Pacific Journal of Mathematics

A COMMUTATION FORMULA FOR ROOT VECTORS IN QUANTIZED ENVELOPING ALGEBRAS

Nanhua Xi

Volume 189 No. 1 May 1999

PACIFIC JOURNAL OF MATHEMATICS
Vol. 189, No. 1, 1999

A COMMUTATION FORMULA FOR ROOT VECTORS IN QUANTIZED ENVELOPING ALGEBRAS

Nanhua Xi

Root vectors are important to understand quantized enveloping algebras. In this paper we establish a commutation formula for root vectors. By means of the formula we show that particular orders on root system are not necessary in constructing some integral bases of a quantized enveloping algebra (Theorem 2.4). Moreover using the formula we can show that certain PBW bases are orthogonal bases of the bilinear form considered by Kashiwara in his work on crystal bases, see 3.9.

In [**CK**] there is a commutation formula for root vectors, our formula here is stronger. For the bilinear form obtained through Drinfeld dual (see [**L5**, **LS**]) Lusztig and Levendorski-Soibelmana showed that certain PBW bases are orthogonal, see loc. cit. However the proofs in [**L5**, **LS**] essentially rely on the property [**L5**, 38.2.1] which does not hold for the bilinear form in [**K**], so it is not easy to use the methods of [**L5**, **LS**] to prove Theorem 3.9.

The paper is organized as follows. In Section 1 we fix some notation. In Section 2 we establish the commutation formula, then prove Theorem 2.4 and state two conjectures. In Section 3 we show that certain PBW bases are orthogonal bases of the bilinear form considered in [**K**].

1. Preliminaries.

1.1. Let U be the quantized enveloping algebra over $\mathbb{Q}(v)$ (v an indeterminate) corresponding to a Cartan matrix (a_{ij}) of rank n. Then U is an associative $\mathbb{Q}(v)$-algebra with generators $E_i,\ F_i,\ K_i, K_i^{-1}$ $(i = 1, 2, ..., n)$ which satisfy the quantized Serre relations. The algebra U has a Hopf algebra structure. Let $U_{\mathcal{A}}$ be the $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$-subalgebra of U generated by all divided powers $E_i^{(a)},\ F_i^{(a)}$ and $K_i,\ K_i^{-1}$. We refer to [**L2**] for the definitions, noting that for defining the divided powers we need to choose integers $d_i \in \{1, 2, 3\}$ such that $(d_i a_{ij})$ is symmetric. As usual we denote the positive parts and negative parts by $U^+,\ U^+_{\mathcal{A}},\ U^-,\ U^-_{\mathcal{A}}$ respectively.

1.2. Let $R \subset \mathbb{Z}^n$ be the root system with simple roots $\alpha_i = (a_{1i}, a_{2i}, ..., a_{ni})$. For $\mu = (\mu_1, ..., \mu_n) \in \mathbb{Z}^n$, we also write $\langle \mu, \alpha_i^\vee \rangle$ for μ_i. Define $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$

by $s_i\mu = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$. The reflections $s_1, s_2, ..., s_n$ generate the Weyl group W of the root system R. Denote by R^+ the set of positive roots. For $\lambda = \sum_{i=1}^{n} a_i\alpha_i,\ \mu = \sum_{j=1}^{n} b_j\alpha_j$, we define $(\lambda|\mu) = \left\langle \sum_{i=1}^{n} a_i\alpha_i, \sum_{i=1}^{n} d_j b_j \alpha_j^\vee \right\rangle$. We have $(\lambda|\mu) = (\mu|\lambda)$. The form $(\cdot|\cdot)$ is non-degenerate and is W-invariant.

Let T_i be the automorphisms $T'_{i,-1}$ of U in [**L5**, 37.1.3]. For each $w \in W$ we define T_w as in [**L5**]. We shall write $\Omega, \Psi : U \to U^{\text{opp}}$ the $\mathbb{Q}$-algebra homomorphisms defined by

$$\Omega E_i = F_i, \quad \Omega F_i = E_i, \quad \Omega K_i = K_i^{-1}, \quad \Omega v = v^{-1};$$

$$\Psi E_i = E_i, \quad \Psi F_i = F_i, \quad \Psi K_i = K_i^{-1}, \quad \Psi v = v.$$

We have $\Psi T_i \Psi = T_i^{-1}$, $\Omega T_i = T_i \Omega$. Let $\Omega' : U \to U$ be the $\mathbb{Q}(v)$-algebra automorphism defined by

$$\Omega' E_i = F_i, \quad \Omega' F_i = E_i, \quad \Omega' K_i = K_i^{-1}.$$

2. The commutation formula.

2.1. Let $s_{i_1} s_{i_2} \cdots s_{i_\nu}$ be a reduced expression of the longest element w_0 of W, thus $\nu = |R^+|$. We have a bijection $[1, \nu] \to R^+$ defined by

$$j \to s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}).$$

This gives rise to a total order on R^+. If $\beta \in R^+$ corresponds to j, we set $w_\beta = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}$. Then define

$$E_\beta^{(a)} = T_{w_\beta}(E_{i_j}^{(a)}) \in U^+, \quad F_\beta^{(a)} = T_{w_\beta}(F_{i_j}^{(a)}) \in U^-.$$

We have $E_\beta^{(a)} \in U_{\mathcal{A}}^+$ and $F_\beta^{(a)} \in U_{\mathcal{A}}^- \quad (a \in \mathbb{N})$.

(a) Let $\mathbf{i} = (i_1, \cdots, i_\nu)$. It is known the following elements

$$E_{\mathbf{i}}^A = E_{i_1}^{(a_1)} T_{i_1}(E_{i_2}^{(a_2)}) \cdots T_{i_1} T_{i_2} \cdots T_{i_{\nu-1}}(E_{i_\nu}^{(a_\nu)}), \quad A = (a_1, \ldots, a_\nu) \in \mathbb{N}^\nu,$$

form an $\mathcal{A}$-base of $U_{\mathcal{A}}$, see [**DL**].

(b) Let $w, u \in W$ such that $l(wu) = l(w) + l(u)$ and let $s_{i_1} \cdots s_{i_k}$ be a reduced expression of u. Let $U_{\mathcal{A},u}^+$ be the $\mathcal{A}$-submodule of $U_{\mathcal{A}}^+$ generated by the elements $E_{i_1}^{(a_1)} T_{i_1}(E_{i_2}^{(a_2)}) \cdots T_{i_1} \cdots T_{i_{k-1}}\ (E_{i_k}^{(a_k)})$ $(a_1, \cdots, a_k \in \mathbb{N})$. Then $T_w(U_{\mathcal{A},u}^+)$ is contained in $U_{\mathcal{A}}^+$ and $U_{\mathcal{A},u}^+$ is independent of the choice of the reduced expression. See [**DL**, **L2**].

2.2. We have seen that for each reduced expression $s_{i_1} \cdots s_{i_\nu}$ of w_0, one can construct an $\mathcal{A}$-basis $\{E_{\mathbf{i}}^A\}_{A \in \mathbb{N}^\nu}$ of $U_{\mathcal{A}}^+$. Note that the element $E_{\mathbf{i}}^A$ is a product of some divided powers of root vectors and the order of the factors in the product is determined by the reduced expression. We will show that we can arrange the product in any fixed total order on R^+ (see Theorem

2.4). For the purpose we need the following result, which is stronger than [**CK**, Lemma 1.7].

Theorem 2.3. *Let $s_{i_1}\cdots s_{i_k}$ be a reduced expression and $1\le j<k$. Let $\beta_m = s_{i_1}\cdots s_{i_{m-1}}\ (\alpha_{i_m})$ $(1\le m\le k)$ and define $E^{(a)}_{\beta_m} = T_{i_1}\cdots T_{i_{m-1}}(E^{(a)}_{i_m})$ $(a\in\mathbb{N})$. Then we have*

$$(*)\qquad E^{(a)}_{\beta_k}E^{(b)}_{\beta_j} - v^{ab(\beta_j|\beta_k)}E^{(b)}_{\beta_j}E^{(a)}_{\beta_k} = \sum_{\substack{a_j,\dots,a_k\in\mathbb{N}\\ a_j<b\\ a_k<a}} \rho(a_j,a_{j+1},\dots,a_k)E^{(a_j)}_{\beta_j}E^{(a_{j+1})}_{\beta_{j+1}}\cdots E^{(a_k)}_{\beta_k},$$

where $\rho(a_j,a_{j+1},\dots,a_k)\in\mathcal{A}$. Note that we have $a_j\beta_j+\cdots+a_k\beta_k = a\beta_k+b\beta_j$ if $\rho(a_j,a_{j+1},\dots,a_k)\neq 0$.

Proof. We use induction on $k-j$ to prove the theorem. We may assume that $j=1$. To see this, apply $T^{-1}_{i_{j-1}}\cdots T^{-1}_{i_1}$ to the wanted identity $(*)$.

Let u be the shortest element of the coset $s_{i_1}\cdots s_{i_{k-1}}\langle s_{i_{k-1}}, s_{i_k}\rangle$ (we use $\langle s_{i_{k-1}}, s_{i_k}\rangle$ for the subgroup of W generated by $s_{i_{k-1}}, s_{i_k}$). Then $s_{i_1}\cdots s_{i_{k-1}} = uu'$, where $u'\in\langle s_{i_{k-1}}, s_{i_k}\rangle$ and $l(uu') = l(u)+l(u')$. Note that $u(\alpha_{i_k})$ and $u(\alpha_{i_{k-1}})$ are contained in R^+.

When $u=e$ is the neutral element of W, the required identity $(*)$ follows from the formulas in [**L2**].

From now on, we suppose that $l(u)\ge 1$. Assume that $(*)$ is true if j,k are replaced by j',k' respectively with $1\le j'<k'\le k$ and $k'-j'<k-j$, and assume that the Cartan matrix includes no factors of type G_2.

Case A. $u = s_{i_1}\cdots s_{i_m}$ and $u' = s_{i_{m+1}}\cdots s_{i_{k-1}}$ for some $m\in[1,k-2]$.

