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We prove a functional version of the Brunn-Minkowski inequality for restricted sums obtained by Szarek and Voiculescu.

We only consider Lebesgue-measurable subsets of $\mathbb{R}^n$, and for $A \subset \mathbb{R}^n$, we denote its volume by $|A|$. If $A, B \subset \mathbb{R}^n$, their Minkowski sum is defined by

$$A + B = \{x + y,\, (x, y) \in A \times B\}.$$

The classical Brunn-Minkowski inequality provides a lower bound for its volume.

Theorem 1. *Let A, B be compact, non void subsets of $\mathbb{R}^n$, one has*

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$

In their study of the free analogue of the entropy power inequality [**SV**], Szarek and Voiculescu define the notion of restricted Minkowski sum of A and B with respect to $\Theta \subset A \times B$:

$$A +_\Theta B = \{x + y,\, (x, y) \in \Theta\},$$

and show that an analogue of the Brunn-Minkowski inequality holds:

Theorem 1′. *There exists a positive constant c such that for all $\rho \in]0, 1[$, $n \in \mathbb{N}$, for all $A, B \subset \mathbb{R}^n$ and $\Theta \subset A \times B$ such that:*

$$\rho \leq \left(\frac{|A|}{|B|}\right)^{\frac{1}{n}} \leq \rho^{-1} \quad \text{and} \quad \frac{|\Theta|}{|A|.|B|} \geq 1 - c\min(\rho\sqrt{n}, 1),$$

one has

$$|A +_\Theta B|^{\frac{2}{n}} \geq |A|^{\frac{2}{n}} + |B|^{\frac{2}{n}}.$$

It is well known that the Brunn-Minkowski inequality can be derived from the Prékopa-Leindler inequality [**Pré**], [**Lei**], which we recall here:

Theorem 2. *Let f, g be non-negative functions in $L_1(\mathbb{R}^n)$ and $\lambda \in]0,1[$, let H be a measurable function on $\mathbb{R}^n$ such that*

$$H(x) \geq \sup\{f^\lambda(u)g^{1-\lambda}(v),\, (u,v) \in \mathbb{R}^n \times \mathbb{R}^n \ \text{ and } \ x = \lambda u + (1-\lambda)v\},$$

then

$$\int_{\mathbb{R}^n} H(x)\, dx \geq \left(\int f\right)^\lambda \left(\int g\right)^{1-\lambda}.$$

We show that a corresponding restricted version of this statement holds.

Theorem 2′. *There exist positive scalars c and n_0 such that for all $0 < \varepsilon \leq 1/2$, for all $\lambda \in [\varepsilon, 1-\varepsilon]$ and for all $n \geq n_0$, if f, g are non-negative functions in $L_1(\mathbb{R}^n)$ and if Θ is a measurable subset of $\mathbb{R}^{2n}$ such that*

$$\frac{\displaystyle\int_\Theta f(x)g(y)\, dx\, dy}{\left(\int f\right)\left(\int g\right)} \geq \frac{1}{2} + \frac{c}{\sqrt{\varepsilon}} \cdot \frac{\log n}{\sqrt{n}},$$

then

$$\int_{\mathbb{R}^n} K(x)\, dx \geq \left(\int f\right)^\lambda \left(\int g\right)^{1-\lambda},$$

as soon as the function K satisfies:

$$K(x) \geq \sup\{f^\lambda(u)g^{1-\lambda}(v),\, (u,v) \in \Theta \ \text{ and } \ x = \sqrt{\lambda}\, u + \sqrt{1-\lambda}\, v\}.$$

