ϵ is weak* closed, convex, bounded with $x_1^* \in \operatorname{ran} T \cap \operatorname{int} C_\epsilon$. Also, $x_0^* \in (\operatorname{int} C_\epsilon) \setminus T x_0$. Local maximal monotonicity (via Proposition 2.7.(iv)) and Fact 1.1 yield

$$0 > \frac{1}{2} \inf_{x \in X : Tx \in C_\epsilon} \langle Tx - x_0^*, x - x_0 \rangle = \inf_{x \in X} q(x) + \left\langle -\frac{1}{2}x_0^*, x \right\rangle + \iota_{C_\epsilon}(Tx)$$

$$\geq - \inf_{x^{**} \in X^{**}} \left\{ q^* \left(-T^* x^{**} + \frac{1}{2}x_0^* \right) + \iota_{C_\epsilon}^*(x^{**}) \right\}.$$

Now pick $x^{**} := \frac{1}{2}x_0^{**}$; then, using the fact that $q^*(0) = 0$,

$$0 < q^* \left(-\frac{1}{2}T^* x_0^{**} + \frac{1}{2}x_0^* \right) + \iota_{C_\epsilon}^* \left(\frac{1}{2}x_0^{**} \right) = \iota_{C_\epsilon}^* \left(\frac{1}{2}x_0^{**} \right)$$

$$= \epsilon \left\| \frac{1}{2}x_0^{**} \right\| + \max \left\{ \left\langle \frac{1}{2}x_0^{**}, x_0^* \right\rangle, \left\langle \frac{1}{2}x_0^{**}, x_1^* \right\rangle \right\}.$$

Multiply by 2 and let ϵ tend to 0 to obtain $0 \leq \max\{\langle x_0^{**}, T^* x_0^{**} \rangle, \langle x_0^{**}, T x_1 \rangle\}$. Since $\langle x_0^{**}, T x_1 \rangle = \langle T^* x_0^{**}, x_1 \rangle = \langle x_0^*, x_1 \rangle < 0$, we obtain $\langle x_0^{**}, T^* x_0^{**} \rangle \geq 0$.

"(vi)$\Rightarrow$(iv)": In view of Proposition 2.7.(iv), let us fix a weak* closed convex bounded subset C of X^* with $\operatorname{ran} T \cap \operatorname{int} C \neq \emptyset$, $x_0 \in X$, $x_0^* \in (\operatorname{int} C) \setminus T x_0$. Let

$$p := \inf_{x \in X : Tx \in C} \frac{1}{2}\langle Tx - x_0^*, x - x_0 \rangle.$$

Clearly, $p < +\infty$ and our aim is $p < 0$. We thus can assume WLOG that $p > -\infty$, hence p is finite. Let $f(x) := q(x) + \frac{1}{2}\langle -x_0^* - T^* x_0, x \rangle + \frac{1}{2}\langle x_0^*, x_0 \rangle$,

$\forall x \in X$, and let $g := \iota_C$. Then, using Fact 1.1,

$$p = \inf_{x \in X} f(x) + g(Tx) = - \inf_{x^{**} \in X^{**}} \{f^*(-T^*x^{**}) + g^*(x^{**})\}$$

$$= \frac{1}{2}\langle x_0^*, x_0 \rangle - \inf_{x^{**} \in X^{**}} \left\{ q^* \left(-T^*x^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \iota_C^*(x^{**}) \right\}.$$

Moreover: The last infimum is *attained* (by Fact 1.1 and $\operatorname{ran} T \cap \operatorname{int} C \neq \emptyset$), say at some $x_0^{**} \in X^{**}$. Thus the proof of "(vi)$\Rightarrow$(iv)" would be complete after reaching the following

(Aim) $\qquad \frac{1}{2}\langle x_0^*, x_0 \rangle < q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \iota_C^*(x_0^{**}).$

By assumption, $0 \leq \langle S^*(x_0 - 2x_0^{**}), x_0 - 2x_0^{**} \rangle$, which is equivalent to

$$\left\langle -T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0, x_0 - 2x_0^{**} \right\rangle - \frac{1}{2}\langle x_0 - 2x_0^{**}, P^*(x_0 - 2x_0^{**}) \rangle$$

$$\geq \frac{1}{2}\langle x_0^*, x_0 \rangle - \langle x_0^{**}, x_0^* \rangle.$$

