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REMARK ON THE $L_q - L_\infty$ ESTIMATE OF THE STOKES SEMIGROUP IN A 2-DIMENSIONAL EXTERIOR DOMAIN

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We proved $L_q - L_\infty$ type estimates of the Stokes semigroup in a 2-dimensional exterior domain. Our proof is based on the investigation of the asymptotic behavior of the resolvent of the Stokes operator near the origin.

1. Introduction.

Let Ω be an unbounded domain in the 2-dimensional Euclidean space $\mathbb{R}^2$ having a compact and smooth boundary $\partial\Omega$ contained in the ball $B_{b_0} = \{x \in \mathbb{R}^2 \mid |x| \le b_0\}$. In $(0,\infty)\times\Omega$, we consider the nonstationary Stokes initial boundary value problem concerning the velocity field $\mathbf{u} = \mathbf{u}(t,x) = {}^t(u_1, u_2)$ and the scalar pressure $\mathfrak{p} = \mathfrak{p}(t,x)$:

$$\text{(NS)}\qquad \begin{aligned} &\partial_t\mathbf{u} - \Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,\infty)\times\Omega,\\ &\mathbf{u} = \mathbf{0} \quad \text{on } (0,\infty)\times\partial\Omega, \qquad \mathbf{u}\to\mathbf{0} \quad \text{as } |x|\to\infty,\\ &\mathbf{u}(0,x) = \mathbf{f}(x) \quad \text{in } \Omega, \end{aligned}$$

where $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbb{R}^2$, $\nabla = (\partial_1, \partial_2)$ with $\partial_j = \partial/\partial x_j$ is the gradient, and $\nabla\cdot\mathbf{u} = \mathrm{div}\mathbf{u} = \partial_1 u_1 + \partial_2 u_2$ is the divergence of $\mathbf{u}$.

We consider this problem in the Lebesgue space $L_r(\Omega)$ for $1 < r \le \infty$ with norm $\|\cdot\|_r$. Let $\mathbb{J}_q(\Omega)$ denote the closure in $L_q(\Omega)\times L_q(\Omega)$ of all solenoidal vector fields with compact support. If we introduce the Stokes operator $\mathbb{A}$, we can reduce (NS) to the following problem (NS′):

$$\text{(NS}'\text{)}\qquad \begin{aligned} &\partial_t\mathbf{u} + \mathbb{A}\mathbf{u} = \mathbf{0} \quad \text{in } (0,\infty)\times\Omega,\\ &\mathbf{u}(0,x) = \mathbf{f}(x) \quad \text{in } \Omega. \end{aligned}$$

According to the result of [**4**], we know that $-\mathbb{A}$ generates an analytic semigroup $e^{-t\mathbb{A}}$ in a 2-dimensional exterior domain.

It is important to investigate the decay property of the analytic semigroup $e^{-t\mathbb{A}}$ in terms of various L_p norms. In fact, Kato [**12**] proved a global in time existence theorem of solutions to Navier-Stokes equation in $\mathbb{R}^n$ by using so called L_q - L_r estimates of $e^{-t\mathbb{A}}$. This work was extended by Iwashita [**11**] to the exterior domain in $\mathbb{R}^n$ ($n \ge 3$) case. The restriction that $n \ge 3$ in [**11**] essentially came from the continuity of the Stokes resolvent at the origin. And therefore, his proof does not seem to be applied directly to the 2-dimensional case, because the 2-dimensional Stokes resolvent has the logarithmic singularity at the origin. Borchers and Varnhorn [**4**] overcame this difficulty first by using the Stokes potentials to show the L_p boundedness of Stokes semigroup in 2-dimensional exterior domain.

On the other hand, in our previous study [**7**], we extended Iwashita's result to 2-dimensional exterior domain case in the same spirit as in Iwashita, which goes back to Shibata [**22**].

In [**7**] we obtained the following $L_q - L_r$ estimates of the Stokes semigroup in a 2-dimensional exterior domain.

Theorem 1.1 ($L_q - L_r$ estimates, cf. [**6**, **7**])**.**
(1) Let $1 < q \leq r < \infty$. Then the following estimate holds for any $\mathbf{f} \in \mathbb{J}_q(\Omega)$:

$$\|e^{-t\mathbb{A}}\mathbf{f}\|_r \leq C_{q,r} t^{-\left(\frac{1}{q}-\frac{1}{r}\right)} \|\mathbf{f}\|_q, \quad t > 0. \tag{1.1}$$

(2) Let $1 < q \leq r \leq 2$. Then, for $\mathbf{f} \in \mathbb{J}_q(\Omega)$

$$\|\nabla e^{-t\mathbb{A}}\mathbf{f}\|_r \leq C_{q,r} t^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \|\mathbf{f}\|_q, \quad t > 0. \tag{1.2}$$

And let $1 < q \leq r$ and $2 < r < \infty$, then, for $\mathbf{f} \in \mathbb{J}_q(\Omega)$

$$\|\nabla e^{-t\mathbb{A}}\mathbf{f}\|_r \leq \begin{cases} C_{q,r} t^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \|\mathbf{f}\|_q, & 0 < t < 1, \\ C_{q,r} t^{-\frac{1}{q}} \|\mathbf{f}\|_q, & t \geq 1. \end{cases} \tag{1.3}$$

Theorem 1.1 does not include the case that $r = \infty$. Our purpose in this study is to obtain the $L_q - L_\infty$ estimate for the Stokes semigroup in a 2-dimensional exterior domain. Our main result of this paper is the following theorem.

Theorem 1.2 ($L_q - L_\infty$ estimate)**.** *Let* $1 < q < \infty$. *Then for* $\mathbf{f} \in \mathbb{J}_q(\Omega)$ *we have*

$$\|e^{-t\mathbb{A}}\mathbf{f}\|_\infty \leq C t^{-\frac{1}{q}} \|\mathbf{f}\|_q \quad \forall t > 0. \tag{1.4}$$

If we try to obtain the L_q - L_∞ estimate by combining the L_q - L_r estimates in $\mathbb{R}^n$ and a local energy decay theorem (such combination was used in [**7**]), we could only obtain

$$\|e^{-t\mathbb{A}}\|_\infty \leq C t^{-\frac{1}{q}} \log t \, \|f\|_q.$$

In this paper, to avoid $\log t$ we will go back to the representation formula of solutions to the resolvent equation. And then, by combining several known results concerning the estimates of Stokes resolvent in $\mathbb{R}^n$ and the asymptotic behavior of Stokes resolvent in the exterior domain near the boundary which was obtained in [**7**], we will be able to show Theorem 1.2. We would like to note that if we apply the known estimations of the Stokes resolvent to the representation formula due to Borchers and Varnhorn [**4**] we can also prove Theorem 1.2. Therefore, the proof itself is not so surprizing if we know how to prove the theorem, but we believe that it is worth while giving the proof of Theorem 1.2, because the result itself is very important. Especially, applying Theorem 1.2 we can show L_∞ estimate of solutions to the Navier-Stokes equations in the 2-dimensional exterior domain.

