Pacific Journal of Mathematics

EQUIVARIANT EMBEDDINGS OF STEIN DOMAINS SITTING INSIDE OF COMPLEX SEMIGROUPS

Bernhard Krötz

Volume 189 No. 1 May 1999

PACIFIC JOURNAL OF MATHEMATICS
Vol. 189, No. 1, 1999

EQUIVARIANT EMBEDDINGS OF STEIN DOMAINS SITTING INSIDE OF COMPLEX SEMIGROUPS

Bernhard Krötz

In this paper we prove an equivariant version of Hörmanders embedding theorem for Stein manifolds. More concretely, let G be a connected Lie group sitting in its complexification $G_{\mathbb{C}}$ and $D \subseteq G_{\mathbb{C}}$ a $G \times G$-invariant Stein domain. Under slight obstructions on D we construct a Hilbert space $\mathcal{H}$ equipped with a unitary $G \times G$-action and a holomorphic equivariant closed embedding $e\colon D \to \mathcal{H}^* \backslash \{0\}$.

Introduction.

An interesting problem in the field of equivariant complex analysis is: Given a connected Lie group G sitting in its universal complexification $G_{\mathbb{C}}$, how do the $G \times G$-invariant Stein domains in $G_{\mathbb{C}}$ look like. K.-H. Neeb has shown in [**Ne98**] that all domains of the form

$$D = G \exp_{G_{\mathbb{C}}}(iD_h),$$

where $D_h \subseteq \mathfrak{g}$ is a $\mathrm{Ad}(G)$-invariant convex domain consisting of elliptic elements, i.e., all operators $i \operatorname{ad} X$, $X \in D_h$, are diagonalizble over the reals, are Stein manifolds. Moreover there is also strong evidence for that these D exhaust up to multiplication with $N_{G_{\mathbb{C}}}(G)$ all proper bi-invariant Stein domains in $G_{\mathbb{C}}$ (cf. [**GG77**], [**Ne98**]).

By Hörmander's Embedding Theorem one knows that every Stein manifold of dimension n can be embedded biholomorphically as a closed submanifold of $\mathbb{C}^{2n+1}$ (cf. [**Hö73**]). Now the natural question is: Given a biinvariant Stein domain $D = G \exp_{G_{\mathbb{C}}}(iD_h)$ in $G_{\mathbb{C}}$, does there exist a $G\times G$-equivariant embedding into some complex Hilbert space $\mathcal{H}$ endowed with a unitary $G \times G$-action. In this paper we show that under quite natural assumptions the answer is affirmative. More concretely, if $\mathrm{Ad}(G)$ is closed in $\mathrm{Aut}(\mathfrak{g})$, the center $Z(G)$ is compact and the convex domain D_h is pointed, then there exists a positive definite biinvariant holomorphic kernel K on D, such that the map

$$e_K\colon D \to \mathcal{H}_K^* \backslash \{0\}, \quad z \mapsto K_z$$

defines a $G \times G$-equivariant closed embedding. Here $\mathcal{H}_K$ denotes the reproducing kernel Hilbert space and $K_z\colon \mathcal{H}_K \to \mathbb{C}$, $f \mapsto f(z)$ the point evaluations corresponding to K.

Our method to construct such a kernel K is to sum up kernels K^λ associated to unitary highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ of G over a certain lattice $\Gamma \subseteq i\mathfrak{t}^*$, where $\mathfrak{t}$ denotes a compactly embedded Cartan subalgebra of $\mathfrak{g}$. More precisely, we set

$$K = \sum_{\lambda\in\Gamma} \|\lambda\|^N K^\lambda$$

with $\|\cdot\|$ denoting a norm on $i\mathfrak{t}^*$ and $N \in \mathbb{N}$. These kernels K have the important property of tending to infinity at the boundary of D, i.e.,

$$\lim_{z\to\partial D} K(z,z) = \infty;$$

a result which is crucial for veryfying the closedness of the map e_K.

We think that our results are a little bit surprising and we do not really understand what is actually going on. For instance, what is the reason for that one has to exclude zero in $\mathcal{H}_K^*$ to achieve the closedness of the map e_K, or, is it possible to find an equivariant closed holomorphic embedding $D \to E$ into a complex topological vector space endowed with a continuous $G \times G$-action. We hope that our results give rise to a further discussion leading to a better understanding of these phenomena.

I. The boundary behaviour of bi-invariant kernels.

In this first section we characterize the boundary behaviour of biinvariant holomorphic positive definite kernels on a bi-invariant domain $D = G\,\mathrm{Exp}(iD_h)$ by means of the boundary behaviour on the abelian submanifold $D_T\colon = T\,\mathrm{Exp}(i(D_h\cap\mathfrak{t}))$. If the convex invariant set $D_h \subseteq \mathfrak{g}$ is a pointed cone, we show that $\lim_{z\to\partial D} K(z,z) = \infty$ if and only if $\lim_{z\to\partial D_T} K(z,z) = \infty$. As abelian domains are comparable easily to deal with contrary to the highly non-commutative bi-invariant domains D, this result allows us in the sequel to make quite explicit computations.

Definition I.1. Let V be a finite dimensional real vector space and V^* its dual.

(a) For each subset $E \subseteq V$ we define its *dual cone* by $E^\star\colon = \{\alpha \in V^*\colon (\forall x \in E)\ \alpha(x) \geq 0\}$. We note that $E^\star$ is a convex closed subcone of V^*.

(b) For a convex subset $E \subseteq V$ we set

$$H(E)\colon = \{x \in V : x+E = E\}, \qquad \text{and} \qquad \lim E\colon = \{x \in V : x+E \subseteq E\}.$$

We call $H(E)$ the *edge* and $\lim E$ the *limit cone* of E. Note that $H(E)$ is a vector space, $H(E) = H(\overline{E})$ if E is open and that $\lim E$ is a convex cone in V.

(c) A convex set E is called *pointed* if it contains no affine lines. Note that if E is open or closed then E is pointed if and only if its edge is zero. □

Definition I.2. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$.

(a) An element $X \in \mathfrak{g}$ is called *elliptic* if $\operatorname{ad} X$ operates semisimply with purely imaginary spectrum. A convex cone $W \subseteq \mathfrak{g}$ is said to be *elliptic* if $W^0 \neq \emptyset$ and all $X \in W^0$ are elliptic.

(b) For a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ we write $\operatorname{Inn}(\mathfrak{a})\colon = \langle e^{\operatorname{ad} \mathfrak{a}} \rangle \subseteq \operatorname{Aut}(\mathfrak{g})$ for the corresponding group of inner automorphisms. A subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is said to be *compactly embedded* if $\operatorname{Inn}(\mathfrak{a})$ is relatively compact in $\operatorname{Aut}(\mathfrak{g})$.

(c) Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan subalgebra and recall that there exists a unique maximal compactly embedded subalgebra $\mathfrak{k}$ containing $\mathfrak{t}$ (cf. [**HHL89**, A.2.40]).

(d) Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a root decomposition as follows. For a linear functional $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$\mathfrak{g}_{\mathbb{C}}^{\alpha}\colon = \{X \in \mathfrak{g}_{\mathbb{C}}\colon (\forall Y \in \mathfrak{t}_{\mathbb{C}})\ [Y, X] = \alpha(Y)X\}$$

and write $\Delta\colon = \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \backslash \{0\}\colon \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$ for the set of roots. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha}$, $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ for all $\alpha \in \Delta$ and $\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$, where $X \to \overline{X}$ denotes complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$.

(e) A root α is said to be *compact* if $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$ and *non-compact* otherwise. We write $\Delta_k$for the set of compact roots and Δ_n for the non-compact ones. If $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a $\mathfrak{k}$-invariant Levi decomposition, then we set

$$\Delta_r\colon = \{\alpha \in \Delta\colon \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{r}_{\mathbb{C}}\} \quad \text{and} \quad \Delta_s\colon = \{\alpha \in \Delta\colon \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{s}_{\mathbb{C}}\}$$

and recall that $\Delta = \Delta_r \dot{\cup} \Delta_s$ (cf. [**Ne99**, Ch. V]).

(f) A positive system Δ^+ of roots is a subset of Δ for which there exists a regular element $X_0 \in i\mathfrak{t}^*$ with $\Delta^+\colon = \{\alpha \in \Delta\colon \alpha(X_0) > 0\}$. A positive system is said to be $\mathfrak{k}$-*adapted* if the set $\Delta_n^+\colon = \Delta_n \cap \Delta^+$ is invariant under the *Weyl group* $\mathcal{W}_{\mathfrak{k}}\colon = N_{\operatorname{Inn}(\mathfrak{k})}(\mathfrak{t})/Z_{\operatorname{Inn}(\mathfrak{k})}(\mathfrak{t})$ acting on $\mathfrak{t}$. We recall from [**Ne99**, Ch. V] that there exists a $\mathfrak{k}$-adapted positive system if and only if $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$. In this case we call $\mathfrak{g}$ *quasihermitian*. In this case it is easy to see that $\mathfrak{s}$ is quasihermitian too, and so all simple ideals of $\mathfrak{s}$ are either compact or hermitian.