Let us return to the example given in [**SV**] to show that the condition on the ratio

$$\theta = \frac{\displaystyle\int_\Theta f(x)g(y)\, dx\, dy}{\left(\int f\right)\left(\int g\right)}$$

is asymptotically optimal. Let B_2^n be the Euclidean unit ball in $\mathbb{R}^n$ and let

$$\Theta = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n,\, \langle x, y\rangle \leq 0\},$$

then $|\Theta \cap (B_2^n \times B_2^n)| = 1/2\, |B_2^n|^2$ and the Θ-restricted sum of a ball of radius r_1 and a ball of radius r_2 is a ball of radius $\sqrt{r_1^2 + r_2^2}$. In particular, for all $\lambda \in [0,1]$,

$$\sqrt{\lambda} B_2^n +_\Theta \sqrt{1-\lambda} B_2^n = B_2^n.$$

The conclusion of Theorem 2′ applied when f and g are the characteristic function of B_2^n would be

$$\left|\sqrt{\lambda} B_2^n +_\Theta \sqrt{1-\lambda} B_2^n\right| \geq |B_2^n|,$$

and actually the equality holds. It is then clear that the conclusion of Theorem 2′ becomes false for ratios $\theta < 1/2$.

We shall first show that Theorem 2′ implies Theorem 1′, maybe with different conditions on the parameters. Let A, B be two subsets of $\mathbb{R}^n$, let $\Theta \subset A \times B$ such that

$$\rho := \left(\frac{|A|}{|B|}\right)^{\frac{1}{n}} \leq 1.$$

Assume that the ratio $\theta = \dfrac{|\Theta|}{|A|.|B|}$ is larger than $\dfrac{1}{2} + c\sqrt{\dfrac{1+\rho^2}{\rho^2}} \cdot \dfrac{\log n}{\sqrt{n}}$. Let us define the set

$$\tilde{\Theta} = \left\{ \left(\frac{a}{|A|^{\frac{1}{n}}}, \frac{b}{|B|^{\frac{1}{n}}} \right) \in \mathbb{R}^{2n},\ (a,b) \in \Theta \right\}.$$

Let

$$\tilde{A} = \frac{A}{|A|^{\frac{1}{n}}} \quad \text{and} \quad \tilde{B} = \frac{B}{|B|^{\frac{1}{n}}}$$

and let f and g be the characteristic functions of $\tilde{A}$ and $\tilde{B}$. A simple change of variables gives that

$$\frac{\int_{\tilde{\Theta}} f(x)g(y)\,dx\,dy}{\left(\int f\right)\,\left(\int g\right)} = \frac{|\Theta|}{|A|.|B|} = \theta,$$

so we can apply Theorem 2′ to f and g, with $\lambda = \dfrac{|A|^{\frac{2}{n}}}{|A|^{\frac{2}{n}} + |B|^{\frac{2}{n}}} = \dfrac{\rho^2}{1+\rho^2}$ and get

$$\left| \sqrt{\lambda}\,\tilde{A} +_{\tilde{\Theta}} \sqrt{1-\lambda}\,\tilde{B} \right| \geq 1,$$

where

$$\begin{aligned} \sqrt{\lambda}\,\tilde{A} +_{\tilde{\Theta}} \sqrt{1-\lambda}\,\tilde{B} &= \left\{ \sqrt{\lambda}\frac{a}{|A|^{\frac{1}{n}}} + \sqrt{1-\lambda}\frac{b}{|B|^{\frac{1}{n}}},\ (a,b) \in \Theta \right\} \\ &= \left\{ \frac{a+b}{\sqrt{|A|^{\frac{2}{n}} + |B|^{\frac{2}{n}}}},\ (a,b) \in \Theta \right\} \\ &= \frac{A +_{\Theta} B}{\sqrt{|A|^{\frac{2}{n}} + |B|^{\frac{2}{n}}}}. \end{aligned}$$

Hence, we obtain

$$|A +_{\Theta} B|^{\frac{2}{n}} \geq |A|^{\frac{2}{n}} + |B|^{\frac{2}{n}}.$$

Our method is based on an observation of Brascamp and Lieb [**BL1**]: the Prékopa-Leindler inequality is a limit case of the reverse sharp form of Young's convolution inequality. We will first prove a restricted form of

Young's inequality and its converse, using a modification of the method we developed in [**Bar**], and then take the limits in certain parameters. Our proof of Young's inequality is based on measure-preserving mappings between measures. We use them in order to build a suitable change of variables which makes the problem simpler; then a simple arithmetico-geometric inequality gives the result. Now, we have to work with functions on $\mathbb{R}^n$, because the set Θ makes it difficult to use the classical tensorisation argument. In general, given two probability on $\mathbb{R}^n$, there are several measure-preserving mappings between them; for our purpose, the mapping built by Knothe in [**Kno**] fits:

Lemma 1. *Let f, F be positive continuous functions on $\mathbb{R}^n$ such that $\int f = \int F$. There exists a differentiable map $u : \mathbb{R}^n \to \mathbb{R}^n$ such that for $x \in \mathbb{R}^n$*

$$\det(du(x)) \cdot f(u(x)) = F(x), \tag{1}$$

and for all $i \leq n$ and all$(x_i)_{i=1}^n \in \mathbb{R}^n$,

$$u((x_i)_{i=1}^n) = (u_1(x_1), u_2(x_1, x_2), \dots, u_n(x_1, \dots, x_n)),$$

where for all $x_1, \dots, x_{i-1}$, the function $u_i(x_1, \dots, x_{i-1}, \cdot)$ is increasing on $\mathbb{R}$. In particular $du(x)$ has always a lower triangular matrix with positive diagonal (in the canonical basis).

We also need a version of the arithmetico-geometric inequality for matrices of the previous form:

Lemma 2. *Let M, N be lower triangular $n \times n$-matrices with non-negative diagonal and let $t \in [0, 1]$, then*

$$\det(tM + (1-t)N) \geq (\det M)^t (\det N)^{1-t}.$$

The first step of the proof is the following restricted version of Young's inequality. For $t > 1$, we denote by t' the real number such that $1/t + 1/t' = 1$.

Lemma 3. *Let f, F, g, G be positive continuous functions on $\mathbb{R}^n$, of integral 1 and dominated by some Gaussian function. Let u and v denote the measure preserving mappings obtained when applying Lemma 1 to (f, F) and (g, G) and let T be the bijective map of $\mathbb{R}^n \times \mathbb{R}^n$ defined by $T(x, y) = (u(x), v(y))$.*

Let $p, q, r \geq 1$ such that $1/p + 1/q = 1 + 1/r$. We set

$$c = \sqrt{r'/q'} \text{ and } s = \sqrt{r'/p'},$$

and notice that $c^2 + s^2 = 1$. Then

$$\int f(x)g(y)\mathbf{1}_{T\Theta}(x, y)\,dx\,dy = \int F(X)G(Y)\mathbf{1}_{\Theta}(X, Y)\,dX\,dY, \tag{2}$$

and

$$\left(\int\left(\int f^{\frac{1}{p}}(cx-sy)g^{\frac{1}{q}}(sx+cy)\mathbf{1}_{T\Theta}(cx-sy,\,sx+cy)dx\right)^r dy\right)^{\frac{1}{r}}$$
$$\leq \int\left(\int F^{\frac{r}{p}}(cX-sY)G^{\frac{r}{q}}(sX+cY)\mathbf{1}_{\Theta}(cX-sY,sX+cY)\,dY\right)^{\frac{1}{r}} dX.$$

Proof. Equality (2) is a consequence of the measure-preserving properties of u and v. We give a detailed proof of the inequality. Let R be the rotation of matrix $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ in the canonical basis. We are going to use the change of variable in $\mathbb{R}^{2n}$ given by the function $\Phi = ({}^tR\otimes I_n)T(R\otimes I_n)$, where I_n is the identity map on $\mathbb{R}^n$. More precisely $(x,y)=\Phi(X,Y)$ means

$$\begin{aligned} x &= c\,u(cX-sY)+s\,v(sX+cY)\\ y &= -s\,u(cX-sY)+c\,v(sX+cY). \end{aligned}$$

It is clear that Φ is a differentiable bijection of $\mathbb{R}^{2n}$. Its jacobian at the point (X,Y) is

$$J\Theta(X,Y) = \det(du(cX-sY))\det(dv(sX+cY)).$$

We want an upper estimate for the integral (finite by assumption)

$$I = \left(\int\left(\int f^{\frac{1}{p}}(cx-sy)g^{\frac{1}{q}}(sx+cy)\mathbf{1}_{T\Theta}(cx-sy,\,sx+cy)\,dx\right)^r dy\right)^{\frac{1}{r}}.$$