When $u'(\alpha_{i_k})$ is a simple root α_i, then $\beta_k = u(\alpha_i)$. Moreover, we have $E^{(a)}_{\beta_k} = T_u(E^{(a)}_i)$. Note that $l(u)\le k-2$. By induction hypothesis we see

$$\text{(a)}\qquad E^{(a)}_{\beta_k}E^{(b)}_{\beta_1} - v^{ab(\beta_1|\beta_k)}E^{(b)}_{\beta_1}E^{(a)}_{\beta_k} = \sum_{\substack{a_1,\dots,a_m,a_k\in\mathbb{N}\\ a_1<b\\ a_k<a}} \rho(a_1,\dots,a_m,a_k)E^{(a_1)}_{\beta_1}\cdots E^{(a_m)}_{\beta_m}E^{(a_k)}_{\beta_k},$$

where $\rho(a_1,\dots,a_m,a_k)\in\mathcal{A}$. Thus the desired identity $(*)$ is true in this case.

Now assume that $u'(\alpha_{i_l})$ is not a simple root. We have the following cases.

(1) $u' = s_{i_{k-1}}$ and $\langle\alpha_{i_k},\alpha^\vee_{i_{k-1}}\rangle = -1$, then $u = s_{i_1}\cdots s_{i_{k-2}}$.

(2) $u' = s_{i_k}s_{i_{k-1}}$, and $\langle \alpha_{i_k}, \alpha_{i_{k-1}}^\vee \rangle = -1$, $\langle \alpha_{i_{k-1}}, \alpha_{i_k}^\vee \rangle = -2$, that is, $\alpha_{i_{k-1}}$ is a long root and α_{i_k} is a short root. We have $d_{i_{k-1}} = 2$, $d_{i_k} = 1$, and $u = s_{i_1} \cdots s_{i_{k-3}}$.

(3) $u' = s_{i_{k-1}}$ and $\langle \alpha_{i_k}, \alpha_{i_{k-1}}^\vee \rangle = -2$, $\langle \alpha_{i_{k-1}}, \alpha_{i_k}^\vee \rangle = -1$. Then $d_{i_{k-1}} = 1$, $d_{i_k} = 2$, and $u = s_{i_1} \cdots s_{i_{k-2}}$.

(4) $u' = s_{i_k}s_{i_{k-1}}$ and $\langle \alpha_{i_k}, \alpha_{i_{k-1}}^\vee \rangle = -2$, $\langle \alpha_{i_{k-1}}, \alpha_{i_k}^\vee \rangle = -1$. Then $d_{i_{k-1}} = 1$, $d_{i_k} = 2$, and $u = s_{i_1} \cdots s_{i_{k-3}}$.

Define $\alpha = u(\alpha_{i_{k-1}})$ and $\gamma = u(\alpha_{i_k})$, they are positive roots. Set $E_\alpha = T_u(E_{i_{k-1}})$ and $E_\gamma = T_u(E_{i_k})$. We have $E_\alpha,\ E_\gamma \in U_{\mathcal{A}}^+$. In cases (1) and (3), we have $\alpha = \beta_{k-1}$ and $E_\alpha = E_{\beta_{k-1}}$. In cases (2) and (4), we have $\gamma = \beta_{k-2}$ and $E_\gamma = E_{\beta_{k-2}}$.

By induction hypothesis we get

(b) $E_\alpha E_{\beta_1} - v^{(\beta_1|\alpha)} E_{\beta_1} E_\alpha = \sum_{a_2,\dots,a_{k-2}\in\mathbb{N}} \rho'(a_2,\dots,a_{k-2}) E_{\beta_2}^{(a_2)} \cdots E_{\beta_{k-2}}^{(a_{k-2})}$,

where $\rho'(a_2,\dots,a_{k-2}) \in \mathcal{A}$. We shall simply write X for the right hand side of the above identity. Then $E_\alpha E_{\beta_1} - v^{(\beta_1|\alpha)} E_{\beta_1} E_\alpha = X$. Note that $a_2\beta_2 + \cdots + a_{k-2}\beta_{k-2} = \beta_1 + \alpha$ if $\rho'(a_2,\dots,a_{k-2}) \neq 0$. Moreover, for cases (2) and (4), $a_{k-2} = 0$ if $\rho'(a_2,\dots,a_{k-2}) \neq 0$.

(c) $E_\gamma E_{\beta_1} - v^{(\beta_1|\gamma)} E_\gamma E_{\beta_1} = \sum_{a_2,\dots,a_{k-2}\in\mathbb{N}} \rho''(a_2,\dots,a_{k-2}) E_{\beta_2}^{(a_2)} \cdots E_{\beta_{k-2}}^{(a_{k-2})}$,

where $\rho''(a_2,\dots,a_{k-2}) \in \mathcal{A}$. We shall simply write Y for the right hand side of the above identity. Then $E_\gamma E_{\beta_1} - v^{(\beta_1|\gamma)} E_{\beta_1} E_\gamma = Y$. Note that $\alpha_2\beta_2 + \cdots + a_{k-2}\beta_{k-2} = \beta_1 + \gamma$ if $\rho'(a_2,\dots,a_{k-2}) \neq 0$. Moreover, for cases (2) and (4), $a_{k-2} = 0$ if $\rho''(a_2,\dots,a_{k-2}) \neq 0$.

Now assume that we are in case (1), then

(d)
$$\begin{aligned} E_{\beta_k} &= T_u T_{i_{k-1}}(E_{i_k}) = T_u(E_{i_k}E_{i_{k-1}} - v^{-d}E_{i_{k-1}}E_{i_k}) \\ &= E_\gamma E_{\beta_{k-1}} - v^{-d} E_{\beta_{k-1}} E_\gamma, \end{aligned}$$

where $d = d_{i_{k-1}}$.

Therefore we have

(e)

$$\begin{aligned} E_{\beta_k}E_{\beta_1} &= E_\gamma E_{\beta_{k-1}} E_{\beta_1} - v^{-d} E_{\beta_{k-1}} E_\gamma E_{\beta_1} \\ &= E_\gamma(v^{(\beta_1|\beta_{k-1})} E_{\beta_1} E_{\beta_{k-1}} + X) - v^{-d} E_{\beta_{k-1}}(v^{(\beta_1|\gamma)} E_{\beta_1} E_\gamma + Y) \\ &= v^{(\beta_1|\beta_{k-1})}(v^{(\beta_1|\gamma)} E_{\beta_1} E_\gamma + Y) E_{\beta_{k-1}} + E_\gamma X \\ &\quad -v^{-d} v^{(\beta_1|\gamma)}(v^{(\beta_1|\beta_{k-1})} E_{\beta_1} E_{\beta_{k-1}} + X) E_\gamma - v^{-d} E_{\beta_{k-1}} Y. \end{aligned}$$

Repeatedly using induction hypothesis we get

(f) $E_\gamma X = v^{(\gamma|\beta_1+\beta_{k-1})} X E_\gamma + \sum_{b_2,\dots,b_{k-2}\in\mathbb{N}} \xi'(b_2,\dots,b_{k-2}) E_{\beta_2}^{(b_2)} \cdots E_{\beta_{k-2}}^{(b_{k-2})}$,
where $\xi'(b_2,\dots,b_{k-2}) \in \mathcal{A}$.

(g) $E_{\beta_{k-1}} Y = \sum_{\substack{b_2,\dots,b_{k-1}\in\mathbb{N} \\ b_{k-1}\le 1}} \xi''(b_2,\dots,b_{k-1}) E_{\beta_2}^{(b_2)} \cdots E_{\beta_{k-1}}^{(b_{k-1})}$,
where $\xi''(b_2,\dots,b_{k-1}) \in \mathcal{A}$.

We have

(h) $(\gamma|\beta_1+\beta_{k-1}) = (\beta_1|\gamma) + (\beta_{k-1}|\gamma) = (\beta_1|\gamma) + (\alpha_{i_{k-1}}|\alpha_{i_k}) = (\beta_1|\gamma) - d$. Moreover $\beta_k = \beta_{k-1} + \gamma$.

Combining (e)-(h) we get

(i) $E_{\beta_k} E_{\beta_1} - v^{(\beta_1|\beta_k)} E_{\beta_1} E_{\beta_k} = \sum_{a_1,\dots,a_{k-1}\in\mathbb{N}} \eta(a_2,\dots,a_{k-1}) E_{\beta_2}^{(a_2)} \cdots E_{\beta_{k-1}}^{(a_{k-1})}$,
where $\eta(a_2,\dots,a_{k-1}) \in \mathcal{A}$.

Using induction on a, b, and using (i) and induction hypothesis repeatedly, we see

(j) $E_{\beta_k}^{(a)} E_{\beta_1}^{(b)} - v^{ab(\beta_1|\beta_k)} E_{\beta_1}^{(b)} E_{\beta_k}^{(a)} = \sum_{\substack{a_1,\cdots,a_k\in\mathbb{N} \\ a_1<b \\ a_k<a}} \rho(a_1,\cdots,a_k) E_{\beta_1}^{(a_1)} \cdots E_{\beta_k}^{(a_k)}$,

where $\rho(a_1,\dots,a_k) \in \mathcal{A}$ (here we need 2.1 (a)). Thus in case (1) the identity $(*)$ is true.

Now assume that we are in case (2). Then

(k) $E_{\beta_k} = T_u T_{i_k} T_{i_{k-1}}(E_{i_k}) = T_u(E_{i_{k-1}} E_{i_k} - v^{-2} E_{i_k} E_{i_{k-1}}) = E_\alpha E_{\beta_{k-2}} - v^{-2} E_{\beta_{k-2}} E_\alpha$.

As a similar argument for case (1) we see that the identity $(*)$ is true in this case.

