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We introduce a notion of finite representability of dual Banach spaces in their subspaces preserving duality (f.d.-r in short) which arises in a natural way in situations such as the principle of local reflexivity. We give a characterization for the f.d.-r. which yields a version of the principle of local reflexivity, and can be applied to the study of the duality theory for ultrapowers of operators. For example, we show that the kernel $\ker({T^{**}}_{\mathfrak{U}})$ of an ultrapower of the second conjugate of an operator T is finitely representable in $\ker(T_{\mathfrak{U}})$, and $\ker({T_{\mathfrak{U}}}^{*})$ is f.d.-r. in $\ker({T^{*}}_{\mathfrak{U}})$. Moreover, we study the duality properties of three semigroups of operators related with the superreflexivity and the finite representability of c_0 and ℓ_1 in a Banach space.

1. Introduction.

We introduce the concept of *finite representability preserving duality* (f.d.-r. in short) of a Banach space in its subspaces, and the *polar property* and the $\mathcal{A}$*-polar property* for subspaces of a dual Banach space, where $\mathcal{A}$ stands for a class of operators. Given a subspace Z of a dual space X^*, we show in Theorem 4 that X^* is f.d.-.r. in Z if and only if Z has the polar property. As a consequence, if a subspace Z of a dual space X^* has the polar property, then Z is norming on X, but the converse implication is not true.

We give some consequences of the $\mathcal{A}$-polar property in Theorem 7, from which it follows a version of the principle of local reflexivity: X^{**} is f.d.-r. in X [**8**], in such a way that given an operator $T \in \mathcal{B}(X, Y)$ we obtain estimates for the norm of the restrictions of T^{**} to finite dimensional subspaces in terms of the norm of the corresponding restrictions of T.

In Section 3 we apply these results to the study of the duality theory for ultrapowers of operators. From Theorem 4 it easily follows that the dual space ${X_{\mathfrak{U}}}^*$ of the ultrapower $X_{\mathfrak{U}}$ of X is f.d.-r. in ${X^*}_{\mathfrak{U}}$, a result first proved by Heinrich [**5**]. Moreover, the general character of our result allows us to obtain some consequences which do not follow from the principle of local reflexivity or the results of [**5**]. Namely, in Theorem 11 we show that for every $T \in \mathcal{B}(X, Y)$ the kernel of the conjugate operator ${T_{\mathfrak{U}}}^*$ is f.d.-r. in the kernel of ${T^*}_{\mathfrak{U}}$, where $T_{\mathfrak{U}}$ is an ultrapower of T, and the kernel $\ker({T^{**}}_{\mathfrak{U}})$ is finitely representable in $\ker(T_{\mathfrak{U}})$.

We study the semigroups ${\mathcal{W}^{up}}_+$, ${\mathcal{U}^{up}}_+$ and ${\mathcal{R}^{up}}_+$, introduced in [**13**, **4**]. Denoting any of them by $\mathcal{A}_+$, we show that an operator T belongs to $\mathcal{A}_+$ if and only if the second conjugate T^{**} belongs to $\mathcal{A}_+$. Moreover, ${T_{\mathfrak{U}}}^* \in \mathcal{A}_+$ if and only if ${T^*}_{\mathfrak{U}} \in \mathcal{A}_+$. Observe that ${T_{\mathfrak{U}}}^*$ is an extension of ${T^*}_{\mathfrak{U}}$. Finally, we give a proof of the fact, first proved in [**13**], that $T^* \in {\mathcal{W}^{up}}_+$ if and only if $\ker({T_{\mathfrak{U}}}^*) = \ker({T^*}_{\mathfrak{U}})$.

We use standard notations: X and Y are Banach spaces, B_X the closed unit ball of X, and S_X the unit sphere of X. The class of (bounded linear) operators from X to Y is $\mathcal{B}(X,Y)$, the dual of X is X^*, and given an operator $T \in \mathcal{B}(X,Y)$, we denote by $T^* : Y^* \longrightarrow X^*$ its conjugate operator, and by $R(T)$, $\ker(T)$ and $\operatorname{coker}(T) := Y/\overline{R(T)}$ the range, the kernel and the cokernel of T. Observe that $\operatorname{coker}(T)^*$ can be identified with $\ker(T^*)$.

Given a class of operators $\mathcal{A}$ and a pair of Banach spaces X and Y, we denote by $\mathcal{A}(X,Y)$ the component $\mathcal{A} \cap \mathcal{B}(X,Y)$ of all operators of $\mathcal{A}$ between X and Y. Also, $F^\perp$ is the annihilator of a subspace F, and $\mathbb{N}$ is the set of all positive integers. We identify X with a subspace of X^{**}. Given a subset A of X, we denote by $\langle A \rangle$ the closure of the span of A in X.

The letters $\mathfrak{U}$, $\mathfrak{V}$,... will be reserved to denote ultrafilters. An ultrafilter $\mathfrak{U}$ on a set I is said to be *countably incomplete* if there is a countable partition $\{I_n : n \in \mathbb{N}\}$ of I such that for every positive integer n, we have that $I_n \notin \mathfrak{U}$. Every infinite set admits a countably incomplete ultrafilter. Throughout the paper, we assume that the ultrafilters are countably incomplete.