(g) We associate to a positive system Δ^+ the convex cones

$$C_{\min}\colon = \operatorname{cone}\{i[\overline{X_\alpha}, X_\alpha]\colon X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}, \alpha \in \Delta_n^+\},$$

and $C_{\max}\colon = (i\Delta_n^+)^\star = \{X \in \mathfrak{t}\colon (\forall \alpha \in \Delta_n^+)\ i\alpha(X) \geq 0\}$. Note that both $C_{\min}$ and $C_{\max}$ are closed convex cones in $\mathfrak{t}$.

(h) Write $p_{\mathfrak{t}}\colon \mathfrak{g} \to \mathfrak{t}$ for the orthogonal projection along $[\mathfrak{t}, \mathfrak{g}]$ and set $\mathcal{O}_X\colon = \operatorname{Inn}(\mathfrak{g}).X$ for the adjoint orbit through $X \in \mathfrak{g}$. We define the *minimal* and *maximal cone* associated to Δ^+ by

$$W_{\min}\colon = \{X \in \mathfrak{g}\colon p_{\mathfrak{t}}(\mathcal{O}_X) \subseteq C_{\min}\} \qquad \text{and}$$
$$W_{\max}\colon = \{X \in \mathfrak{g}\colon p_{\mathfrak{t}}(\mathcal{O}_X) \subseteq C_{\max}\}$$

and note that both cones are convex closed and $\operatorname{Inn}(\mathfrak{g})$-invariant. □

From now on we assume that $\mathfrak{g}$ contains a compactly embedded Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and that there exists an elliptic cone $W \subseteq \mathfrak{g}$. Then there exists a $\mathfrak{k}$-adapted positive system Δ^+ such that

$$C_{\min} \subseteq W \cap \mathfrak{t} \subseteq C_{\max}$$

holds and $W_{\max}$ is an elliptic cone (cf. [**Ne96b**, Th. II.11]). Moreover, we have $W_{\min} \cap \mathfrak{t} = C_{\min}$ and $W_{\max} \cap \mathfrak{t} = C_{\max}$ (cf. [**Ne97**, Lemma I.1]).

Definition I.3. (a) Let $W \subseteq \mathfrak{g}$ be a closed elliptic cone. Let $\tilde{G}$, resp. $\tilde{G}_{\mathbb{C}}$, be the simply connected Lie groups associated to $\mathfrak{g}$, resp. $\mathfrak{g}_{\mathbb{C}}$, and set $G_1 := \langle \exp \mathfrak{g} \rangle \subseteq \tilde{G}_{\mathbb{C}}$. Then Lawson's Theorem (cf. [**HiNe93**, Th. 7.34, 35]) says that the subset $\Gamma_{G_1}(W) := G_1 \exp(iW)$ is a closed subsemigroup of $G_{\mathbb{C}}$ and the polar map

$$G_1 \times W \to \Gamma_{G_1}(W), \quad (g, X) \mapsto g \exp(iX)$$

is a homeomorphism.

Now the universal covering semigroup $\Gamma_{\tilde{G}}(W) := \tilde{\Gamma}_{G_1}(W)$ has a similar structure. We can lift the exponential function $\exp\colon \mathfrak{g} + iW \to \Gamma_{G_1}(W)$ to an exponential mapping $\operatorname{Exp}\colon \mathfrak{g} + iW \to \Gamma_{\tilde{G}}(W)$ with $\operatorname{Exp}(0) = \mathbf{1}$ and thus obtain a polar map

$$\tilde{G} \times W \to \Gamma_{\tilde{G}}(W), \quad (g, X) \mapsto g \operatorname{Exp}(iX)$$

which is a homeomorphism.

If G is a connected Lie group associated to $\mathfrak{g}$, then $\pi_1(G)$ is a discrete central subgroup of $\Gamma_{\tilde{G}}(W)$ and we obtain a covering homomorphism $\Gamma_{\tilde{G}}(W) \to \Gamma_G(W) := \Gamma_{\tilde{G}}(W)/\pi_1(G)$ (cf. [**HiNe93**, Ch. 3]). It is easy to see that there is also a polar map $G \times W \to \Gamma_G(W), (g, X) \mapsto g \operatorname{Exp}(iX)$ which is a homeomorphism. The semigroups of the type $\Gamma_G(W)$ are called *complex Ol'shanskiĭ semigroups*.

The subset $\Gamma_G(W^0) \subseteq \Gamma_G(W)$ is an open semigroup carrying a complex manifold structure such that semigroup multiplication is holomorphic. Moreover there is an involution on $\Gamma_G(W)$ given by

$$^*\colon \Gamma_G(W) \to \Gamma_G(W), s = g \operatorname{Exp}(iX) \mapsto s^* = \operatorname{Exp}(iX) g^{-1}$$

which is antiholomorphic on $\Gamma_G(W^0)$ (cf. [**HiNe93**, Th. 9.15] for a proof of all that). Thus $\Gamma_G(W)$ is an involutive semigroup.

(b) A *bi-invariant domain* $D \subseteq \Gamma_G(W^0_{\max})$ is an open connected $G \times G$ bi-invariant subset of $\Gamma_G(W^0_{\max})$. Note that

$$D = G \operatorname{Exp}(iD_h) = G \operatorname{Exp}(i\mathcal{D})G,$$

where $D_h \subseteq W^0_{\max}$ and $\mathcal{D} = D_h \cap \mathfrak{t}$. Recall that D is a Stein manifold if and only if D_h is convex (cf. [**Ne98**, Th. 6.1]). In this case D is called a *bi-invariant Stein domain*. We call D *pointed* if D_h is pointed in $\mathfrak{g}$. The *boundary* of a left G-invariant subset $E = G\,\mathrm{Exp}(iE_h) \subseteq \Gamma_G(W_{\max})$ is defined as $\partial E \colon = G\,\mathrm{Exp}(i\partial E_h)$. Note that $\partial D = \overline{D}\backslash D$ for every bi-invariant domain D, where the closure $\overline{D}$ is taken in $\Gamma_G(W_{\max})$. □

Lemma I.4. *Let $W \subseteq \mathfrak{g}$ be an invariant elliptic pointed convex cone and set $C \colon = W \cap \mathfrak{t}$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in W^0 converging to $X \in \partial W$. Then there exists a subsequence $(X_{n_k})_{k\in N}$ and a sequence $(Y_{n_k})_{k\in\mathbb{N}}$ in C^0 with $Y_{n_k} \in \mathrm{Inn}(\mathfrak{g}).X_{n_k}$ and $Y_{n_k} \to Y \in \partial C$.*

Proof. W.l.o.g. we may assume that W is closed. According to [**HiNe93**, Th. 7.27], we can reconstruct W^0 from C^0, i.e., we have $W^0 = \mathrm{Inn}(\mathfrak{g}).C^0$. In particular, we find a sequence $(Y_n)_{n\in\mathbb{N}}$ in C^0 and a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{Inn}(\mathfrak{g})$ such that $g_n.X_n = Y_n$. We claim that $(Y_n)_{n\in\mathbb{N}}$ is bounded.

The Convexity Theorem for Adjoint Orbits (cf. [**KrNe96**, Th. VIII.9]) implies that

$$p_{\mathfrak{t}}(X_n) \in \mathrm{conv}(\mathcal{W}_{\mathfrak{k}}.Y_n) + C_{\min} \subseteq C \tag{1.1}$$

for all $n \in \mathbb{N}$.

As C is pointed, a sequence $(Z_n)_{n\in\mathbb{N}}$ in C is unbounded if and only if $\overline{\lim}_{n\to\infty}\alpha(Z_n) = \infty$ holds for one $\alpha \in \mathrm{int}\, C^\star$. Thus if $(Y_n)_{n\in\mathbb{N}}$ is unbounded, then (1.1) together with the invariance of C under $\mathcal{W}_{\mathfrak{k}}$ implies that $(p_{\mathfrak{t}}(X_n))_{n\in\mathbb{N}}$ is unbounded. But this contradicts the fact that $(p_{\mathfrak{t}}(X_n))_{n\in\mathbb{N}}$ being a continuous image of a Cauchy sequence is bounded, proving the claim.

Let now $(Y_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence of $(Y_n)_{n\in\mathbb{N}}$ and $Y = \lim_{k\to\infty} Y_{n_k}$ the corresponding limit in C. It remains to show that $Y \in \partial C$. To obtain a contradiction we assume that $Y \in C^0$.