On the other hand, Theorem 3.6.(iii) gives a bounded net (x_α) in X such that $x_\alpha \xrightarrow{w*} x_0 - 2x_0^{**}$ and $Px_\alpha \to P^*(x_0 - 2x_0^{**})$; thus altogether

$$q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right)$$

$$\geq \lim_\alpha \left\langle -T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0, x_\alpha \right\rangle - \frac{1}{2}\langle x_\alpha, Px_\alpha \rangle$$

$$= \left\langle -T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0, x_0 - 2x_0^{**} \right\rangle$$

$$\quad - \frac{1}{2}\langle x_0 - 2x_0^{**}, P^*(x_0 - 2x_0^{**}) \rangle$$

$$\geq \frac{1}{2}\langle x_0^*, x_0 \rangle - \langle x_0^{**}, x_0^* \rangle.$$

Consequently, since x_0^* is in the *interior* of C,

$$\frac{1}{2}\langle x_0^*, x_0 \rangle \leq q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \langle x_0^{**}, x_0^* \rangle$$

$$< q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \iota_C^*(x_0^{**}),$$

which is what we aimed for.

We just proved that (i)-(vi) are equivalent for an *arbitrary* continuous linear operator $\tilde{T}$ from X to X^*. If we apply this to $\tilde{T} = S$, then the remaining items are readily seen to be equivalent as well. $\qquad\square$

Remark 4.2. Gossez [**13**, End of Section 2] found the following question interesting:

> Suppose that T is a closed densely defined linear monotone operator from X to X^* and that T^* is monotone. Is T_1 maximal monotone?

He then proved that the answer is "yes" if T is continuous and skew. We are now able to give an affirmative answer to this question *provided that T is merely continuous*: Indeed, this follows from Theorem 4.1."(v)$\Rightarrow$(i)" and Proposition 2.7.(i).

Theorem 4.3. *Suppose T is a continuous linear operator from X to X^* with symmetric part P and skew part S. Then* TFAE:

- (i) T and $T^*|_X$ are monotone and of *dense* type.
- (ii) T and $T^*|_X$ are monotone and of type (NI).
- (iii) T and $T^*|_X$ are *locally* maximal monotone.
- (iv) T^* and $(T^*|_X)^*$ are monotone.
- (v) T is monotone and weakly compact.
- (vi) P is monotone and S is weakly compact.
- (vii) P is monotone and S^* is skew.
- (viii) P is monotone and $S, -S$ are of *dense* type.
- (ix) P is monotone and $S, -S$ are of type (NI).
- (x) P is monotone and $S, -S$ are *locally* maximal monotone.

Proof. Applying Theorem 4.1 to $T = P + S$ and $T^*|_X = P - S$ yields the equivalence of (i), (ii), (iii), (iv), (vii), (viii), (ix), and (x). Now (v) $\Leftrightarrow$ (vi) (by weak compactness of P; see Theorem 3.6.(ii)) $\Leftrightarrow$ (vii) (by Theorem 3.7.(v)); so (i)-(x) are all equivalent. $\qquad\square$

In hindsight, we can interpret monotonicity of the conjugate of the skew part of a given continuous linear monotone operator as "one half of weak compactness".

5. Examples and concluding remarks.

Suppose we are given a continuous linear monotone operator T from X to X^* with skew part S. In view of Theorem 4.1 and Theorem 4.3, the following three mutually exclusive alternatives are conceivable:

- T is "good": Both S^* and $-S^*$ are monotone.
- T is "so-so": Either S^* or $-S^*$ is monotone but not both.
- T is "bad": Neither S^* nor $-S^*$ is monotone.

A priori, it is not clear that "so-so" or "bad" operators exist. However, this is indeed the case and we will now systematically recover two classical examples.

Theorem 5.1. *Suppose T is a continuous linear operator from X to X^* with skew part S and there exists some $e \in X^*$ such that*

$$e \notin \operatorname{cl} \operatorname{ran} T \quad \textit{and} \quad \langle Tx, x \rangle = \langle e, x \rangle^2, \quad \forall x \in X.$$

Then T is monotone but S^* is not.

If $\operatorname{ran} T^* = \operatorname{ran} T^*|_X$, equivalently, if $(\operatorname{ran} T)^* \subseteq X$, then $-S^*$ is monotone.

Proof. T is obviously monotone.