Now assume that we are in case (3). Then

(l)

$$\begin{aligned} E_{\beta_k} = T_u T_{i_{k-1}}(E_{i_k}) &= T_u(E_{i_k} E_{i_{k-1}}^{(2)} - v^{-1} E_{i_{k-1}} E_{i_k} E_{i_{k-1}} + v^{-2} E_{i_{k-1}}^{(2)} E_{i_k}) \\ &= E_\gamma E_{\beta_{k-1}}^{(2)} - v^{-1} E_{\beta_{k-1}} E_\gamma E_{\beta_{k-1}} + v^{-2} E_{\beta_{k-1}}^{(2)} E_\gamma. \end{aligned}$$

We have

(m)
$$\begin{aligned}&E_\gamma E^{(2)}_{\beta_{k-1}}E_{\beta_1}\\&=\frac{1}{[2]}E_\gamma E_{\beta_{k-1}}(v^{(\beta_1|\beta_{k-1})}E_{\beta_1}E_{\beta_{k-1}}+X)\\&=\frac{1}{[2]}v^{(\beta_1|\beta_{k-1})}E_\gamma(v^{(\beta_1|\beta_{k-1})}E_{\beta_1}E_{\beta_{k-1}}+X)E_{\beta_{k-1}}+\frac{1}{[2]}E_\gamma E_{\beta_{k-1}}X\\&=v^{2(\beta_1|\beta_{k-1})}(v^{(\beta_1|\gamma)}E_{\beta_1}E_\gamma+Y)E^{(2)}_{\beta_{k-1}}+\frac{1}{[2]}v^{(\beta_1|\beta_{k-1})}E_\gamma XE_{\beta_{k-1}}\\&\quad+\frac{1}{[2]}E_\gamma E_{\beta_{k-1}}X,\end{aligned}$$

(n)
$$\begin{aligned}&E_{\beta_{k-1}}E_\gamma E_{\beta_{k-1}}E_{\beta_1}\\&=E_{\beta_{k-1}}E_\gamma(v^{(\beta_1|\beta_{k-1})}E_{\beta_1}E_{\beta_{k-1}}+X)\\&=v^{(\beta_1|\beta_{k-1})}E_{\beta_{k-1}}(v^{(\beta_1|\gamma)}E_{\beta_1}E_\gamma+Y)E_{\beta_{k-1}}+E_{\beta_{k-1}}E_\gamma X\\&=v^{(\beta_1|\beta_{k-1}+\gamma)}(v^{(\beta_1|\beta_{k-1})}E_{\beta_1}E_{\beta_{k-1}}+X)E_\gamma E_{\beta_{k-1}}\\&\quad+v^{(\beta_1|\beta_{k-1})}E_{\beta_{k-1}}YE_{\beta_{k-1}}+E_{\beta_{k-1}}E_\gamma X,\end{aligned}$$

(o)
$$\begin{aligned}&E^{(2)}_{\beta_{k-1}}E_\gamma E_{\beta_1}\\&=E^{(2)}_{\beta_{k-1}}(v^{(\beta_1|\gamma)}E_{\beta_1}E_\gamma+Y)\\&=\frac{1}{[2]}v^{(\beta_1|\gamma)}E_{\beta_{k-1}}(v^{(\beta_1|\beta_{k-1})}E_{\beta_1}E_{\beta_{k-1}}+X)E_\gamma+E^{(2)}_{\beta_{k-1}}Y\\&=\frac{1}{[2]}v^{(\beta_1|\beta_{k-1}+\gamma)}(v^{(\beta_1|\beta_{k-1})}E_{\beta_1}E_{\beta_{k-1}}+X)E_{\beta_{k-1}}E_\gamma\\&\quad+\frac{1}{[2]}v^{(\beta_1|\gamma)}E_{\beta_{k-1}}XE_\gamma+E^{(2)}_{\beta_{k-1}}Y.\end{aligned}$$

Using induction hypothesis repeatedly we see

(p)
$$\begin{aligned}E_\gamma X&=v^{(\gamma|\beta_1+\beta_{k-1})}XE_\gamma\\&\quad+\sum_{b_2,\dots,b_{k-2}\in\mathbb{N}}\xi'(b_2,\dots,b_{k-2})E^{(b_2)}_{\beta_2}\cdots E^{(b_{k-2})}_{\beta_{k-2}},\\E_{\beta_{k-1}}X&=v^{(\beta_{k-1}|\beta_1+\beta_{k-1})}XE_{\beta_{k-1}}\\&\quad+\sum_{b_2,\dots,b_{k-2}\in\mathbb{N}}\xi''(b_2,\dots,b_{k-2})E^{(b_2)}_{\beta_2}\cdots E^{(b_{k-2})}_{\beta_{k-2}},\end{aligned}$$

where $\xi'(b_2, \dots, b_{k-2}),\ \xi''(b_2, \dots, b_{k-2}) \in \mathcal{A}$. We shall simply write X', X'' for the second terms of the right hand sides of the above two identities respectively. Then $E_\gamma X = v^{(\gamma|\beta_1+\beta_{k-1})} X E_\gamma + X'$, $E_{\beta_{k-1}} X = v^{(\beta_{k-1}|\beta_1+\beta_{k-1})} X E_{\beta_{k-1}} + X''$.

Using (p) and induction hypothesis repeatedly, we get

$$
\text{(q)}\quad
\begin{aligned}
E_\gamma E_{\beta_{k-1}} X &= E_\gamma (v^{(\beta_{k-1}|\beta_1+\beta_{k-1})} X E_{\beta_{k-1}} + X'') \\
&= v^{(\beta_{k-1}+\gamma|\beta_1+\beta_{k-1})} X E_\gamma E_{\beta_{k-1}} + v^{(\gamma|\beta_1+2\beta_{k-1})} X'' E_\gamma \\
&\quad + \sum_{c_2,\dots,c_{k-1}\in\mathbb{N}} \eta'(c_2,\dots,c_{k-1}) E_{\beta_2}^{(c_1)} \cdots E_{\beta_{k-1}}^{(c_{k-1})}, \\
E_{\beta_{k-1}} E_\gamma X &= E_{\beta_{k-1}} (v^{(\gamma|\beta_1+\beta_{k-1})} X E_\gamma + X') \\
&= v^{(\gamma+\beta_{k-1}|\beta_1+\beta_{k-1})} X E_{\beta_{k-1}} E_\gamma + v^{(\gamma|\beta_1+\beta_{k-1})} X'' E_\gamma \\
&\quad + \sum_{c_2,\dots,c_{k-1}\in\mathbb{N}} \eta''(c_2,\dots,c_{k-1}) E_{\beta_2}^{(c_1)} \cdots E_{\beta_{k-1}}^{(c_{k-1})},
\end{aligned}
$$

$$
- v^{-1+(\beta_1|\beta_{k-1})} E_{\beta_{k-1}} Y E_{\beta_{k-1}} + E_{\beta_{k-1}}^{(2)} Y = \sum_{c_2,\dots,c_{k-1}} \eta'''(c_2,\dots,c_{k-1}) E_{\beta_2}^{(c_2)} \cdots E_{\beta_{k-1}}^{(c_{k-1})},
$$

where $\eta'(c_2, \dots, c_{k-1}),\ \eta''(c_2, \dots, c_{k-1}),\ \eta'''(c_2, \dots, c_{k-1}) \in \mathcal{A}$.

Moreover we have

(r) $\beta_k = \gamma + 2\beta_{k-1}$ and $(\beta_{k-1}|\beta_{k-1}) = 2$, $\ (\gamma|\beta_{k-1}) = (\alpha_{i_k}|\alpha_{i_{k-1}}) = -2$.

Combining (l)-(r) we see

(s) $E_{\beta_k} E_{\beta_1} - v^{(\beta_1|\beta_k)} E_{\beta_1} E_{\beta_k} = \sum_{a_2,\dots,a_{k-1}\in\mathbb{N}} \eta(a_2,\dots,a_{k-1}) E_{\beta_2}^{(a_2)} \cdots E_{\beta_{k-1}}^{(a_{k-1})}$,

where $\eta(a_2, \cdots, a_{k-1}) \in \mathcal{A}$.

Using induction on a, b, and using (s) and induction hypothesis repeatedly, we see

(t) $E_{\beta_k}^{(a)} E_{\beta_1}^{(b)} - v^{ab(\beta_1|\beta_k)} E_{\beta_1}^{(b)} E_{\beta_k}^{(a)} = \sum_{\substack{a_1,\dots,a_k\in\mathbb{N} \\ a_1<b \\ a_k<a}} \rho(a_1,\dots,a_k) E_{\beta_1}^{(a_1)} \cdots E_{\beta_k}^{(a_k)}$,

where $\rho(a_1, \dots, a_k) \in \mathcal{A}$ (here we need 2.1 (a)). Thus in case (3) the identity ($*$) is true.

Now assume that we are in case (4), then

$$\text{(u)}\qquad \begin{aligned} E_{\beta_k} &= T_u T_{i_k} T_{i_{k-1}}(E_{i_k}) \\ &= T_u \left(E^{(2)}_{i_{k-1}} E_{i_k} - v^{-1} E_{i_{k-1}} E_{i_k} E_{i_{k-1}} + v^{-2} E_{i_k} E^{(2)}_{i_{k-1}} \right) \\ &= E^{(2)}_{\alpha} E_{\beta_{k-2}} - v^{-1} E_\alpha E_{\beta_{k-2}} E_\alpha + v^{-2} E_{\beta_{k-2}} E^{(2)}_{\alpha} \\ &\quad \left(E^{(2)}_\alpha = T_u \left(E^{(2)}_{i_{k-1}} \right) \right). \end{aligned}$$

As a similar argument for case (3) we see that the identity (∗) is true in this case.

Thus we proved the theorem for Case A.

Case B. $u = s_{j_1} s_{j_2} \cdots s_{j_m},\ u' = s_{j_{m+1}} \cdots s_{j_{k-1}}$, and $j_1 = i_1,\ j_{k-1} = i_{k-1}$. Define $\gamma_p = s_{j_1} \cdots s_{j_{p-1}}\ (\alpha_{j_p})$ $(2 \le p \le k-1)$ and ${E'}^{(a)}_{\gamma_p} = T_{j_1} \cdots T_{j_{p-1}} (E^{(a)}_{j_p})$ $(a \in \mathbb{N})$.

According to the arguments in Case A we get

$$\text{(v)}\qquad \begin{aligned} & E^{(a)}_{\beta_k} E^{(b)}_{\beta_1} - v^{ab(\beta_1|\beta_k)} E^{(b)}_{\beta_1} E^{(a)}_{\beta_k} \\ &= \sum_{\substack{a_1,\dots,a_k \in \mathbb{N} \\ a_1 < b \\ a_k < a}} \rho'(a_1, \dots, a_k) E^{(a_1)}_{\beta_1} {E'}^{(a_2)}_{\gamma_2} \cdots {E'}^{(a_{k-1})}_{\gamma_{k-1}} E^{(a_k)}_{\beta_k}, \end{aligned}$$

where $\rho'(a_1, \dots, a_k) \in \mathcal{A}$.