Namely, let us consider the Navier-Stokes equation in a 2-dimensional exterior domain:

$$\text{(NL)}\quad \begin{aligned} &\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0} \ \text{ and } && \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,\infty)\times\Omega,\\ &\mathbf{u} = \mathbf{0} \quad \text{on } (0,\infty)\times\partial\Omega, && \mathbf{u}(0,x) = \mathbf{f}(x) \quad \text{in } \Omega. \end{aligned}$$

In 1993, for (NL) Kozono and Ogawa [**16**] proved a unique existence theorem of global strong solution $\mathbf{u}(t)$ with initial data in $L_2(\Omega)$, which satisfies the following decay rate:

$$\text{(D)}\quad \begin{aligned} &\|\mathbf{u}(t)\|_q = o\left(t^{-\left(\frac12-\frac1q\right)}\right) \ \ 2\le q<\infty, \quad \|\mathbf{u}(t)\|_\infty = o\left(t^{-\frac12}\sqrt{\log t}\right),\\ &\|\nabla\mathbf{u}(t)\|_2 = o\left(t^{-\frac12}\right) \end{aligned}$$

as $t\to\infty$. They did not use $L_q - L_r$ type estimate of the Stokes semigroup in the 2-dimensional exterior domain. Their proof was based on the argument due to Masuda and some sharp interpolation inequalities like Gagriardo-Nirenberg type. Compared with the Kato's result [**12**] in $\mathbb{R}^2$ case, the L_∞ estimate of solution is worse. In fact, according to Kato [**12**], applying Theorems 1.1 and 1.2, we can easily obtain the L_∞ estimate as follows:

Theorem 1.3. *Let* $\mathbf{u}(t)$ *be the solution obtained in* [**16**] *with initial data* $\mathbf{f}\in\mathbb{J}_2(\Omega)$. *Then, we have* (D) *and the following* L_∞ *estimates:*

$$(\mathrm{D}_\infty)\qquad \|\mathbf{u}(t)\|_\infty = o\left(t^{-\frac12}\right).$$

Finally we collect the symbols used throughout this paper. To denote the special sets, we use the following symbols:

$$D_b = \{x\in\mathbb{R}^2 \mid b-1\le|x|\le b\},\ \ S_b = \{x\in\mathbb{R}^2 \mid |x| = b\},\ \ \Omega_b = \Omega\cap B_b.$$

Let $W_q^m(D)$ denote the Sobolev space of order m on a domain D in the L_q sense and $\|\cdot\|_{m,q,D}$ its usual norm. For simplicity, we use the following abbreviation:

$$\|\cdot\|_{q,D} = \|\cdot\|_{0,q,D}, \quad \|\cdot\|_{m,q} = \|\cdot\|_{m,q,\Omega}, \quad \|\cdot\|_q = \|\cdot\|_{0,q,\Omega}.$$

Moreover, we put

$$\begin{aligned} &L_{q,b}(D) = \{u\in L_q(D) \mid u(x) = 0 \ \ \forall x\notin B_b\},\\ &W_{q,b}^m(D) = \{u\in W_q^m(D) \mid u(x) = 0 \ \ \forall x\notin B_b\},\\ &W_{q,\mathrm{loc}}^m(\mathbb{R}^2) = \{u\in\mathcal{S}' \mid \partial_x^\alpha u\in L_q(B_b) \ {}^\forall\alpha, |\alpha|\le m \ \text{ and } {}^\forall b>0\}, \end{aligned}$$

$$W^m_{q,\mathrm{loc}}(D) = \{u \mid {}^\exists U \in W^m_{q,\mathrm{loc}}(\mathbb{R}^2) \ \text{such that} \ u = U \ \text{on} \ D\},$$
$$L_{q,\mathrm{loc}}(D) = W^0_{q,\mathrm{loc}}(D),$$
$$\dot{W}^m_q(D) = \text{the completion of } C^\infty_0(D) \text{ with respect to } \|\cdot\|_{m,q,D},$$
$$\dot{W}^m_{q,a}(D) = \left\{u \in \dot{W}^m_q(D) \mid \int_D u(x)dx = 0\right\},$$
$$\hat{W}^m_q(D) = \{u \in W^m_{q,\mathrm{loc}}(D) \mid \|\partial^m_x u\|_{q,D} < \infty\},$$
$$(\mathbf{u}, \mathbf{v})_D = \int_D \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)} dx, \quad (\cdot,\cdot) = (\cdot,\cdot)_\Omega.$$

To denote function spaces of 2-dimensional column vector–valued functions, we use the blackboard bold letters. For example, $\mathbb{L}_q(D) = \{\mathbf{u} = {}^t(u_1, u_2) \mid u_j \in L_q(D), j = 1, 2\}$. Likewise for $\mathbb{C}^\infty_0(D)$, $\mathbb{L}_{q,b}(D)$, $\mathbb{W}^m_{q,\mathrm{loc}}(D)$, $\mathbb{L}_{q,\mathrm{loc}}(D)$, $\mathbb{W}^m_q(D)$, $\mathbb{W}^m_{q,b}(D)$, $\dot{\mathbb{W}}^m_q(D)$ and $\hat{\mathbb{W}}^m_q(D)$. Moreover, we put

$$\mathbb{J}_q(D) = \text{the completion in } \mathbb{L}_q(D) \quad \text{of the set } \ \{\mathbf{u} \in \mathbb{C}^\infty_0(D) | \nabla \cdot \mathbf{u} = 0 \ \text{ in } D\},$$
$$\mathbb{G}_q(D) = \{\nabla p \mid p \in \hat{W}^1_q(D)\}.$$

We know that the Banach space $\mathbb{L}_q(D)$ admits the Helmholtz decomposition: $\mathbb{L}_q(D) = \mathbb{J}_q(D) \oplus \mathbb{G}_q(D)$, where $\oplus$ denotes the direct sum. Let $\mathbb{P}_D$ be a continuous projection from $\mathbb{L}_q(D)$ onto $\mathbb{J}_q(D)$. The Stokes operator $\mathbb{A}_D$ is defined by $\mathbb{A}_D = -\mathbb{P}_D\Delta$ with dense domain $\mathcal{D}_q(\mathbb{A}_D) = \mathbb{J}_q(D) \cap \dot{\mathbb{W}}^1_q(D) \cap \mathbb{W}^2_q(D)$. For simplicity, we write: $\mathbb{P} = \mathbb{P}_\Omega$, $\mathbb{A} = \mathbb{A}_\Omega$. It is known that $-\mathbb{A}$ generates an analytic semigroup $e^{-t\mathbb{A}}$ in $\mathbb{J}_q(\Omega)$ [**9**, **4**, **25**]. To denote various constants we use the same letter C, and by $C_{A,B,\cdots}$ we denotes the constant depending on the quantities A, B, $\cdots$. The constants C and $C_{A,B,,\cdots}$ may change from line to line. For two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y and $\|\cdot\|_{\mathcal{L}(X,Y)}$ means its operator norm. In particular, we put $\mathcal{L}(X) = \mathcal{L}(X,X)$. $\mathcal{A}(I,X)$ denotes the set of all X-valued analytic functions in I.