We write $\mathrm{Sl}(W)$ for the special automorphism group of the cone W and note that $\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{Sl}(W)$ (cf. [**HiNe93**, Prop. 7.3(v)]). Then [**HiNe93**, Prop. 1.11] implies that there exists a convergent subsequence of $(g_{n_k})_{k\in\mathbb{N}}$ in $\mathrm{Sl}(W)$ which we also denote by $(g_{n_k})_{k\in\mathbb{N}}$. Write g for the corresponding limit. Then

$$X = \lim_{k\to\infty} g_{n_k}^{-1}.Y_{n_k} = g^{-1}.Y.$$

Since $\mathrm{Sl}(W).W^0 = W^0$ and $Y \in W^0$, this implies that $X \in W^0$; a contradiction, concluding the proof of the lemma. □

Definition I.5. Let M be a complex manifold and $\mathrm{Hol}(M)$ denote the space of holomorphic functions on M. We write $\overline{M}$ for M equipped with the opposite complex structure.

(a) A function $K \in \mathrm{Hol}(M \times \overline{M})$ is called a *holomorphic positive definite kernel* if for every sequence $z_1, \ldots, z_n$ in M the matrix $\big(K(z_i, z_j)\big)_{i,j}$ is positive semi-definite. We write $\mathcal{P}(M^2)$ for the convex cone of all holomorphic positive definite kernels on M. Note that every $K \in \mathcal{P}(M^2)$ satsifies the inequality

$$(\forall z, w \in M) \quad |K(z,w)| \leq \sqrt{K(z,z)}\sqrt{K(w,w)}. \tag{1.2}$$

Recall that $K \in \mathcal{P}(M^2)$ if and only if there exists a Hilbert space $\mathcal{H} \subseteq \operatorname{Hol}(M)$ with continuous point evaluations $K_z\colon \mathcal{H} \to \mathbb{C},\ f \mapsto f(z)$ such that $K(z,w) = \langle K_w, K_z\rangle$ holds for all $(z,w) \in M \times M$ (cf. [**Ne99**, Ch. II]). In this case we also write $\mathcal{H}_K$ instead of $\mathcal{H}$ and refer to $\mathcal{H}_K$ as the *reproducing kernel Hilbert space* corresponding to K.

(b) An *involutive semigroup* is a semigroup S together with an involutive antiautomorphism ${}^*\colon S \to S$, i.e., $(s^*)^* = s$ and $(st)^* = t^*s^*$ holds for all $s, t \in S$.

A mapping $\alpha\colon S \to \mathbb{R}^+$ is called an *absolute value* if $\alpha(s^*) = \alpha(s)$ and $\alpha(st) \leq \alpha(s)\alpha(t)$ hold for all $s, t \in S$. We denote by $\mathcal{A}(S)$ the collection of all absolute values on S.

(c) Let S be an involutive semigroup acting on M from the left by holomorphic mappings. A positive definite kernel K is said to be *S-invariant* if $K(s.z, w) = K(z, s^*.w)$ holds for all $s \in S$, $z, w \in M$. We write $\mathcal{P}_S(M^2)$ for the subcone of $\mathcal{P}(M^2)$ of all S-invariant elements.

(d) An S-invariant positve definite kernel $K \in \mathcal{P}_S(M^2)$ is called *α-bounded* for some $\alpha \in \mathcal{A}(S)$ if

$$K(s.z, s.z) \leq \alpha(s)K(z,z)$$

holds for all $z \in M$, $s \in S$. The set of all α-bounded positve definite kernels is denoted by $\mathcal{P}_S(M^2, \alpha)$. Note that each $K \in \mathcal{P}_S(M^2, \alpha)$ gives rise to an *involutive representation* of S given by

$$\pi_K\colon S \to B(\mathcal{H}_K), \quad \big(\pi_K(s).f\big)(z) = f(s^*.z),$$

i.e., $(\pi_K, \mathcal{H}_K)$ is a representation of S satisfying $\pi_K(s^*) = \pi_K(s)^*$ for all $s \in S$ (cf. [**Ne99**, Ch. II]). □

We equip $G \times G$ with the involution $(g_1, g_2)^* = (g_1^{-1}, g_2^{-1})$ for $g_1, g_2 \in G$. Note that every $K \in \mathcal{P}_{G\times G}$ is trivially α-bounded with $\alpha = \mathbf{1}$, and that

$$\pi_K\colon G \times G \to U(\mathcal{H}_K), \quad \big(\pi_K(g_1, g_2).f\big)(z) = f(g_1^{-1} z g_2)$$

is a unitary representation of $G \times G$ (cf. [**Kr97**, Lemma III.6]).

Lemma I.6. *Let V be a finite dimensional real vector space, $V^\sharp \colon = V \oplus \mathbb{R}$ and $E \subseteq V$ a convex subset. Set $E^\sharp \colon = \mathbb{R}^+(E \times \{1\})$. Then $E^\sharp$ is a convex subcone of $V^\sharp$ and the following assertions hold:*

(i) *E is closed in $E^\sharp$.*
(ii) *E is pointed if and only if $E^\sharp$ is pointed.*
(iii) *If E is open or closed, then $\partial E^\sharp =]0, \infty[.(\partial E \times \{1\}) \cup (\lim E \times \{0\})$.*

Proof. (i) This is clear.

(ii) If $E^\sharp$ is not pointed, then there exists a non-zero element $y = (x, r) \in V^\sharp$ such that $\mathbb{R}y \subseteq E^\sharp$. In view of $E^\sharp = \mathbb{R}^+(E \times \{1\})$, we must have $r = 0$. Now $\mathbb{R}y + E^\sharp = E^\sharp$ implies that $\mathbb{R}x + E = E$, i.e., E is not pointed.

Conversely, if E is not pointed, then there exists a non-zero element $x \in V$ such that $\mathbb{R}x \subseteq H(E)$. Then $\mathbb{R}(x, 0) \subseteq H(E^\sharp)$, i.e., C is not pointed.

(iii) Note that $]0,\infty[(E^0 \times \{1\})$ is open and that $]0,\infty[(\overline{E}\times\{1\})$ is closed in $V^\sharp \backslash (V \times \{0\})$. Hence $\partial E^\sharp \cap V^\sharp \backslash (V \times \{0\}) =]0,\infty[(\partial E \times \{1\})$. By the definition of $E^\sharp$ we have $(x, 0) \in \partial E^\sharp$ if and only if there exists a sequence of positive real numbers $(\lambda_n)_{n\in\mathbb{N}}$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\lim_{n\to\infty} \lambda_n x_n = x$. In view of [**Ne99**, Prop. III.1.5(iii)], this means that $x \in \lim E$, concluding the proof of (iii). □

Lemma I.7. *Let V be a finite dimensional real vector space and $E \subseteq V$ a convex set. Further let $V_1 \colon = V/H(E)$, denote $q \colon V \to V_1$ the corresponding quotient homomorphism and set $E_1 \colon = q(E)$. Then we have $q(\partial E) = \partial E_1$.*

Proof. As $E + H(E) = E$ it follows that $E^0 + H(E) = E^0$ and $\overline{E} + H(E) = H(E)$. Thus $q(\overline{E}) = \overline{E}_1$, and $q(E^0) = E_1^0$ since q is an open mapping. This proves the lemma. □

Proposition I.8. *Let $\mathfrak{g}^\sharp = \mathfrak{g} \oplus \mathbb{R}$, $G^\sharp = G \times \mathbb{R}$, and*

$$D^\sharp \colon = \Gamma_{G^\sharp}\left(D_h^\sharp\right) \subseteq \Gamma_{G^\sharp}\left(W_{\max}^0 \oplus \mathbb{R}\right) \cong \Gamma_G(W_{\max}^0) \oplus \mathbb{C}.$$

(i) *The map $j \colon D \to D^\sharp$, $s \mapsto (s, i)$ is a $G \times G$-equivariant holomorphic closed embedding inducing a map*

$$\mathcal{P}_{G^\sharp \times G^\sharp}\left({D^\sharp}^2\right) \to \mathcal{P}_{G\times G}(D^2), \quad K^\sharp \mapsto K := K^\sharp \circ j.$$

(ii) *Let $K \in \mathcal{P}_{G\times G}(D^2)$. Then the following statements are equivalent:*
 (a) $\lim_{z\to\partial D} K(z, z) = \infty$.
 (b) $\lim_{X\to\partial D_h} K(\mathrm{Exp}(iX), \mathrm{Exp}(iX)) = \infty$. *Moreover, if $K = K^\sharp \circ j$ with some $K^\sharp \in \mathcal{P}_{G^\sharp\times G^\sharp}({D^\sharp}^2)$, then (a)-(b) are implied by*
 (c) $\lim_{X\to\partial D_h^\sharp} K^\sharp(\mathrm{Exp}(iX), \mathrm{Exp}(iX)) = \infty$.