Let $Px := \langle e, x \rangle e$, $\forall x \in X$; then $\langle P^* x^{**}, x \rangle = \langle x^{**}, Px \rangle = \langle x^{**}, e \rangle \langle e, x \rangle$ and hence $P^* x^{**} = \langle x^{**}, e \rangle e$, $\forall x^{**} \in X^{**}$. So P is symmetric. Consider now $S := T - P$. Then $\langle Sx, x \rangle = \langle Tx, x \rangle - \langle Px, x \rangle = \langle Tx, x \rangle - \langle e, x \rangle^2 = 0$, $\forall x \in X$, thus S is skew. Since $T = P + S$, the symmetric (resp. skew) part of T is P (resp. S) by Proposition 3.5. Because $e \notin \operatorname{cl} \operatorname{ran} T = {}^\perp(\ker T^*)$, there exists some $x_0^{**} \in \ker T^*$ with $\langle x_0^{**}, e \rangle \neq 0$. Hence

$$\langle S^* x_0^{**}, x_0^{**} \rangle = \langle T^* x_0^{**}, x_0^{**} \rangle - \langle P^* x_0^{**}, x_0^{**} \rangle = 0 - \langle x_0^{**}, e \rangle^2 < 0;$$

so S^* is not monotone.

"If" part: First note that since $\operatorname{ran} T \subseteq X^*$, the Hahn/Banach Theorem allows us to identify $(\operatorname{ran} T)^*$ with $\{x^{**}|_{\operatorname{ran} T} : x^{**} \in X^{**}\}$. We thus derive the "equivalently" part as follows.

$$
\begin{aligned}
\operatorname{ran} T^* = \operatorname{ran}(T^*|_X) &\Leftrightarrow \forall x^{**} \in X^{**} \quad \exists \hat{x} \in X \ : \ T^* x^{**} = T^* \hat{x} \\
&\Leftrightarrow \forall x^{**} \in X^{**} \quad \exists \hat{x} \in X \quad \forall x \in X \ : \ \langle x^{**} - \hat{x}, Tx \rangle = 0 \\
&\Leftrightarrow \forall x^{**} \in X^{**} \quad \exists \hat{x} \in X \ : \ x^{**}|_{\operatorname{ran} T} = \hat{x}|_{\operatorname{ran} T} \\
&\Leftrightarrow \forall z^* \in (\operatorname{ran} T)^* \quad \exists \hat{x} \in X \subseteq X^{**} \ : \ z^* = \hat{x}|_{\operatorname{ran} T} \\
&\Leftrightarrow (\operatorname{ran} T)^* \subseteq X.
\end{aligned}
$$

Now fix an arbitrary $x^{**} \in X^{**}$. Then there exists $\hat{x} \in X \subseteq X^{**}$ with $T^* x^{**} = T^* \hat{x}$. Thus we have $\langle S^* x^{**}, x \rangle = \langle T^* x^{**}, x \rangle - \langle P^* x^{**}, x \rangle = \langle T^* \hat{x}, x \rangle - \langle P^* x^{**}, x \rangle$, $\forall x \in X$; hence

$$S^* x^{**} = T^* \hat{x} - P^* x^{**} = T^* \hat{x} - \langle x^{**}, e \rangle e.$$

Because $T^*|_X = P - S = 2P - T$, we further obtain

$$
\begin{aligned}
\langle S^* x^{**}, x^{**} \rangle &= \langle T^* \hat{x}, x^{**} \rangle - \langle x^{**}, e \rangle^2 = 2\langle P\hat{x}, x^{**} \rangle - \langle T\hat{x}, x^{**} \rangle - \langle x^{**}, e \rangle^2 \\
&= 2\langle P\hat{x}, x^{**} \rangle - \langle \hat{x}, T^* x^{**} \rangle - \langle x^{**}, e \rangle^2 \\
&= 2\langle P\hat{x}, x^{**} \rangle - \langle \hat{x}, T^* \hat{x} \rangle - \langle x^{**}, e \rangle^2 \\
&= 2\langle e, \hat{x} \rangle \langle x^{**}, e \rangle - \langle e, \hat{x} \rangle^2 - \langle x^{**}, e \rangle^2 = -\langle e, x^{**} - \hat{x} \rangle^2 \leq 0;
\end{aligned}
$$

consequently, $-S^*$ is monotone. $\qquad\square$

Here is the announced example of a "so-so" operator.

Example 5.2 (Gossez). Define the map G from ℓ_1 to ℓ_∞ by

$$(Gx)_n := -\sum_{k<n} x_k + \sum_{k>n} x_k, \qquad \forall x = (x_k) \in \ell_1, n \in \mathbb{N}.$$

Then: G and $-G$ are continuous linear skew operators from ℓ_1 to $\ell_1^* = \ell_\infty$.
  G^* is not monotone whereas $-G^*$ is; consequently:
  G is neither of type (NI) nor locally maximal monotone;
  $-G$ is both of dense type and locally maximal monotone.

