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Gerhard Larcher and Gottlieb Pirsic

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# BASE CHANGE PROBLEMS FOR GENERALIZED WALSH SERIES AND MULTIVARIATE NUMERICAL INTEGRATION

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**We recall the notion of Walsh functions over a finite abelian group as it was given for example in Larcher, Niederreiter and Schmid, 1996. These function systems play an important role for various "digital lattice rules" in multivariate numerical integration. We consider the following problem:**

**Assume, that a function $f$ can be represented by a Walsh-series over a group $G_1$ with a certain speed of convergence. Take another group $G_2$. What can be said about the speed of convergence of the Walsh-series of $f$ over $G_2$?**

**Answers to this question are essential for certain numerical integration error estimates. We are able to give some results, partly best possible ones.**

**A connection of the above problem to "digital differentiability" of functions and applications to numerical integration are given. Open problems are stated.**

## 1. Introduction.

The classical Walsh function system $\{\mathrm{wal}_n | n = 0, 1, 2, \dots\}$, $\mathrm{wal}_n : [0, 1) \to \mathbb{C}$ (in the Paley enumeration) can be defined in the following way:

For a non-negative integer $n$ and a real $x$ in $[0, 1)$ let

$$n = n_v \cdot 2^{v-1} + \cdots + n_1 \qquad \text{and}$$
$$x = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots$$

(with $x_i \neq 1$ for inifinitely many $i$) be the digit representation in base 2 . Then

$$\mathrm{wal}_n(x) := (-1)^{x_1 \cdot n_1 + \cdots + x_v \cdot n_v}.$$

The set $\{\mathrm{wal}_n | n = 0, 1, 2, \dots\}$ is a complete orthonormal function system in $L^2([0, 1))$. (See for example [**23**].)

There exist various generalizations of this concept in the literature. Chrestenson [**1**] studied Walsh functions in an arbitrary base $b$. ("Representation in base 2" is replaced by "representation in base $b$" and "$-1$" is replaced by "$e^{\frac{2\pi i}{b}}$".) Vilenkin [**26**] introduced the now so called Vilenkin systems, and Onneweer studied his concept of Rademacher and Walsh functions over groups [20].

Motivated by problems in multivariate quasi-Monte Carlo integration and also by investigations on the distribution of certain number-theoretical point sets (see [7], and also [6] and [8]), in [9] Walsh-function systems $W_{G,\varphi}$ over finite abelian groups $G$ of, say, order $b$, and with respect to certain bijections $\varphi$ between the "digits" $\{0, 1, \ldots, b-1\}$ and the group $G$, were introduced and used for applications. See also [21], [27], [10]. This concept contains the classical Walsh systems in any base $b$ (use $G = \mathbb{Z}_b$ and $\varphi$ the "identity"). In some sense our concept, which is presented in Defintion 2 and in Definition 3 in the next section of this paper, is more general than the systems of Onneweer and of Vilenkin. In other aspects, however, their systems are more general than ours. For some details, see the examples in Section 2.

Solutions (or good estimates) concerning the following problem are of great interest for applications:

**Problem.** Let a finite group $G$ of order $b$ and a corresponding bijection $\varphi$ be given. Assume, that the function $f : [0, 1) \to \mathbb{C}$ can be represented by a series of Walsh-functions from $W_{G,\varphi} := \{ {}_{G,\varphi}\mathrm{wal}_n | n = 0, 1, 2, \ldots \}$, say, $f(x) = \sum_{n=0}^{\infty} \widehat{f}(n) \cdot {}_{G,\varphi}\mathrm{wal}_n(x)$. Assume further, that this Walsh series has a certain speed of convergence in the following way:

There are $\alpha > 1$ and $C > 0$, such that $|\hat{f}(n)| \leq C \cdot n^{-\alpha}$ for all $n = 1, 2, \ldots$. We say "$f$ belongs to ${}_{G,\varphi}E^{\alpha}(C)$".

Let now $H$ be another finite abelian group with, say, order $c$ and $\psi$ a corresponding bijection. We now ask: is there a $\beta > 1$ and a $C' > 0$, such that $f \in {}_{H,\psi}E^{\beta}(C')$?

That is: We are looking for the following "base change coefficient":

**Definition 1.** For given finite abelian groups $G, H$ and corresponding bijections $\varphi$ and $\psi$ and for $\alpha > 1$ let the base change coefficient $\beta(G, \varphi, H, \psi, \alpha)$ be defined by

$$\beta(G, \varphi, H, \psi, \alpha) := \sup\{\beta > 1 | \quad \text{for all } C > 0 \text{ there is a } C' > 0 \text{ with } {}_{G,\varphi}E^{\alpha}(C) \subseteq {}_{H,\psi}E^{\beta}(C')\}.$$

(We set $\beta(G, \varphi, H, \psi, \alpha) = 1$ if there is no such $\beta$.)

Until now, an exact solution to this question was given only in [12] for the case $G = \mathbb{Z}_2, H = \mathbb{Z}_{2^h}$, and $\varphi, \psi$ identities. We obtained

$$\beta(\mathbb{Z}_2, \mathrm{id}, \mathbb{Z}_{2^h}, \mathrm{id}, \alpha) = \alpha - \beta_h \quad \text{with}$$

$$\beta_h = \frac{h-1}{2h} + \frac{\sum_{k=0}^{h-2} \log \sin\left(\frac{\pi}{4} + \frac{\pi}{2}\left\{\frac{4^{[h/2]}-1}{3 \cdot 2^{k+1}}\right\}\right)}{h \cdot \log 2}.$$

We have $0 \le \beta_h < 1/2$ for all $h$ and

$$\lim_{h\to\infty} \beta_h = \frac{1}{2} + \frac{\log \sin \frac{5\pi}{12}}{\log 2} = 0.4499\ldots .$$

Partial results for $G = \mathbb{Z}_{2^h}$ and $H = \mathbb{Z}_2$ (the problem is not "commutative"!) were given in [**13**].

It is the aim of this paper to give in some sense "exact" solutions to the above problem for many cases and good estimates for $\beta$ in all other cases.

Further we demonstrate a connection of our problem to questions concerning the "digital differentiability" of a function. This is done in analogy to the connection between the differentiability of functions in the usual sense and the speed of convergence of their Fourier-series. Finally we give applications of our results to error estimation in quasi-Monte Carlo integration.

In Section 2 we shall recall the concept of Walsh functions over groups and give their basic properties.

In Section 3 we shall recall the notion of the "digital derivative" and give the connection to our problem.

In Sections 4, 5 and 6 we give solutions to our problem, respectively estimates concerning the quantity $\beta$.

In Sections 4 and 5, especially, we give in some sense "exact" answers for the following cases (let $G$ be of order $b$ and let $H$ be of order $c$):

**Case 1:** $b \nmid c^N$ for all positive integers $N$.

**Case 2:** $b^M = c^N$ for some positive integers $M, N$.

In Section 6 we give estimates of $\beta$ for the remaining

**Case 3:** $b | c^N$ for some positive integer $N$ but $b^M \neq c^N$ for all positive integers $M$ and $N$.

In a short Section 7 for the sake of the reader we will summarize the results on the base change coefficient given in Sections 4, 5 and 6.

In Section 8, as a consequence of the results in Sections 4, 5 and 6 we give the main error estimate in the theory of "digital lattice rules" in its, until now, most general form.

Finally in Section 9 we state some of the most interesting open problems in the field.

## 2. Walsh functions over groups.

We recall the definitions for the concept of Walsh functions over a finite abelian group given in [**9**] and in [**21**].

Let $G$ be a finite abelian group of order $b$ . Let

$$G \cong \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_m}$$

and

$$\varphi : \{0, 1, \ldots , b-1\} \to G \cong \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_m}$$

$$k \mapsto \varphi(k) := (\varphi_1(k), \ldots, \varphi_m(k))$$

be a bijection with $\varphi(0) = 0$.

For $g = (g_1, \ldots, g_m) \in G$ let the character $\chi_g \in \hat{G}$ be defined by

$$\chi_g(y) := \prod_{l=1}^{m} e^{\frac{2\pi i \cdot g_l \cdot y_l}{b_l}}$$

for $y = (y_1, \ldots, y_m) \in G$. Of course $\hat{G} = \{\chi_g | g \in G\}$.

**Definition 2.** For a non-negative integer $n$ with $b$-adic representation $n = n_v \cdot b^{v-1} + \cdots + n_1$ we define the function ${}_{G,\varphi}\mathrm{wal}_n : [0,1) \to \mathbb{C}$ in the following way:

$${}_{G,\varphi}\mathrm{wal}_n(x) := \prod_{j=1}^{v} \chi_{\varphi(n_j)}(\varphi(x_j)),$$

where $x = \sum_{j=1}^{\infty} x_j \cdot b^{-j}$ is the $b$-adic representation of $x$ (with inifinitely many of the $x_j$ different from $b-1$).

**Definition 3.** The set

$$W_{G,\varphi} := \{{}_{G,\varphi}\mathrm{wal}_n | n = 0, 1, \ldots\}$$

is called the Walsh function system over $G$ with respect to $\varphi$.

In most cases we will write ${}_G\mathrm{wal}$ instead of ${}_{G,\varphi}\mathrm{wal}$, if it is clear which bijection $\varphi$ we use.

Note, that in classical Fourier theory, i.e. representation with respect to the characters $e^{2\pi ikx}$, $k \in \mathbb{Z}$ of the Torus group $\mathbb{T}$, also something similar to the bijections $\varphi$ occurs, as we have to identify the unit interval with the torus.

**Examples:**

a) Let $G = \mathbb{Z}_b$ and $\varphi$ be the "identity" between $\{0, \ldots, b-1\}$ and $\mathbb{Z}_b$. We then obtain the classical systems of Walsh-Paley and of Chrestenson.

b) In [20] Onneweer defines Walsh functions on the infinite product of groups. A "continuation" of his functions to $[0,1)$ is easily possible in an obvious way.

Our Walsh functions are products of characters on one fixed group. Onneweer's functions are products of characters on possibly different groups. In this sense Onneweer's concept is more general, however his groups must be of prime order, and he just uses identities for $\varphi$. In this sense our concept is more general.

c) The bijection $\varphi$ indeed strongly influences the structure of $W_{G,\varphi}$. Consider for example $W_{\mathbb{Z}_4,\mathrm{id}}$ and $W_{\mathbb{Z}_4,\varphi}$ with the transposition $\varphi = (1\,2)$ on $\mathbb{Z}_4$. The first four functions in each of the systems can be illustrated by the following diagrams. (Different grey tones represent different function values and the same grey tones correspond to the same function values on each side.)