(iii) *Let $\mathfrak{a} \colon = H(D_h)$ be the edge of D_h, $\mathfrak{g}_1 \colon = \mathfrak{g}/\mathfrak{a}$ and $q \colon \mathfrak{g} \to \mathfrak{g}_1$ the corresponding quotient morphism. Further let $A \colon = \langle \exp(\mathfrak{a})\rangle$, $G_1 \colon = G/A$ and $D_1 \colon = G_1 \mathrm{Exp}(iD_{h,1})$. Then the quotient morphism $\tilde{q} \colon D \to D_1$ induced by q gives rise to an injection*

$$\tilde{q}^* : \mathcal{P}_{G_1\times G_1}(D_1^2) \to \mathcal{P}_{G\times G}(D^2), \quad K_1 \mapsto K := K_1 \circ \tilde{q}$$

and the following statements are equivalent:
 (a) $\lim_{X\to\partial D_h} K(\mathrm{Exp}(iX), \mathrm{Exp}(iX)) = \infty$.
 (b) $\lim_{X\to\partial D_{h,1}} K_1(\mathrm{Exp}(iX), \mathrm{Exp}(iX)) = \infty$.
 (c) $\lim_{X\to\partial \mathcal{D}_1} K_1(\mathrm{Exp}(iX), \mathrm{Exp}(iX)) = \infty$.

Proof. (i) Obviously, j is holomorphic and $G \times G$-equivariant. Further the closedness of j follows from Lemma I.6(i) together with the Polar Decomposition of a complex Ol'shanskiĭ semigroup. The second assertion is clear.

(ii) (a)⇒(b) is obvious.
(b) ⇒(a): This follows from the biinvariance of K.

The remaining statement follows from Lemma I.6(iii).

(iii) (a) $\iff$ (b): In view of $q(\partial D_h) = \partial D_{h,1}$ (cf. Lemma I.7), this follows from the equivalence of (a) and (b) in (ii).
(b)⇒(c) is clear.
(c)⇒(b) follows from the $\mathrm{Inn}(\mathfrak{g})$-invariance of the map

$$D_{h,1} \to \mathbb{R}^+, \quad X \mapsto K_1(\mathrm{Exp}(iX), \mathrm{Exp}(iX))$$

together with Lemma I.4. □

II. A kernel tending to infinity at the boundary.

In this section we construct a kernel $K \in \mathcal{P}_{G\times G}(D^2)$ tending to infinity at the boundary. In view of Proposition I.8, this reduces to the case where $D = \Gamma_G(W)$ is a pointed complex Ol'shanskiĭ semigroup. But Proposition I.8(iii) tells us even more: We only have to check that $\lim_{X\mapsto \partial C} K(\mathrm{Exp}(iX), \mathrm{Exp}(iX)) = \infty$. To start out we need some notation concerning highest weight representations and their associated characters.

Definition II.1. Let Δ^+ be a positive system.

(a) For a $\mathfrak{g}_\mathbb{C}$-module V and $\beta \in (\mathfrak{t}_\mathbb{C})^*$ we write $V^\beta := \{v \in V : (\forall X \in \mathfrak{t}_\mathbb{C}) X.v = \beta(X)v\}$ for the *weight space of weight* β and $\mathcal{P}_V = \{\beta \colon V^\beta \neq \{0\}\}$ for the set of weights of V.

(b) Let V be a $\mathfrak{g}_\mathbb{C}$-module and $v \in V^\lambda$ a $\mathfrak{t}_\mathbb{C}$-weight vector. We say that v is a *primitive element of* V (with respect to Δ^+) if $\mathfrak{g}_\mathbb{C}^\alpha.v = \{0\}$ holds for all $\alpha \in \Delta^+$.

(c) A $\mathfrak{g}_\mathbb{C}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.

(d) Let $\lambda \in i\mathfrak{t}^*$ be dominant integral w.r.t. Δ_k^+ and $F(\lambda)$ the corresponding highest weight module for $\mathfrak{k}_\mathbb{C}$. Assume that Δ^+ is $\mathfrak{k}$-adapted and set $\mathfrak{p}^\pm = \bigoplus_{\alpha\in\Delta_n^\pm} \mathfrak{g}_\mathbb{C}^\alpha$. We define the *generalized Verma module* by

$$N(\lambda) \colon = \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{k}_\mathbb{C}+\mathfrak{p}^+)} F(\lambda).$$

Note that $N(\lambda)$ is a highest weight module for $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ with highest weight λ. We denote by $L(\lambda)$ the unique irreducible quotient of $N(\lambda)$.

(e) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We write K for the analytic subgroup of G corresponding to $\mathfrak{k}$. Let $(\pi, \mathcal{H})$ be a unitary representation of G. A vector $v \in \mathcal{H}$ is called *K-finite* if it is contained in a finite dimensional K-invariant subspace. We write $\mathcal{H}^{K,\omega}$ for the space of analytic K-finite vectors.

(f) An irreducible unitary representation $(\pi, \mathcal{H})$ of G is called a *highest weight representation* w.r.t. Δ^+ with highest weight $\lambda \in i\mathfrak{t}^*$ if $\mathcal{H}^{K,\omega}$ is a highest weight module for $\mathfrak{g}_{\mathbb{C}}$ w.r.t. Δ^+ and highest weight λ. We say that the irreducible highest weight module $L(\lambda)$ is *unitarizable* if there exists a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $\tilde{G}$ with $\mathcal{H}_\lambda^{K,\omega} \cong L(\lambda)$ as $\mathfrak{g}_{\mathbb{C}}$-modules. We write $HW(G, \Delta^+) \subset i\mathfrak{t}^*$ for the set of highest weights corresponding to unitary highest weight representations of G w.r.t. Δ^+ and write $HW(\Delta^+) := HW(\tilde{G}, \Delta^+)$ for the set of all highest weights w.r.t. Δ^+ which correspond to a unitarizable $L(\lambda)$.

(g) Let $\lambda \in HW(\Delta^+)$. We call λ *singular* if the natural map $N(\lambda) \to L(\lambda)$ has a non-trivial kernel and *non-singular* otherwise. □

For each unitary representation $(\pi, \mathcal{H})$ of G we write $(\pi^*, \mathcal{H}^*)$ for the corresponding dual representation. Let $B_2(\mathcal{H})$ be the space of Hilbert Schmidt operators on $\mathcal{H}$. We define a representation of $G \times G$ on $B_2(\mathcal{H})$ by

$$\pi^c \colon G \times G \to U(B_2(\mathcal{H})), \quad \pi^c(g_1, g_2).A := \pi(g_2)A\pi(g_1)^*.$$

Note that there is a canonical isomorphism between $(\pi^* \otimes \pi, \mathcal{H}^* \widehat{\otimes} \mathcal{H})$ and $(\pi^c, B_2(\mathcal{H}))$.

Now we fix a positive system Δ^+ associated to $W_{\max}$.

Recall from [**HiNe96**, Th. 3.6, Th. B] that each highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G extends to an holomorphic representation of $\Gamma_G(W_{\max})$ denoted by the same symbol. Moreover all operators $\pi_\lambda(s), s \in \Gamma_G(W_{\max}^0)$, are of trace class (cf. [**Ne94**, Th. III.8]), so that $\Theta_\lambda(s) := \operatorname{tr} \pi_\lambda(s)$ makes sense for all $s \in \Gamma_G(W_{\max}^0)$. We call Θ_λ the *character* of $(\pi_\lambda, \mathcal{H}_\lambda)$ and note that Θ_λ is holomorphic on $\Gamma_G(W_{\max}^0)$ (cf. [**Ne94**, Th. IV.11]).

Associated to a $\mathfrak{k}$-adapted positive system, we define the function

$$\phi \colon iC_{\max}^0 \to \mathbb{R}^+, \quad X \mapsto \frac{1}{\prod_{\alpha \in \Delta_n^+} \left(1 - e^{-\alpha(X)}\right)^{m_\alpha}},$$

where $m_\alpha := \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^\alpha$ for all $\alpha \in \Delta$.

Lemma II.2. *Let $\lambda \in HW(G, \Delta^+)$ be non-singular and $(\pi_\lambda, \mathcal{H}_\lambda)$ an associated highest weight representation of G. If Θ_λ^K denotes the character of $F(\lambda)$, then*

$$\Theta_\lambda(\operatorname{Exp} X) = \phi(X)\Theta_\lambda^K(\operatorname{Exp}(X))$$

for all $X \in iC_{\max}^0$.

Proof. [**Kr97**, Lemma IV.8(i)]. □

Proposition II.3. *Let K be a connected Lie group with compact Lie algebra $\mathfrak{k}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$, Δ_k^+ be a positive system of roots and $\lambda \in i\mathfrak{t}^*$ be a dominant analytically integral element. Further let Θ_λ^K denote* *the character of the holomorphic representation* $(\pi_\lambda^K, F(\lambda))$ *of* $K_\mathbb{C}$ *and set* $d_\lambda := \dim F(\lambda)$.