Noting that $j_1 = i_1$, by 2.1 (b) we see that the $\mathcal{A}$-submodule of $U^+_{\mathcal{A}}$ generated by the elements ${E'}^{(a_2)}_{\gamma_2} \cdots {E'}^{(a_{k-1})}_{\gamma_{k-1}}$ $(a_2, \dots, a_{k-1} \in \mathbb{N})$ is equal to the $\mathcal{A}$-submodule of $U^+_{\mathcal{A}}$ generated by the elements $E^{(a_2)}_{\beta_2} \cdots E^{(a_{k-1})}_{\beta_{k-1}}$ $(a_2, \dots, a_{k-1} \in \mathbb{N})$. Therefore we have

$$\text{(w)}\ E^{(a)}_{\beta_k} E^{(b)}_{\beta_1} - v^{ab(\beta_1|\beta_k)} E^{(b)}_{\beta_1} E^{(a)}_{\beta_k} = \sum_{\substack{a_1,\dots,a_k \in \mathbb{N} \\ a_1 < b \\ a_k < a}} \rho(a_1, \dots, a_k) E^{(a_1)}_{\beta_1} \cdots E^{(a_k)}_{\beta_k},$$

where $\rho(a_1, \dots, a_k) \in \mathcal{A}$.

Hence the identity (∗) is true for Case B.

Case C. $u = s_{j_1} s_{j_2} \cdots s_{j_m},\ u' = s_{j_{m+1}} \cdots s_{j_{k-1}}$, and $j_1 \neq i_1,\ j_{k-1} = i_{k-1}$.

In this case uu' has a reduced expression of the form $s_{p_1} s_{p_2} \cdots s_{p_{k-1}}$ such that $p_1 = i_1,\ p_{k-1} = i_{k-1}$, and one of the following three cases happens.

(5) $\langle \alpha_{p_1}, \alpha^\vee_{p_2} \rangle = 0$,

(6) $p_1 = p_3$ and $\langle \alpha_{p_1}, \alpha^\vee_{p_2} \rangle \langle \alpha_{p_2}, \alpha^\vee_{p_1} \rangle = 1$,

(7) $p_1 = p_3,\ p_2 = p_4$, and $\langle \alpha_{p_1}, \alpha^\vee_{p_2} \rangle \langle \alpha_{p_2}, \alpha^\vee_{p_1} \rangle = 2$.

Define $p_k = i_k$. We set, for case (5),

$$\gamma_1 = \alpha_{p_1}, \quad \gamma_3 = s_{p_1}(\alpha_{p_3}), \quad \gamma_h = s_{p_1}s_{p_3}\cdots s_{p_{h-1}}(\alpha_{p_h}) \quad (4 \le h \le k),$$
$$E'^{(a)}_{\gamma_1} = E^{(a)}_{p_1}, \quad E'^{(a)}_{\gamma_3} = T_{p_1}(E^{(a)}_{p_3}),$$
$$E'^{(a)}_{\gamma_h} = T_{p_1}T_{p_3}\cdots T_{p_{h-1}}(E^{(a)}_{p_h}) \quad (4 \le h \le k), \quad a \in \mathbb{N};$$

for case (6),

$$\gamma_1 = \alpha_{p_2}, \quad \gamma_4 = s_{p_2}(\alpha_{p_4}), \quad \gamma_h = s_{p_2}s_{p_4}\cdots s_{p_{h-1}}(\alpha_{p_h}) \quad (5 \le h \le k),$$
$$E'^{(a)}_{\gamma_1} = E^{(a)}_{p_2}, \quad E'^{(a)}_{\gamma_4} = T_{p_2}(E^{(a)}_{p_4}),$$
$$E'^{(a)}_{\gamma_h} = T_{p_2}T_{p_4}\cdots T_{p_{h-1}}(E^{(a)}_{p_h}) \quad (5 \le h \le k), \quad a \in \mathbb{N};$$

for case (7),

$$\gamma_1 = \alpha_{p_1}, \quad \gamma_5 = s_{p_1}(\alpha_{p_5}), \quad \gamma_h = s_{p_1}s_{p_5}\cdots s_{p_{h-1}}(\alpha_{p_h}) \quad (6 \le h \le k),$$
$$E'^{(a)}_{\gamma_1} = E^{(a)}_{p_1}, \quad E'^{(a)}_{\gamma_5} = T_{p_1}(E^{(a)}_{p_5}),$$
$$E'^{(a)}_{\gamma_h} = T_{p_1}T_{p_5}\cdots T_{p_{h-1}}(E^{(a)}_{p_h}) \quad (6 \le h \le k), \quad a \in \mathbb{N}.$$

By induction hypothesis we get:

(x1) For case (5), (since $s_{p_1}s_{p_3}\cdots s_{p_k}$ is a reduced expression),

$$E'^{(a)}_{\gamma_k}E'^{(b)}_{\gamma_1} - v^{ab(\gamma_1|\gamma_k)}E'^{(b)}_{\gamma_1}E'^{(a)}_{\gamma_k} = \sum_{\substack{a_1,a_3,\dots,a_k\in\mathbb{N}\\ a_1<b\\ a_k<a}} \rho'(a_1,a_3,\dots,a_k)E'^{(a_1)}_{\gamma_1}E'^{(a_3)}_{\gamma_3}\cdots E'^{(a_k)}_{\gamma_k},$$

where $\rho'(a_1,a_3,\dots,a_k) \in \mathcal{A}$.

(x2) For case (6), (since $s_{p_1}s_{p_4}\cdots s_{p_k}$ is a reduced expression),

$$E'^{(a)}_{\gamma_k}E'^{(b)}_{\gamma_1} - v^{ab(\gamma_1|\gamma_k)}E'^{(b)}_{\gamma_1}E'^{(a)}_{\gamma_k} = \sum_{\substack{a_1,a_4,\dots,a_k\in\mathbb{N}\\ a_1<b\\ a_k<a}} \rho'(a_1,a_4,\dots,a_k)E'^{(a_1)}_{\gamma_1}E'^{(a_4)}_{\gamma_4}\cdots E'^{(a_k)}_{\gamma_k},$$

where $\rho'(a_1,a_4,\dots,a_k) \in \mathcal{A}$.

(x3) For case (7), (since $s_{p_1}s_{p_5}\cdots s_{p_k}$ is a reduced expression),

$$E'^{(a)}_{\gamma_k}E'^{(b)}_{\gamma_1} - v^{ab(\gamma_1|\gamma_k)}E'^{(b)}_{\gamma_1}E'^{(a)}_{\gamma_k} = \sum_{\substack{a_1,a_5,\dots,a_k\in\mathbb{N}\\ a_1<b\\ a_k<a}} \rho'(a_1,a_5,\dots,a_k)E'^{(a_1)}_{\gamma_1}E'^{(a_5)}_{\gamma_5}\cdots E'^{(a_k)}_{\gamma_k},$$

where $\rho'(a_1, a_5, \dots, a_k) \in \mathcal{A}$.

Note that we always have $(\gamma_1|\gamma_k) = (\beta_1|\beta_k)$ since $(\cdot|\cdot)$ is W-invariant and $p_1 = i_1$. Recall that $T_w(E_i) = E_j$ if $w(\alpha_i) = \alpha_j$ (see [**L5**]). Applying T_{p_2} (resp. $T_{p_2}T_{p_1}$; $T_{p_2}T_{p_1}T_{p_2}$) to the identity in (x1) (resp. (x2); (x3)) and using 2.1 (b) (see the argument for Case B) we get

(y) $E^{(a)}_{\beta_k} E^{(b)}_{\beta_1} - v^{ab(\beta_1|\beta_k)} E^{(b)}_{\beta_1} E^{(a)}_{\beta_k} = \sum_{\substack{a_1,\dots,a_k \in \mathbb{N} \\ a_1 < b \\ a_k < a}} \rho(a_1, \dots, a_k) E^{(a_1)}_{\beta_1} \cdots E^{(a_k)}_{\beta_k},$

where $\rho(a_1, \dots, a_k) \in \mathcal{A}$.

Thus the identity $(*)$ is true for Case C.

The theorem is proved. □

Theorem 2.4. *Keep the notation in 2.1. Then:*

(i) *The elements*

$$\prod_{\beta \in R^+} E^{(a_\beta)}_\beta \quad (a_\beta \in \mathbb{N})$$

form an $\mathcal{A}$-basis of $U^+_{\mathcal{A}}$. Where the factors in the product are written in a given total order on R^+.

(ii) *The elements*

$$\prod_{\beta \in R^+} F^{(a_\beta)}_\beta \quad (a_\beta \in \mathbb{N})$$

form an $\mathcal{A}$-basis of $U^-_{\mathcal{A}}$. Where the factors in the product are written in a given total order on R^+.

Proof. We only need to prove (i) since $\Omega(E^{(a_\beta)}_\beta) = F^{(a_\beta)}_\beta$ and $\Omega U^+_{\mathcal{A}} = U^-_{\mathcal{A}}$.

Define the lexicographical order $>$ on $\mathbb{N}^{|R^+|}$ such that

$$(1, 0, \dots, 0) > (0, 1, \dots, 0) > \cdots > (0, \dots, 0, 1).$$

Using Theorem 2.3 repeatedly we see

$$\prod_{\beta \in R^+} E^{(a_\beta)}_\beta = v^p E^A_{\mathbf{i}} + \sum_{\substack{B \in \mathbb{N}^{|R^+|} \\ A > B}} \rho_B E^B_{\mathbf{i}}, \quad \rho_B \in \mathcal{A},$$

where $p \in \mathbb{Z}$ and $A = (a_{\beta_1}, a_{\beta_2}, \dots, a_{\beta_\nu})$ (we define $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ and $\nu = |R^+|$). Noting that $\rho_B = 0$ if $\sum_{i=1}^{\nu} b_i \beta_i \neq \sum_{\beta \in R^+} a_\beta \beta$ (here $B = (b_1, \dots, b_\nu)$), we see $E^A_{\mathbf{i}}$ is an $\mathcal{A}$-linear combination of the elements $\prod_{\beta \in R^+} E^{(c_\beta)}_\beta$ $(c_\beta \in \mathbb{N})$. Since for any λ in $\mathbb{N}R^+$, the number

$$\#\{E^A_{\mathbf{i}} \mid A = (a_1, \dots, a_\nu) \in \mathbb{N}^\nu \text{ such that } a_1\beta_1 + \cdots + a_\nu\beta_\nu = \lambda\}$$

is equal to

$$\#\left\{\prod_{\beta\in R^+} E_\beta^{(a_\beta)} \;\middle|\; \sum_{\beta\in R^+} a_\beta\beta = \lambda\right\},$$

by 2.1 (a), the elements

$$\prod_{\beta\in R^+} E_\beta^{(a_\beta)} \quad (a_\beta \in \mathbb{N})$$

form an $\mathcal{A}$-basis of $U_{\mathcal{A}}^+$.