2. Preliminaries.

Let us first consider the stationary Stokes equation in $\mathbb{R}^2$:

$$(2.1) \qquad (\lambda - \Delta)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^2.$$

When $\lambda \in \Sigma = \mathbb{C} \setminus \{\lambda \le 0\}$, put

$$A_\lambda \mathbf{f} = \mathcal{F}^{-1}\left[\frac{(1 - P(\xi))\hat{\mathbf{f}}(\xi)}{|\xi|^2 + \lambda}\right](x) = E_\lambda * \mathbf{f},$$

$$\Pi\mathbf{f} = \mathcal{F}^{-1}\left[\frac{\xi\cdot\hat{\mathbf{f}}(\xi)}{i|\xi|^2}\right](x) = \mathbf{p}*\mathbf{f}$$

for $\mathbf{f}\in\mathbb{L}_q(\mathbb{R}^2)$, where $i=\sqrt{-1}$, $P(\xi)=(\xi_j\xi_k/|\xi|^2)_{j,k=1,2}$,

$$\hat{\mathbf{f}}(\xi)=\int_{\mathbb{R}^2}e^{-ix\cdot\xi}\mathbf{f}(x)dx,\quad \mathcal{F}^{-1}\mathbf{f}(x)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{i\xi\cdot x}\mathbf{f}(\xi)d\xi$$

and

$$\begin{aligned}
E_\lambda &= E_\lambda(x) = (E^\lambda_{jk}(x))_{j,k=1,2},\\
E^\lambda_{jk}(x) &= (2\pi)^{-1}\left\{\delta_{jk}K_0(\sqrt{\lambda}|x|)-\lambda^{-1}\partial_j\partial_k\left(\log|x|+K_0(\sqrt{\lambda}|x|)\right)\right\}\\
&= (2\pi)^{-1}\left\{\delta_{jk}e_1(\sqrt{\lambda}|x|)+\frac{x_jx_k}{|x|^2}e_2(\sqrt{\lambda}|x|)\right\}, \qquad (2.2)\\
\mathbf{p}(x) &= (p_1(x),p_2(x)) = \frac{1}{2\pi}\left(\frac{x_1}{|x|^2},\frac{x_2}{|x|^2}\right).
\end{aligned}$$

Here, K_n ($n\in\mathbb{N}\cup\{0\}$) denotes the modified Bessel function of order n and

$$\begin{aligned}
e_1(\kappa) &= K_0(\kappa)+\kappa^{-1}K_1(\kappa)-\kappa^{-2}\\
&= -\frac{1}{2}\left(\gamma+\frac{1}{2}-\log 2+\log\kappa\right)+O(\kappa^2)\log\kappa \quad \text{as } \kappa\to 0,
\end{aligned}$$

where γ is Euler's constant,

$$\begin{aligned}
e_2(\kappa) &= -K_0(\kappa)-2\kappa^{-1}K_1(\kappa)+2\kappa^{-2}\\
&= \frac{1}{2}+O(\kappa^2)\log\kappa \quad \text{as } \kappa\to 0.
\end{aligned}$$

These are calculated in [**4**, **25**]. Then, for $1<q<\infty$ and any integer $m\ge 0$, by the L_q boundedness of Fourier multiplier (cf. [**10**, Theorem 7.9.5]), we have

(2.3)

$$A_\lambda\in\mathcal{A}(\Sigma,\mathcal{L}(\mathbb{W}^{2m}_q(\mathbb{R}^2),\mathbb{W}^{2m+2}_q(\mathbb{R}^2))),\quad \Pi\in\mathcal{L}(\mathbb{W}^{2m}_q(\mathbb{R}^2),\hat{W}^{2m+1}_q(\mathbb{R}^2)),$$

and the pair of $\mathbf{u}=A_\lambda\mathbf{f}$ and $\mathfrak{p}=\Pi\mathbf{f}$ solves (2.1) for $\lambda\in\Sigma$. When $\mathbf{f}\in\mathbb{L}_{q,b}(\mathbb{R}^2)$, we have

$$A_\lambda\mathbf{f}=O(|x|^{-2}),\quad \Pi\mathbf{f}=O(|x|^{-1})\quad \text{as } |x|\to\infty. \qquad (2.4)$$

For $\lambda=0$, put

$$A_0\mathbf{f}=E_0*\mathbf{f}\quad \text{for } \mathbf{f}\in\mathbb{W}^{2m}_q(\mathbb{R}^2), \qquad (2.5)$$

where

$$E_0=E_0(x)=(E^0_{jk}(x))_{j,k=1,2};$$

$$E^0_{jk}(x) = \frac{1}{4\pi}\left\{-\delta_{jk}\log|x| + \frac{x_j x_k}{|x|^2}\right\}$$

(cf. [**8**, IV.2]). Then the pair of $\mathbf{u} = A_0\mathbf{f}$ and $\mathfrak{p} = \Pi\mathbf{f}$ solves (2.1) for $\lambda = 0$. We have the following facts for $1 < q < \infty$:

$$A_0 \in \mathcal{L}(\mathbb{W}^{2m}_q(\mathbb{R}^2), \hat{\mathbb{W}}^{2m+2}_q(\mathbb{R}^2));\tag{2.6}$$
$$A_0\mathbf{f} = O(\log|x|) \quad \text{as } |x| \to \infty \text{ for } \mathbf{f} \in \mathbb{L}_{q,b}(\mathbb{R}^2).$$

From (2.2) and (2.5), it follows that

$$E_\lambda(x) = E_0(x) - \frac{1}{4\pi}(c + \log\sqrt{\lambda})I_2 + H_\lambda(x),\tag{2.7}$$

where I_2 is the 2×2 identity matrix, $H_\lambda(x) = O(\lambda|x|^2)\log(\sqrt{\lambda}|x|)$ and $c = \gamma + \frac{1}{2} - \log 2$.

From the above facts, we have the following lemmas.

Lemma 2.1. *Let* $1 < q < \infty$.

(1) *For* $\mathbf{f} \in \mathbb{L}_q(\mathbb{R}^2)$, *we have*

$$\|A_\lambda\mathbf{f}\|_{\infty,\mathbb{R}^2} \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_{q,\mathbb{R}^2}.\tag{2.8}$$

(2) *For* $\mathbf{f} \in \mathbb{L}_q(\mathbb{R}^2)$ *and* $\operatorname{supp}\mathbf{f} \subset \{y \in \mathbb{R}^2 \mid |y| \ge R\}$,

$$\|\nabla A_\lambda\mathbf{f}\|_{\infty,\{|x|\le R-1\}} \le Cd(\lambda)\|\mathbf{f}\|_q \quad \text{as } |\lambda| \to 0,\tag{2.9}$$

where

$$d(\lambda) = \begin{cases} |\lambda|^{\frac{1}{q}-\frac{1}{2}} & q > 2, \\ |\log\lambda| & q = 2, \\ 1 & q < 2. \end{cases}$$

Remark.

$$d(\lambda)|\log\lambda| \le C|\lambda|^{\frac{1}{q}-1}.\tag{2.10}$$

Proof. (2.8) is obtained by Young's convolution theorem (cf. Proposition 4.1 of [**4**]). When we estimate $\nabla A_\lambda\mathbf{f}(x)$ for $|x| \le R - 1$, because of the support of $\mathbf{f}$ we have $|x - y| \ge 1$.