Proof. Consider the map T from ℓ_1 to ℓ_∞ given by

$$(Tx)_n := x_n + 2\sum_{k>n} x_k, \qquad \forall x = (x_k) \in \ell_1, n \in \mathbb{N}.$$

Then T is linear, continuous (in fact, $\|T\| = 2$), and $\operatorname{ran} T \subseteq c_0 \subseteq \ell_\infty$. Let $e := (1, 1, 1, \dots) \in \ell_\infty = \ell_1^*$. Then $e \notin \operatorname{cl} \operatorname{ran} T \subseteq \operatorname{cl} c_0 = c_0$ and for every $x \in \ell_1$,

$$\langle Tx, x \rangle = \sum_n x_n \left(x_n + \sum_{k>n} 2x_k \right) = \sum_n x_n^2 + \sum_n \sum_{k>n} 2x_n x_k$$

$$= \sum_n x_n^2 + \sum_{n \neq k} x_n x_k$$

$$= \sum_{n,k} x_n x_k = \left(\sum_n x_n \cdot 1 \right) \left(\sum_k x_k \cdot 1 \right) = \langle e, x \rangle^2.$$

The proof of Theorem 5.1 shows that the symmetric part P of T is given by $Px = \langle e, x \rangle e$, $\forall x \in \ell_1$, and the skew part of T is $S := T - P$. Now for all $x \in \ell_1$, $n \in \mathbb{N}$:

$$(Sx)_n = (Tx)_n - (Px)_n = x_n + 2\sum_{k>n} x_k - \sum_k x_k$$

$$= -\sum_{k<n} x_k + \sum_{k>n} x_k = (Gx)_n;$$

hence $S = G$. By Theorem 5.1, G^* is not monotone. Hence (Theorem 4.1) G is neither of type (NI) nor locally maximal monotone. Because $(\operatorname{ran} T)^* \subseteq c_0^* = \ell_1$, Theorem 5.1 yields that $-G^*$ is monotone. By Theorem 4.1, $-G$ is of dense type and locally maximal monotone. $\qquad \square$

Somewhat surprisingly, the "continuous" version of the (negative) Gossez operator yields a "bad" operator.

Example 5.3 (Fitzpatrick and Phelps). Define the map F from $L_1[0,1]$ to $L_\infty[0,1]$ by

$$(Fx)(t) := \int_0^t x(s)ds - \int_t^1 x(s)ds, \qquad \forall x \in L_1[0,1], t \in [0,1].$$

Then F, $-F$ are continuous linear skew operators from $L_1[0, 1]$ to $L_1^*[0, 1] = L_\infty[0, 1]$.

Neither F^* nor $-F^*$ is monotone; consequently:

F and $-F$ are not of type (NI) nor locally maximal monotone.

Proof. Step 1. Define the map T from $L_1[0, 1]$ to $L_\infty[0, 1]$ by

$$(Tx)(t) := 2 \int_0^t x(s) \, ds, \qquad \forall x \in L_1[0, 1], t \in [0, 1].$$

Then T is linear and continuous (with $\|T\| = 2$). The range of T is contained in the subspace $C_{0,0}$ of $L_\infty[0, 1]$ that consists of all equivalence classes that contain a continuous function vanishing at 0. Let e denote the equivalence class in $L_\infty[0, 1]$ that contains the constant function 1. Then the distance from e to any member in $C_{0,0}$ is at least 1; thus certainly $e \notin \operatorname{cl} \operatorname{ran} T$. Also, for every $x \in L_1[0, 1]$,

$$\begin{aligned}
\langle Tx, x \rangle &= 2 \int_0^1 \left(\int_0^t x(s) \, ds \right) x(t) \, dt = 2 \iint_{s \le t} x(s) x(t) \, ds \, dt \\
&= \iint_{[0,1] \times [0,1]} x(s) x(t) \, ds \, dt \\
&= \left(\int_0^1 x(s) \cdot 1 \, ds \right) \left(\int_0^1 x(t) \cdot 1 \, dt \right) = \langle x, e \rangle^2.
\end{aligned}$$

Then (see again the proof of Theorem 5.1) the positive part P of T is given by $Px := \langle e, x \rangle e$, $\forall x \in L_1[0, 1]$. The skew part S of T is given by

$$\begin{aligned}
(Sx)(t) &:= (Tx)(t) - (Px)(t) = 2 \int_0^t x(s) \, ds - \langle e, x \rangle e(t) \\
&= 2 \int_0^t x(s) \, ds - \int_0^1 x(s) \, ds \\
&= \int_0^t x(s) \, ds - \int_t^1 x(s) \, ds,
\end{aligned}$$

for every $x \in L_1[0, 1]$, $t \in [0, 1]$; consequently, $S = F$. Now Theorem 5.1 and Theorem 4.1 imply that F^* is not monotone and F is neither of type (NI) nor locally maximal monotone.