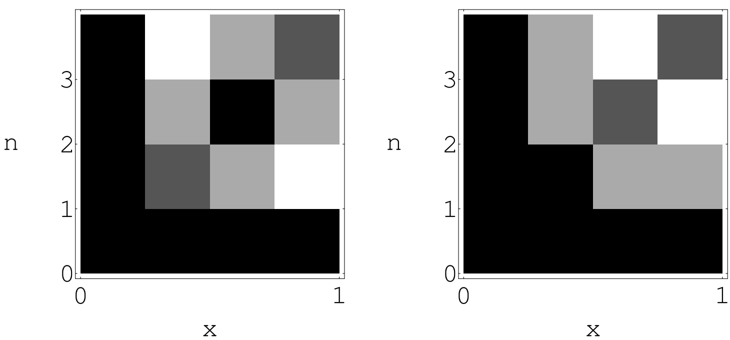


**Figure 1.** ${}_{\mathbb{Z}_4,\mathrm{id}}\mathrm{wal}_n, n = 0, 1, 2, 3$ and ${}_{\mathbb{Z}_4,\varphi}\mathrm{wal}_n, n = 0, 1, 2, 3$.

d) Concerning the connection between our concept and Vilenkin systems, see [**21**]. Again we have the situation, that both concepts have a "non-empty intersection" but neither is a "sub-concept" of the other.

Of course one may ask, why we did not extend the investigations of this paper also to Onneweer systems and Vilenkin systems. There are two reasons for this: The problem studied in this paper was motivated by investigations on numerical integration by digital nets, initiated for example in [**9**]. The methods developed there were restricted to the classes $W_{G,\varphi}$. They cannot be extended in a reasonable way to, say, Onneweer systems.

It should without problems be possible to obtain results for Onneweer and Vilenkin systems with the methods and results of this paper (or in an analogous way). However, since these systems are based on sequences of different groups of possibly different orders, the conditions for non-trivial connections between two different systems would be quite technical and quite restrictive. So we have concentrated ourselves in this paper to the more natural (and for our applications more important) classes $W_{G,\varphi}$.

In the following, we will give some basic properties of Walsh functions over groups, which will be used later on.

For $G, \varphi$ with $|G| = b$ given, we define digital summation $\oplus = \oplus_{G,\varphi}$ on $\mathbb{R}_0^+$ in the following way: For $u, v \in \mathbb{R}_0^+$ let

$$u = \sum_{i=w}^{\infty} u_i \cdot b^{-i} \text{ and } v = \sum_{i=w}^{\infty} v_i \cdot b^{-i}$$

be the $b$-adic representations of $u$ and $v$. Then

$$u \oplus v := \sum_{i=w}^{\infty} z_i \cdot b^{-i}$$

with $z_i := \varphi^{-1}\left(\varphi\left(u_i\right) + \varphi\left(v_i\right)\right)$ for $i = w, w+1, \ldots$ . Note that $\varphi(0) = 0$!

Further define

$$\ominus u := \sum_{i=w}^{\infty} \varphi^{-1}\left(-\varphi\left(u_i\right)\right) \cdot b^{-i},$$

which is the additive inverse of $u$, since $u \oplus (\ominus u) = 0$.

The following properties (compare these with the analogous properties of the function class $\{e^{2\pi i k x};\, k \in \mathbb{Z}\}$) are easily checked by insertion of the definitions. (Note, how the proper definition of digital summation enabled us to carry the character properties over to $\mathbb{R}_0^+$.)

**Lemma 1.** *For $G, \varphi$ given, and $\oplus := \oplus_{G,\varphi}$ we have:*

- **a)** ${}_G\mathrm{wal}_p(x) \cdot {}_G\mathrm{wal}_q(x) = {}_G\mathrm{wal}_{p\oplus q}(x)$ *for all* $p, q = 0, 1, 2, \ldots$ *and* $x \in [0, 1)$.
- **b)** ${}_G\mathrm{wal}_n(x) \cdot {}_G\mathrm{wal}_n(y) = {}_G\mathrm{wal}_n(x \oplus y)$ *for all* $n = 0, 1, 2, \ldots$ *and* $x, y \in [0, 1)$.
- **c)** $\overline{{}_G\mathrm{wal}_n(x)} = \frac{1}{{}_G\mathrm{wal}_n(x)} = {}_G\mathrm{wal}_{\ominus n}(x) = {}_G\mathrm{wal}_n(\ominus x)$.

We omit the easy proof.

The next lemma proves, that integrals of Walsh functions over certain intervals can be omitted.

**Lemma 2.** *Let $m > 0$ and $n, a$ be integers with $b^{m-1} \le n < b^m$ and $0 \le a < b^{m-1}$ . Then*

$$\int_{\frac{a}{b^{m-1}}}^{\frac{a+1}{b^{m-1}}} {}_G\mathrm{wal}_n(x)\, dx = 0.$$

*Proof.*

$$\int_{\frac{a}{b^{m-1}}}^{\frac{a+1}{b^{m-1}}} {}_G\mathrm{wal}_n(x)\, dx = {}_G\mathrm{wal}_n\left(\frac{a}{b^{m-1}}\right) \cdot \int_0^{\frac{1}{b^{m-1}}} {}_G\mathrm{wal}_n(x)\, dx$$

and (with $n = \sum_{i=1}^{m} n_i \cdot b^i$ and $x = \sum_{i=m}^{\infty} x_i \cdot b^{-i}$)

$$\begin{aligned}\int_0^{\frac{1}{b^{m-1}}} {}_G\mathrm{wal}_n(x)\, dx &= \int_0^{\frac{1}{b^{m-1}}} \chi_{\varphi(n_m)}\left(\varphi\left(x_m\right)\right) dx \\ &= \frac{1}{b^m} \cdot \sum_{g \in G} \chi_{\varphi(n_m)}(g) = 0\end{aligned}$$

since $\varphi(n_m) \neq 0$. □

From Lemma 1 and Lemma 2 we immediately obtain the orthonormality of $W_{G,\varphi}$:

**Lemma 3.**

**a)**
$$\int_0^1 {}_G\mathrm{wal}_n(x)\, dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

**b)**
$$\int_0^1 {}_G\mathrm{wal}_n(x) \cdot \overline{{}_G\mathrm{wal}_m(x)}\, dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}.$$

Indeed, $W_{G,\varphi}$ also is a *complete* orthonormal system in $L^2([0,1))$. We do not use the completeness in the following. A proof can be found in [**21**].

For Walsh functions, some Dirichlet kernels take a very simple form:

**Lemma 4.** *For any non-negative integer $v$ we have*
$$\sum_{j=0}^{b^v-1} {}_G\mathrm{wal}_j(x) = \begin{cases} b^v & \text{for } x \in \left[0, \frac{1}{b^v}\right) \\ 0 & \text{otherwise} \end{cases}.$$

*Proof.* For a non-negative integer $j$ with $b$-adic representation $j = \sum_{i=1}^{v} j_i \cdot b^{i-1}$ let $d_i(j) := j_i$ denote the $i$-th digit of $j$.

Let $x = \sum_{i=1}^{\infty} x_i \cdot b^{-i}$. Then
$$\begin{aligned}
\sum_{j=0}^{b^v-1} {}_G\mathrm{wal}_j(x) &= \sum_{j=0}^{b^v-1} \prod_{i=1}^{v} \chi_{\varphi(d_i(j))}(\varphi(x_i)) \\
&= \sum_{j_1,\ldots,j_v=0}^{b-1} \prod_{i=1}^{v} \chi_{\varphi(j_i)}(\varphi(x_i)) \\
&= \prod_{i=1}^{v} \sum_{g \in G} \chi_g(\varphi(x_i)) \\
&= \begin{cases} b^v & \text{if } x_i = 0 \text{ for } i = 1, \ldots, v \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

The result follows. □

The next lemma provides a useful method to lower the index of a Walsh function, when it is of a special form.

**Lemma 5.** *Let $l, m, M$ be arbitrary positive integers. Let $d := b^M$ and let $j$ and $n$ be integers with $0 \leq j, n < d$. Then*
$$ {}_G\mathrm{wal}_{j \cdot d^{m-1}}\left(\frac{n}{d^l}\right) = \begin{cases} 1 & \text{if } l \neq m \\ {}_G\mathrm{wal}_j\left(\frac{n}{d}\right) & \text{if } l = m \end{cases}.$$

*Proof.* For $M = 1$ we have

$$ {}_{G}\mathrm{wal}_{j\cdot b^{m-1}}\left(\frac{n}{b^{l}}\right) = \begin{cases} 1 & \text{if } l \neq m \\ \chi_{\varphi(j)}\left(\varphi(n)\right) & \text{if } l = m \end{cases}. $$

If $m = l$ then moreover

$$ \chi_{\varphi(j)}\left(\varphi(n)\right) = {}_{G}\mathrm{wal}_{j}\left(\frac{n}{d}\right). $$

Therefore for $M \geq 2$ we get: Let $n = \sum_{i=1}^{M} n_i \cdot b^{i-1}$ and $j = \sum_{i=1}^{M} j_i \cdot b^{i-1}$, then ${}_{G}\mathrm{wal}_{j\cdot d^{m-1}}\left(\frac{n}{d^{l}}\right) =$

$$ \begin{aligned} &= {}_{G}\mathrm{wal}_{j_1\cdot b^{M\cdot m-M}+\cdots+j_M\cdot b^{M\cdot m-1}}\left(\frac{n_M}{b^{M\cdot l-M+1}}+\cdots+\frac{n_1}{b^{M\cdot l}}\right) \\ &= {}_{G}\mathrm{wal}_{j_1}\left(\frac{n_M}{b}\right)\cdot {}_{G}\mathrm{wal}_{j_2}\left(\frac{n_{M-1}}{b}\right)\cdot\cdots\cdot {}_{G}\mathrm{wal}_{j_M}\left(\frac{n_1}{b}\right) \end{aligned} $$

if $l = m$, and 1 if $l \neq m$ by Lemma 1 and since the result holds for $M = 1$.

On the other hand

$$ \begin{aligned} {}_{G}\mathrm{wal}_{j}\left(\frac{n}{d}\right) &= {}_{G}\mathrm{wal}_{j_1+\cdots+j_M\cdot b^{M-1}}\left(\frac{n_1+\cdots+n_M\cdot b^{M-1}}{b^{M}}\right) \\ &= {}_{G}\mathrm{wal}_{j_1}\left(\frac{n_M}{b}\right)\cdot\cdots\cdot {}_{G}\mathrm{wal}_{j_M}\left(\frac{n_1}{b}\right) \end{aligned} $$

and the result follows. □

A function $f : [0,1) \to \mathbb{C}$ of the form

$$ f(x) = \sum_{n=0}^{\infty} \hat{f}_G(n) \cdot {}_{G}\mathrm{wal}_{n}(x) $$

with certain $\hat{f}_G(n) \in \mathbb{C}$, shall be called a Walsh series over $G$ with respect to $\varphi$.