$$\begin{aligned}\nabla A_\lambda\mathbf{f}(x) &= \int_{\mathbb{R}^2}\nabla E_\lambda(x-y)\mathbf{f}(y)dy \\ &= \int_{|x-y|\ge\frac{1}{\sqrt{|\lambda|}}} + \int_{1\le|x-y|\le\frac{1}{\sqrt{|\lambda|}}}\nabla E_\lambda(x-y)\mathbf{f}(y)dy \equiv I + II;\end{aligned}$$

$$|I| \le \int_{|x-y|\ge\frac{1}{\sqrt{|\lambda|}}}\frac{C\sqrt{|\lambda|}}{(\sqrt{|\lambda|}|x-y|)^3}|\mathbf{f}(y)|dy$$

$$\le \frac{C}{|\lambda|}\left(\int_{|x-y|\ge\frac{1}{\sqrt{|\lambda|}}}\frac{1}{|x-y|^{3q'}}dy\right)^{\frac{1}{q'}}\|\mathbf{f}\|_q$$

$$\le C|\lambda|^{\frac{1}{q}-\frac{1}{2}}\|\mathbf{f}\|_q;$$

$$|II| \le \int_{1\le|x-y|\le\frac{1}{\sqrt{|\lambda|}}}\frac{C}{|x-y|}|\mathbf{f}(y)|dy$$

$$\le \left(\int_{1\le|x-y|\le\frac{1}{\sqrt{|\lambda|}}}\frac{1}{|x-y|^{q'}}dy\right)^{\frac{1}{q'}}\|\mathbf{f}\|_q$$

$$\le \begin{cases} C|\lambda|^{\frac{1}{q}-\frac{1}{2}}\|\mathbf{f}\|_q & q>2,\\ C|\log\lambda|\|\mathbf{f}\|_q & q=2,\\ C\|\mathbf{f}\|_q & q<2,\end{cases}$$

where $1/q+1/q'=1$, which implies (2.9). □

Lemma 2.2.

$$\sup_{|x-y|\ge 1}|E_\lambda(x-y)|\le C|\log\lambda|; \tag{2.11}$$

$$\sup_{|x-y|\ge 1}|\nabla E_\lambda(x-y)|\le C; \tag{2.12}$$

$$\sup_{|x-y|\ge 1}|\mathbf{p}(x-y)|\le C. \tag{2.13}$$

Proof. In view of the form of the fundamental solution, we estimate it dividing the region into two parts: One is $|x-t|\ge 1/\sqrt{|\lambda|}$ and the other is $1\le|x-y|\le 1/\sqrt{|\lambda|}$. In the former case we know that $E_\lambda(x-y)$ is bounded from the definition (2.2). In the latter case $|E_\lambda(x-y)|$ behaves like $|\log\lambda|$ in view of (2.7), thus we have (2.11). We obtain (2.12) in the same way. (2.13) is trivial. □

We prepare the following formula.

Lemma 2.3 (Green's first and second identity)**.** *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be divergence free vector functions. Then, we have the following two formula:*

$$\begin{aligned}&\int_\Omega\langle(\lambda-\Delta)\mathbf{u}+\nabla\mathfrak{p},\mathbf{v}\rangle dx\\ &\quad=\int_{\partial\Omega}\langle(-\nabla\mathbf{u}+\mathfrak{p}I_2)\mathbf{n},\mathbf{v}\rangle dS+\int_\Omega\langle\lambda\mathbf{u},\mathbf{v}\rangle dx+\int_\Omega\langle\nabla\mathbf{u},\nabla\mathbf{v}\rangle dx,\end{aligned} \tag{2.14}$$

provided that $|\nabla\mathbf{u}||\mathbf{v}|=o(|x|^{-1})$ *and* $|\mathfrak{p}||\mathbf{v}|=o(|x|^{-1})$*;*

$$\int_\Omega\langle(\lambda-\Delta)\mathbf{u}+\nabla\mathfrak{p},\mathbf{v}\rangle dx-\int_\Omega\langle\mathbf{u},(\lambda-\Delta)\mathbf{v}-\nabla\mathfrak{q}\rangle dx \tag{2.15}$$

$$= \int_{\partial\Omega} \langle (-\nabla \mathbf{u} + \mathfrak{p} I_2)\mathbf{n}, \mathbf{v}\rangle dS - \int_{\partial\Omega} \langle \mathbf{u}, (-\nabla \mathbf{v} - \mathfrak{q} I_2)\mathbf{n}\rangle dS,$$

provided that $|\nabla \mathbf{u}||\mathbf{v}| = o(|x|^{-1})$, $|\mathfrak{p}||\mathbf{v}| = o(|x|^{-1})$, $|\mathbf{u}||\nabla \mathbf{v}| = o(|x|^{-1})$ *and* $|\mathbf{u}||\mathfrak{q}| = o(|x|^{-1})$.

Here we have put $\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{j=1}^2 a_j b_j$ *for* $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^2$ *and* $\langle A, B\rangle = \sum_{j,k=1}^2 A_{jk}B_{jk}$ *for* 2×2 *matrices* A, B. *The vector* $\mathbf{n} = \mathbf{n}(x)$ *denotes the exterior normal on* $\partial\Omega$ *and* dS *is the surface element of* $\partial\Omega$.

Let D be a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂D and $\Sigma_0 = \Sigma \cup \{0\}$.

Lemma 2.4 (Bogovskiĭ, cf. [**1**, **2**]). *Let* $1 < q < \infty$ *and let* m *be an integer* ≥ 0. *Then, there exists a linear bounded operator* $\mathbf{B} : \dot{W}_{q,a}^m(D) \longrightarrow \dot{\mathbb{W}}_q^{m+1}(D)$ *such that*

$$\nabla \cdot \mathbf{B}[f] = f \quad \text{in } D, \quad \|\mathbf{B}[f]\|_{m+1,q,D} \leq C_{q,m,D}\|f\|_{m,q,D}. \tag{2.16}$$

Let us consider the stationary problem for the Stokes equation with parameter $\lambda \in \Sigma$ in Ω:

$$\text{(S)} \qquad (\lambda - \Delta)\mathbf{u} + \nabla \mathfrak{p} = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega.$$

In terms of the Stokes operator $\mathbb{A}$, (S) is written in the form:

$$(S') \qquad (\lambda + \mathbb{A})\mathbf{u} = \mathbf{f}.$$

Giga [**9**] proved that Σ belongs to the resolvent set $\rho(\mathbb{A})$ of $\mathbb{A}$ and for any $\gamma > 0$ and $0 < \tau < \pi$

$$\|(\lambda + \mathbb{A})^{-1}\|_{\mathcal{L}(\mathbb{J}_q(\Omega))} \leq C_{q,\delta}|\lambda|^{-1}, \tag{2.17}$$

when $|\lambda| \geq \gamma$, $|\arg\lambda| \leq \tau$ for any $0 < \tau < \pi$. Borchers and Varnhorn [**4**] proved that (2.17) is also valid in a punctured sectorial neighborhood of the origin by classical potential theory.