The theorem is proved.

From the above proof we see the following:

Corollary 2.5. *Keep the notation in Theorem 2.4 and its proof. If* $A = (a_{\beta_1}, \dots, a_{\beta_\nu})$ *is minimal in the set*

$$\left\{(b_1, \dots, b_\nu) \in \mathbb{N}^\nu \;\middle|\; \sum_{i=1}^{\nu} b_i\beta_i = \sum_{\beta\in R^+} a_\beta\beta\right\},$$

then $\prod_{\beta\in R^+} E_\beta^{(a_\beta)} = v^p E_{\mathbf{i}}^A$. *That is, for all* $\beta, \gamma \in R^+$ *we have* $E_\beta^{(a_\beta)} E_\gamma^{(a_\gamma)} = v^q E_\gamma^{(a_\gamma)} E_\beta^{(a_\beta)}$ *for some* $q \in \mathbb{Z}$. (*Of course, many* a_β *are* 0 *in this case.*)

2.6. We would like to state two conjectures, one describes the root vectors intrinsically. The conjectures might be helpful for constructing an $\mathcal{A}$-basis of the $\mathcal{A}$-form of the quantized enveloping algebra of a symmetrizable Kac-Moody algebra. For $\lambda \in \mathbb{N}R^+$, we denote by U_λ^+ the set $\{x \in U^+ \mid K_i x K_i^{-1} = v^{(\lambda|\alpha_i)} x\}$ and let $U_{\mathcal{A},\lambda}^+ = U_\lambda \cap U_{\mathcal{A}}$. We also write $U_{-\lambda}^-$ for $\Omega(U_\lambda^+)$.

Conjecture A. Let $\alpha \in R^+$ and set $d_\alpha = d_i$ if $w(\alpha_i) = \alpha$ for some $w \in W$. Let $E \in U_{\mathcal{A},\alpha}^+$. If $E^{(a)} = E^a/[a]_{d_\alpha}^! \in U_{\mathcal{A}}^+$ for all $a \in \mathbb{N}$, then there exists a simple root α_j and $u \in W$, $f \in \mathcal{A}$, such that $u(\alpha_j) = \alpha$ and $E = fT_u(\alpha_j)$. (We refer to [**L2**] for the definition of $[a]_{d_\alpha}^!$.) For type A_2, the conjecture is true.

Conjecture B. For any $\beta \in R^+$, choose $w_\beta \in W$ and $i_\beta \in [1, n]$ such that $w_\beta(\alpha_{i_\beta}) = \beta$. Define $E_\beta^{(a)} = T_{w_\beta}(E_{i_\beta}^{(a)})$. Then the elements

$$\prod_{\beta\in R^+} E_\beta^{(a_\beta)} \quad (a_\beta \in \mathbb{N})$$

form an $\mathcal{A}$-basis of $U_{\mathcal{A}}^+$. Where the factors in the product are written according to a given total order on R^+.

3. Some orthogonal bases of the bilinear form in [K].

In this section we show that certain PBW bases are orthogonal bases of the bilinear from considered in [K], see Theorem 3.9. For the bilinear form obtained from the Drinfeld dual, a similar result was established in [L5, LS]. Although the difference between the two bilinear forms are small, it is difficult to apply the methods in [L5, LS] for proving Theorem 3.9, since the methods rely on a property ([L5, 38.2.1]) which does not hold for the bilinear form in [K].

3.1. Following Kashiwara [K, Prop. 3.4.4] we define a bilinear form on U^+.

(a) For each $P \in U^+$ and F_i, there exist unique $P', P'' \in U^+$ such that

$$PF_i - F_iP = \frac{K_iP' - K_i^{-1}P''}{v_i - v_i^{-1}}.$$

(We set $v_i = v^{d_i}$.)

Define $\varphi_i(P) = P''$ and $\psi_i(P) = P'$. We have (cf. [K, Prop. 3.4.4]).

(b) There is a unique symmetric bilinear form $(\ ,\ )$ on U^+ such that $(1,1) = 1$,

$$(E_ix, y) = (x, \varphi_i(y)) \quad \text{for all } i \in [1,n] \text{ and } x, y \in U^+,$$

$$(x, E_iy) = (\varphi_i(x), y) \quad \text{for all } i \in [1,n] \text{ and } x, y \in U^+.$$

We need some preparation for proving Theorem 3.9. Let $\mathcal{X}$ be the set of all sequences $\mathbf{i} = (i_1, \dots, i_\nu)$ in $[1,n]$ such that $s_{i_1} \cdots s_{i_\nu}$ is a reduced expression of the longest element $w_0 \in W$. For $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{X}$, $A = (a_1, \dots, a_\nu) \in \mathbb{N}^\nu$, we shall write

$$E_{\mathbf{i}}^A = E_{i_1}^{(a_1)} T_{i_1}(E_{i_2}^{(a_2)}) \cdots T_{i_1} \cdots T_{i_{\nu-1}}(E_{i_\nu}^{(a_\nu)}),$$

$$F_{\mathbf{i}}^A = T_{i_1} \cdots T_{i_{\nu-1}}(F_{i_\nu}^{(a_\nu)}) \cdots T_{i_1}(F_{i_2}^{(a_2)}) F_{i_1}^{(a_1)} = \Omega(E_{\mathbf{i}}^A),$$

$$\hat{E}_{\mathbf{i}}^A = E_{i_1}^{(a_1)} T_{i_1}^{-1}(E_{i_2}^{(a_2)}) \cdots T_{i_1}^{-1} \cdots T_{i_{\nu-1}}^{-1}(E_{i_\nu}^{(a_\nu)}).$$

For $\lambda = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in \mathbb{Z}R$, we define $K_\lambda = K^{a_1}K^{a_2}\cdots K^{a_n}$.

The following result plays a key role in the proof, which is essentially a variation of Theorem 2.3.

Lemma 3.2. *Let $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{X}$ and let $s, k \in [0, \nu-1]$ such that $s > k$. Set $E = T_{i_1} \cdots T_{i_s}(E_{i_{s+1}})$, $F = T_{i_1} \cdots T_{i_k}(F_{i_{k+1}})$. We have*

$$EF - FE = \sum \sigma(A, \lambda, B) F_{\mathbf{i}}^A K_\lambda E_{\mathbf{i}}^B, \quad \sigma(A, \lambda, B) \in \mathcal{A},$$

where $A=(a_1,\dots,a_\nu)$ and $B=(b_1,\cdots,b_\nu)$ run through a finite subset of $\mathbb{N}^\nu$, λ runs through a finite subset of $\mathbb{N}R^+$, and $a_{k+1}=\cdots=a_\nu=0,\ b_1=\cdots=b_{s+1}=0$ if $\sigma(A,\lambda,B)\neq 0$.

Proof. Set $j_1=i_s,\ j_2=i_{s-1},\ \dots,\ j_s=i_1$. Choose $j_{s+1},\ \dots,\ j_\nu\in[1,n]$ such that $(j_1,\dots,j_\nu)\in\mathcal{X}$ and $s_{j_1}\cdots s_{j_{\nu-1}}(\alpha_{j_\nu})=\alpha_{i_{s+1}}$.

For $m\in[1,\nu],\ a\in\mathbb{N}$, define

$$X_m^{(a)}=T_{j_1}\cdots T_{j_{m-1}}(E_{j_m}^{(a)}),$$

$$X'^{(a)}_m=\Psi(X_m^{(a)})=T_{j_1}^{-1}\cdots T_{j_{m-1}}^{-1}(E_{j_m}^{(a)}).$$

Then $T_{i_s}\cdots T_{i_{k+2}}(E_{i_{k+1}})=X_{s-k}$ and $E_{i_{s+1}}=X_\nu$.

Set $\beta=s_{i_s}\cdots s_{i_{k+2}}(\alpha_{i_{k+1}}),\ \beta'=\alpha_{i_{s+1}}$. Using Theorem 2.3 repeatedly we see

(a)
$$\begin{aligned}&E_{i_{s+1}}X_{s-k}-v^{(\beta|\beta')}X_{s-k}E_{i_{s+1}}\\&=\sum\sigma(a_{\nu-1},\dots,a_{s-k+1})X_{\nu-1}^{(a_{\nu-1})}\cdots X_{s-k+1}^{(a_{s-k+1})},\end{aligned}$$

where $\sigma(a_{\nu-1},\dots,a_{s-k+1})\in\mathcal{A}$, and $a_{\nu-1},\ \dots,\ a_{s-k+1}$ run through a finite subset of $\mathbb{N}$.

Applying Ψ to the identity (a) we get

(b)
$$\begin{aligned}&X'_{s-k}E_{i_{s+1}}-v^{(\beta|\beta')}E_{i_{s+1}}X'_{s-k}\\&=\sum\sigma(a_{\nu-1},\dots,a_{s-k+1})X'^{(a_{s-k+1})}_{s-k+1}\cdots X'^{(a_{\nu-1})}_{\nu-1}.\end{aligned}$$

If $\nu>m\geq s+1$, then we may find $k_{m+1},\dots,k_\nu\in[1,n]$ such that $(k_\nu,\dots,k_{m+1},\ j_1,\dots,j_m)\in\mathcal{X}$. Noting that $s_{k_\nu}\cdots s_{k_{m+1}}s_{j_1}\cdots s_{j_{m-1}}(\alpha_{j_m})$ is a simple root α_j for some $j\in[1,n]$, we see

(c)
$$T_{k_\nu}^{-1}\cdots T_{k_{m+1}}^{-1}T_{j_1}^{-1}\cdots T_{j_{m-1}}^{-1}(E_{j_m})=E_j.$$

Since $s_{j_{s+1}}\cdots s_{j_{m-1}}(\alpha_{j_m})=s_{j_s}\cdots s_{j_1}s_{k_{m+1}}\cdots s_{k_\nu}(\alpha_j)\in R^+$, we have

(d)
$$\begin{aligned}Y_m=T_{j_{s+1}}^{-1}\cdots T_{j_{m-1}}^{-1}(E_{j_m})&=T_{j_s}\cdots T_{j_1}T_{k_{m+1}}\cdots T_{k_\nu}(E_j)\\&=T_{i_1}\cdots T_{i_s}T_{k_{m+1}}\cdots T_{k_\nu}(E_j)\in U^+,\end{aligned}$$

for $s+1\leq m\leq\nu-1$.