In the following we will only deal with absolutely convergent Walsh series. Thus we will always have

$$ \hat{f}_G(n) = \int_0^1 f(x) \cdot \overline{{}_{G}\mathrm{wal}_{n}}(x)\, dx. $$

Recall the definition of the classes ${}_{G,\varphi}E^{\alpha}(C)$ given in Section 1:

$$ f \in {}_{G,\varphi}E^{\alpha}(C) :\Leftrightarrow |\hat{f}_G(n)| \leq C \cdot n^{-\alpha} \text{ for } n = 1, 2, \ldots . $$

## 3. Digital derivatives and speed of convergence of Walsh series.

In [**20**] a derivative for functions defined on groups $G$, that are the direct product of countably many groups of prime order, was introduced. Such functions can be continued to functions on $[0,1)$ quite naturally in the same manner as it was done for our Walsh functions $W_{G,\varphi}$. In this sense the following notion of a (digital) derivative with respect to $G$ and $\varphi$ is an analog to Onneweer's concept of a derivative.

**Definition 4.** For $G$ of order $b$, a corresponding bijection $\varphi$, a function $f:[0,1)\to\mathbb{C}$, for $x\in[0,1)$ and positive integers $n$ let

$$\mathrm{d}_n f(x) := \sum_{j=0}^{n} b^{j-1} \sum_{k=0}^{b-1} k \sum_{l=0}^{b-1} \overline{{}_{G,\varphi}\mathrm{wal}_l\left(\frac{k}{b}\right)} \cdot f\left(x \oplus \frac{l}{b^{j+1}}\right).$$

We say, that $f$ is digitally differentiable in $x$ with respect to $G$ and $\varphi$ if

$$f^{[1]}(x) := \lim_{n\to\infty} \mathrm{d}_n f(x)$$

exists. $f^{[1]}(x)$ is called the digital derivative of $f$ in $x$.

Higher derivatives $f^{[n]}(x)\,;\, n=2,3,\dots$ can be defined in the usual way.

A function $f\in L_p([0,1))\,,\ 1\le p<\infty$ is called strongly differentiable with respect to $G$ and $\varphi$ if there exists $g\in L_p([0,1))$ with

$$\lim_{n\to\infty} \left\|\mathrm{d}_n f - g\right\|_p = 0.$$

We then denote $g$ by $\mathbf{d}f$.

**Example.** As an illustration for the difficulty in combining this concept of a derivative with "geometric intuition", consider for example the function $f := 1_{[0,1/8)}$, the indicator function of the interval $[0,1/8)$, and its derivative with respect to $\mathbb{Z}_2$ and identity.

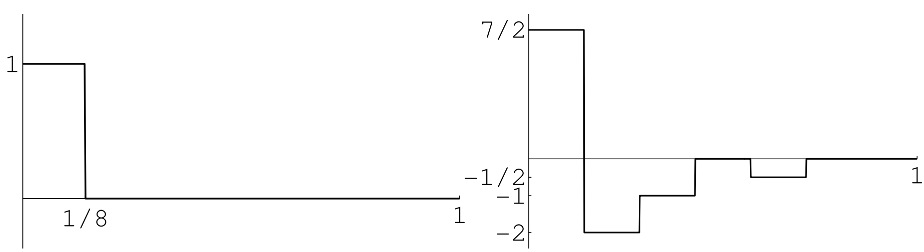


**Figure 2.** The indicator function of $[0,1)$ and its digital derivative with respect to $\mathbb{Z}_2$ and id.

We may state the following question: Assume, that $f:[0,1)\to\mathbb{C}$ is differentiable $\alpha$ times with respect to $G$ and $\varphi$. How often is $f$ differentiable with respect to another group $H$ with corresponding bijection $\psi$?

Or: Given $G,\varphi,H,\psi$ as above and a positive integer $\alpha$. What can be said about the quantity

> $\gamma(G,\varphi,H,\psi,\alpha) := \sup\{\gamma|$ if $f$ is $\alpha$ times differentiable with respect to $G$ and $\varphi$, then $f$ is at least $\gamma$ times differentiable with respect to $H$ and $\psi\}$?

A certain connection between $\beta(G,\varphi,H,\psi,\alpha)$ as defined in Section 1 and $\gamma(G,\varphi,H,\psi,\alpha)$ is given by the subsequent Theorem 1.

First we need two further lemmata. They already show, that the digital derivative plays the same role for Walsh functions over groups as the usual derivative does for exponential function, i.e. for Fourier analysis.

**Lemma 6.** *For $G$ of order $b$ and $\varphi$ fixed, ${}_G\mathrm{wal}_w$ is strongly differentiable with $\mathbf{d}\left({}_G\mathrm{wal}_w\right) = w \cdot {}_G\mathrm{wal}_w$.*

*Proof.* Let $0 \le l < b$. Let $w = \sum_{i=1}^{\infty} w_i \cdot b^{i-1}$ and $j$ be a non-negative integer. Then

$$\begin{aligned} {}_G\mathrm{wal}_w\left(\frac{l}{b^{j+1}}\right) &= \prod_{v=1}^{m} e^{\frac{2\pi i\cdot\varphi_v(w_{j+1})\cdot\varphi_v(l)}{b_v}} \\ &= {}_G\mathrm{wal}_l\left(\frac{w_{j+1}}{b}\right). \end{aligned}$$

(Here we used the representation $G \cong \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_m}$ for $G$ again.)

So

$$\begin{aligned} &\mathrm{d}_n\left({}_G\mathrm{wal}_w(x)\right) \\ &= \sum_{j=0}^{n} b^{j-1} \sum_{k=0}^{b-1} k \sum_{l=0}^{b-1} \overline{{}_G\mathrm{wal}_l\left(\frac{k}{b}\right)} \cdot {}_G\mathrm{wal}_w\left(x \oplus \frac{l}{b^{j+1}}\right) \\ &= {}_G\mathrm{wal}_w(x) \cdot \sum_{j=0}^{n} b^{j-1} \sum_{k=0}^{b-1} k \sum_{l=0}^{b-1} {}_G\mathrm{wal}_l\left(\frac{w_{j+1} \ominus k}{b}\right) = w \cdot {}_G\mathrm{wal}_w(x) \end{aligned}$$

by Lemma 4 and for all $n$ large enough. □

**Lemma 7.** *If $f : [0,1) \to \mathbb{R}$ is strongly differentiable, then $\widehat{\mathbf{d}f}_G(m) = m \cdot \widehat{f}_G(m)$ for all $m$.*

*Proof.* Note that for positive integers $j, l$ we have

$$\int_0^1 f\left(x \oplus \frac{l}{b^{j+1}}\right) \cdot \overline{{}_G\mathrm{wal}_m(x)}\, dx = \int_0^1 f(x) \cdot \overline{{}_G\mathrm{wal}_m\left(x \ominus \frac{l}{b^{j+1}}\right)}\, dx.$$

So

$$\begin{aligned} &\widehat{\mathrm{d}_n f}_G(m) \\ &= \int_0^1 \sum_{j=0}^{n} b^{j-1} \sum_{k=0}^{b-1} k \sum_{l=0}^{b-1} \overline{{}_G\mathrm{wal}_l\left(\frac{k}{b}\right)} \cdot f\left(x \oplus \frac{l}{b^{j+1}}\right) \cdot \overline{{}_G\mathrm{wal}_m(x)}\, dx \\ &= \int_0^1 f(x) \cdot \overline{{}_G\mathrm{wal}_m(x)}\, dx \cdot \sum_{j=0}^{n} b^{j-1} \sum_{k=0}^{b-1} k \sum_{l=0}^{b-1} {}_G\mathrm{wal}_l\left(\frac{m_{j+1} \ominus k}{b}\right) \end{aligned}$$

$$= \hat{f}_G(m) \cdot \sum_{j=0}^{n} b^{j-1} \sum_{k=0}^{b-1} k \sum_{l=0}^{b-1} {}_G\mathrm{wal}_l\left(\frac{m_{j+1} \ominus k}{b}\right)$$
$$= \hat{f}_G(m) \cdot m,$$

with $m = \sum_{j=1}^{\infty} m_j \cdot b^{j-1}$, by Lemma 4, for all $n$ large enough. □

As is the aim of this section, a certain connection between the quantities $\beta$ and $\gamma$ is now given by the following result. (Compare with analogous results for Fourier series as they appear for example in [5] in a higher dimensional form.)

**Theorem 1.**

**a)** *Let $\alpha > 1$ be an integer. If a Walsh series $f$ over $G$ and $\varphi$ is $\alpha$ times strongly differentiable as a function in $L_1([0,1))$, then $f \in {}_{G,\varphi}E^{a}(C)$ for some $C > 0$.*

**b)** *If for some integer $\alpha > 2$ we have $f \in {}_{G,\varphi}E^{\alpha}(C)$ for some $C > 0$, then $f$ is at least $\alpha - 2$ times strongly differentiable with respect to $G$ and $\varphi$.*

*Proof.* **a)** Let $g := \mathbf{d}^{[\alpha]} f \in L_1([0,1))$. So $\widehat{g}_G(m)$ exists for all $m$ and, by Lemma 7, equals $m^{\alpha} \cdot \widehat{f}_G(m)$.

Further $|m^{\alpha} \cdot \hat{f}_G(m)| \le \int_0^1 |g(x)|\, dx =: C$ for all $m$ and therefore $f \in {}_{G,\varphi}E^{a}(C)$.

**b)** If for $\alpha > 2$ we have $f \in {}_{G,\varphi}E^{a}(C)$ then $\sum_{n=0}^{\infty} n^{\alpha-2} \cdot \widehat{f}_G(n) \cdot {}_G\mathrm{wal}_n(x)$ is absolutely convergent and therefore, by Lemma 7, the strong $\alpha$-th derivative of $f$ with respect to $G$ and $\varphi$.

□

Consequently we have, for example, the following relation between the quantities $\beta$ and $\gamma$: If $\alpha > 1$ and $\beta(G,\varphi,H,\psi,\alpha) > 1$, then $\gamma(G,\varphi,H,\psi,\alpha) \le \beta(G,\varphi,H,\psi,\alpha)$.

## 4. The base change coefficient in the case: $|G| \nmid |H|^N$ for all positive $N$.

In this case there is no "convergence connection" between the Walsh series representations over $G$ and $H$. We have the following result.

**Theorem 2.** *If $G, H$ are such, that for all positive integers $N$ we have $|G| \nmid |H|^N$, then $\beta(G,\varphi,H,\psi,\alpha) = 1$ for all $\alpha > 1$.*

*Proof.* With $b := |G|$, let

$$f(x) := \sum_{i=0}^{b-1} {}_G\mathrm{wal}_i(x) = \begin{cases} b & \text{for } x \in [0,1/b) \\ 0 & \text{otherwise.} \end{cases}$$

So for all $\alpha > 1$ we have $f \in {}_G E^{\alpha}(b^{\alpha})$. We show, that for all $\epsilon > 0$ and all $K > 0$, $f \notin {}_H E^{1+\epsilon}(K)$.