Moreover, contracting the domain of $(\lambda + \mathbb{A})^{-1}$ from $\mathbb{J}_q(\Omega)$ to $\mathbb{J}_{q,b}(\Omega)$, we investigated the asymptotic behavior of $(\lambda + \mathbb{A})^{-1}$ as $|\lambda| \to 0$ (cf. [**6**, **7**]). Put $\Sigma_{\tau,\varepsilon} = \{\lambda \in \Sigma \mid |\arg\lambda| \leq \tau, |\lambda| \leq \varepsilon\}$.

Proposition 2.5 (cf. [**7**, Proposition 3.6] and [**6**, Corollary 3.8]). *Let* $1 < q < \infty$ *and* m *be any integer* ≥ 0. *There exist operator valued functions* R_λ *and* P_λ *possessing the following properties:*

(1)
$$R_\lambda \in \mathcal{A}(\Sigma, \mathcal{L}(\mathbb{W}_{q,b}^{2m}(\Omega), \mathbb{W}_q^{2m+2}(\Omega_b)));$$
$$P_\lambda \in \mathcal{A}(\Sigma, \mathcal{L}(\mathbb{W}_{q,b}^{2m}(\Omega), W_q^{2m+1}(\Omega_b))),$$

(2) *the pair of* $\mathbf{u} = R_\lambda \mathbf{f}$ *and* $\mathfrak{p} = P_\lambda \mathbf{f}$ *is a solution to* (S) *and we have*

$$R_\lambda = (\lambda + \mathbb{A})^{-1} \quad \text{on } \mathbb{J}_{q,b}(\Omega) \quad \text{for } \lambda \in \Sigma. \tag{2.18}$$

Moreover, for any $0 < \tau < \pi$*, there exists an* $\varepsilon = \varepsilon(\tau)$ *such that*

$$\begin{pmatrix} R_\lambda \\ P_\lambda \end{pmatrix} \mathbf{f} = \begin{pmatrix} V_0 \\ Q_0 \end{pmatrix} \mathbf{f} + (\log \lambda)^{-1} \begin{pmatrix} V_\lambda \\ Q_\lambda \end{pmatrix} \mathbf{f} \qquad \text{as } \lambda \in \Sigma_{\tau,\varepsilon}, \tag{2.19}$$

where V_0*,* Q_0 *are independent of* λ *and there exist a constant* C *which does not depend on* λ *such that*

$$\|V_0\mathbf{f}\|_{q,2,\Omega_b} + \|Q_0\mathbf{f}\|_{q,1,\Omega_b} \le C\|\mathbf{f}\|_q;$$
$$\|V_\lambda\mathbf{f}\|_{q,2,\Omega_b} + \|Q_\lambda\mathbf{f}\|_{q,1,\Omega_b} \le C\|\mathbf{f}\|_q.$$

If we put $\mathbf{u}_0 = V_0\mathbf{f}$ *and* $\mathfrak{q}_0 = Q_0\mathbf{f}$*, then* $(\mathbf{u}_0, \mathfrak{q}_0)$ *is a unique solution to the problem:*

$$-\Delta\mathbf{u}_0 + \nabla\mathfrak{q}_0 = \mathbf{f} \quad \text{and} \quad \nabla\cdot\mathbf{u}_0 = 0 \quad \text{in } \Omega, \qquad \mathbf{u}_0 = \mathbf{0} \text{ on } \partial\Omega, \tag{2.20}$$
$$\mathbf{u}_0(x) = O(1), \quad \mathfrak{q}_0(x) = O(|x|^{-1}) \quad \text{as } |x| \to \infty.$$

Moreover, $\mathbf{u}_0$ *and* $\mathfrak{q}_0$ *satisfy the following behavior:*
(2.21)

$$\mathbf{u}_0(x) = O(1), \quad \nabla\mathbf{u}_0(x) = O(|x|^{-2}) \quad \text{and} \quad \mathfrak{q}_0(x) = O(|x|^{-2}) \quad \text{as } |x| \to \infty.$$

Remark. (2.21) follows from the proof of this Proposition 3.6 in [**6**] or [**7**].

3. Proof of Theorem 1.2.

In this section, we shall prove Theorem 1.2. A main part of our proof is an analysis of the resolvent of stationary problem (S) (i.e. (S′)) near $\lambda = 0$.

Proof of Theorem 1.2. Since the semigroup $e^{-t\mathbb{A}}$ admits the representation:

$$e^{-t\mathbb{A}} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + \mathbb{A})^{-1} d\lambda, \quad t > 0. \tag{3.1}$$

Here the curve $\Gamma \subset \mathbb{C}$ consists of three curves $\Gamma_1^{\pm}$ and Γ_0, where

$$\Gamma_1^{\pm} = \{\lambda \in \mathbb{C} \mid \arg\lambda = \pm 3\pi/4,\ |\lambda| \ge \varepsilon\},$$
$$\Gamma_0 = \Gamma_2^{+} \cup \Gamma_3 \cup \Gamma_2^{-},$$
$$\Gamma_2^{\pm} = \{\lambda \in \mathbb{C} \mid \arg\lambda = \pm 3\pi/4,\ 2/t \le |\lambda| \le \varepsilon\},$$
$$\Gamma_3 = \{\lambda \in \mathbb{C} \mid |\lambda| = 2/t,\ -3\pi/4 \le \arg\lambda \le 3\pi/4\}.$$

We shall estimate

$$J_1^{\pm}(t)\mathbf{f} = \frac{1}{2\pi i}\int_{\Gamma_1^{\pm}} e^{\lambda t}(\lambda + \mathbb{A})^{-1}\mathbf{f} d\lambda, \quad J_0(t)\mathbf{f} = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t} R_\lambda \mathbf{f} d\lambda.$$

To obtain

$$\|J_1^{\pm}(t)\mathbf{f}\|_\infty \le C_{q,\varepsilon} e^{-\frac{\varepsilon}{2\sqrt{2}}t}\|\mathbf{f}\|_q, \tag{3.2}$$

we use the estimate

$$\|(\lambda + \mathbb{A})^{-1}\mathbf{f}\|_\infty \le C_q\|(\lambda + \mathbb{A})^{-1}\mathbf{f}\|_{2,q} \le C_{q,\varepsilon}\|\mathbf{f}\|_q \quad \text{as } \lambda \in \Gamma_1^{\pm}.$$

To estimate $J_0(t)\mathbf{f}$, it is enough to show:

Proposition 3.1. *Let* $1 < q < \infty$*. Then we have*

$$\|(\lambda + \mathbb{A})^{-1}\mathbf{f}\|_\infty \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q \quad \text{as } |\lambda| \to 0 \tag{3.3}$$

for any $\mathbf{f} \in \mathbb{J}_q(\Omega)$*.*

In fact, we know the following lemma (see [**24**, p. 370, Lemma 8]).