Step 2.   Here we define the map T by $(Tx)(t) := 2 \int_t^1 x(s) \, ds$, $\forall x \in L_1[0, 1]$, $t \in [0, 1]$. We let e be as in Step 1 and check analogously: T is continuous, linear, $e \notin \operatorname{cl} \operatorname{ran} T$, and $\langle Tx, x \rangle = \langle x, e \rangle^2$, $\forall x \in L_1[0, 1]$. This time, however, the skew part of T is equal to $-F$. We deduce as in Step 1 that $-F^*$ is not monotone and that $-F$ is neither of type (NI) nor locally maximal monotone. □

Remark 5.4. Let G be the operator from Example 5.2. Gossez [13] proved that G is not of dense type whereas Phelps [22, Example 4.5] showed that G

is not locally maximal monotone. Let F be the operator from Example 5.3. Fitzpatrick and Phelps [**9**, Example 3.2] showed that F is not locally maximal monotone.

We observe that our discussion of G and F via Theorem 5.1 is much simpler.

We conclude by reporting on two sets of results that are closely connected to the present paper. We omit the proofs as we think the results are not in their final form; nonetheless, the interested reader is able to find the details in [**3**] or in [**2**].

Remark 5.5 (conjugate monotone spaces). We say that X is a *conjugate monotone space* (*cms*), if the conjugate of every continuous linear monotone operator from X to X^* is monotone as well. Thus X is a conjugate monotone space precisely when it does not admit "so-so" or "bad" operators; equivalently, when every continuous linear monotone operator from X to X^* is weakly compact.

It is clear that reflexive spaces are (cms). Also, one can use Example 5.2 and Example 5.3 to show that none of ℓ_1, $L_1[0,1]$, ℓ_∞^*, $L_\infty^*[0,1]$ is (cms).

A result relying on deeper Banach space theory states that for a Banach *lattice* X, TFAE:

 (i) X is (cms).
 (ii) X is *(w)*, *i.e.*, every continuous linear operator from X to X^* is weakly compact.
 (iii) X does not contain a complemented copy of ℓ_1.

As a consequence, the classical spaces c_0, c, ℓ_∞, $L_\infty[0,1]$, and $C[0,1]$ are all (cms). It would be interesting to know whether or not (i)–(iii) are still equivalent for Banach *spaces*. (Note that (i) always implies (iii).)

Remark 5.6 (some nonlinear results). Recall that the *duality map*, denoted J, is the subdifferential map of the function $\frac{1}{2}\|\cdot\|^2$. Suppose T is a continuous linear monotone operator from X to X^* and $\lambda > 0$. Then the operator $T + \lambda J$ is called a *regularization* of T. Regularizations of monotone operators constitute perhaps the simplest class of *nonlinear* monotone operators. Then one can show that TFAE:

 (i) T is of dense type.
 (ii) $\operatorname{cl} \operatorname{ran}(T + \lambda J) = X^*$, $\forall \lambda > 0$.
 (iii) $T + \lambda J$ is of range-dense type, $\forall \lambda \geq 0$.

If the underlying space X is *rugged* (*i.e.*, $\operatorname{cl} \operatorname{span} \operatorname{ran}(J - J) = X^*$), then (i)-(iii) are also equivalent to:

 (iv) $\operatorname{cl} \operatorname{ran}(T + \lambda J)$ is convex, $\forall \lambda > 0$.
 (v) $T + \lambda J$ is locally maximal monotone, $\forall \lambda > 0$.

The full equivalence of (i)–(v) thus holds for the following rugged spaces: c_0, c, ℓ_1, ℓ_∞, $L_1[0,1]$, $L_\infty[0,1]$, $C[0,1]$.

Conclusion.

Maximal monotonicities of dense type, range-dense type, or type (NI), and local maximal monotoncity all coincide: • For subdifferentials of convex functions (Fact 2.5); • in reflexive spaces; • for continuous linear monotone operators (Theorem 4.1). These monotonicities always hold for subdifferentials of convex functions. They may well be absent for continuous linear monotone operators (Example 5.2 and Example 5.3); however, in reflexive and most of the classical nonreflexive spaces, they are automatic (Remark 5.5).

The question whether or not the monotonicities all coincide for a general maximal monotone operator remains open.

Some preliminary results (Remark 5.6) seem to indicate that this may well be the case.

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