Let $c := |H|$ and $1/b = \sum_{i=1}^{\infty} \kappa_i \cdot c^{-i}$ be the representation of $1/b$ in base $c$. This representation is periodic and non-terminating. So for $k = \sum_{i=1}^{m} k_i \cdot c^{i-1}$ we have

$$\hat{f}_H(k) = \int_0^{1/b} \overline{{}_H\mathrm{wal}_k}(x)\, dx = \int_{\sum_{i=1}^{m-1} \kappa_i \cdot c^{-i}}^{1/b} \overline{{}_H\mathrm{wal}_k}(x)\, dx$$

by Lemma 2. Therefore

$$\begin{aligned}\hat{f}_H(k) &= \overline{{}_H\mathrm{wal}_k}\left(\sum_{i=1}^{m-1} \frac{\kappa_i}{c^i}\right) \cdot \int_0^{\sum_{i=m}^{\infty} \kappa_i c^{-i}} \overline{{}_H\mathrm{wal}_n}(x)\, dx \\ &= \overline{{}_H\mathrm{wal}_k}\left(\sum_{i=1}^{m-1} \frac{\kappa_i}{c^i}\right) \cdot \left(\frac{1}{c^m} \sum_{j=0}^{\kappa_m - 1} \overline{\chi_{\varphi(k_m)}}(\varphi(j)) + \right. \\ &\quad \left. + \sum_{i=m+1}^{\infty} \frac{\kappa_i}{c^i} \cdot \chi_{\varphi(k_m)}(\varphi(\kappa_m))\right) =: \frac{1}{c^m} \cdot T(k).\end{aligned}$$

Since the sequence of the $\kappa_i$ is periodic, $T(k)$ attains only finitely many different values.

A finite Walsh polynomial over $H$ is certainly continuous in $x = 1/b$ (the number $b$ contains prime factors not dividing $c!$) and therefore cannot represent $f$.

So there is a $T > 0$ with $T = |T(k)|$ for infinitely many $k$. And for these $k$ we have

$$\left|\hat{f}_H(k)\right| \geq \frac{1}{c^m} T \geq \frac{T}{c} \cdot \frac{1}{k}.$$

The result follows. □

## 5. The base change coefficient in the case: $|G|^M = |H|^N$ for some positive $M$ and $N$.

In this case the "convergence connection" is non-trivial and we can give the exact form of $\beta(G, \varphi, H, \psi, \alpha)$. This quantity $\beta$ shows great similarities to a quantity studied in [**27**], which measures a certain distance between groups.

Before we state the main results in Theorem 3 and Theorem 4, we give some auxiliary technical results, which will be needed in the proofs. The first lemma in this section establishes the fact, that in the currently considered case the scalar product of Walsh functions is non-zero only for constrained index ranges.

**Lemma 8.** *Let $G, H$ be groups with order $b$ and $c$. Let the corresponding bijections $\varphi$ and $\psi$ be fixed. Assume, that there exist $v, w \geq 1$, such that*

$b^v | c^{w-1}$. *Then for all* $j, k$ *with* $b^{v-1} \le j < b^v$ *we have*

$$\gamma(j,k) := \int_0^1 {}_G\mathrm{wal}_j(x) \cdot {}_H\overline{\mathrm{wal}_k}(x)\, dx = 0.$$

*Proof.* Note that ${}_G\mathrm{wal}_j(x)$ is constant on $\left[\frac{l}{b^v}, \frac{l+1}{b^v}\right)$ for all $0 \le l < b^v$. So by Lemma 2 and with $A := c^{w-1}/b^v$ we have

$$\gamma(j,k) = \sum_{l=0}^{b^{v-1}} {}_G\mathrm{wal}_j\left(\frac{l}{b^v}\right) \cdot \int_{\frac{A\cdot l}{c^{w-1}}}^{\frac{A\cdot l}{c^{w-1}}+\frac{A}{c^{w-1}}} {}_H\mathrm{wal}_k(x)\, dx = 0.$$

□

In the next lemma we see, how we can estimate $\beta(G, \varphi, H, \psi, \alpha)$ by estimating certain sums involving the $\gamma(j,k)$ defined in the last lemma.

Let $G, H, \varphi, \psi, b, c$ be as above ($b^v | c^{w-1}$ for some positive $v, w$) and let $\gamma$ be defined like in Lemma 8.

For non-negative integers $m, k$ let

$$P(m) := \min\{n : b^m | c^n\}$$
$$Q(m) := \min\left\{n : k < c^{P(n)}\right\}.$$

**Lemma 9.** *Let* $\alpha, \beta > 1$ *and* $C_1, C_2 > 0$ *be reals and* $f \in {}_GE^\alpha(C_1)$, *such that*

$$\left|\sum_{j=b^{Q(k)-1}}^{\infty} \widehat{f}_G(j) \cdot \gamma(j,k)\right| < \frac{C_2}{k^\beta}$$

*for all* $k \ge 1$. *Then for all* $x \in [0,1)$, *we have*

$$f(x) = \hat{f}_G(0) + \sum_{k=1}^{\infty}\left(\sum_{j=b^{Q(k)-1}}^{\infty} \hat{f}_G(j) \cdot \gamma(j,k)\right) \cdot {}_H\mathrm{wal}_k,$$

*therefore*

$$\hat{f}_H(k) = \sum_{j=b^{Q(k)-1}}^{\infty} \hat{f}_G(j) \cdot \gamma(j,k)$$

*for* $k > 0$ *and* $f \in {}_HE^\beta(C_2)$.

*Proof.* By Lemma 8 for $j$ with $b^{m-1} \le j < b^m$ we have

$$_G\mathrm{wal}_j(x) = \sum_{k=0}^{c^{P(m)}-1} \gamma(j,k) \cdot {}_H\mathrm{wal}_k(x).$$

So (since $\gamma(j,0) = 0$ for $j > 1$ )

$$\begin{aligned}
f(x) &= \sum_{j=0}^{\infty} \hat{f}_G(j) \cdot {}_G\mathrm{wal}_j(x) \\
&= \hat{f}_G(0) + \sum_{m=1}^{\infty} \sum_{j=b^{m-1}}^{b^m-1} \hat{f}_G(j) \sum_{k=0}^{c^{P(m)}-1} \gamma(j,k) \cdot {}_H\mathrm{wal}_k(x) \\
&= \hat{f}_G(0) + \sum_{m=1}^{\infty} \sum_{k=0}^{c^{P(m)}-1} \left( \sum_{j=b^{m-1}}^{b^m-1} \hat{f}_G(j) \cdot \gamma(j,k) \right) \cdot {}_H\mathrm{wal}_k(x) \\
&= \hat{f}_G(0) + \sum_{k=1}^{\infty} \sum_{\substack{m, \\ c^{P(m)}-1>k}} \left( \sum_{j=b^{m-1}}^{b^m-1} \hat{f}_G(j) \cdot \gamma(j,k) \right) \cdot {}_H\mathrm{wal}_k(x) \\
&= \hat{f}_G(0) + \sum_{k=1}^{\infty} \left( \sum_{j=b^{Q(k)-1}}^{\infty} \hat{f}_G(j) \cdot \gamma(j,k) \right) \cdot {}_H\mathrm{wal}_k(x) .
\end{aligned}$$

$\square$

For the remaining part of this chapter, let $M, N$ be positive integers, such that $b^M = c^N =: d$. We show, that in this case, the $\gamma(j,k)$ can be evaluated in terms of $\gamma(j_l, k_l)$ with only finitely many indices $j_l, k_l$.

**Lemma 10.** *Let $j, k$ be positive integers with $j, k < d^L$ and let $j = \sum_{l=1}^{L} j_l \cdot d^{l-1}$, $k = \sum_{l=1}^{L} k_l \cdot d^{l-1}$ be their $d$-adic representations. Then*

$$\gamma(j,k) = \prod_{l=1}^{L} \gamma(j_l, k_l) .$$

*Proof.* In the following we use Lemma 5:

$$\begin{aligned}
\gamma(j,k) &= \int_0^1 {}_G\mathrm{wal}_j(x) \cdot \overline{{}_H\mathrm{wal}_k}(x)\, dx \\
&= \frac{1}{d^L} \sum_{n=0}^{d^L-1} {}_G\mathrm{wal}_j\left(\frac{n}{d^L}\right) \cdot \overline{{}_H\mathrm{wal}_k}\left(\frac{n}{d^L}\right) \\
&= \frac{1}{d^L} \sum_{n_1,\dots,n_L=0}^{d-1} \left( \prod_{m=1}^{L} {}_G\mathrm{wal}_{j_m \cdot d^{m-1}}\left(\frac{n_m}{d^m}\right) \right) \\
&\qquad \cdot \left( \prod_{m=1}^{L} \overline{{}_H\mathrm{wal}_{k_m \cdot d^{m-1}}}\left(\frac{n_m}{d^m}\right) \right)
\end{aligned}$$

$$= \prod_{m=1}^{L} \left( \frac{1}{d} \sum_{n=0}^{d-1} {}_G\mathrm{wal}_{j_m \cdot d^{m-1}} \left( \frac{n}{d^m} \right) \cdot \overline{{}_H\mathrm{wal}_{k_m \cdot d^{m-1}}} \left( \frac{n}{d^m} \right) \right)$$
$$= \prod_{m=1}^{L} \left( \frac{1}{d} \sum_{n=0}^{d-1} {}_G\mathrm{wal}_{j_m} \left( \frac{n}{d} \right) \cdot \overline{{}_H\mathrm{wal}_{k_m}} \left( \frac{n}{d} \right) \right) = \prod_{l=1}^{L} \gamma \left( j_l, k_l \right).$$

□

We define a quantity, using those finitely many $\gamma(j,k)$, which will help us in estimating the sum of the above Lemma 9.

**Definition 5.** Let $G$ and $H$ be finite abelian groups of order $b$ and $c$. Let $\varphi, \psi$ be corresponding bijections. Assume, there are positive integers $M, N$, such that $b^M = c^N =: d$. Then

$$\beta_{G,H,\varphi,\psi} := \log_d \left( \max_{k=0,\dots,d-1} \sum_{j=0}^{d-1} \left| \gamma \left( j,k \right) \right| \right).$$

($\log_d$ denotes logarithm to base $d$.)

The quantity $\beta_{G,H,\varphi,\psi}$ also was studied in [**27**].