(i) $(\forall X \in i\mathfrak{t})\ e^{\lambda(X)} \leq \Theta_\lambda^K(\exp_{K_\mathbb{C}}(X)) \leq d_\lambda \sup_{w\in\mathcal{W}_k} e^{\lambda(w.X)}$.

(ii) *If* $\|\cdot\|$ *denotes a norm on* $i\mathfrak{t}^*$, *then there exists a constant* $c > 0$ *and an element* $n \in \mathbb{N}$ *such that*

$$d_\lambda \leq c\|\lambda\|^n + c$$

holds for all integral elements λ.

Proof. (i) Let $\{v_j : 1 \leq j \leq d_\lambda\}$ be an orthonormal basis of weight vectors of $F(\lambda)$. For each j let α_j be the weight corresponding to v_j. Then we have for all $X \in \mathfrak{t}_\mathbb{C}$ that

$$\Theta_\lambda^K(\exp_{K_\mathbb{C}}(X)) = \sum_{j=1}^{d_\lambda} \langle \pi_\lambda^K(\exp_{K_\mathbb{C}}(X)).v_j, v_j\rangle = \sum_{j=1}^{d_\lambda} e^{\alpha_j(X)}. \tag{2.1}$$

Since $\lambda = \alpha_j$ for some $1 \leq j \leq d_\lambda$, this proves the first inequality.

To prove the second inequality, we first observe that both $\Theta_\lambda^K(\exp_{K_\mathbb{C}}(X))$ and $d_\lambda \sup_{w\in\mathcal{W}_k} e^{\lambda(w.X)}$ considered as functions of $X \in i\mathfrak{t}^*$ are invariant under the Weyl group $\mathcal{W}_k$. Thus we may assume that $X \in -(\Delta_k^+)^\star := \{Y \in i\mathfrak{t}^* : (\forall \alpha \in \Delta_k^+)\alpha(X) \leq 0\}$. Since $\lambda(X) = \sup_{1\leq j\leq d_\lambda} \alpha_j(X)$ whenever $X \in -(\Delta_k^+)^\star$, (2.1) implies that $\Theta_\lambda^K(\exp_{K_\mathbb{C}}(X)) \leq d_\lambda e^{\lambda(X)}$, concluding the proof of (i).

(ii) This is a direct consequence of the Weyl Dimension Formula. □

Corollary II.4. *If* $\lambda \in HW(G, \Delta^+)$ *is non-singular, then there exist a constant* $c > 0$ *and* $n \in \mathbb{N}$ *such that*

$$e^{\lambda(X)} \leq \Theta_\lambda(\mathrm{Exp}(X)) \leq (c\|\lambda\|^n + c)\phi(X) \sup_{w\in\mathcal{W}_k} e^{\lambda(w.X)}$$

holds for all $X \in iC^0_{\max}$.

Proof. As $1 = \inf_{X\in iC^0_{\max}} \phi(X)$, the corollary follows from Lemma II.2 and Proposition II.3. □

Lemma II.5. *Let* V *be a finite dimensional real vector space,* $C \subseteq V$ *a convex cone with non-empty interior,* $\Gamma \subseteq V$ *a lattice and* $Q \subseteq V$ *a compact subset. Then*

$$\Gamma(C, Q) := \{\gamma \in \Gamma : \gamma \in C, \gamma + Q \subseteq C\}$$

is an additive subsemigroup of Γ, $\overline{\mathbb{R}^+\Gamma(C,Q)}$ *is a closed convex cone and the following equalities hold:*

(i) $\overline{\mathbb{R}^+\Gamma(C,Q)} = \overline{C}$.

(ii) $\Gamma(C,Q)^\star = C^\star$.

Proof. First we show that $\Gamma(C,Q)$ is an additive semigroup. Let $\gamma_1, \gamma_2 \in \Gamma(C,Q)$. Then $\gamma_1 + \gamma_2 \in C$ since C is a convex cone and further we have for all $x \in Q$

$$\gamma_1 + \gamma_2 + x = \gamma_1 + \underbrace{(\gamma_2 + x)}_{\in C} \in C + C = C,$$

proving that $\Gamma(C,Q)$ is an additive semigroup.

It follows in particular that $\mathbb{Q}^+\Gamma(C,Q)$ is an additive semigroup and hence the same holds for $\overline{\mathbb{R}^+\Gamma(C,Q)} = \overline{\mathbb{Q}^+\Gamma(C,Q)}$. This proves that $\overline{\mathbb{R}^+\Gamma(C,Q)}$ is a closed convex cone.

(i) Since $\Gamma(C,Q) \subseteq C$ we obtain in particular that $\overline{\mathbb{R}^+\Gamma(C,Q)} \subseteq \overline{C}$.

To prove the converse inclusion, we assume that $\overline{\mathbb{R}^+\Gamma(C,Q)} \neq \overline{C}$. Then we find an open ball $B \subseteq V$ such that $B \subseteq C^0 \backslash \overline{\mathbb{R}^+\Gamma(C,Q)}$. Since $C^0 \backslash \overline{\mathbb{R}^+\Gamma(C,Q)}$ is an open cone, this implies in particular that

$$(\forall \lambda > 0)\ \lambda B \cap \Gamma(C,Q) = \emptyset. \tag{2.2}$$

If λ is sufficiently large, then we have $\lambda B \subseteq C - x$ for all $x \in Q$, because $B + \frac{1}{\lambda}x$ is contained in C for sufficiently large λ and Q is compact. Let $\lambda_0 > 0$ such that $\lambda B \subseteq C - x$ for all $x \in Q$ and $\lambda > \lambda_0$.

In view of this and $\Gamma(C,Q) = \Gamma \cap C \cap \bigcap_{x \in Q}(C - x)$, (2.2) implies in particular that

$$(\forall \lambda > \lambda_0)\ \Gamma \cap C \cap \lambda B = \Gamma(C,Q) \cap \lambda B = \emptyset;$$

a contradiction, concluding the proof of (i).

(ii) This follows from (i) and $\overline{\mathbb{R}^+\Gamma(C,Q)}^\star = \Gamma(C,Q)^\star$. □

Let $\mathfrak{t}_e := \{X \in \mathfrak{t} : \exp(X) = \mathbf{1}\}$ and note that $\mathfrak{t}_e$ is a lattice in span $\{\mathfrak{t}_e\}$. Hence we find a lattice $\Gamma \in i\mathfrak{t}^*$ which is contained in the set $\{\alpha \in i\mathfrak{t}^* \colon (\forall X \in \mathfrak{t}_e)\ \alpha(X) \in 2\pi i\mathbb{Z}\}$. From now on we fix a lattice $\Gamma \subseteq i\mathfrak{t}^*$ having this property.

Lemma II.6. *Let $W \subseteq \mathfrak{g}$ be a pointed closed $\mathrm{Inn}(\mathfrak{g})$-invariant cone with non-empty interior, Δ^+ a positive system satisfying $C_{\min} \subseteq C = W \cap \mathfrak{t} \subseteq C_{\max}$ and $\rho \colon = \frac{1}{2}\sum_{\alpha \in \Delta^+} m_\alpha \alpha$. Then*

$$\Gamma(i \operatorname{int} C^\star, 2\rho) := \{\lambda \in \Gamma \colon \lambda \in i \operatorname{int} C^\star, \lambda + 2\rho \in i \operatorname{int} C^\star\}$$

consists of non-singular G-analytically integral elements.

Proof. First note that Γ consists of G-analytically integral elements, hence the same holds for the subset $\Gamma(i \operatorname{int} C^\star, 2\rho)$. We claim that $\Gamma(i \operatorname{int} C^\star, 2\rho) \subseteq \Gamma(i \operatorname{int} C^\star, \rho)$. In fact, Lemma II.5 implies that $\Gamma(i \operatorname{int} C^\star, 2\rho)$ is an additive semigroup, so that $\lambda \in \Gamma(i \operatorname{int} C^\star, 2\rho)$ implies that $2\lambda \in \Gamma(i \operatorname{int} C^\star, 2\rho)$ which means that $2\lambda + 2\rho \in i \operatorname{int} C^\star$ or equivalently $\lambda + \rho \in \Gamma(i \operatorname{int} C^\star, \rho)$. This proves the claim.