By our choice on $j_1,\dots,j_\nu$ we may require that $k_{m+1}=i_{s+1}$. By (d) and 2.1 (b) we see

(e)
$$Y_m=T_{j_{s+1}}^{-1}\cdots T_{j_{m-1}}^{-1}(E_{j_m})=\sum\sigma'(A)E_{\mathbf{i}}^A,\quad \sigma'(A)\in\mathcal{A},$$

where $A=(a_1,\dots,a_\nu)$ runs through a finite subset of $\mathbb{N}^\nu$ and $a_1=\cdots=a_{s+1}=0$ if $\sigma'(A)\neq 0$.

Define $\beta_m=s_{i_1}\cdots s_{i_{m-1}}(\alpha_{i_m})$ for $1\le m\le k+1$, and set $Z_m^{(a)}=T_{i_1}\cdots T_{i_{m-1}}\ (F_{i_m}^{(a)})$ for $1\le m\le k+1,\ a\in\mathbb{N}$. For $m\ge s+1,\ a\in\mathbb{N}$, we set $Y_m^{(a)}=T_{j_{s+1}}^{-1}\cdots T_{j_{m-1}}^{-1}\ (E_{j_m}^{(a)})\in U^+$.

Applying $T_{i_1}\cdots T_{i_s}$ to the identity (b) we get

$$\text{(f)}\quad \begin{aligned}&-K_{\beta_{k+1}}^{-1}FE+v^{(\beta|\beta')}EK_{\beta_{k+1}}^{-1}F\\ &=\sum\sigma(a_{\nu-1},\dots,a_{s-k+1})v^cK_{-\lambda}Z_k^{(a_{s-k+1})}\cdots Z_1^{(a_s)}Y_{s+1}^{(a_{s+1})}\cdots Y_{\nu-1}^{(a_{\nu-1})},\end{aligned}$$

where c is a suitable integer depending on $i_1,\ \dots,\ i_k,\ a_{s-k+1},\ \dots,\ a_s$, and $\lambda=a_{s-k+1}\beta_k+a_{s-k+2}\beta_{k-1}+\cdots+a_s\beta_1\in\mathbb{N}R^+$.

Since $(\beta_{k+1}|s_{i_1}\cdots s_{i_s}(\alpha_{i_{s+1}}))=-(s_{i_s}\cdots s_{i_{k+2}}(\alpha_{i_{k+1}})|\alpha_{i_{s+1}})=-(\beta|\beta')$, we see

$$\text{(g)}\qquad EK_{\beta_{k+1}}^{-1}=v^{-(\beta|\beta')}K_{\beta_{k+1}}^{-1}E.$$

Obviously we have

$$\text{(h)}\qquad \beta_{k+1}\ge\lambda\quad\text{if }\ \sigma(a_{\nu-1},\dots,a_{s-k+1})\neq 0.$$

Combining (e)-(h) and Theorem 2.3 we see the lemma is true.

3.3. Let $\beta=\sum a_i\alpha_i\in\mathbb{N}R^+$. We define $\sigma(\beta)=\prod_i(v_i-v_i^{-1})^{a_i}$, $d'_\beta=\sum a_id_i-d_\beta$ if $\beta\in R^+$. Let a be an integer and b,d positive integers, set

$$\{a\}_d=\frac{1-v^{-2ad}}{1-v^{-2d}},\quad \{b\}_d^!=\{b\}_d\{b-1\}_d\cdots\{1\}_d,\quad \{0\}_d^!=1,$$

$$\{-b\}_d^!:=(-1)^b\{b\}_d^!,\quad \begin{Bmatrix}a\\ b\end{Bmatrix}_d:=\prod_{h=1}^{b}\frac{1-v^{-2(a-h+1)d}}{1-v^{-2hd}},\quad \begin{Bmatrix}a\\ 0\end{Bmatrix}_d:=1.$$

We have

$$\text{(a)}\qquad \begin{Bmatrix}a+b\\ b\end{Bmatrix}_d=\frac{\{a+b\}_d^!}{\{a\}_d^!\{b\}_d^!}\quad\text{for } a,b\in\mathbb{N}.$$

We shall omit the subscript d if $d=1$.

$$\text{(b)}\qquad \sigma(\beta+\gamma)=\sigma(\beta)\sigma(\gamma)\quad\text{for }\ \beta,\gamma\in\mathbb{N}R^+.$$

Recall for $\lambda = a_1\alpha_1+\cdots+a_n\alpha_n$, we write K_λ for $K_1^{a_1}\cdots K_n^{a_n}$. For $\beta \in R^+$, we shall write Σ_β (resp. Σ'_β) for any element of U of the form

$$\sum uK_\lambda x \quad \left(\text{resp.} \sum uK_\lambda x\right),$$

where u runs through a finite subset of U^-, x runs through a finite subset of U^+, and λ runs through the set $\{\sum b_i\alpha_i \in \mathbb{Z}R \mid |b_i| \leq a_i \text{ for all } i \text{ and } \sum b_i\alpha_i \neq \pm\beta\}$ (resp. $\{\sum b_i\alpha_i \in \mathbb{Z}R \mid |b_i| \leq a_i \text{ for all } i\}$). The following assertions (c) and (d) are obvious.

(c) $$\Sigma_\beta + \Sigma_\beta = \Sigma_\beta \quad \text{and} \quad \Sigma'_\beta + \Sigma'_\beta = \Sigma'_\beta \quad \text{for } \beta \in \mathbb{N}R^+$$

(d) $$\begin{aligned} &\Sigma_\beta\Sigma_\gamma = \Sigma_{\beta+\gamma}, \quad \Sigma'_\beta\Sigma_\gamma = \Sigma_{\beta+\gamma}, \\ &\Sigma_\beta\Sigma'_\gamma = \Sigma_{\beta+\gamma}, \quad \text{for } \beta,\gamma \in \mathbb{N}R^+. \end{aligned}$$

Lemma 3.4. *Let $\beta \in \mathbb{N}R^+$ and let $u \in U^-_{-\beta}$ be a monomial of $F_1, \ldots, F_n$. Then for any $x \in U^+$, there exist unique $x_1, x_2 \in U^+$ such that*

$$xu - ux = \frac{K_\beta x_1 + (-1)^{ht(\beta)}K_\beta^{-1}x_2}{\sigma(\beta)} + \Sigma_\beta.$$

Proof. We use induction on $\mathrm{ht}(\beta)$. When $\mathrm{ht}(\beta) = 0, 1$, the lemma is just 3.1 (a). Assume that $\mathrm{ht}(\beta) \geq 2$ and $u = F_iu'$. By induction hypothesis we get

$$\begin{aligned}
&xu - ux \\
&= (xF_i - F_ix)u' + F_i(xu' - u'x) \\
&= \frac{K_iy_1 - K_i^{-1}y_2}{\sigma(\alpha_i)}u' + F_i\left(\frac{K_{\beta-\alpha_i}x'_1 + (-1)^{\mathrm{ht}(\beta-\alpha_i)}K^{-1}_{\beta-\alpha_i}x'_2}{\sigma(\beta-\alpha_i)} + \Sigma_{\beta-\alpha_i}\right) \\
&= \frac{1}{\sigma(\alpha_i)}\left\{K_i\left(u'y_1 + \frac{K_{\beta-\alpha_i}z_1 + (-1)^{\mathrm{ht}(\beta-\alpha_i)}K^{-1}_{\beta-\alpha_i}z_2}{\sigma(\beta-\alpha_i)} + \Sigma_{\beta-\alpha_i}\right)\right. \\
&\quad \left. -K_i^{-1}\left(u'y_2 + \frac{K_{\beta-\alpha_i}z'_1 + (-1)^{\mathrm{ht}(\beta-\alpha_i)}K^{-1}_{\beta-\alpha_i}z'_2}{\sigma(\beta-\alpha_i)} + \Sigma_{\beta-\alpha_i}\right)\right\} + \Sigma_\beta,
\end{aligned}$$

where $y_1,\ y_2,\ x'_1,\ x'_2,\ z_1,\ z_2,\ z'_1,\ z'_2$ are elements of U^+.

We have $K_iK^{-1}_{\beta-\alpha_i} = \Sigma_\beta$, $K_i^{-1}K_{\beta-\alpha_i} = \Sigma_\beta$, and $K_i\Sigma_{\beta-\alpha_i} = \Sigma_\beta$, $K_i^{-1}\Sigma_{\beta-\alpha_i} = \Sigma_\beta$ (cf. 3.3 (d)). By (b), (c) and (d) in 3.3 we get

$$xu - ux = \frac{K_\beta z_1 + (-1)^{\mathrm{ht}(\beta)}K_\beta^{-1}z'_2}{\sigma(\beta)} + \Sigma_\beta.$$

The uniqueness of $x_1 = z_1,\ x_2 = z'_2$ follows from PBW theorem (see [**L2**]). The lemma is proved.

Proposition 3.5. *Let $\beta, \gamma \in \mathbb{N}R^+$ such that $\beta - \gamma \in \mathbb{N}R^+$, and let $x \in U^+_\beta$, $y \in U^+_\gamma$, $z \in U^+_{\beta-\gamma}$. Let $\xi_1, \xi_2 \in U^+$ be such that (see Lemma 3.4)*

$$x\Omega'(y) - \Omega'(y)x = \frac{K_\gamma \xi_1 + (-1)^{ht(\gamma)} K_\gamma^{-1} \xi_2}{\sigma(\gamma)} + \Sigma_\gamma.$$

(See 1.2 for the definition of Ω'.) Then $(x, yz) = (\xi_2, z)$. In particular, if $\beta = \gamma$ and $z = 1$, then $(x, y) = \xi_2$.