Lemma 3.2. *Suppose that* $\omega(\lambda)$ *is analytic function in* $\Sigma_{\gamma,\varepsilon}$ *and has the estimate* $|\omega(\lambda)| \le C|\lambda|^r|\log\lambda|^s$ *as* $|\lambda| \to 0$*. Then as* $t \to \infty$

$$\left|\int_{\Gamma_0} e^{\lambda t}\omega(\lambda)d\lambda\right| \le Ct^{-r-1}(\log t)^s, \tag{3.4}$$

where Γ_0 *is the same contour as in* (3.1)*.*

Combining (3.3) and Lemma 3.2, we easily see that

$$\|J_0(t)\mathbf{f}\|_\infty \le Ct^{-\frac{1}{q}}\|\mathbf{f}\|_q \quad \text{as } t \to \infty. \tag{3.5}$$

Therefore from (3.2) and (3.5) Theorem 1.2 follows. □

Proof of Proposition 3.1. Our proof of (3.3) is based on the result of Proposition 2.5. As stated in the introduction, if we estimate directly the representation formula of the Stokes resolvent by potentials which was proved by Borchers & Varnhorn [**4**], we can also obtain the estimate (3.3). But now we shall show (3.3) without using potentials.

Put $\mathbf{u} = (\lambda + \mathbb{A})^{-1}\mathbf{f}$ for $\mathbf{f} \in \mathbb{J}_q(\Omega)$. In view of the result of Proposition 2.5, we divide Ω into two parts:

$$\Omega = \Omega_b \cup \{|x| \ge b\}.$$

At first, we shall prove

$$\|\mathbf{u}\|_{\infty,\Omega_b} \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q \quad \text{as } |\lambda| \to 0 \text{ for } \mathbf{f} \in \mathbb{J}_q(\Omega). \tag{3.6}$$

Since the support $\mathbf{f}$ is not compact, we shall employ the cut-off technique. Put

$$\mathbf{v} = \mathbf{u} - \psi A_\lambda \iota\mathbf{f} + \mathbf{B}[(\nabla\psi)\cdot A_\lambda\iota\mathbf{f}], \tag{3.7}$$

where $\psi(x) = 0$ for $|x| \le b-4$ and $= 1$ for $|x| \ge b-3$, and $\iota\mathbf{f}$ is the extension of $\mathbf{f}$ to whole $\mathbb{R}^2$ by the relation: $\iota\mathbf{f}(x) = \mathbf{f}(x)$ for $x \in \Omega$ and $\iota\mathbf{f}(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega$. Then $\mathbf{v}$ satisfies the following equations with some $\mathfrak{q}$:

$$(\lambda - \Delta)\mathbf{v} + \nabla\mathfrak{q} = F_\lambda \quad \text{and} \quad \nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \tag{3.8}$$
$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega,$$

where

$$F_\lambda = (\Delta\psi)A_\lambda\iota\mathbf{f} - \Delta\mathbf{B}[(\nabla\psi)\cdot A_\lambda\iota\mathbf{f}](1-\psi)\mathbf{f} + 2(\nabla\psi\cdot\nabla)A_\lambda\iota\mathbf{f}$$

$$- (\nabla\psi)\Pi\iota\mathbf{f} + \lambda\mathbf{B}[(\nabla\psi)\cdot A_\lambda\iota\mathbf{f}].$$

By Lemmas 2.1 and 2.4 we have

$$\|F_\lambda\|_q \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q. \tag{3.9}$$

From Proposition 2.5 and (3.9) it follows that

$$\|\mathbf{v}\|_{q,2,\Omega_b} \le C\|F_\lambda\|_q \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q. \tag{3.10}$$

Therefore by Lemmas 2.1 and 2.4, and (3.10), then

$$\begin{aligned}\|\mathbf{u}\|_{\infty,\Omega_b} &\le \|\mathbf{v}\|_{\infty,\Omega_b} + \|\psi A_\lambda\iota\mathbf{f}\|_{\infty,\Omega_b} + \|\mathbf{B}[\nabla\psi\cdot A_\lambda\iota\mathbf{f}]\|_{\infty,\Omega_b}\\ &\le C(\|\mathbf{v}\|_{q,2,\Omega_b} + \|A_\lambda\iota\mathbf{f}\|_\infty + \|\mathbf{B}[\nabla\psi\cdot A_\lambda\iota\mathbf{f}]\|_{q,2,\Omega_b})\\ &\le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q.\end{aligned}$$

Thus we have (3.6).

It remains to estimate $\mathbf{u}(x)$ for $|x| \ge b$:

$$\|\mathbf{u}\|_{\infty,\{|x|\ge b\}} \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q \quad \text{as } |\lambda| \to 0 \text{ for } \mathbf{f} \in \mathbb{J}_q(\Omega). \tag{3.11}$$

We divide $\mathbf{f} \in \mathbb{J}_q(\Omega)$ into two parts:

$$\mathbf{f} = (1-\varphi)\mathbf{f} + \varphi\mathbf{f} \equiv \mathbf{f}^1 + \mathbf{f}^2,$$

where $\varphi(x) = 1$ for $|x| \le b-2$ and $= 0$ for $|x| \ge b-1$. We divide $\mathbf{u}$ into two parts: $\mathbf{u} = \mathbf{u}^1 + \mathbf{u}^2$, where for $j = 1, 2$ with some pressure $\mathfrak{p}^j$

$$(\lambda-\Delta)\mathbf{u}^j + \nabla\mathfrak{p}^j = \mathbf{f}^j \quad \text{and} \quad \nabla\cdot\mathbf{u}^j = 0 \quad \text{in } \Omega, \tag{S^j}$$
$$\mathbf{u}^j = \mathbf{0} \quad \text{on } \partial\Omega.$$

At first we shall prove

$$\|\mathbf{u}^1\|_{\infty,\{|x|\ge b\}} \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q \quad \text{as } |\lambda| \to 0. \tag{3.12}$$

Put

$$\mathbf{v}^1 = \mathbf{u}^1 - \psi A_\lambda\iota\mathbf{f}^1 + \mathbf{B}[(\nabla\psi)\cdot A_\lambda\iota\mathbf{f}^1], \tag{3.13}$$

where $\psi(x)$ is the same function as (3.7). When $|x| \ge b$, $\mathbf{u}^1(x) = \psi(x)A_\lambda\iota\mathbf{f}^1(x) + \mathbf{v}^1(x)$ and the estimate of the first term is obtained by (2.8). Thus we shall estimate $\mathbf{v}^1$, which satisfies the following equations with some pressure $\mathfrak{q}^1$:

$$(\lambda-\Delta)\mathbf{v}^1 + \nabla\mathfrak{q}^1 = F^1_\lambda + G^1_\lambda \quad \text{and} \quad \nabla\cdot\mathbf{v}^1 = 0 \quad \text{in } \Omega,$$
$$\mathbf{v}^1 = \mathbf{0} \quad \text{on } \partial\Omega,$$

where

$$F^1_\lambda = (\Delta\psi)A_\lambda\iota\mathbf{f}^1 - \Delta\mathbf{B}[(\nabla\psi)\cdot A_\lambda\iota\mathbf{f}^1];$$
$$G^1_\lambda = (1-\psi)\mathbf{f}^1 + 2(\nabla\psi\cdot\nabla)A_\lambda\iota\mathbf{f}^1 - (\nabla\psi)\Pi\iota\mathbf{f}^1 + \lambda\mathbf{B}[(\nabla\psi)\cdot A_\lambda\iota\mathbf{f}^1].$$