Using the $(L^r, L^{r'})$-duality, there exists a positive function h on $\mathbb{R}^n$ such that $\|h\|_{r'}=1$ and

$$I = \iint f^{\frac{1}{p}}(cx-sy)g^{\frac{1}{q}}(sx+cy)\mathbf{1}_{T\Theta}(cx-sy,\,sx+cy)h(y)\,dx\,dy.$$

By the change of variable $(x,y)=\Phi(X,Y)$, we obtain that I is equal to

$$\begin{aligned} \iint & f^{\frac{1}{p}}(u(cX-sY))g^{\frac{1}{q}}(v(sX+cY))h(-s\,u(cX-sY)+c\,v(sX+cY))\\ &\cdot \mathbf{1}_{T\Theta}(u(cX-sY),v(sX+cY))\\ &\cdot \det(du(cX-sY))\det(dv(sX+cY))\,dX\,dY. \end{aligned}$$

In order to shorten the formulas, denote

$$\begin{aligned} U &= u(cX-sY), & V &= v(sX+cY),\\ U' &= \det(du(cX-sY)), & V' &= \det(dv'(sX+cY)). \end{aligned}$$

Noticing that the definition of T implies $\mathbf{1}_{T\Theta}(u(cX-sY),v(sX+cY)) = \mathbf{1}_{\Theta}(cX-sY,sX+cY)$, and using the differential formulas

$$\det(du(x)).f(u(x)) = F(x),$$

$$\det(dv(x)).g(u(x)) = G(x),$$

we get

$$I = \iint f^{\frac{1}{p}}(u(cX-sY))g^{\frac{1}{q}}(v(sX+cY))\mathbf{1}_\Theta(cX-sY, sX+cY) \cdot h(-sU+cV)U'V'\,dX\,dY$$

$$= \int\Big(\int F^{\frac{1}{p}}(cX-sY)G^{\frac{1}{q}}(sX+cY)\mathbf{1}_\Theta(cX-sY, sX+cY) \cdot h(-sU+cV)(U')^{\frac{1}{p'}}(V')^{\frac{1}{q'}}\,dY\Big)dX.$$

Using Hölder's inequality for the integral in Y with parameters r and r', one has:

$$I \le \int\Big(\Big(\int F^{\frac{r}{p}}(cX-sY)G^{\frac{r}{q}}(sX+cY)\mathbf{1}_\Theta(cX-sY, sX+cY)\,dY\Big)^{\frac{1}{r}} \cdot \Big(\int h^{r'}(-sU+cV)(U')^{\frac{r'}{p'}}(V')^{\frac{r'}{q'}}\,dY\Big)^{\frac{1}{r'}}\,dX.$$

Let $H(X) = \int h^{r'}(-sU+cV)(U')^{\frac{r'}{p'}}(V')^{\frac{r'}{q'}}\,dY$, then

$$H(X) = \int h^{r'}(a(X,Y))(\det du(cX-sY))^{s^2}(\det dv(sX+cY))^{c^2}\,dY,$$

where

$$a(X,Y) = -s\,u(cX-sY) + c\,v(sX+cY).$$

It is clear that the partial differential of a with respect to Y is

$$d_Y a(X,Y) = s^2 du(cX-sY) + c^2 dv(sX+cY).$$

By the arithmetico-geometric inequality stated in Lemma 2,

$$\det(d_Y a(X,Y)) \ge (\det du(cX-sY))^{s^2}(\det dv(sX+cY))^{c^2},$$

hence

$$H(X) \le \int h^{r'}(a(X,Y))\det d_Y a(X,Y)\,dY \le \int h^{r'} = 1,$$

where we use the fact that $a(X,Y)$ is an injective function of Y (indeed, u and v are by definition increasing for the lexicographic order on $\mathbb{R}^n$). This proves that

$$I \le \int\Big(\int F^{\frac{r}{p}}(cX-sY)G^{\frac{r}{q}}(sX+cY)\mathbf{1}_\Theta(cX-sY, sX+cY)\,dY\Big)^{\frac{1}{r}}\,dX.$$

$\square$

We are going to take a limit in r to obtain an inequality similar to the Prékopa-Leindler inequality. To simplify the notations, we set $\kappa = 1-\lambda$.