Further, $C_{\min} \subseteq C$ implies that $i \operatorname{int} C^\star \subseteq i \operatorname{int} C^\star_{\min}$ and we therefore get

$$\Gamma(i \operatorname{int} C^\star, 2\rho) \subseteq \{\lambda \in \Gamma \colon \lambda + \rho \in i \operatorname{int} C^\star_{\min}\}.$$

In view of [**Ne99**, Ch. IX], this implies that all elements of $\Gamma(i\operatorname{int} C^\star, 2\rho)$ are non-singular. ☐

Lemma II.7. *Let V be a finite dimensional real vector space, $C \subseteq V$ a convex pointed cone with non-empty interior, $\Gamma \subseteq V^*$ a lattice and $Q \subseteq V^*$ a compact subset. We fix a norm $\|\cdot\|$ on V^* with $\|\gamma\| \geq 1$ for all $\gamma \in \Gamma\backslash\{0\}$ and consider for each $N \in \mathbb{N}_0$ the mapping*

$$F^N \colon V \to \mathbb{R}^+ \cup \{\infty\}, \quad x \mapsto \sum_{\gamma\in\Gamma(\operatorname{int} C^\star, Q)} \|\gamma\|^N e^{-\gamma(X)}.$$

(i) *For all $N \in \mathbb{N}_0$ the series defining F^N converges compactly on C^0. In particular, $F^N|_{C^0}$ is continuous.*
(ii) *For all $N \geq 1$ we have $\lim_{\substack{x\to\partial C\\ X\in C^0}} F^N(x) = \infty$.*

Proof. (i) Since $C^\star$ is pointed we find for every $x \in C^0$ a constant $C_x > 0$ such that $\|\alpha\| \leq C_x\alpha(x)$ holds for every $\alpha \in C^\star$. Thus we find for every compact subset $K \subseteq C^0$ a constant $C_K > 0$ such that

$$(\forall x \in K)(\forall \alpha \in C^\star)\ \|\alpha\| \leq C_K\alpha(x).$$

This in turn implies that

$$\sup_{x\in K} \sum_{\gamma\in\Gamma(\operatorname{int} C^\star,Q)} \|\gamma\|^N e^{-\gamma(x)} \leq \sum_{\gamma\in\Gamma(\operatorname{int} C^\star,Q)} \|\gamma\|^N e^{-\frac{1}{C_K}\|\gamma\|} < \infty,$$

proving (i).

(ii) As $\|\gamma\| \geq 1$ for all $\gamma \in \Gamma\backslash\{0\}$, we have $F^1 \leq F^N$ for all $N \in \mathbb{N}$ and hence we only have to prove the assertion for $N = 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C^0 converging to $x \in \partial C$. Then there exists an element $0 \neq \alpha \in \partial C^\star$ such that $\alpha(x) = 0$. Choose $\alpha_n \in \mathbb{R}^+\alpha$ with $\alpha_n(x_n) = 1$. We claim that $\lim_{n\to\infty} \|\alpha_n\| = \infty$. Indeed, otherwise we find a number $L > 0$ and a subsequence $(\alpha_{n_k})_{k\in\mathbb{N}}$ such that $\alpha_{n_k} \in [0, L]\alpha$. But then

$$1 = \alpha_{n_k}(x_{n_k}) \leq L\alpha(x_{n_k}) \to 0$$

yields a contradiction and proves the claim.

Next we claim that there exists a constant $c > 0$ and elements $\gamma_n \in \Gamma(\operatorname{int} C^\star, Q)$ such that $\|\gamma_n - \alpha_n\| < c$ for all $n \in \mathbb{N}$. Let $r > 0$ such that $B(\beta, r) \colon = \{\mu \in V^* \colon \|\beta - \mu\| \leq r\}$ intersects Γ for all $\beta \in V^*$. Now choose $\mu \in \operatorname{int} C^\star$ such that $B(\mu + t\alpha, r) + Q \subseteq \operatorname{int} C^\star$ holds for all $t \geq 0$. Then we find elements $\gamma_n \in \Gamma(\operatorname{int} C^\star, Q) \cap B(\mu + \alpha_n, r)$. These elements γ_n satisfy $\|\gamma_n - \alpha_n\| \leq c$ with $c = r + \|\mu\|$, proving our second claim.

Now we get

$$F_1(x_n) = \sum_{\gamma\in\Gamma(\operatorname{int} C^\star, Q)} \|\gamma\| e^{-\gamma(x_n)} \geq \|\gamma_n\| e^{-(\gamma_n(x_n)-\alpha_n(x_n))} e^{-\alpha_n(x_n)}$$
$$\geq \frac{\|\gamma_n\|}{e} e^{-\|\gamma_n-\alpha_n\|\cdot\|x_n\|} \geq \frac{1}{e} e^{-c\|x_n\|}(\|\alpha_n\| - c),$$

and so

$$\lim_{n\to\infty} F_1(x_n) \geq \lim_{n\to\infty} \frac{1}{e} e^{-c\|x_n\|}(\|\alpha_n\| - c) = \infty.$$

This proves (ii). □

Theorem II.8. *Let $S = \Gamma_G(W)$ be a pointed complex Ol'shanskiĭ semigroup. Then for all $N \in \mathbb{N}$ the prescription*

$$K^N : S^0 \times S^0 \to \mathbb{C}, \quad (z, w) \mapsto \sum_{\lambda\in\Gamma(i\operatorname{int} C^\star, 2\rho)} \|\lambda\|^N \Theta_\lambda(zw^*)$$

defines an element of $\mathcal{P}_{G\times G}(S^{0\,2})$ satisfying

$$\lim_{z\mapsto\partial S^0} K^N(z, z) = \infty.$$

Proof. First we show that $K^N \in \mathcal{P}_{G\times G}(S^{0\,2})$ for all $N \in \mathbb{N}$. Since all kernels

$$K^\lambda : S^0 \times S^0 \to \mathbb{C}, \quad (z, w) \mapsto \Theta_\lambda(zw^*)$$

belong to $\mathcal{P}_{G\times G}(S^{0\,2})$ (cf. [**Ne94**, Th. IV.11]), we only have to show that the series defining K^N converges uniformly on compact subsets. In view of (1.2), a series of holomorphic positive definite kernels on a complex manifold converges compactly if and only if it converges uniformly on compact subsets on the diagonal. Therefore the bi-invariance of the kernels K^λ together with the Polar Decomposition of S^0 imply that it suffices to prove the compact convergence of the series defining the function

$$\Theta^N : iW^0 \to \mathbb{R}^+, \quad X \mapsto \sum_{\lambda\in\Gamma(i\operatorname{int} C^\star, 2\rho)} \|\lambda\|^N \Theta_\lambda(\operatorname{Exp}(X)).$$

If $C(W^0)^G$ denotes the $\operatorname{Ad}(G)$-invariant continuous functions on W^0 endowed with the topology of uniform convergence on compact subsets, then [**Ne96b**, Prop. III.6] entails that the restriction mapping $C(W^0)^G \to C(C^0)^{\mathcal{W}_\mathfrak{k}}$ is an isomorphism of Fréchet spaces. Thus we have reduced the problem to showing the compact convergence of the series defining $\Theta^N\mid_{C^0}$. In view of Corollary II.4, we therefore have to prove that the series defined by

$$F^N\mid_{iC^0} : iC^0 \to \mathbb{R}^+, \quad X \mapsto \sum_{\lambda\in\Gamma(i\operatorname{int} C^\star, 2\rho)} \|\lambda\|^N e^{\lambda(X)}$$

converges compactly. But this is exactly the contents of Lemma II.7(i).

In order to prove the second assertion, Proposition I.8(iii) implies that we only have to check that $\lim_{\substack{X \mapsto \partial C \\ X \in C^0}} K^N(\mathrm{Exp}(iX), \mathrm{Exp}(iX)) = \infty$. According to Corollary II.4, this follows from $\lim_{\substack{X \to \partial C \\ X \in C^0}} F^N(X) = \infty$ which is the contents of Lemma II.7(ii). This proves the theorem. □

Corollary II.9. *Let $D \subseteq \Gamma_G(W^0_{\max})$ be a bi-invariant Stein domain. Then there exists a kernel $K \in \mathcal{P}_{G\times G}(D^2)$ such that*

$$\lim_{z \mapsto \partial D} K(z, z) = \infty.$$

Proof. In view of Proposition I.8(ii), we may assume that $D_h = D^\sharp_h$ is a cone, and hence Proposition I.8(iii) implies that we even may assume that $D_h = D^\sharp_{h,1}$ is a pointed cone. Now the corollary follows from Theorem II.8. □

Remark II.10. One can modify Γ a little bit without affecting the contents Theorem II.8 as follows. If one takes $\Gamma' = \Gamma \cup F$, where $F \subseteq HW(G, \Delta^+)$ is a finite set of highest weights, then Theorem II.8 remains true with Γ replaced by Γ'.

III. The equivariant embedding theorem.

In this section we apply the results of Section II to construct a kernel $K \in \mathcal{P}_{G\times G}(D^2)$ such that the map

$$e_K : D \mapsto \mathcal{H}^*_K \backslash \{0\}, \quad z \mapsto K_z$$

defines a $G \times G$-equivariant holomorphic embedding with closed image.