The following theorem is the main result of this section:

**Theorem 3.** *Let $G$ and $H$ be finite abelian groups of order $b$ and $c$. Let $\varphi, \psi$ be corresponding bijections. Assume, there are positive integers $M, N$, such that $b^M = c^N =: d$. Then for all $\alpha > 1 + \beta_{G,H,\varphi,\psi}$ we have*

$$\beta \left( G, H, \varphi, \psi, \alpha \right) = \alpha - \beta_{G,H,\varphi,\psi}.$$

*Proof.* Let $f \in {}_{G,\varphi}E^{\alpha}\left(C\right)$. Let $d^{L-1} \leq k < d^L, k = \sum_{i=1}^{L} k_i \cdot d^{i-1}$. By Lemma 8 we have $\gamma\left(j,k\right) = 0$ if $j < d^{L-1}$ or $j \geq d^L$. By $l\left(j\right)$ we denote the $l$-th digit of a non-negative integer $j$ in base $d$. Then by Lemma 9 and Lemma 10:

$$\begin{aligned} \left| \hat{f}_H \left( k \right) \right| &= \left| \sum_{j=0}^{d^L-1} \hat{f}_G \left( j \right) \cdot \gamma \left( j,k \right) \right| \\ &= \left| \sum_{j=0}^{d^L-1} \hat{f}_G \left( j \right) \cdot \left( \prod_{l=1}^{L} \gamma \left( l(j), k_l \right) \right) \right| \\ &\leq \frac{C}{d^{(L-1)\cdot\alpha}} \sum_{j_1,\dots j_L=0}^{d-1} \prod_{l=1}^{L} \left| \gamma \left( l(j), k_l \right) \right| \\ &\leq \frac{C}{d^{(L-1)\cdot\alpha}} \left( \max_{k=0,\dots,d-1} \sum_{j=0}^{d-1} \left| \gamma \left( j,k \right) \right| \right)^L \end{aligned}$$

$$\leq C \cdot d^{\alpha} \cdot k^{\beta_{G,H,\varphi,\psi}-\alpha}$$

and therefore (again using Lemma 9) $f \in {}_{H,\psi}E^{\alpha-\beta_{G,H,\varphi,\psi}}(C \cdot d^{\alpha})$ so that $\beta(G,H,\varphi,\psi,\alpha) \geq \alpha - \beta_{G,H,\varphi,\psi}$.

On the other hand, consider the function

$$f(x) = \sum_{j=1}^{\infty} \hat{f}_G(j) \cdot {}_{G,\varphi}\mathrm{wal}_j(x)$$

with $\hat{f}_G(j)$ of the following form: Let $k_0$ be such, that

$$\sum_{j=0}^{d-1} |\gamma(j,k_0)| = \max_{k=0,\dots,d-1} \sum_{j=0}^{d-1} |\gamma(j,k)|.$$

For $L \geq 1$ let $K(L) := k_0 + k_0 \cdot d + \cdots + k_0 \cdot d^{L-1}$ and for $d^{L-1} \leq j < d^L$ we set

$$\hat{f}_G(j) := \frac{1}{d^{L\cdot\alpha}} \cdot \frac{|\gamma(j,K(L))|}{\gamma(j,K(L))}.$$

Then $f \in {}_{G,\varphi}E^{\alpha}(1)$.

Further for all $L \geq 1$ we have

$$\begin{aligned}\left|\hat{f}_H(K(L))\right| &= \left|\sum_{j=0}^{d^L-1} \hat{f}_G(j) \cdot \gamma(j,K(L))\right| \\ &= \frac{1}{d^{L\cdot\alpha}} \sum_{j_1,\dots,j_L=0}^{d-1} \prod_{l=1}^{L} |\gamma(j_l,k_0)| \\ &= \frac{1}{d^{L\cdot\alpha}} \prod_{l=1}^{L} \sum_{j=0}^{d-1} |\gamma(j,k_0)| \geq \frac{1}{d^{\alpha-\beta_{G,H,\varphi,\psi}}} \cdot \frac{1}{K(L)^{\alpha-\beta_{G,H,\varphi,\psi}}},\end{aligned}$$

so that $\beta(G,\varphi,H,\psi,\alpha) \leq \alpha - \beta_{G,H,\varphi,\psi}$, and the result follows. □

The constant $\beta_{G,H,\varphi,\psi}$ is computable in finitely many steps for every $G,H,\varphi,\psi$, and, as already mentioned in Section 1, $\beta_{G,H,\varphi,\psi}$ was explicitely computed for $G = \mathbb{Z}_2, H = \mathbb{Z}_{2^h}$ and $\varphi,\psi$ identities in [**12**].

In the following we give an estimate for the quantity $\beta(G,H,\varphi,\psi,\alpha)$ based on Theorem 3.

Let $G,H,\varphi,\psi,d$ be as above. We say $G$ and $H$ are of comparable order with common multiple $d = |G|^M = |H|^N$. We define the bijection $\tau : G^M \to H^N$ "induced by $\varphi$ and $\psi$": Let

$$\bar{\varphi} : \{0,\dots,b^M-1\} \to G^M,$$

$$x = a_M \cdot b^{M-1} + \cdots + a_1 \mapsto \bar{\varphi}(x) := (\varphi(a_M),\dots,\varphi(a_1)).$$

Let

$$\bar{\psi} : \{0, \ldots, c^N - 1\} \to H^N,$$

$$y = e_N \cdot c^{N-1} + \cdots + e_1 \mapsto \bar{\psi}(y) := (\psi(e_N), \ldots, \psi(e_1)).$$

Then $\tau := \bar{\psi} \circ \bar{\varphi}^{-1}$.

**Theorem 4.** *Let $G$ and $H$ be of comparable order. Let $\varphi$ and $\psi$ be corresponding bijections and $\tau$ the bijection induced by $\varphi$ and $\psi$. Then for all $\alpha > 1$ we have*

$$\alpha - 1/2 < \beta(G, \varphi, H, \psi, \alpha) \leq \alpha$$

*with equality on the right side if and only if $\tau$ is a group isomorphism.*

*Proof.* We have $\beta_{G,H,\varphi,\psi} \geq 0$ by definition. By Theorem 1 in [**27**], $\beta_{G,H,\varphi,\psi} = 0$ if and only if $\tau$ is a group isomorphism. From Theorem 3 the right side of the inequality follows.

Now let $\gamma(l) := (\gamma(0,l), \ldots, \gamma(d-1,l)) \in \mathbb{C}^d$. By Cauchy's inequality we have

$$\begin{aligned} \beta_{G,H,\varphi,\psi} &= \log_d \left( \max_{l=0,\ldots,d-1} \|\gamma(l)\|_1 \right) \leq \log_d \left( \max_{l=0,\ldots,d-1} \sqrt{d} \cdot \|\gamma(l)\|_2 \right) \\ &= \log_d \sqrt{d} = 1/2. \end{aligned}$$

Moreover we never have $\beta_{G,H,\varphi,\psi} = 1/2$, since $\gamma(0,l) = 0$ for $l > 0$ and $\|\gamma(0)\|_1 = 1$, so that in the above application of Cauchy's inequality the inequality is strict. Again from Theorem 3 the result follows. ☐

## 6. The base change coefficient in the case: $|G|$ divides $|H|^N$ for some positive $N$, but $|G|^M \neq |H|^N$ for all positive $M$ and $N$.

In this section we will always assume, that the orders $b$ and $c$ of the groups $G$ and $H$ satisfy $b|c^N$ for some positive integer $N$, that is: Let $c$ have canonical prime factorization $c = \prod_{i=1}^r p_i^{\nu_i}$ with $\nu_i \geq 1$ for $i = 1, \ldots, r$ , then $b = \prod_{i=1}^r p_i^{\mu_i}$ with $\mu_i \geq 0$ for $i = 1, \ldots, r$.

**Definition 6.** Let $\rho = \rho(b,c) := \min\{\nu_i/\mu_i : i = 1, \ldots, r\}$.

**Definition 7.** Let

$$\sigma_w := \max_{c^{w-1} \leq k < c^w} \sum_{v=[(w-1)\cdot\rho]+1}^{\infty} \frac{1}{b^{v\cdot\alpha}} \sum_{j=b^{\nu-1}}^{b^{\nu}-1} |\gamma(j,k)|$$

and

$$\lambda = \lambda(G, \varphi, H, \psi, \alpha) := -\limsup_{w\to\infty} \frac{\log_c \sigma_w}{w}.$$

With this notation, we have:

**Theorem 5.** *If* $\lambda(G,\varphi,H,\psi,\alpha) > 1$ *then*

$$\beta(G,\varphi,H,\psi,\alpha) = \lambda(G,\varphi,H,\psi,\alpha).$$

*Proof.* For $f \in {}_{G,\varphi}E^{\alpha}(C)$ and for $k$ with $c^{w-1} \le k < c^{w}$ and since $\lambda > 1$ we have by Lemma 9:

$$\begin{aligned}\left|\hat{f}_H(k)\right| &= \left|\sum_{j=b^{Q(k)-1}}^{\infty} \hat{f}_G(j)\cdot\gamma(j,k)\right| \\ &\le C\cdot b^{\alpha}\cdot\sum_{v=Q(k)}^{\infty}\frac{1}{b^{v\cdot\alpha}}\sum_{j=b^{v-1}}^{b^{v}-1}|\gamma(j,k)|.\end{aligned}$$

Remember that $P(m) := \min\{n : b^m | c^n\}$, $Q(k) := \min\left\{n : k < c^{P(n)}\right\}$, so

$$P(m) = \left\lceil m\cdot\max_{i=1,\dots,r}\frac{\mu_i}{\nu_i}\right\rceil$$

and therefore

$$\begin{aligned}Q(k) = Q\left(c^{w-1}\right) &= \min\left\{n : w \le \left\lceil m\cdot\max_{i=1,\dots,r}\frac{\mu_i}{\nu_i}\right\rceil\right\} \\ &= [(w-1)\cdot\rho] + 1.\end{aligned}$$

(Here by $\lceil y\rceil$ we denote the smallest integer larger or equal to $y$.) Consequently

$$\left|\hat{f}_H(k)\right| \le C\cdot b^{\alpha}\cdot\sigma_w \le C'(\epsilon)\cdot\frac{1}{k^{\lambda-\epsilon}}$$

for all $\epsilon > 0$ and a suitable $C'(\epsilon) > 0$, so $\beta(G,\varphi,H,\psi,\alpha) \ge \lambda(G,\varphi,H,\psi,\alpha)$.

Now let $\epsilon > 0$ be given. Let $1 \le \omega_1 < \omega_2 < \dots$ be any sequence $\omega$ of positive integers. Then we define a function $f^{(\omega)}$ in the following way:

Let $k_i$ with $c^{\omega_i - 1} \le k_i < c^{\omega_i}$ be such, that

$$\sum_{v=[(\omega_i-1)\rho]+1}^{[(\omega_{i+1}-1)\rho]}\frac{1}{b^{v\alpha}}\sum_{j=b^{v-1}}^{b^{v}-1}|\gamma(j,k)|$$

attains its maximum for $k = k_i$. Then set

$$\widehat{f_G^{(\omega)}}(j) := \frac{1}{b^{v\alpha}}\cdot\frac{|\gamma(j,k_i)|}{\gamma(j,k_i)}$$

for all $j$ with $b^{[(\omega_i-1)\rho]} \le b^{v-1} \le j < b^{v} \le b^{[(\omega_{i+1}-1)\rho]}$, and $\widehat{f_G^{(\omega)}}(j) = 0$ for all other $j$.

Thus $f^{(\omega)} \in {}_{G,\varphi}E^{\alpha}(1)$.