Lemma 4. *Let f, g, F, G be as in Lemma 3. Let $\Theta \subset \mathbb{R}^{2n}$ and denote $\theta = \int_\Theta F(X)G(Y)\,dX\,dY$. Then*

$$\begin{aligned}\int \sup_{X=\sqrt{\lambda}u+\sqrt{\kappa}v} & F^\lambda(u)G^\kappa(v)\mathbf{1}_\Theta(u,v)\,dX \\ \geq \inf_A \sup_{y\in\mathbb{R}^n} & \int f^\lambda\left(\sqrt{\lambda}\,x - \sqrt{\kappa}\,y\right) g^\kappa\left(\sqrt{\kappa}\,x + \sqrt{\lambda}\,y\right) \\ & \cdot \mathbf{1}_A\left(\sqrt{\lambda}\,x - \sqrt{\kappa}\,y, \sqrt{\kappa}\,x + \sqrt{\lambda}\,y\right)\,dx,\end{aligned}$$

where the infimum is over the sets $A \subset \mathbb{R}^{2n}$ such that $\int_A f(x)g(y)\,dxdy \geq \theta$.

Proof. This lemma is a limit case of Lemma 3. For $r > 1$, we set

$$p_r = \frac{r}{\lambda(r+1)}\,, \qquad q_r = \frac{r}{\kappa(r+1)}\,.$$

Then $1/p_r + 1/q_r = 1 + 1/r$ and when r is large enough $p_r, q_r > 1$. We apply Lemma 3 with f, g, F, G for this triple and take the limit when r tends to $+\infty$. Notice that

$$\frac{1}{p_r} \to \lambda, \qquad \frac{1}{q_r} \to \kappa,$$

and

$$c_r = \sqrt{\frac{r'}{q_r'}} = \sqrt{\frac{1-q_r^{-1}}{1-r^{-1}}} \to \sqrt{\lambda}, \qquad s_r \to \sqrt{\kappa}\,.$$

Our strong domination hypothesis ensures that the r-norms tend to essential suprema when r tends to infinity. We get:

$$\begin{aligned}\sup_{y\in\mathbb{R}^n} & \int f^\lambda\left(\sqrt{\lambda}\,x - \sqrt{\kappa}\,y\right) g^\kappa\left(\sqrt{\kappa}\,x + \sqrt{\lambda}\,y\right) \\ & \cdot \mathbf{1}_{T\Theta}\left(\sqrt{\lambda}\,x - \sqrt{\kappa}\,y, \sqrt{\kappa}\,x + \sqrt{\lambda}\,y\right)\,dx \\ \leq & \int_{\mathbb{R}^n} \sup_{Y\in\mathbb{R}^n} F^\lambda\left(\sqrt{\lambda}\,X - \sqrt{\kappa}\,Y\right) G^\kappa\left(\sqrt{\kappa}\,X + \sqrt{\lambda}\,Y\right) \\ & \cdot \mathbf{1}_\Theta\left(\sqrt{\lambda}\,X - \sqrt{\kappa}\,Y, \sqrt{\kappa}\,X + \sqrt{\lambda}\,Y\right)\,dX.\end{aligned}$$

Noticing that $\left\{\begin{array}{l} u=\sqrt{\lambda}\,X-\sqrt{\kappa}\,Y \\ v=\sqrt{\kappa}\,X+\sqrt{\lambda}\,Y \end{array}\right.$ is equivalent to $\left\{\begin{array}{l} X=\sqrt{\lambda}\,u+\sqrt{\kappa}\,v \\ Y=-\sqrt{\kappa}\,u+\sqrt{\lambda}\,v \end{array}\right.$, we can rewrite the second member of the previous inequality as

$$\int \sup_{X=\sqrt{\lambda}\,u+\sqrt{\kappa}\,v} F^{\lambda}(u)G^{\kappa}(v)\mathbf{1}_{\Theta}(u,v)\,dX.$$

By equality (2) in Lemma 3, we have $\int_{T\Theta} f(x)g(y)\,dxdy=\theta$, which leads to the conclusion. □

To finish the proof of Theorem 2′, we have to estimate the infimum given in the previous lemma for two specific functions f and g.