Proposition III.1. *Let M be a complex manifold, S an involutive semigroup acting on M by holomorphic maps and $K \in \mathcal{P}_S(M^2)$. Then*

$$e_K : M \to \mathcal{H}^*_K, \quad z \mapsto K_z$$

is an S-equivariant holomorphic map.

Proof. From the S-invariance of K it follows that

$$e_K(s.z) = K_{s.z} = \pi_K(s).K_z = \pi_K(s).e_K(z)$$

for all $s \in S$ and $z \in M$, proving the S-equivariance of the map e_K. It remains to show that e_K is holomorphic. As $\mathcal{H}_K$ is a Hilbert space, e_K is holomorphic if and only if the following conditions are satisfied:

(1) e_K is locally bounded.
(2) There exists a total subset $T \subseteq \mathcal{H}_K$ such that the mappings $M \to \mathbb{C}$, $z \mapsto K_z(f) = \langle f, K_z\rangle$ are holomorphic for all $f \in T$.

If $Q \subseteq D$ is a compact subset, then we have

$$\sup_{z\in Q} \|K_z\|^2 = \sup_{z\in Q} K(z,z) < \infty,$$

proving (1). Finally $T = \{K_w : w \in M\}$ is a total subset in $\mathcal{H}_K$ and $z \mapsto K_z(K_w) = \langle K_w, K_z\rangle = K(z,w)$ is holomorphic because $K \in \mathrm{Hol}(M \times \overline{M})$. This proves (2) and concludes the proof of the lemma. □

Corollary III.2. *If $K \in \mathcal{P}_{G\times G}(D^2)$ is non-zero, then $K_z \neq 0$ for all $z \in D$ and the mapping*

$$e_K : D \to \mathcal{H}_K^* \backslash \{0\}, \quad z \mapsto K_z$$

is $G \times G$-equivariant and holomorphic.

Proof. It follows from [**Ne97**, Lemma III.6] that $K_z \neq 0$ for all $z \in D$ and thus the map e_K is well defined. Now the assertions follow from Proposition III.1. □

Definition III.3. A connected Lie group is called a (CA)-*Lie group* if $\mathrm{Ad}(G) \subseteq \mathrm{Aut}(\mathfrak{g})$ is closed. Note that all connected reductive and nilpotent Lie groups are (CA)-Lie groups. □

The next lemma is our key observation. The proof depends heavily on the special choice of the lattice $\Gamma(i \operatorname{int} C^\star, 2\rho)$ and is a little bit tricky.

Lemma III.4. *Let $S = \Gamma_G(W)$ be a pointed complex Ol'shanskiĭ semigroup and suppose that G is a* (CA)*-Lie group. Let K^N, $N \in \mathbb{N}$, be as in Theorem II.8 and $(K^N_{s_n})_{n\in\mathbb{N}}$ a convergent sequence in $\mathcal{H}_{K^N}$ with limit different from zero. Further let Z denote the center of G.*

(i) *The set $\{s_n Z : n \in \mathbb{N}\}$ is relatively compact in S^0/Z.*
(ii) *If, in addition, Z is compact, then $\{s_n : n \in \mathbb{N}\}$ is relatively compact in S^0.*

Proof. (i) For simplicity, we write K instead of K^N. Since $K = \sum_\lambda \|\lambda\|^N K^\lambda$ is a direct sum of positive definite kernels corresponding to inequivalent irreducible unitary representations of $G \times G$, it follows in particular that $\mathcal{H}_K = \widehat{\bigoplus}_\lambda \mathcal{H}_{K^\lambda}$ (cf. [**Ne99**, Th. I.11, Rem. I.12(a)]). Thus $(K_{s_n})_{n\in\mathbb{N}}$ being a convergent sequence with non-zero limit implies in particular that all sequences $(K^\lambda_{s_n})_{n\in\mathbb{N}}$ are convergent and at least one limit $f^\lambda = \lim_{n\to\infty} K^\lambda_{s_n}$ is different from zero.

Step 1: The set $\{s_n Z : n \in \mathbb{N}\}$ is relatively compact in S/Z.

Let $\lambda \in \Gamma(i \operatorname{int} C^\star, 2\rho)$ be such that $(K^\lambda_{s_n})_{n\in\mathbb{N}}$ converges with limit different from zero. Let

$$\chi_\lambda : Z \to \mathrm{S}^1, \quad z = \exp(X) \mapsto e^{-\lambda(X)}$$

and note that χ_λ is an element of $\widehat{Z}$.

The *Bergman space* corresponding to the character χ_λ is defined as

$$\mathcal{B}^2(S/Z,\chi_\lambda)=\Big\{f\in \mathrm{Hol}(S^0)\colon (\forall z\in Z, s\in S^0)\ f(sz)=\chi_\lambda(z)^{-1}f(s),$$
$$\|f\|_2^2\colon =\int_{S^0/Z}|f(s)|^2\ d\mu_{S^0/Z}(sZ)<\infty\Big\},$$

where $\mu_{S^0/Z}$ denotes the canonical left S-invariant measure on S^0/Z (cf. [**Kr98**, Sect. II]). Recall from [**Kr98**, Prop. II.4, Th. IV.5] that $\mathcal{B}^2(S/Z,\chi_\lambda)$ is a closed subspace of the Hilbert space $L^2(S/Z,\chi_\lambda)$ and that there exists a positive constant $c>0$ such that the prescription

$$\mathcal{H}_{K^\lambda}\to\mathcal{B}^2(S/Z,\chi_\lambda),\quad K_z^\lambda\mapsto cK_z^\lambda$$

defines an $S\times S$-equivariant isometric embedding.

We obtain in particular that $(K^\lambda_{s_n})_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{B}^2(S/Z,\chi_\lambda)$ with limit $f^\lambda\neq 0$. To obtain a contradiction, we now assume that there exists a subsequence of $(s_nZ)_{n\in\mathbb{N}}$ leaving every compact subset of S/Z. To avoid further notation we denote this subsequence again by $(s_nZ)_{n\in\mathbb{N}}$. Note that $(s_n^*Z)_{n\in\mathbb{N}}$ also leaves every compact subset of S/Z, since the involution on S induces an involution ${}^*\colon S/Z\to S/Z$. We write

$$\rho\colon S\to B(\mathcal{B}^2(S/Z,\chi_\lambda)),\quad (\rho(s).f)(z)=f(zs)$$

for the right regular representation of S on $\mathcal{B}^2(S/Z,\chi_\lambda)$ and note that $(\rho,\mathcal{B}^2(S/Z,\chi_\lambda))$ is a holomorphic contraction representation of S (cf. [**Kr98**, Prop. II.4]). Let $s_0\in S^0$. Then $\rho(s_0).K^\lambda_{s_n}\to\rho(s_0).f^\lambda$ and $\rho(s_0).f^\lambda\neq 0$ since $\rho(s_0)$ is an injective operator. It follows in particular that there exists a convergent subsequence of $(\rho(s_0).K^\lambda_{s_n})_{n\in\mathbb{N}}$ converging to $\rho(s_0).f^\lambda$ pointwise. Note that $|K_s^\lambda|\in C_0(S/Z)\mid_{S^0/Z}$ for all $s\in S^0$ (cf. [**Kr98**, Prop. II.4]). Thus we obtain from

$$\begin{aligned}\rho(s_0).K^\lambda_{s_n}(z)&=K^\lambda_{s_n}(zs_0)=K^\lambda(zs_0,s_n)=K^\lambda(z,s_ns_0^*)\\&=K^\lambda(s_n^*z,s_0^*)=K^\lambda_{s_0^*}(s_n^*z)\end{aligned}$$

for all $z\in S^0$ that $\rho(s_0).K^\lambda_{s_n}\to 0$ pointwise. This is a contradiction to $\rho(s_0).f^\lambda\neq 0$ and proves our first step.

Step 2: Every cluster point of $(s_nZ)_{n\in\mathbb{N}}$ lies in S^0/Z.

As the Polar Decomposition of S is inherited by S/Z, i.e., the mapping

$$G/Z\times W\to S/Z,\quad (gZ,X)\mapsto g\,\mathrm{Exp}(iX)Z$$

is a homeomorphism, we can write $s_n=g_n\,\mathrm{Exp}(X_n)$, where $g_n\in G$ and $X_n\in iW^0$. According to Step 1, we now may assume that both $(g_nZ)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ converge. Let $X\in iW$ be the limit of $(X_n)_{n\in\mathbb{N}}$. Note that it suffices to show that $X\in iW^0$. As $(K_{s_n})_{n\in\mathbb{N}}$ is convergent with non-zero limit, $(\|K_{s_n}\|^2)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}^+$ with positive limit. The bi-invariance of K further implies that

$$\begin{aligned}\|K_{s_n}\|^2 = K(s_n, s_n) &= K(g_n \operatorname{Exp}(X_n), g_n \operatorname{Exp}(X_n)) \\ &= K(\operatorname{Exp}(X_n), \operatorname{Exp}(X_n)),\end{aligned}$$

so that Step 2 follows from Theorem II.8.