Proof. We may assume that y is a monomial $E_{i_1} \cdots E_{i_k}$. Repeatedly using the properties in the definition of the bilinear form we get the proposition.

Corollary 3.6. *Let $\beta \in R^+$ and F a root vector corresponding to $-\beta$. Then for any $x \in U^+$ there exist unique $x_1, x_2 \in U^+$ such that*

$$xF - Fx = \frac{K_\beta x_1 + (-1)^{ht(\beta)} K_\beta^{-1} x_2}{\sigma(\beta)} + \Sigma_\beta.$$

We shall write $\varphi_F(x) = x_2$ and $\psi_F(x) = x_1$.

Proof. Since F is a $\mathbb{Q}(v)$-linear combination of monomials of $F_1, \ldots, F_n$ with degree $-\beta$, the corollary follows from Lemma 3.4.

Proposition 3.7. *Let $\beta \in R^+$ and F a root vector corresponding to $-\beta$. Then for any $x, y \in U^+$ we have (see 3.3 for the definition of d'_β)*

$$(x, Ey) = (-1)^{ht(\beta)-1} v^{-d'_\beta} (\varphi_F(x), y),$$

where $E = \Omega(F) \in U^+$.

Proof. Let $s_{i_1} \cdots s_{i_k}$ be a reduced expression of $w \in W$ such that

$$F = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}).$$

We use induction on $k = l(w)$ to prove the proposition. When $k = 1$, then $F = F_{i_1}$, the proposition is just a property of the bilinear form (,) since $d'_\beta = 0$ in this case. Assume the proposition is true when $l(w) \leq k - 1$.

Now assume that $k = l(w) \geq 2$. Let u be the shortest element of the coset $w\langle s_{i_{k-1}}, s_{i_k}\rangle$, then $w = uu'$ for some $u' \in \langle s_{i_{k-1}}, s_{i_k}\rangle$ and $l(w) = l(u) + l(u')$. Moreover $l(us_{i_{k-1}}) = l(us_{i_k}) = l(u) + 1 \leq k - 1$. If $u'(\alpha_{i_k})$ is a simple root α_j, then $j = i_k$ or i_{k-1} and $F = T_u(F_j)$. By induction hypothesis, the proposition is true in this case.

Suppose that $\gamma = u'(\alpha_{i_k})$ is not a simple root, then we have the following cases.

(1) $u' = s_{i_{k-1}}$ and $\gamma = \alpha_{i_{k-1}} + \alpha_{i_k}$,

(2) $u' = s_{i_{k-1}}$ and $\gamma = 2\alpha_{i_{k-1}} + \alpha_{i_k}$,

(3) $u' = s_{i_{k-1}}$ and $\gamma = 3\alpha_{i_{k-1}} + \alpha_{i_k}$,

(4) $u' = s_{i_k} s_{i_{k-1}}$ and $\gamma = \alpha_{i_{k-1}} + \alpha_{i_k}$,

(5) $u' = s_{i_k}s_{i_{k-1}}$ and $\gamma = 2\alpha_{i_{k-1}} + \alpha_{i_k}$,

(6) $u' = s_{i_k}s_{i_{k-1}}$ and $\gamma = \alpha_{i_{k-1}} + 2\alpha_{i_k}$, (type G_2)

(7) $u' = s_{i_k}s_{i_{k-1}}$ and $\gamma = 3\alpha_{i_{k-1}} + 2\alpha_{i_k}$,

(8) $u' = s_{i_{k-1}}s_{i_k}s_{i_{k-1}}$ and $\gamma = \alpha_{i_{k-1}} + 2\alpha_{i_k}$,

(9) $u' = s_{i_{k-1}}s_{i_k}s_{i_{k-1}}$ and $\gamma = 3\alpha_{i_{k-1}} + 2\alpha_{i_k}$,

(10) $u' = s_{i_k}s_{i_{k-1}}s_{i_k}s_{i_{k-1}}$ and $\gamma = \alpha_{i_{k-1}} + \alpha_{i_k}$,

(11) $u' = s_{i_k}s_{i_{k-1}}s_{i_k}s_{i_{k-1}}$ and $\gamma = 3\alpha_{i_{k-1}} + \alpha_{i_k}$.

Case (1). Let $\beta_1 = u(\alpha_{i_k})$, $\beta_2 = u(\alpha_{i_{k-1}})$, then $\beta_1, \beta_2 \in R^+$ and $\beta = \beta_1 + \beta_2$. We have $T_{i_{k-1}}(E_{i_k}) = E_{i_k}E_{i_{k-1}} - v^{-d}E_{i_{k-1}}E_{i_k}$ and

$$E = T_{i_1}\cdots T_{i_{k-1}}(E_{i_k}) = T_u(E_{i_k})T_u(E_{i_{k-1}}) - v^{-d}T_u(E_{i_{k-1}})T_u(E_{i_k}),$$

where $d = -d_{i_{k-1}}a_{i_{k-1},i_k} = d_{i_{k-1}}$.

Let $E' = T_u(E_{i_k})$, $E'' = T_u(E_{i_{k-1}})$, $F' = \Omega(E') = T_u(F_{i_k})$, $F'' = \Omega(E'') = T_u(F_{i_{k-1}})$. Then $E = E'E'' - v^{-d}E''E'$ and $F = F''F' - v^dF'F''$. By induction hypothesis, we have

(a) $(x, E'E''y) = (-1)^{\mathrm{ht}(\beta_1)-1+\mathrm{ht}(\beta_2)-1}v^{-d'_{\beta_1}-d'_{\beta_2}}(\varphi_{F''}(\varphi_{F'}(x)), y)$,

(b) $(x, E''E'y) = (-1)^{\mathrm{ht}(\beta_2)-1+\mathrm{ht}(\beta_1)-1}v^{-d'_{\beta_1}-d'_{\beta_2}}(\varphi_{F'}(\varphi_{F''}(x)), y)$.

Recall that we have

$$\begin{aligned}
&xF' - F'x = \frac{K_{\beta_1}\psi_{F'}(x) + (-1)^{\mathrm{ht}(\beta_1)}K_{\beta_1}^{-1}\varphi_{F'}(x)}{\sigma(\beta_1)} + \Sigma_{\beta_1},\\
&\varphi_{F'}(x)F'' - F''\varphi_{F'}(x)\\
&= \frac{K_{\beta_2}\psi_{F''}(\varphi_{F'}(x)) + (-1)^{\mathrm{ht}(\beta_2)}K_{\beta_2}^{-1}\varphi_{F''}(\varphi_{F'}(x))}{\sigma(\beta_2)} + \Sigma_{\beta_2},\\
&xF'' - F''x = \frac{K_{\beta_2}\psi_{F''}(x) + (-1)^{\mathrm{ht}(\beta_2)}K_{\beta_2}^{-1}\varphi_{F''}(x)}{\sigma(\beta_2)} + \Sigma_{\beta_2},\\
&\varphi_{F''}(x)F' - F'\varphi_{F''}(x)\\
&= \frac{K_{\beta_1}\psi_{F'}(\varphi_{F''}(x)) + (-1)^{\mathrm{ht}(\beta_1)}K_{\beta_1}^{-1}\varphi_{F'}(\varphi_{F''}(x))}{\sigma(\beta_1)} + \Sigma_{\beta_1}.
\end{aligned}$$

Using 3.3 (b)-3.3 (d) and Corollary 3.6 repeatedly, we get

$$\begin{aligned}
&xF - Fx \\
&= xF''F' - v^d xF'F'' - F''F'x + v^d F'F''x \\
&= (xF'' - F''x)F' + F''(xF' - F'x) \\
&\quad -v^d(xF' - F'x)F'' - v^d F'(xF'' - F''x) \\
&= \frac{K_\beta}{\sigma(\beta)}(\psi_{F'}(\psi_{F''}(x)) - v^d\psi_{F''}(\psi_{F'}(x))) \\
&\quad + \frac{(-1)^{\mathrm{ht}(\beta)}K_\beta^{-1}}{\sigma(\beta)}(\varphi_{F'}(\varphi_{F''}(x)) - v^d\varphi_{F''}(\varphi_{F'}(x))) + \Sigma_\beta.
\end{aligned}$$

Therefore we have

(c) $\varphi_F(x) = \varphi_{F'}\varphi_{F''}(x) - v^d\varphi_{F''}\varphi_{F'}(x)$.

Since $d_\beta = d_{i_k} = d_{\beta_1}$, so, $d'_\beta = d'_{\beta_1} + d'_{\beta_2} + d_{\beta_2}$. Note that $d_{\beta_2} = d_{i_{k-1}} = d$. Hence

$$\begin{aligned}
(x, Ey) &= (x, E'E''y - v^{-d}E''E'y) \\
&= (-1)^{\mathrm{ht}(\beta)}v^{-d'_\beta+d}(\varphi_{F''}(\varphi_{F'}(x)), y) \\
&\quad -(-1)^{\mathrm{ht}(\beta)}v^{-d'_\beta+d-d}(\varphi_{F'}(\varphi_{F''}(x)), y) \\
&= (-1)^{\mathrm{ht}(\beta)-1}v^{-d'_\beta}(\varphi_F(x), y).
\end{aligned}$$

We may deal with other cases similarly. The proposition is proved.

Corollary 3.8. *Let* $\mathbf{i} = (i_1, \cdots, i_\nu) \in \mathcal{X}$ *and let* $F = T_{i_1}\cdots T_{i_{k-1}}(E_{i_k})$, $E = \Omega(F)$, $A = (a_1, \ldots, a_\nu) \in \mathbb{N}^\nu$. *Then*

(a) $\varphi_F(E_{\mathbf{i}}^A) = 0$ *if* $a_1 = \cdots = a_k = 0$,

(b) $\varphi_F(E_{\mathbf{i}}^A) = (-1)^{ht(\beta)-1}\dfrac{v_{i_k}^{a_k-1}\sigma(\beta)}{\sigma(\alpha_{i_k})}E_{\mathbf{i}}^{A'}$, *if* $a_1 = \cdots = a_{k-1} = 0$,

where $A' = (0, \cdots, 0, a_k - 1, a_{k+1}, \cdots a_\nu)$, $\beta = s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k})$.