Lemma 5. *Let F,G be as in Lemma 3, then*

$$\int \sup_{X=\sqrt{\lambda}u+\sqrt{\kappa}\,v} F^{\lambda}(u)G^{\kappa}(v)\mathbf{1}_{\Theta}(u,v)\,dX$$
$$\geq \mathbb{E}\left(\exp\left(\sqrt{\lambda\kappa}\sum_{i=1}^{n} X_i\right)\mathbf{1}_{\{\sum X_i\leq M_{n,\theta}\}}\right),$$

where $(X_i)_{i=1}^n$ is a sequence of i.i.d. random variables, their common law being the law of a difference of squares of two independent Gaussian variables $N(0,1/\sqrt{2})$ and the number $M_{n,\theta}$ satisfies $\mathbb{P}(\sum X_i\leq M_{n,\theta})=\theta$.

Proof. We apply Lemma 4 with

$$f(x)=g(x)=\pi^{-n/2}e^{-x^2}.$$

We denote by γ_n the probability measure on $\mathbb{R}^n$ with the previous density. We want a lower estimate of

$$\mathcal{I}=\inf_{\gamma_{2n}(A)=\theta}\sup_{y\in\mathbb{R}^n}\int \exp\left(-\lambda\left(\sqrt{\lambda}\,x-\sqrt{\kappa}\,y\right)^2-\kappa\left(\sqrt{\kappa}\,x+\sqrt{\lambda}\,y\right)^2\right)$$
$$\cdot\mathbf{1}_A\left(\sqrt{\lambda}\,x-\sqrt{\kappa}\,y,\sqrt{\kappa}\,x+\sqrt{\lambda}\,y\right)\pi^{-n/2}\,dx.$$

Since the condition on A is rotation invariant, we can replace A by B such that $(x,y)\in B$ if and only if $(\sqrt{\lambda}\,x-\sqrt{\kappa}\,y,\sqrt{\kappa}\,x+\sqrt{\lambda}\,y)\in A$. Hence

$$\mathcal{I}=\inf_{\gamma_{2n}(B)=\theta}\sup_{y\in\mathbb{R}^n}\int \exp\left(-\lambda\left(\sqrt{\lambda}\,x-\sqrt{\kappa}\,y\right)^2-\kappa\left(\sqrt{\kappa}\,x+\sqrt{\lambda}\,y\right)^2\right)$$
$$\cdot\mathbf{1}_B(x,y)\pi^{-n/2}\,dx$$
$$=\inf_{\gamma_{2n}(B)=\theta}\sup_{y\in\mathbb{R}^n}\int \exp\left(x^2-\lambda\left(\sqrt{\lambda}\,x-\sqrt{\kappa}\,y\right)^2-\kappa\left(\sqrt{\kappa}\,x+\sqrt{\lambda}\,y\right)^2\right)$$
$$\cdot\mathbf{1}_B(x,y)\,d\gamma_n(x),$$
$$\geq\inf_{\gamma_{2n}(B)=\theta}\iint \exp\left(x^2-\lambda\left(\sqrt{\lambda}\,x-\sqrt{\kappa}\,y\right)^2-\kappa\left(\sqrt{\kappa}\,x+\sqrt{\lambda}\,y\right)^2\right)$$

$$\cdot\, \mathbf{1}_B(x,y)\, d\gamma_n(x) d\gamma_n(y).$$

The matrix of the quadratic form on $\mathbb{R}^{2n}$, $Q(x,y) = x^2 - \lambda(\sqrt{\lambda}\, x - \sqrt{\kappa}\, y)^2 - \kappa(\sqrt{\kappa}\, x + \sqrt{\lambda}\, y)^2$ in a suitable orthonormal basis is

$$\sqrt{\lambda\kappa} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},$$

where I_n is the identity $n \times n$ matrix. Hence, by rotation invariance of the Gaussian measure

$$\mathcal{I} \geq \inf_{\gamma_{2n}(B)=\theta} \iint_B \exp\left(\sqrt{\lambda\kappa}(x^2 - y^2)\right) d\gamma_n(x) d\gamma_n(y).$$