On the other hand

$$\left|\hat{f}_H(k_i)\right| \geq \sum_{v=[(\omega_i-1)\rho]+1}^{[(\omega_{i+1}-1)\rho]} \frac{1}{b^{v\alpha}} \sum_{j=b^{v-1}}^{b^v-1} |\gamma(j,k_i)| - \sum_{j=b^{[(\omega_{i+1}-1)\rho]}}^{\infty} \frac{1}{j^\alpha}.$$

We have

$$\left|\sigma_{\omega_i} - \sum_{v=[(\omega_i-1)\rho]+1}^{[(\omega_{i+1}-1)\rho]} \frac{1}{b^{v\alpha}} \sum_{j=b^{v-1}}^{b^v-1} |\gamma(j,k_i)|\right| \leq \sum_{j=b^{[(\omega_{i+1}-1)\rho]}}^{\infty} \frac{1}{j^\alpha}$$

and therefore

$$\left|\widehat{f_H^{(\omega)}}(k_i)\right| \geq \sigma_{\omega_i} - 2\sum_{j=b^{[(\omega_{i+1}-1)\rho]}}^{\infty} \frac{1}{j^\alpha}.$$

Let now the sequence $\omega$ be such, that $\lambda \geq -\frac{\log_c \sigma_{\omega_i}}{\omega_i} - \epsilon$ for all $i$ and such, that

$$\sum_{j=b^{[(\omega_{i+1}-1)\rho]}}^{\infty} \frac{1}{j^\alpha} < \frac{1}{4}\cdot\frac{1}{c^{\omega_i\cdot(\lambda-\epsilon)}}$$

for all $i$. Then

$$\left|\widehat{f_H^{(\omega)}}(k_i)\right| \geq \frac{1}{c^{\omega_i\cdot(\lambda-\epsilon)}} - \frac{2}{4}\cdot\frac{1}{c^{\omega_i\cdot(\lambda-\epsilon)}} \geq \frac{1}{2\cdot c^{(\lambda-\epsilon)}}\cdot\frac{1}{k_i^{\lambda-\epsilon}}$$

for all $i$.

Therefore $\beta(G,\varphi,H,\psi,\alpha) \leq \lambda(G,\varphi,H,\psi,\alpha)$ and the result follows. $\square$

Although we now have an exact formula for $\beta$, it can, however, not be computed in finite time. Consequently, until now we do not even know, for example, the value of $\beta(\mathbb{Z}_2,\mathrm{id},\mathbb{Z}_6,\mathrm{id},\alpha)$. But we can use Theorem 5 to obtain good estimates for $\beta(G,\varphi,H,\psi,\alpha)$. Using the trivial estimate $|\gamma(j,k)| \leq 1$ together with Theorem 5 would lead to $\beta(G,\varphi,H,\psi,\alpha) \geq (\alpha-1)\cdot\rho\cdot\log(b)/\log(c)$. An upper bound and a sharper lower bound is given by the following result.

**Theorem 6.** *Let $G$ and $H$ be of order $b$ and $c$ and let $\varphi$ and $\psi$ be suitable bijections. Assume, that there is a positive integer $N$, such that $b|c^N$. Let $\rho = \rho(b,c)$ be defined as in Definition 6. Then with $\theta := \rho\cdot\log(b)/\log(c)$ we have*

$$\alpha\cdot\theta - \min(\theta/2, 2\theta-1) \leq \beta(G,\varphi,H,\psi,\alpha) \leq \alpha\cdot\theta + (1-\theta).$$

**Remark.** Note that $0 < \theta \leq 1$ and that the upper and the lower bound differ at most by 2/3. If $b$ and $c$ are as in Section 5 then $\theta = 1$ and we obtain the bounds from Theorem 4.

*Proof of the theorem.* By Cauchy's inequality we have for $c^{w-1} \leq k < c^w$ and with

$$\sigma_w(k) := \sum_{v=[(w-1)\rho]+1}^{\infty} \frac{1}{b^{v\alpha}} \sum_{j=b^{v-1}}^{b^v-1} |\gamma(j,k)|,$$

that

$$|\sigma_w(k)| \leq \sum_{v=[(w-1)\rho]+1}^{\infty} \frac{b^{v/2}}{b^{v\alpha}} \left( \sum_{j=b^{v-1}}^{b^v-1} |\gamma(j,k)|^2 \right)^{1/2}.$$

By the Bessel inequality for $L_2([0,1))$ with the orthonormal basis $W_{G,\varphi}$

$$\begin{aligned} \sum_{j=b^{v-1}}^{b^v-1} |\gamma(j,k)|^2 &= \sum_{j=b^{v-1}}^{b^v-1} \left| \int_0^1 {}_H\mathrm{wal}_k(x) \cdot \overline{{}_G\mathrm{wal}_j(x)}\, dx \right|^2 \\ &\leq \sum_{j=0}^{b^v-1} \left| \int_0^1 {}_H\mathrm{wal}_k(x) \cdot \overline{{}_G\mathrm{wal}_j(x)}\, dx \right|^2 \leq \| {}_H\mathrm{wal}_k \|_2^2 \\ &= \int_0^1 {}_H\mathrm{wal}_k(x) \cdot \overline{{}_H\mathrm{wal}_k(x)}\, dx = 1 \end{aligned}$$

for all $v$.

So

$$|\sigma_w(k)| \leq \sum_{v=[(w-1)\rho]+1}^{\infty} \frac{b^{v/2}}{b^{v\alpha}} \leq C_1(b,\alpha) \cdot \frac{1}{b^{w\cdot\rho\cdot(\alpha-1/2)}}$$

for all $w$ and $k$ (and a constant $C_1(b,\alpha)$ depending only on $b$ and $\alpha$) and we obtain $\lambda \geq (\alpha - 1/2) \cdot \theta$.

To obtain a further lower bound for $\lambda$ we estimate the coefficients $\gamma(j,k)$ "individually": For $k$ given with $c^{w-1} \leq k < c^w$, for $v$ with

$$[(w-1)\rho] + 1 \leq v \leq (w-1) \cdot \frac{\log c}{\log b}$$

and for $j$ with $b^{v-1} \leq j < b^v$ we have $1/c^{w-1} \leq 1/b^v$ and therefore by Lemma 2 and since ${}_G\mathrm{wal}_j$ is constant on intervals of the form $[A/b^v, (A+1)/b^v)$:

$$\begin{aligned} |\gamma(j,k)| &= \left| \int_0^1 {}_G\mathrm{wal}_j(x) \cdot \overline{{}_H\mathrm{wal}_k(x)}\, dx \right| \\ &= \left| \sum_{a=0}^{b^v-1} \int_{\frac{1}{c^{w-1}}\left\lceil \frac{a\cdot c^{w-1}}{b^v} \right\rceil}^{\frac{1}{c^{w-1}}\left\lceil \frac{a\cdot c^{w-1}}{b^v} \right\rceil} {}_G\mathrm{wal}_j(x) \cdot \overline{{}_H\mathrm{wal}_k(x)}\, dx \right| \leq \frac{b^v}{c^{w-1}}. \end{aligned}$$

So

$$\begin{aligned}\sigma_w &\le \sum_{v=[(w-1)\rho]+1}^{\left[(w-1)\cdot\frac{\log c}{\log b}\right]} \frac{1}{c^{w-1}}\cdot\frac{1}{b^{v(\alpha-2)}} + \sum_{v=\left[(w-1)\cdot\frac{\log c}{\log b}\right]+1}^{\infty} \frac{1}{b^{v(\alpha-1)}}\\ &\le C_2(b,c,\alpha)\cdot\left(\frac{1}{c^w}\cdot\frac{1}{b^{w\cdot\rho\cdot(\alpha-2)}}+\frac{1}{c^{w\cdot(\alpha-1)}}\right)\\ &\le C_2(b,c,\alpha)\cdot\left(\frac{1}{c^{w\cdot(1+\theta\cdot(\alpha-2))}}+\frac{1}{c^{w\cdot(\alpha-1)}}\right)\\ &\le C_2(b,c,\alpha)\cdot\frac{1}{c^{w\cdot(1+\theta\cdot(\alpha-2))}}\end{aligned}$$

with the constant $C_2(b,c,\alpha)$ depending only on $b,c$ and $\alpha$, and since $\theta\le 1$.

From the definition of $\lambda(G,\varphi,H,\psi,\alpha)$ we immediately get $\lambda\ge\theta\cdot\alpha-2\theta+1$ and by Theorem 5 the lower bound for $\beta(G,\varphi,H,\psi,\alpha)$ follows.

To show the upper bound it suffices to prove the following: For any positive integer $w$ let $v:=[(w-1)\cdot\rho]+1$. Then there is a $k$ with $c^{w-1}\le k<c^w$, such that

$$\sum_{j=b^{v-1}}^{b^v-1}|\gamma(j,k)|\ge\frac{b^{v-1}}{c^w}.$$

Since then for all $w$

$$\sigma_w\ge C_3(b,c,\alpha)\cdot\frac{1}{b^{w\cdot\rho\cdot\alpha}}\cdot\frac{b^{w\cdot\rho}}{c^w}\ge C_3(b,c,\alpha)\cdot\frac{1}{c^{w\cdot(1+\theta\cdot(\alpha-1))}}$$

(here again the constant is depending only on $b,c$ and $\alpha$ ), and by Theorem 5 the upper bound follows.

Now by Lemma 8 and 4:

$$\begin{aligned}\sum_{j=b^{v-1}}^{b^v-1}|\gamma(j,k)| &= \sum_{j=0}^{b^v-1}|\gamma(j,k)|\\ &\ge\left|\int_0^1 {}_H\mathrm{wal}_k(x)\cdot\sum_{j=0}^{b^v-1}\overline{{}_G\mathrm{wal}_j(x)}\,dx\right|\\ &= b^v\cdot\left|\int_0^{b^{-v}} {}_H\mathrm{wal}_k(x)\;dx\right|.\end{aligned}$$

Because of the special form of $v$ the fraction $A:=c^{w-1}/b^{v-1}$ is an integer, but $A/b$ is not. Let the integer $E$ be such, that

$$\frac{E}{c^{w-1}}+y=\frac{1}{b^v},$$

where $0 < y < 1/c^{w-1}$. Then

$$E = \frac{A}{b} - c^{w-1} \cdot y$$

and

$$\frac{1}{b} \cdot \frac{1}{c^{w-1}} \leq y \leq \left(1 - \frac{1}{b}\right) \cdot \frac{1}{c^{w-1}}.$$

Now by Lemma 2

$$\begin{aligned} \left| \int_0^{b^{-v}} {}_H\mathrm{wal}_k\,(x)\ dx \right| &= \left| \int_{\frac{E}{c^{w-1}}}^{\frac{E}{c^{w-1}}+y} {}_H\mathrm{wal}_k\,(x)\ dx \right| \\ &= \left| \int_0^{y} {}_H\mathrm{wal}_k\,(x)\ dx \right| \\ &= \frac{1}{c^{w-1}} \left| \int_0^{y \cdot c^{w-1}} {}_H\mathrm{wal}_{k_w}\,(x)\ dx \right|. \end{aligned}$$

Here

$$\frac{1}{b} \leq z := y \cdot c^{w-1} \leq 1 - \frac{1}{b}$$

and $k_w \neq 0$ is such, that $k = k_w \cdot c^{w-1} + \cdots + k_1$.