Proof. (a) follows from Lemma 3.2 and the definition of φ_F (see Corollary 3.6). (b) follows from the definition of φ_F and the following identity

$$E^{(a)}F = FE^{(a)} + \frac{K_\beta v_{i_k}^{1-a} - K_\beta^{-1}v_{i_k}^{a-1}}{v_{i_k} - v_{i_k}^{-1}}E^{(a-1)}, \quad a \in \mathbb{N}.$$

Theorem 3.9. *Let* $\mathbf{i} = (i_1, \ldots, i_\nu) \in \mathcal{X}$ *and* $A = (a_1, \ldots, a_\nu)$, $B = (b_1, \ldots, b_\nu)$ *be elements of* $\mathbb{N}^\nu$. *Then*

(a) $(E_{\mathbf{i}}^A, E_{\mathbf{i}}^B) = 0$ *if* $A \neq B$.

(b) $(E_{\mathbf{i}}^A, E_{\mathbf{i}}^A) = \prod_{k=1}^{\nu} \frac{\xi(\beta_k)^{a_k}}{\{a_k\}_{d_k''}^!}$,

where $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$, $d_k'' = d_{i_k}$ *and* $\xi(\beta_k) = \frac{\prod_i (1 - v_i^{-2})^{c_i}}{(1 - v_{i_k}^{-2})}$ *if* $\beta_k = \sum_i c_i \alpha_i$. *In particular,* $E_{\mathbf{i}}^A \in \mathcal{L}_{\mathcal{B}}$ (*see* 3.11 *for definition*).

Proof. Repeatedly use Prop. 3.7 and Corollary 3.8.

Corollary 3.10. *Let* $x \in U^+$. *Then*

(a) $(x, x) \neq 0$ *if* $x \neq 0$. *In particular,* $(\ ,\ )$ *is non-degenerate* [**K**, Corollary 3.4.8].

(b) *For any* $\mathbf{i} \in \mathcal{X}$ *we have* $x = \sum_{A \in \mathbb{N}^\nu} \frac{(x, E_{\mathbf{i}}^A)}{(E_{\mathbf{i}}^A, E_{\mathbf{i}}^A)} E_{\mathbf{i}}^A$.

3.11. Let $\mathcal{B}$ be the subring of $\mathbb{Q}(v)$ consisting of all rational functions which are regular at $v^{-1} = 0$ (i.e. $v = \infty$). Define $\mathcal{L}_{\mathcal{B}} = \{x \in U^+ \mid (x, x) \in \mathcal{B}\}$. The $\mathcal{B}$-submodule $\mathcal{L}_{\mathcal{B}}$ of U^+ is crucial for discussing canonical bases.

Corollary 3.12. *For any* $\mathbf{i} \in \mathcal{X}$, *the elements* $E_{\mathbf{i}}^A$ $(A \in \mathbb{N}^\nu)$ *form a* $\mathcal{B}$*-basis of* $\mathcal{L}_{\mathcal{B}}$.

Proof. Let $\xi \in \mathbb{Q}(v)$, then $\xi \in \mathcal{B}$ if and only if $\xi^2 \in \mathcal{B}$. The corollary then follows from Theorem 3.9 and 2.1 (a).

Corollary 3.13. *Let* $\mathbf{i}, \mathbf{j} \in \mathcal{X}$ *and let* $A \in \mathbb{N}^\nu$. *Write*

$$E_{\mathbf{i}}^A = \sum_{B \in \mathbb{N}^\nu} \xi_B E_{\mathbf{j}}^B, \quad \xi_B \in \mathcal{A},$$

then there exists a unique $B_0 \in \mathbb{N}^\nu$ *such that* $\xi_{B_0} \in \pm 1 + v^{-1}\mathbb{Z}[v^{-1}]$, *and* $\xi_B \in v^{-1}\mathbb{Z}[v^{-1}]$ *if* $B \neq B_0$ (*see* [**L3**, Prop. 2.3]).

Proof. By Corollary 3.12 we see that $\xi_B \in \mathcal{A} \cap \mathcal{B} = \mathbb{Z}[v^{-1}]$. By Theorem 3.9 (a) we know

$$\left(E_{\mathbf{i}}^A, E_{\mathbf{i}}^A\right) = \sum_{B \in \mathbb{N}^\nu} \xi_B^2 \left(E_{\mathbf{j}}^B, E_{\mathbf{j}}^B\right).$$

By Theorem 3.9 (b), the values of $(E_{\mathbf{i}}^A, E_{\mathbf{i}}^A)$, $(E_{\mathbf{j}}^B, E_{\mathbf{j}}^B)$ at $v^{-1} = 0$ are 1. So there is a unique $B_0 \in \mathbb{N}^\nu$ such that $\xi_{B_0}^2|_{v^{-1}} = 1$, and $\xi_B^2|_{v^{-1}=0} = 0$ if $B \neq B_0$.

The corollary is proved.

Corollary 3.14. (a) *Let* $\mathcal{L}$ *be the* $\mathbb{Z}[v^{-1}]$*-submodule of* $\mathcal{L}_{\mathcal{B}}$ *spanned by the elements* $E_{\mathbf{i}}^A$, $\mathbf{i} \in \mathcal{X}$, $A \in \mathbb{N}^\nu$. *Then* $\mathcal{L}$ *is a free* $\mathbb{Z}[v^{-1}]$*-module and for any* $\mathbf{i} \in \mathcal{X}$, *the elements* $E_{\mathbf{i}}^A$ $(A \in \mathbb{N}^\nu)$ *form a* $\mathbb{Z}[v^{-1}]$*-basis of* $\mathcal{L}$.

(b) *Let* $x \in U^+_{\mathcal{A}}$. *Then* $x \in \mathcal{L}$ *if and only if* $x \in \mathcal{L}_{\mathcal{B}}$, *i.e.* $(x,x) \in \mathcal{B}$. *See* [**L3**].

3.15. The $\mathbb{Z}[v^{-1}]$-module $\mathcal{L}$ can be defined through Kashiwara's operators $\tilde{e}_i, \tilde{f}_i : U^+ \to U^+$, which are defined as follows

$$\tilde{e}_i : \sum_{A\in\mathbb{N}^\nu} \xi_A E^A_{\mathbf{i}} \to \sum_{A\in\mathbb{N}^\nu} \xi_A E^{A+A_1}_{\mathbf{i}}, \quad \xi_A \in \mathbb{Q}(v),$$
$$\tilde{f}_i : \sum_{A\in\mathbb{N}^\nu} \xi_A E^A_{\mathbf{i}} \to \sum_{A\in\mathbb{N}^\nu} \xi_A E^{A-A_1}_{\mathbf{i}}, \quad \xi_A \in \mathbb{Q}(v),$$

where $\mathbf{i} = (i_1, \cdots, i_\nu) \in \mathcal{X}$ such that $i_1 = i$ and $A_1 = (1, 0, \cdots, 0)$.

Obviously we have

(a) $\tilde{e}_i$ and $\tilde{f}_i$ map $\mathcal{L}$ to $\mathcal{L}$ for all $i \in [1, n]$,

(b) $\tilde{f}_i\tilde{e}_i$ =id for all $i \in [1, n]$,

(c) $U^+ = \ker\tilde{f}_i \oplus \mathrm{im}\tilde{e}_i$ for each i in $[1, n]$,

(d) $\ker\tilde{f}_i = \ker\varphi_i$. In particular $\bigcap_{i=1}^{n} \ker\tilde{f}_i = \mathbb{Q}(v).1$ (cf. [**K**, Lemma 3.4.7]).

Proposition 3.16. *Let* $\mathcal{L}'$ *be the* $\mathbb{Z}[v^{-1}]$*-submodule of* U^+ *generated by the elements* $\tilde{e}_{i_1} \cdots \tilde{e}_{i_k}(1)$ $(i_1, \cdots, i_k \in [1, n]$ *and* $k \in \mathbb{N})$. *Then we have* $\mathcal{L}' \subseteq \mathcal{L}$. (*See* [**L4**, Theorem 2.3 (a)].)

Proof. Using Corollary 3.13 and the definition of $\tilde{e}_i$ we see $\mathcal{L}' \subseteq \mathcal{L}$.

It is not difficult to prove that $\mathcal{L}' = \mathcal{L}$, see [**L4**] or [**X3**].

Acknowledgement. The author was supported by the National Natural Science Foundation of China (No. 19425003).

References

[CK] C. De Concini and V.G. Kac, *Representations of quantum groups at roots of* 1, in 'Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory', Actes du colloque en l'honneur de Jacques Dixmier, edited by A. Connes, M. Duflo, A. Joseph and R. Rentschler, Progress in Mathematics 92, Birkháuser, Boston-Basel-Berlin, (1990), 471-506.

[DL] M. Dyer and G. Lusztig, Appendix to [**L2**].

[K] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J., **63** (1991), 465-516.

[LS] Levendorski-Soibelman, *Some applications of quantum Weyl groups*, J. Geom. and Physics, **7** (1990), 241-254.

[L1] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, Jour. Amer. Math. Soc., **3** (1990), 257-296.

[L2] ———, *Quantum groups at roots of* 1, Geom. Ded., **35** (1990), 89-114.

[L3] ———, *Canonical bases arising from quantized enveloping algebras*, Jour. Amer. Math. Soc., **3** (1990), 447-498.

[L4] ———, *Canonical bases arising from quantized enveloping algebras*, II, Progr. Theor. Phys. Suppl., **102** (1990), 175-201.

[L5] ———, *Introduction to quantum groups*, Progress in Mathematics 110, Birkháuser, Boston-Basel-Berlin, 1993.

[X1] N. Xi, *Root vectors in quantum groups*, Comm. Math. Helv., **69** (1994), 612-639.

[X2] ———, *Irreducible modules of quantized enveloping algebras at roots of* 1, Publ. RIMS. Kyoto Univ., **32** (1996), 235-276.

[X3] ———, *Bases of quantized enveloping algebras*, Preprint, 1995.

Received September 26, 1997.

Academia Sinica
Beijing 100080
China
E-mail address: nanhua@math08.math.ac.cn

University of California
Riverside, CA 92521