This is exactly

$$\mathcal{J} = \iint \exp\left(\sqrt{\lambda\kappa}(x^2 - y^2)\right) \mathbf{1}_{\{x^2 - y^2 \leq M_{n,\theta}\}}\, d\gamma_n(x) d\gamma_n(y),$$

where $M_{n,\theta}$ is such that $\gamma_{2n}(\{x^2 - y^2 \leq M_{n,\theta}\}) = \theta$. We get the conclusion of the lemma by rewriting this with $X_i = x_i^2 - y_i^2$, where x_i and y_i are the i^{th} coordinates of x and y. □

We are going to use the central-limit theorem in the rather precise form of the Berry-Essen theorem. [**Fel**].

Theorem 3. *Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables, let*

$$m = \mathbb{E}X_i, \qquad \sigma = \left(\mathbb{E}X_i^2\right)^{\frac{1}{2}} \quad \text{and} \quad \beta = \mathbb{E}|X_i|^3.$$

For all $t \in \mathbb{R}$, let

$$F_n(t) = \mathbb{P}\left(\frac{\sum_{i=1}^n X_i - nm}{\sigma\sqrt{n}} < t\right)$$

and

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-s^2/2}\, ds.$$

There exists a universal constant $c > 0$ such that for all t and for all n,

$$|F_n(t) - G(t)| \leq \frac{c\beta}{\sigma^3\sqrt{n}}.$$

Proof of Theorem 2′. By homogeneity, we may assume $\int F = \int G = 1$. Comparing the assertions of Lemma 5 and Theorem 2′, we see that to prove the latter, it is enough to show that the expectation from the former is ≥ 1 provided the parameter $\theta = \int_\Theta F(x)G(y)dxdy$ exceeds

$$\frac{1}{2} + \frac{c}{\sqrt{\varepsilon}} \frac{\log n}{\sqrt{n}}.$$

To this end, we apply Theorem 3 to the variables X_i defined in Lemma 5, and notice that $m=0$ and β, σ and c are universal constants. We set

$$\xi_n = \frac{\log n}{\sigma\sqrt{\lambda\kappa n}} \quad \text{and} \quad \alpha = \frac{c\beta}{\sigma^3}.$$

We fix λ and prove that, for n large enough and for

$$\theta = G(\xi_n) + \frac{\alpha}{\sqrt{n}},$$

the quantity

$$\mathcal{J} = \mathbb{E}\left(\exp\left(\sqrt{\lambda\kappa}\sum_{i=1}^n X_i\right)\mathbf{1}_{\{\sum X_i \le M_{n,\theta}\}}\right)$$

is larger than 1.

As $\mathbb{E}X_i = 0$, we get from the Berry-Essen theorem

$$\mathbb{P}\left(\sum_{i=1}^n X_i < \xi_n\sigma\sqrt{n}\right) \le G(\xi_n) + \frac{c\beta}{\sigma^3\sqrt{n}} = \theta,$$

so $M_{n,\theta} \ge \xi_n\sigma\sqrt{n}$. We set $Z_n = \dfrac{\sum_{i=1}^n X_i}{\sigma\sqrt{n}}$, it is clear that

$$\mathcal{J} \ge \mathbb{E}\left(\exp\left(\sigma\sqrt{\lambda\kappa n} Z_n\right)\mathbf{1}_{\{Z_n \le \xi_n\}}\right).$$

Let $n_1(\lambda)$ be the smallest integer n such that $\xi_n \le 1$, notice that it is a non-increasing function of $\lambda \in]0, 1/2]$. We work with $n \ge n_1(\lambda)$. When n is large, Z_n behaves like a normal Gaussian g. So we can almost estimate this expectation by replacing Z_n by g.