By Lemma 4

$$\begin{aligned} \sum_{l=1}^{c-1} \int_0^z {}_H\mathrm{wal}_l\,(x)\ dx &= \int_0^z \left( \sum_{l=0}^{c} {}_H\mathrm{wal}_l\,(x) \right) - 1\, dx \\ &= \begin{cases} c \cdot z - z & \text{if } \ z \leq 1/c \\ 1 - z & \text{if } \ z > 1/c \end{cases}. \end{aligned}$$

So there exists an $l \in \{1, \ldots, c-1\}$, such that

$$\left| \int_0^z {}_H\mathrm{wal}_l\,(x)\ dx \right| \geq \frac{1}{c} \cdot \min\,(c \cdot z - z, 1 - z) \geq \frac{1}{b \cdot c}.$$

Hereby we have shown the existence of a $k$ with $c^{w-1} \leq k < c^w$ and with

$$\sum_{j=b^{v-1}}^{b^v - 1} |\gamma\,(j, k)| \geq \frac{b^{v-1}}{c^w}.$$

This completes the proof. $\square$

## 7. Summary of the base change results.

In this section, for the sake of the reader, we collect and summarize the results on the base change coefficient given in Sections 4-6.

Given two finite abelian groups, $G, H$, we distinguished three cases, according to the relations between the group orders:

1) If $|G|$ did not divide any (integral) power of $|H|$, we found, that the relation between the convergence classes was quite bad. We obtained the non-improvable result
$$\beta\,(G, \varphi, H, \psi, \alpha) = 1$$
for all $\alpha > 1$.
2) If some power of $|G|$ equalled some power of $|H|$, the results showed very good relations between the convergence classes:
$$\beta\,(G, \varphi, H, \psi, \alpha) = \alpha - \beta_{G,H,\varphi,\psi},$$
where $\beta_{G,H,\varphi,\psi}$ was a finitely computable constant (see Definition 5). Estimates of $\beta_{G,H,\varphi,\psi}$ lead to an estimate of $\beta(G, \varphi, H, \psi, \alpha)$:
$$\alpha - 1/2 < \beta\,(G, \varphi, H, \psi, \alpha) \leq \alpha.$$
3) If $|G|$ divided some power of $|H|$, we had to formulate the result using the, in general, not finitely computable constant $\lambda$ (see Definition 7). Then
$$\beta\,(G, \varphi, H, \psi, \alpha) = \lambda\,(G, \varphi, H, \psi, \alpha)\,.$$
We were able to give good estimates for this constant, leading to the inequalities
$$\alpha \cdot \theta - \min\,(\theta/2, 2\theta - 1) \leq \beta\,(G, \varphi, H, \psi, \alpha) \leq \alpha \cdot \theta + (1 - \theta),$$
where $\theta$ is a constant in $(0, 1]$, depending only on the group orders (see Definition 6 and the formulation of Theorem 6).

Note, that the second case is actually a special case of the third: It corresponds to the value $\theta = 1$. The resulting bounds for $\beta(G, \varphi, H, \psi, \alpha)$ coincide.

## 8. An application to quasi-Monte Carlo integration.

In a series of papers (see for example [**14**], [**11**], [**9**], [**15**],...) a so-called "digital lattice rule" for the numerical integration of functions defined on the $s$-dimensional unit cube $[0, 1)^s$ was developed. The essential observation of this method is, that certain classes of functions $f : [0, 1)^s \to \mathbb{C}$ can be approximately integrated with the help of so-called "digital nets" in a much more accurate way than with other methods. The main result of this "digital lattice rule" is an integration error estimate, which was given in improved and generalized form in [**14**], [**9**] and [**27**].

In this section we will give just the necessary preliminaries to state and prove the most general and sharpest form of this error estimate until now, based on Theorems 3 and 6. The error estimate is then given in Theorem 7.

We begin by extending the concept of Walsh systems over groups to arbitrary dimensions.

**Definition 8.** Let $G$ be a finite abelian group and $\varphi$ a corresponding bijection. Let $W_{G,\varphi}$ be the system of Walsh functions over $G$ and $\varphi$. For an integer $s \geq 2$ and non-negative integers $n_1, \ldots, n_s$ let

$$_{G,\varphi}\mathrm{wal}_{n_1,\ldots,n_s} : [0,1)^s \to \mathbb{C}$$

be defined by

$$_{G,\varphi}\mathrm{wal}_{n_1,\ldots,n_s}(x_1,\ldots,x_s) := \prod_{i=1}^{s} {}_{G,\varphi}\mathrm{wal}_{n_i}(x_i),$$

and

$$W^s_{G,\varphi} := \{{}_{G,\varphi}\mathrm{wal}_{n_1,\ldots,n_s} : n_1,\ldots,n_s \geq 0\}$$

is called the system of $s$-dimensional Walsh functions over $G$ and $\varphi$.

**Definition 9.** For $G$ and $\varphi$ given, for an integer $s \geq 2$ and real numbers $\alpha > 1$ and $C > 0$ let $_{G,\varphi}E^{\alpha}_s(C)$ denote the class of all functions $f : [0,1)^s \to \mathbb{C}$ which are representable by an $s$-dimensional Walsh series

$$f(x_1,\ldots,x_s) = \sum_{n_1,\ldots,n_s=0}^{\infty} \hat{f}_G(n_1,\ldots,n_s) \cdot {}_G\mathrm{wal}_{n_1,\ldots,n_s}(x_1,\ldots,x_s)$$

with Walsh coefficients $\hat{f}_G(n_1,\ldots,n_s) \in \mathbb{C}$ satisfying

$$\left|\hat{f}_G(n_1,\ldots,n_s)\right| \leq C \cdot (\overline{n_1} \cdots \overline{n_s})^{-\alpha}$$

for all $(n_1,\ldots,n_s) \neq (0,\ldots,0)$. (Here $\overline{n} := \max(1,n)$.)

**Definition 10.** Let $G$ and $H$ be finite abelian groups and $\varphi, \psi$ corresponding bijections. For an arbitrary dimension $s$ and real numbers $\alpha > 1$ we define: $\beta(G,\varphi,H,\psi,\alpha,s) := \sup\Big\{\beta > 1 :$ for all $C > 0$ there is a $C' > 0$ with $_{G,\varphi}E^{\alpha}_s(C) \subseteq {}_{H,\psi}E^{\beta}_s(C')\Big\}$.

Again $\beta(G,\varphi,H,\psi,\alpha,s) := 1$ if no such $\beta$ exists.

For this multi-dimensional extension we can show:

**Lemma 11.** $\beta(G,\varphi,H,\psi,\alpha,s) = \beta(G,\varphi,H,\psi,\alpha)$.

*Proof.* The proof is exactly the same as the proof of Lemma 2 in [12]. We omit the obvious adaptions. □

The concept of digital $(t, m, s)$-nets – these are point sets in the $s$-dimensional unit cube of a special structure – was introduced by Niederreiter ([**9**], [**16**], see also [**17**]) and was subsequently investigated in detail by various authors. (Special examples of digital $(t, m, s)$-nets already can be found in Sobol' [**25**] and Faure [**4**].)

**Definition 11.** Let $b \geq 2,\ s \geq 1$, and $0 \leq t \leq m$ be integers. Then a point set $P = \{\mathbf{x}_0 \dots, \mathbf{x}_{N-1}\}$ consisting of $N = b^m$ points of $[0,1)^s$ forms a $(t, m, s)$-net in base $b$ if the number of $n$ with $0 \leq n \leq N-1$, for which $\mathbf{x}_n$ is in the subinterval $J$ of $[0,1)^s$, is $b^t$ for all $J = \prod_{i=1}^{s} \left[a_i b^{-d_i}, (a_i+1)b^{-d_i}\right)$ with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$, and with $s$-dimensional volume $b^{t-m}$.

**Definition 12.** Let $b \geq 2,\ s \geq 1$, and $m \geq 1$ be integers. We consider the following construction principle for point sets $P$ consisting of $b^m$ points in $[0,1)^s$. We choose:

(i) A commutative ring $R$ with identity and $|R| = b$;
(ii) a bijection $\tau : R \to Z_b = \{0, 1, \dots, b-1\}$ with $\tau(0) = 0$;
(iii) elements $c_{jr}^{(i)} \in R$ for $1 \leq i \leq s$, $1 \leq j \leq m$, and $0 \leq r \leq m-1$.

For $n = 0, 1, \dots, b^m - 1$ let

$$n = \sum_{r=0}^{m-1} a_r(n) b^r \qquad \text{with all } a_r(n) \in Z_b$$

be the digit expansion of $n$ in base $b$. We put

$$x_n^{(i)} = \sum_{j=1}^{m} y_{nj}^{(i)} b^{-j} \qquad \text{for } 0 \leq n < b^m \ \text{ and } \ 1 \leq i \leq s,$$

with

$$y_{nj}^{(i)} = \tau\left(\sum_{r=0}^{m-1} c_{jr}^{(i)} \tau^{-1}(a_r(n))\right) \in Z_b$$

for $0 \leq n < b^m, 1 \leq i \leq s, 1 \leq j \leq m$.

If for some integer $t$ with $0 \leq t \leq m$ the point set

$$\mathbf{x}_n = \left(x_n^{(1)}, \dots, x_n^{(s)}\right) \ \in [0,1)^s \qquad \text{for } n = 0, 1, \dots, b^m - 1$$

is a $(t, m, s)$-net in base $b$, then it is called a digital $(t, m, s)$-net constructed over R with respect to the bijection $\tau$.

The most powerful construction methods for digital $(t, m, s)$-nets of high quality (i.e. with small $t$) are based on methods from algebraic geometry. See for example [**18**], [**19**].

**Lemma 12.** *Let $R$ be a finite commutative ring of order $c$ with additive group $H$ and let $\psi : \{0,\dots,c-1\} \to R$ be a bijection with $\psi(0) = 0$. Let $\mathbf{x}_0,\dots \mathbf{x}_{N-1}$ be a digital $(t,m,s)$-net constructed over $R$ with respect to $\tau = \psi^{-1}$. Then for all $\alpha > 1, C > 0$ we have*

$$\left| \int_{[0,1)^s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right| \le K \cdot c^{t\alpha} \cdot \frac{(\log N)^{s-1}}{N^{\alpha}}$$

*for all $f \in {}_{H,\psi}E_s^{\alpha}(C)$, where $K$ is a constant depending only on $s, c, C$ and $\alpha$.*

*Proof.* This is Theorem 1 in [9], stated in a slightly simplified form. □

In this result for the construction of the digital point set, a ring has to be used, which is based on the same additive group $G$ as the considered Walsh system. However, only certain rings are well suited for the construction of digital $(t,m,s)$-nets of highest quality.