By Lemmas 2.1 and 2.4 we have

$$\|F^1_\lambda\|_q \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q; \tag{3.14a}$$

$$\|G^1_\lambda\|_q \le Cd(\lambda)\|\mathbf{f}^1\|_q. \tag{3.14b}$$

Put $\mathbf{v}^1 = \mathbf{w}+\mathbf{z}$, where $\mathbf{w}$ and $\mathbf{z}$ are solutions to the following equations with some pressures $\mathfrak{r}$ and $\mathfrak{s}$ respectively:

$$(\lambda-\Delta)\mathbf{w}+\nabla\mathfrak{r} = F^1_\lambda \quad \text{and} \quad \nabla\cdot\mathbf{w}=0 \quad \text{in } \Omega, \qquad \mathbf{w}=\mathbf{0} \quad \text{on } \partial\Omega; \tag{3.15a}$$

$$(\lambda-\Delta)\mathbf{z}+\nabla\mathfrak{s} = G^1_\lambda \quad \text{and} \quad \nabla\cdot\mathbf{z}=0 \quad \text{in } \Omega, \qquad \mathbf{w}=\mathbf{0} \quad \text{on } \partial\Omega. \tag{3.15b}$$

Since F^1_λ and G^1_λ have compact support, from Proposition 2.5, (3.14a) and (3.14b) it follows that

$$\|\mathbf{w}\|_{q,2,\Omega_b} + \|\mathfrak{r}\|_{q,1,\Omega_b} \le C\|F^1_\lambda\|_q \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q; \tag{3.16a}$$

$$\|\mathbf{z}\|_{q,2,\Omega_b} + \|\mathfrak{s}\|_{q,1,\Omega_b} \le C\|G^1_\lambda\|_q \le Cd(\lambda)\|\mathbf{f}^1\|_q. \tag{3.16b}$$

To investigate $\mathbf{w}(x)$ and $\mathbf{z}(x)$ for $|x|\ge b$, we shall represent $\mathbf{w}(x)$ and $\mathbf{z}(x)$ by Green's second identity. Applying (2.15) with $\mathbf{u}=\mathbf{w}$ (resp. $\mathbf{z}$), $\mathfrak{p}=\mathfrak{r}$ (resp. $\mathfrak{s}$), $\mathbf{v}=(E^\lambda_{jk}(x-\cdot))_{j=1,2}$ and $\mathfrak{q}=p_k(x-\cdot)$ $(k=1,2)$, then we have

$$\begin{aligned}\mathbf{w}(x) = &\int_\Omega \langle F^1_\lambda(y), E_\lambda(x-y)\rangle dy \\ &- \int_{\partial\Omega}\langle(-\nabla\mathbf{w}(y)+\mathfrak{r}(y)I_2)\mathbf{n}(y), E_\lambda(x-y)\rangle dS \\ &- \int_{\partial\Omega}\langle\mathbf{w}(y), (-\nabla_y E_\lambda(x-y)+\mathbf{p}(x-y)I_2)\mathbf{n}(y)\rangle dS;\end{aligned} \tag{3.17a}$$

$$\begin{aligned}\mathbf{z}(x) = &\int_\Omega \langle G^1_\lambda(y), E_\lambda(x-y)\rangle dy \\ &- \int_{\partial\Omega}\langle(-\nabla\mathbf{z}(y)+\mathfrak{s}(y)I_2)\mathbf{n}(y), E_\lambda(x-y)\rangle dS \\ &- \int_{\partial\Omega}\langle\mathbf{z}(y), (-\nabla_y E_\lambda(x-y)+\mathbf{p}(x-y)I_2)\mathbf{n}(y)\rangle dS.\end{aligned} \tag{3.17b}$$

Since the supports of F_λ and G_λ are included in D_{b-3} and since $|x|\ge b$, we know that $|x-y|\ge 1$. Then we can use Lemma 2.2. In view of (3.14b), (3.16b), Lemma 2.2 and (2.10), we have

$$\|\mathbf{z}\|_{\infty,\{|x|\ge b\}} \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q \quad \text{as } |\lambda|\to 0. \tag{3.18}$$

On the other hand, if we estimated $\mathbf{w}$ in the same way as above, we would obtain the order $|\log\lambda||\lambda|^{\frac{1}{q}-1}$ which is worse than the expected order $|\lambda|^{\frac{1}{q}-1}$. The main idea to overcome this difficulty is integration by parts.

If we apply integration by parts to the first term of (3.17a), the terms on the boundary do not appear, because the support of F^1_λ is apart from the boundary. Thus by Lemmas 2.1, 2.2 and 2.4 we have

(3.19)

$$\begin{aligned}
&\sup_{|x|\ge b}\left|\int_\Omega \langle F^1_\lambda(y), E_\lambda(x-y)\rangle dy\right| \\
&\le C \sup_{|x-y|\ge 1} |E_\lambda(x-y)| \|\nabla A_\lambda \iota \mathbf{f}^1\|_{\infty,\{|y|\le b-3\}} \\
&+ C \sup_{|x-y|\ge 1} |\nabla E_\lambda(x-y)| \left(\|A_\lambda \iota \mathbf{f}^1\|_{\infty,\{|y|\le b-3\}} + \|\nabla \mathbf{B}[(\nabla\psi)\cdot A_\lambda \iota \mathbf{f}^1]\|_{q,\Omega_{b-3}}\right) \\
&\le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q.
\end{aligned}$$

Next we shall estimate the second term of (3.17a). At first, recall Proposition 2.5 and (3.14a). Then we have

$$\begin{pmatrix}\mathbf{w}\\ \mathfrak{r}\end{pmatrix} = \begin{pmatrix}V_0\\ Q_0\end{pmatrix} F^1_\lambda + (\log\lambda)^{-1}\begin{pmatrix}V_\lambda\\ Q_\lambda\end{pmatrix} F^1_\lambda \quad \text{as } |\lambda|\to 0, \tag{3.20}$$

where

$$\begin{aligned}
&\|V_0 F^1_\lambda\|_{q,2,\Omega_b} + \|Q_0 F^1_\lambda\|_{q,1,\Omega_b} \le C\|F^1_\lambda\|_q \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q; \\
&\|V_\lambda F^1_\lambda\|_{q,2,\Omega_b} + \|Q_\lambda F^1_\lambda\|_{q,1,\Omega_b} \le C\|F^1_\lambda\|_q \le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q.
\end{aligned} \tag{3.21}$$

Put $(\mathbf{w}_0, \mathfrak{r}_0) \equiv (V_0 F^1_\lambda, Q_0 F^1_\lambda)$ and $(\mathbf{w}_1, \mathfrak{r}_1) \equiv (\log\lambda)^{-1}(V_1 F^1_\lambda, Q_1 F^1_\lambda)$. For $(\mathbf{w}_1, \mathfrak{r}_1)$, by Lemma 2.1 and (3.21)

$$\begin{aligned}
&\sup_{|x|\ge b}\left|\int_{\partial\Omega} \langle(-\nabla \mathbf{w}_1(y) + \mathfrak{r}_1(y) I_2)\mathbf{n}(y), E_\lambda(x-y)\rangle dS\right| \\
&\le C \sup_{|x-y|\ge 1} |E_\lambda(x-y)| (\|\mathbf{w}_1\|_{q,2,\Omega_b} + \|\mathfrak{r}_1\|_{q,1,\Omega_b}) \\
&\le C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q.
\end{aligned} \tag{3.22}$$