More precisely, let $d = 2\alpha\sqrt{2\pi e}$ and let $n_2(\lambda)$ be the smallest integer such that $\xi_n/3 \ge d/\sqrt{n}$, it is a non-decreasing function of $\lambda \in]0, 1/2]$. Then for $n > \max(n_1(\lambda), n_2(\lambda))$, one has

$$\mathbb{P}_{Z_n}([t, \xi_n]) \ge \mathbb{P}_g\left(\left[t, \xi_n - \frac{d}{\sqrt{n}}\right]\right).$$

This comes from the Berry-Essen theorem and from the fact that ξ_n stays in $[0, 1]$ where the density of the law of g is bounded from below:

$$\begin{aligned}
\mathbb{P}_{Z_n}([t, \xi_n]) &\ge \mathbb{P}_g([t, \xi_n]) - 2\frac{\alpha}{\sqrt{n}} \\
&= \mathbb{P}_g\left(\left[t, \xi_n - \frac{d}{\sqrt{n}}\right]\right) + \frac{1}{\sqrt{2\pi}}\int_{\xi_n - \frac{d}{\sqrt{n}}}^{\xi_n} e^{-t^2/2} dt - 2\frac{\alpha}{\sqrt{n}} \\
&\ge \mathbb{P}_g\left(\left[t, \xi_n - \frac{d}{\sqrt{n}}\right]\right) + \frac{1}{\sqrt{n}}\left(\frac{d}{\sqrt{2\pi}} e^{-1/2} - 2\alpha\right).
\end{aligned}$$

We are now able to compute a lower estimate of $\mathcal{J}$.

$$\begin{aligned}
\mathcal{J} &\geq \int_{-\infty}^{\xi_n} \exp\left(\sigma\sqrt{\lambda\kappa n}t\right)\, dP_{Z_n}(t) \\
&= \int_{-\infty}^{\xi_n} \sigma\sqrt{\lambda\kappa n}\exp\left(\sigma\sqrt{\lambda\kappa n}t\right)\, P_{Z_n}([t,\xi_n])\, dt \\
&\geq \int_{-\infty}^{\xi_n-\frac{d}{\sqrt{n}}} \sigma\sqrt{\lambda\kappa n}\exp\left(\sigma\sqrt{\lambda\kappa n}t\right)\, P_g\left(\left[t,\xi_n-\frac{d}{\sqrt{n}}\right]\right) dt \\
&= \int_{-\infty}^{\xi_n-\frac{d}{\sqrt{n}}} \exp\left(\sigma\sqrt{\lambda\kappa n}t\right)\, e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}.
\end{aligned}$$

Because of our assumptions on n, we can write:

$$\begin{aligned}
\mathcal{J} &\geq \int_{\xi_n/2}^{2\xi_n/3} \exp\left(\sigma\sqrt{\lambda\kappa n}t\right)\, e^{-t^2/2}\frac{dt}{\sqrt{2\pi}} \\
&\geq \frac{\xi_n}{6}\exp\left(\sigma\sqrt{\lambda\kappa n}\xi_n/2\right)\frac{e^{-1/2}}{\sqrt{2\pi}} \\
&= \frac{\log n}{6\sigma\sqrt{2\pi e\lambda\kappa n}}\exp\left(\frac{\log n}{2}\right) \\
&= \frac{\log n}{6\sigma\sqrt{2\pi e\lambda\kappa}}.
\end{aligned}$$

We denote by $n_3(\lambda)$ the smallest integer n such that the previous quantity is larger than 1. It is a non-decreasing function of $\lambda \in]0,1/2]$.

Eventually, if $\lambda \in [\varepsilon, 1/2]$, then for $n \geq \max(n_1(\varepsilon), n_2(1/2), n_3(1/2))$ the conclusion of Theorem $2'$ holds for

$$\theta = G(\xi_n) + \frac{\alpha}{\sqrt{n}}.$$

As by concavity, $G(t) \leq \dfrac{1}{2} + \dfrac{t}{\sqrt{2\pi}}$ for all t positive, one easily deduces the theorem from the previous result. Theorem $2'$ gives some information only if the quantity $1/2 + c\log n/\sqrt{\varepsilon n}$ is smaller than one. So the condition $n \geq n_1(\varepsilon)$ is implicitely contained in Theorem $2'$. □

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