Therefore it is sometimes more convenient to use a ring $R$ for the construction of the digital net which is based on another additive group $H$ as the considered Walsh system, which is based on, say, a group $G$. So we need a corresponding, more general integration error estimate.

In Theorem 4 in [27] such an estimate was given for the case $|G| = |H|$. We are now able to give an error estimate for the case, that for some positive integer $L$ we have: $|G|$ divides $|H|^L$. Our result - given in the subsequent Theorem - contains the above mentioned result of Wolf in [27] (Th.4).

**Theorem 7.** *Let $G$ and $H$ be finite abelian groups of orders $b$ and $c$, and $\varphi$ and $\psi$ corresponding bijections. Let $R$ be a commutative ring with additive group $H$. Assume, that there exists a positive integer $L$, such that $b|c^L$. Let $\rho(b,c)$ be defined like in Definition 6 and let*

$$\theta := \rho(b,c) \cdot \frac{\log b}{\log c}.$$

*Let $\mathbf{x}_1,\dots,\mathbf{x}_N$ be a digital $(t,m,s)$-net constructed over $R$ with respect to $\tau = \psi^{-1}$.*

**a)** *For all $\alpha > 1/\theta + \min(1/2, 2 - 1/\theta)$ and all $C > 0$ we have*

$$\left| \int_{[0,1)^s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right| \le K \cdot c^{t\cdot(\alpha\cdot\theta - \min(\theta/2, 2\theta-1))} \cdot \frac{(\log N)^{s-1}}{N^{\alpha\cdot\theta - \min(\theta/2, 2\theta-1)}}$$

*for all $f \in {}_{G,\varphi}E_s^{\alpha}(C)$.*

**b)** *Assume, that for some positive integers $M$ and $L$ even $b^M = c^L$ holds. Let $\beta_{G,H,\varphi,\psi}$ be defined like in Definition 5.*

*For all* $\alpha > 1 + \beta_{G,H,\varphi,\psi}$ *and all* $C > 0$ *we have*

$$\left|\int_{[0,1)^s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)\right| \leq K \cdot c^{t\cdot(\alpha-\beta_{G,H,\varphi,\psi})} \cdot \frac{(\log N)^{s-1}}{N^{\alpha-\beta_{G,H,\varphi,\psi}}}$$

*for all* $f \in {}_{G,\varphi}E_s^{\alpha}(C)$.

(*In both cases again the* $K$ *denote constants depending only on* $s, b, c, C$ *and* $\alpha$.)

*Proof.* Note, that in the proofs of the lower bound in Theorem 6 and of Theorem 3 we indeed even have proved a little more. From the proofs we even obtain the following:

- For all $\alpha > 1/\theta + \min(1/2, 2 - 1/\theta)$ and all $C > 0$ we have

$${}_{G,\varphi}E^{\alpha}(C) \subseteq {}_{H,\psi}E^{\alpha\cdot\theta - \min(\theta/2, 2\theta-1)}(C')$$

  for some $C' > 0$;

respectively if $G$ and $H$ are of comparable order:

- For all $\alpha > 1 + \beta_{G,H,\varphi,\psi}$ and all $C > 0$ we have

$${}_{G,\varphi}E^{\alpha}(C) \subseteq {}_{H,\psi}E^{\alpha-\beta_{G,H,\varphi,\psi}}(C')$$

  for some $C' > 0$.

From this and Lemma 12 the assertion of Theorem 7 now immediately follows. □

The results of Theorem 7 (but also the result of Theorem 2, stating that there are no base change connections in Case 1), also are reflected in numerical results. As a small sample, we give in the following some results on the numerical integration of a function $f \in {}_{\mathbb{Z}_2,\mathrm{id}}E_s^3(c)$ (i.e. $b = 2$) with digital nets over $\mathbb{Z}_2, \mathbb{Z}_5, \mathbb{F}_8$ and $\mathbb{Z}_{10}$, and with the Hammersley-Halton sequence.

As can be seen, we obtain excellent results for nets over $\mathbb{Z}_2$ and $\mathbb{F}_8$ (Case 2), good results for $\mathbb{Z}_{10}$ (Case 3) and the "worst" results for $\mathbb{Z}_5$ (Case 1) and the Hammersley-Halton sequence. In the last case (Case 1), the structure of the digital net does not play a role any more in integrating the function $f$. Only the small discrepancy of any digital net of high quality provides an integration error about as small as the error obtained by using the Hammersley-Halton sequence.

In the following table, there are two sections: In the first, the point sets contain between $2^{20}$ and $2^{21}$ points, in the second the range is from $2^{23}$ to $2^{24}$. In each section, the first column describes the point set, in the second the number of integration points $N$ is listed and in the third and fourth column, the integration error of a test function $f$ in 7, resp. 8 variables is shown. The function $f$ is the test function described for example in [**12**].

We use the digital nets over $\mathbb{Z}_5$ and $\mathbb{Z}_{10}$ described in [9], digital nets over $\mathbb{F}_8$ as they are used in [12] and digital nets over $\mathbb{Z}_2$ as they are generated in [24].

| Method | N | dim 7 | dim 8 |
|---|---|---|---|
| $\mathbb{Z}_2$ | 1048576 | 2.3280821e-12 | 1.3543555e-11 |
| $\mathbb{Z}_5$ | 1953125 | 4.3304898e-04 | 2.5758582e-04 |
| $\mathbb{F}_8$ | 2097152 | 2.0078383e-13 | 5.1528231e-10 |
| $\mathbb{Z}_{10}$ | 1000000 | 1.3743740e-05 | 5.0843230e-06 |
| Hamm. | 1048576 | 2.0067839e-03 | 2.2115463e-04 |
| $\mathbb{Z}_2$ | 8388608 | 2.3425705e-14 | 1.7175150e-13 |
| $\mathbb{Z}_5$ | 9765625 | 5.2695372e-05 | 2.3610550e-04 |
| $\mathbb{F}_8$ | 16777216 | 9.9364960e-15 | 1.0041967e-13 |
| $\mathbb{Z}_{10}$ | 10000000 | 1.2159300e-05 | 5.4922667e-06 |
| Hamm. | 8388608 | 7.9642499e-05 | 3.2087933e-05 |

**Table 1.** $N$ denotes the number of integration points, under dim 7 and dim 8 the integration errors of the 7- respective 8-dimensional test function are listed.

Finally we mention, that the above investigations also could be extended to investigations on the connection between Walsh series and Haar series (the problem does not really occur between different classes of Haar series) and the corresponding results could be used to reprove and improve results of Entacher ([2],[3]) on the numerical integration of Haar series. A forthcoming paper concerning this is in preparation. (See also [22].)

## 9. Some open problems.

Of course many questions remain open. In the following we restate just some of the – in our opinion – most challenging problems once more explicitely.

**Problem 1:** In practice, given $G$ and $H$ of comparable order, in the choice of bijections we are quite free. So it is obvious to choose $\varphi$ and $\psi$, such that $\beta_{G,H,\varphi,\psi}$ is minimal. We know (Theorem 4) that

$$0 \leq \beta_{G,H,\varphi,\psi} < \frac{1}{2}.$$

Is

$$\sup_{G,H} \min_{\varphi,\psi} \beta_{G,H,\varphi,\psi} < \frac{1}{2} \ ?$$

In fact we conjecture

$$\sup_{G,H} \min_{\varphi,\psi} \beta_{G,H,\varphi,\psi} = \lim_{h\to\infty} \beta_{\mathbb{Z}_2,\mathbb{Z}_{2^h},\mathrm{id},\mathrm{id}}$$

$$= \frac{1}{2} + \frac{\log \sin \frac{5\pi}{12}}{\log 2} = 0.4499...$$

(See also [**27**].)

**Problem 2:** Is

$$\beta_{\mathbb{Z}_2,\mathbb{Z}_{2^h},\mathrm{id},\mathrm{id}} = \min_{\varphi,\psi} \beta_{\mathbb{Z}_2,\mathbb{Z}_{2^h},\varphi,\psi} \ ?$$

We conjecture, yes.

**Problem 3:** We know a closed form for the exact values of $\beta_{\mathbb{Z}_2,\mathbb{Z}_{2^h},\mathrm{id},\mathrm{id}}$ (see Section 1). However, we do not know a closed form for the exact values of $\beta_{\mathbb{Z}_{2^h},\mathbb{Z}_2,\mathrm{id},\mathrm{id}}$.

**Problem 4:** Of course also in "Case Three", $b|c^N$ for some $N$, a solution in the form of Theorem 3 for groups of comparable order would be desirable. However, this seems to be quite difficult.

Until now we do not even know the exact value of $\beta(G,\varphi,H,\psi,\alpha)$ in "easiest" cases, like, for example, $\beta(\mathbb{Z}_2,\mathrm{id},\mathbb{Z}_6,\mathrm{id},\alpha)$.

**Problem 5:** Any improvement of the inequality given in Theorem 6 would be of interest.

**Problem 6:** Is it possible to give more exact results than Theorems 5 and 6 in the case that $G$ and $H$ are not of comparable order, but $|G|$ and $|H|$ have the same prime factors?

(For example, it might be easier to compute $\beta(\mathbb{Z}_6,\mathrm{id},\mathbb{Z}_{12},\mathrm{id},\alpha)$ than to compute $\beta(\mathbb{Z}_2,\mathrm{id},\mathbb{Z}_6,\mathrm{id},\alpha)$. )

**Problem 7:** Give at least numerical estimates for $\beta(G,\varphi,H,\psi,\alpha)$ for certain examples in "Case Three", maybe based on Theorem 5.

**Problem 8:** Motivated by Theorem 3 for "Case Two" and Theorem 6 it seems that in "Case Three" we should have

$$\beta(G,\varphi,H,\psi,\alpha) = \alpha\cdot\theta + \delta(G,\varphi,H,\psi)$$

with some quantity $\delta(G,\varphi,H,\psi)$ not depending on $\alpha$. Is this true?

**Problem 9:** Concerning the constant $\gamma$ considered in Section 3, we know that $\gamma(G,\varphi,H,\psi,\alpha) \le \beta(G,\varphi,H,\psi,\alpha)$. Can $\gamma$ also be estimated from below by some expression depending on $\beta$?

This is motivated by Theorem 1. However, note that $\beta$ is defined just over Walsh series, not over arbitrary functions as $\gamma$ is.

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Universität Salzburg
Hellbrunnerstrasse 34
A–5020 Salzburg
Austria
*E-mail address*: Gerhard.Larcher@sbg.ac.at

Universität Salzburg
Hellbrunnerstrasse 34
A–5020 Salzburg
Austria
*E-mail address*: pirsic@mat.sbg.ac.at