(ii) This is a direct consequence of (i) and the compactness of Z. □

Lemma III.5. *If $S = \Gamma_G(W)$ is a pointed complex Ol'shanskiĭ semigroup, then there exists a finite set $F \subseteq HW(G, \Delta^+)$ such that all representations $\pi_{K^N} \colon S \to B(\mathcal{H}_{K^N})$, $N \in \mathbb{N}$, associated to $\Gamma' = \Gamma \cup F$ (cf. Remark II.10) are injective.*

Proof. Since S is pointed, it follows from [**Kr98**, Prop. V.7] that π_{K^N} is injective if and only if $\pi_{K^N}|_T$ is injective, where $T = \exp \mathfrak{t}$. Evaluation of the operators $\pi_{K^N}(t)$, $t \in T$, on highest weight vectors v_λ, $\lambda \in \Gamma(i \operatorname{int} C^\star, 2\rho)$, now easily shows how one can choose F to obtain injective representations. For more details we refer to [**Ne96a**, Sect. V]. □

Theorem III.6. *Let $D \subseteq \Gamma_G(W^0_{\max})$ be a pointed bi-invariant Stein domain and suppose that G is a (CA)-Lie group and Z is compact. Then there exists a kernel $K \in \mathcal{P}_{G\times G}(D^2)$ such that the mapping*

$$e_K \colon D \to \mathcal{H}_K^* \backslash \{0\}, \quad z \mapsto K_z$$

defines a $G \times G$-equivariant holomorphic embedding with closed range.

Proof. In view of Proposition I.8(i) and Lemma I.6(ii), we may assume that $D = S^0 = \Gamma_G(W^0)$ is a pointed open complex Ol'shanskiĭ semigroup. Now let Γ' as in Lemma III.5 and $K = K^N$ for some $N \in \mathbb{N}$. As $K \neq 0$, the map e_K is a well defined $G\times G$-equivariant holomorphic map (cf. Corollary III.2). Lemma III.5 implies that the representation $\pi_K \colon S \to B(\mathcal{H}_K)$ is injective so that $\mathcal{H}_K$ separates points by [**Kr98**, Prop. V.10], which in turn means that e_K is injective.

Next we show that $\operatorname{im} e_K$ is closed. In fact, if $K_{s_n} \to f \neq 0$, then Lemma III.4 implies that $(s_n)_{n\in N}$ is a bounded sequence in S^0 with all accumulation points in S^0. Thus we find a convergent subsequence $(s_{n_k})_{k\in\mathbb{N}}$ with limit $s \in S^0$. Now we get

$$f = \lim_{n\to\infty} K_{s_n} = \lim_{k\to\infty} K_{s_{n_k}} = \lim_{k\to\infty} e_K(s_{n_k}) = e_K(s) = K_s,$$

proving the closedness of $\operatorname{im} e_K$.

Finally another easy application of Lemma III.4 shows that e_K is homeomorphic onto its image, concluding the proof of the theorem. □

Example III.7 (The Bergman kernel associated to $\mathrm{Sl}(2,\mathbb{R})$). Let $G = \mathrm{Sl}(2,\mathbb{R})$ and $\mathfrak{g}\colon = \mathfrak{sl}(2,\mathbb{R})$. We choose

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a basis for $\mathfrak{g}$.

Then $\mathfrak{t} := \mathbb{R}U$ is a compactly embedded Cartan subalgebra. Let $\alpha \in i\mathfrak{t}^*$ be defined by $\alpha(U) = -2i$. The root system of $\mathfrak{g}$ is given by $\Delta = \{\pm\alpha\}$ with root spaces $\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathbb{C}(T + iH)$ and $\mathfrak{g}_{\mathbb{C}}^{-\alpha} = \mathbb{C}(T - iH)$. We define a positive system by $\Delta^+ := \{\alpha\}$ and write κ for the Cartan-Killing form on $\mathfrak{g}$. Then the upper light cone

$$\begin{aligned} W &:= \{X = uU + tT + hH : u \geq 0,\ \kappa(X,X) \leq 0\} \\ &= \{X = uU + tT + hH : u \geq 0,\ h^2 + t^2 - u^2 \leq 0\} \end{aligned}$$

is an invariant pointed cone in $\mathfrak{g}$. Moreover, W is up to sign the unique invariant elliptic cone in $\mathfrak{g}$ (cf. [**HiNe93**, Th. 7.25]). Thus up to isomorphism $S := \Gamma_G(W)$ is the unique complex Ol'shanskiĭ semigroup corresponding to G.

In the following we identify $\mathfrak{t}_{\mathbb{C}}$ with $\mathbb{C}$ via the isomorphism $\mathfrak{t}_{\mathbb{C}} \to \mathbb{C},\ \lambda \mapsto \lambda(iU)$. Then $HW(G, \Delta^+) = HW(G, W) = \{\lambda \in \mathbb{Z}\colon \lambda \leq 0\}$, $\lambda + 2\rho \in i \operatorname{int} C^\star$ if and only if $\lambda \leq -3$, and thus $\Gamma(i \operatorname{int} C^\star, 2\rho) = \{\lambda \in \mathbb{Z}\colon \lambda \leq -3\}$. The main point is that $\Gamma(i \operatorname{int} C^\star, 2\rho)$ coincides with the weights in the decomposition of the Bergman kernel B of the *Bergman space*

$$\mathcal{B}^2(S)\colon = \left\{ f \in \mathrm{Hol}(S^0)\colon \|f\|_2^2\colon = \int_{S^0} |f(s)|^2 \, d\mu_{G_{\mathbb{C}}}(s) < \infty \right\}$$

(cf. [**Kr98**, Th. IV.7]). Further one knows that

$$B = \sum_{\lambda \leq -3} \lambda(1+\lambda)^2(4-\lambda^2)K^\lambda$$

(cf. [**Kr98**, Th. IV.7, Ex. IV.8]) so that Theorem II.8 and Theorem III.6 imply that

$$\lim_{z \to \partial S} B(z,z) = \infty$$

and that the map

$$e_B\colon S^0 \to \mathcal{B}^2(S)^* \backslash \{0\}, \quad s \mapsto B_s$$

is a $G \times G$-equivariant holomorphic embedding with closed range. □

Problems III.8. (a) What is the reason for that one has to exclude zero in $\mathcal{H}_K^*$ to obtain the closedness of the map e_K?

(b) Given a pointed biinvariant Stein domain D, does there exist an equivariant closed embedding of D into a complex topological vector space E endowed with a continuous $G \times G$-action?

References

[GG77] M. Gel'fand and S.G. Gindikin, *Complex manifolds whose skeletons are real Lie groups, and analytic discrete series of representations*, Funct. Anal. and Appl., **11** (1977), 19-27.

[HHL89] J. Hilgert, K.H. Hofmann and J.D. Lawson, *Lie Groups, Convex Cones, and Semigroups*, Oxford University Press, 1989.

[HiNe93] J. Hilgert and K.-H. Neeb, *Lie semigroups and their Applications*, Lecture Notes in Math., **1552**, Springer, 1993.

[HiNe96] ———, *Positive definite spherical functions on Ol'shanskiĭ domains*, Mathematik-Bericht, **96/7**, Technische Universität Clausthal.

[Hö73] L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland, 1973.

[Kr97] B. Krötz, *The Plancherel theorem for bi-invariant Hilbert spaces*, submitted.

[Kr98] ———, *On Hardy and Bergman spaces on complex Ol'shanskiĭ semigroups*, Math. Ann., **312** (1998), 13-52.

[KrNe96] B. Krötz and K.-H. Neeb, *On hyperbolic cones and mixed symmetric spaces*, Journal of Lie Theory, **6**(**1**) (1996), 69-146.

[Ne94] K.-H. Neeb, *Holomorphic representation theory* II, Acta Math., **173**(**1**) (1994), 103-133.

[Ne96a] ———, *Coherent states, holomorphic extensions, and highest weight representations*, Pac. J. Math., **174**(**2**) (1996), 497-542.

[Ne96b] ———, *Invariant convex sets and functions in Lie algebras*, Semigroup Forum, **53** (1996), 230-261.

[Ne97] ———, *Representation Theory and Convexity*, submitted.

[Ne98] ———, *On the complex and convex geometry of Ol'shanskiĭ semigroups*, Ann. Inst. Fourier, **48**(**1**) (1998), 149-203.

[Ne99] ———, *Holomorphy and Convexity in Lie Theory*, Expositions in Mathematics, de Gruyter, in preparation.

Received September 22, 1997 and revised November 20, 1997.

TU Clausthal
Erzstrasse 1
D-38678 Clausthal-Zellerfeld
Germany
E-mail address: mabk@math.tu-clausthal.de