Since the first term of (3.20) is $(\mathbf{w}_0, \mathfrak{r}_0)$ which dose not have $(\log\lambda)^{-1}$, we have to treat it more carefully. We know that $\{x\in\Omega \mid |x|\ge b\} \subset \{x\in\Omega \mid \mathrm{dist}(x,\partial\Omega)\ge 1\}$. Put

$$\begin{aligned}
&\{x\in\Omega \mid \mathrm{dist}(x,\partial\Omega)\ge 1\} \\
&= \left\{x\in\Omega \mid \mathrm{dist}(x,\partial\Omega)\ge 1/\sqrt{|\lambda|}\right\} \cup \left\{x\in\Omega \mid 1\le \mathrm{dist}(x,\partial\Omega)\le 1/\sqrt{|\lambda|}\right\} \\
&\equiv A\cup B.
\end{aligned}$$

In A, $E_\lambda(x-y)$ is bounded. Thus from (3.21)

$$\sup_{x\in A}\left|\int_{\partial\Omega} \langle(-\nabla \mathbf{w}_0(y) + \mathfrak{r}_0(y) I_2)\mathbf{n}(y), E_\lambda(x-y)\rangle dS\right| \tag{3.23}$$

$$\leq C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q.$$

But in B, $|E_\lambda(x-y)|$ behaves like $|\log\lambda|$, so that we shall expand $E_\lambda(x-y)$ by (2.7) as follows:

$$E_\lambda(x-y) = E_0(x) - \frac{1}{4\pi}(c+\log\sqrt{\lambda})I_2 + R_\lambda(x,y),$$

where $R_\lambda(x,y) = E_0(x-y) - E_0(x) + H_\lambda(x-y)$. Since the term $R_\lambda(x,y)$ is bounded in B, by (3.21) we have

$$\sup_{x\in B}\left|\int_{\partial\Omega}\langle(-\nabla\mathbf{w}_0(y) + \mathfrak{r}_0(y)I_2)\mathbf{n}(y), R_\lambda(x,y)\rangle dS\right| \leq C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}^1\|_q. \tag{3.24}$$

Thus we should show

$$\sup_{x\in B}\left|\int_{\partial\Omega}\langle(-\nabla\mathbf{w}_0(y) + \mathfrak{r}_0(y)I_2)\mathbf{n}(y), I_2\rangle dS\left(E_0(x) - \frac{1}{4\pi}(c+\log\sqrt{\lambda})I_2\right)\right| \leq C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q. \tag{3.25}$$

At first, we know

$$\sup_{x\in B}\left|E_0(x) - \frac{1}{4\pi}(c+\log\sqrt{\lambda})I_2\right| \leq C|\log\lambda|. \tag{3.26}$$

Next we should note that

$$\int_{\partial\Omega}\langle(-\nabla\mathbf{w}_0(y) + \mathfrak{r}_0(y)I_2)\mathbf{n}(y), I_2\rangle dS = \int_\Omega\langle F^1_\lambda, I_2\rangle dy. \tag{3.27}$$

In fact, from Proposition 2.5 and (2.21) it follows that $(\mathbf{w}_0, \mathfrak{r}_0)$ satisfies the following formulas:

$$\begin{gathered} -\Delta\mathbf{w}_0 + \nabla\mathfrak{r}_0 = F^1_\lambda \quad\text{and}\quad \nabla\cdot\mathbf{w}_0 = 0 \quad\text{in } \Omega. \\ \mathbf{w}_0 = \mathbf{0} \quad\text{on } \partial\Omega, \\ \mathbf{w}_0(x) = O(1),\quad \nabla\mathbf{w}_0(x) = O(|x|^{-2}) \text{ and } \mathfrak{r}_0(x) = O(|x|^{-2}) \quad\text{as } |x|\to\infty. \end{gathered} \tag{3.28}$$

Thus we have (3.27) by integration by parts. If we employ the same argument as (3.19), we have

$$\left|\int_\Omega\langle F^1_\lambda, I_2\rangle dy\right| \leq C\|\nabla A_\lambda\iota\mathbf{f}^1\|_{\infty,\{|x|\leq b-3\}} \leq Cd(\lambda)\|\mathbf{f}^1\|_q. \tag{3.29}$$

From (3.26), (3.27) and (3.29) we have (3.25).

On the last term of (3.17a) by (2.12), (2.13) and (3.21) we have

$$\left|\int_{\partial\Omega}\langle\mathbf{w}(y), (-\nabla_y E_\lambda(x-y) + \mathbf{p}(x-y)I_2)\mathbf{n}(y)\rangle dS\right| \leq C\|\mathbf{w}\|_{q,2,\Omega_b} \leq C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q. \tag{3.30}$$

Thus from (3.19), (3.22), (3.23), (3.24), (3.25) and (3.30) it follows that

$$\|\mathbf{w}(x)\|_{\infty,\{|x|\geq b\}} \leq C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q \quad \text{as } |\lambda| \to 0. \tag{3.31}$$

Thus by (3.13), Lemma 2.1, (3.18) and (3.31) we have

$$\begin{aligned}\|\mathbf{u}^1\|_{\infty,\{|x|\geq b\}} &\leq C(\|\mathbf{w}\|_{\infty,\{|x|\geq b\}} + \|\mathbf{z}\|_{\infty,\{|x|\geq b\}} + \|A_\lambda \iota \mathbf{f}^1\|_\infty) \\ &\leq C|\lambda|^{\frac{1}{q}-1}\|\mathbf{f}\|_q \quad \text{as } |\lambda| \to 0,\end{aligned}$$

which implies (3.12).

For $\mathbf{u}^2$ we obtain

$$\|\mathbf{u}^2\|_{\infty,\{|x|\geq b\}} \leq C|\log\lambda|\|\mathbf{f}^2\| \quad \text{as } |\lambda| \to 0. \tag{3.32}$$

In fact, if we represent $\mathbf{u}^2$ in the same way as in (3.17) we have

$$\begin{aligned}\mathbf{u}^2(x) = &\int_\Omega \langle \mathbf{f}^2(y), E_\lambda(x-y)\rangle dy \\ &- \int_{\partial\Omega} \langle(-\nabla\mathbf{u}^2(y) + \mathfrak{p}^2(y)I_2)\mathbf{n}(y), E_\lambda(x-y)\rangle dS \\ &- \int_{\partial\Omega} \langle \mathbf{u}^2(y), (-\nabla_y E_\lambda(x-y) + \mathbf{p}(x-y)I_2)\mathbf{n}(y)\rangle dS.\end{aligned} \tag{3.33}$$

Since $\operatorname{supp}\mathbf{f}^2 \subset \Omega_{b-1}$, by Proposition 2.5 we have

$$\|\mathbf{u}^2\|_{q,2,\Omega_b} + \|\mathfrak{p}^2\|_{q,1,\Omega_b} \leq C\|\mathbf{f}^2\|_q. \tag{3.34}$$

Therefore by Lemma 2.2 and (3.34) we have (3.32).

Combining (3.6), (3.12) and (3.32), we get (3.3), which completes the proof of Proposition 3.1. □

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