Given a number $0 < \varepsilon < 1$, an operator $T \in \mathcal{B}(X,Y)$ is said to be an *ε-isometry* if $(1+\varepsilon)^{-1} < \|Tx\| < 1+\varepsilon$ for all $x \in S_X$, and if the ε-isometry T is onto, then X is said to be ε-isometric to Y. A Banach space X is said to be *finitely representable in* Y if given $\varepsilon > 0$ and a finite dimensional subspace M of X there exists an ε–isometry $T : M \longrightarrow Y$. We will write X *f.r. in* Y to mean that the space X is finitely representable in Y. Given a number $C > 1$, two sequences (x_n) and (y_n) are said to be *C-equivalent* if for every sequence (a_k) of scalars and every n we have

$$C^{-1} \left\| \sum_{k=1}^{n} a_k x_k \right\| \leq \left\| \sum_{k=1}^{n} a_k y_k \right\| \leq C \left\| \sum_{k=1}^{n} a_k x_k \right\|.$$

2. Finite representability preserving the duality.

Since we need to use ultrapowers, we recall here some definitions and results, and refer to [**5**] for more information. Let I be an infinite set. We denote by $\ell_\infty(I,X)$ the Banach space of bounded families $(x_i)_{i\in I}$ in X with norm $\|(x_i)\| := \sup\{\|x_i\| : i \in I\}$.

Let $\mathfrak{U}$ be an ultrafilter on I and let $N_{\mathfrak{U}}(X)$ be the closed subspace of all families $(x_i) \in \ell_\infty(I, X)$ which converge to 0 following $\mathfrak{U}$. The *ultrapower of X following* $\mathfrak{U}$ is defined as follows:

$$X_{\mathfrak{U}} := \frac{\ell_\infty(I, X)}{N_{\mathfrak{U}}(X)}.$$

The element of $X_{\mathfrak{U}}$ admitting the family $(x_i) \in \ell_\infty(I, X)$ as a representative is denoted by $[x_i]_i$, or simply $[x_i]$ if it does not lead to confusion. The norm of $[x_i]$ in $X_{\mathfrak{U}}$ is given by

$$\|[x_i]\| = \lim_{\mathfrak{U}} \|x_i\|.$$

The ultrapower $X_{\mathfrak{U}}$ canonically contains an isometric copy of X generated by the constant families of $\ell_\infty(I, X)$. We identify this copy with X. An operator $T \in \mathcal{B}(X, Y)$ admits an extension $T_{\mathfrak{U}} \in \mathcal{B}(X_{\mathfrak{U}}, Y_{\mathfrak{U}})$ given by $T_{\mathfrak{U}}[x_i] := [Tx_i]$.

The ultrapower $(X^*)_{\mathfrak{U}}$ is canonically contained in $(X_{\mathfrak{U}})^*$, but in general these spaces do not coincide. Actually, given $(f_i) \in \ell_\infty(I, X^*)$, the class $[f_i] \in {X^*}_{\mathfrak{U}}$ is identified with the element $f \in (X_{\mathfrak{U}})^*$ given by $f([x_i]) := \lim_{\mathfrak{U}} f_i(x_i)$. Heinrich [5] proved that $(X^*)_{\mathfrak{U}}$ coincides with $(X_{\mathfrak{U}})^*$ if and only if X is superreflexive.

In the following definition we introduce a concept of finite representability which is stronger than the usual one: The *finite representability preserving the duality*.

Definition 1. Let Z be a subspace of the dual X^* of a Banach space X. We say that X^* is *finitely representable in Z preserving the duality* (f.d.-r. in short) if for every couple of finite dimensional subspaces F of X^* and G of X, and for every $0 < \varepsilon < 1$, there is an ε-isometry $L : F \longrightarrow Z$ such that $(Lf)(x) = f(x)$ for all $x \in G$ and all $f \in F$.

The celebrated *principle of local reflexivity* states that the second dual X^{**} is f.d.-r. in X. We refer to [10] for an elementary proof of this principle.

A subspace Z of a dual space X^* is said to be *norming* if for every $x \in X$ we have that $\|x\| = \sup\{f(x) : f \in B_Z\}$. It easily follows from the Hahn-Banach theorem that a subspace Z of X^* is norming if and only if $\overline{B_Z}^{\sigma(X^*,X)} = B_{X^*}$ (see [1]).

Let X be a Banach space and k, l two positive integers. A linear function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^l$, represented by a matrix $(a_{ij})_{i=1}^{l}{}_{j=1}^{k}$, induces an operator

$$f_X : X \times \overset{k}{\cdots} \times X \longrightarrow X \times \overset{l}{\cdots} \times X$$

in a natural way $f_X(x_i) := (\sum_{j=1}^k a_{ij} x_j)$. Note that $(f_X)^* = {f^*}_{X^*}$. We denote by $\mathcal{L}$ the class of all these operators.

$$\mathcal{L} := \{f_X \;:\; k, l \in \mathbb{N}, f : \mathbb{R}^k \longrightarrow \mathbb{R}^l \text{ linear, } X \text{ Banach space}\}.$$

Henceforth we will denote by $\ell_1^k(X)$ and $\ell_\infty^k(X)$ the space $X\times \overset{k}{\cdots} \times X$ endowed with the norms $\|(x_j)_{j=1}^k\| := \sum_{j=1}^k \|x_j\|$ and $\|(x_j)_{j=1}^k\| := \sup_{1\leq j\leq k} \|x_j\|$, respectively. Given two subsets $A \subset X$ and $B \subset X^*$, their polar sets are defined as follows:

$$A^\circ := \{f \in X^* : |f(a)| \leq 1 \text{ for all } a \in A\};$$
$$B_\circ := \{z \in X : |f(z)| \leq 1 \text{ for all } f \in B\}.$$

Note that the sets A° and $B_\circ$ are closed in the norm topology. If, in addition, A and B are convex and symmetric then we have $(B_\circ)^\circ = \overline{B}^{\sigma(X^*,X)}$ and $(A^\circ)_\circ = \overline{A}$.

Definition 2. A subspace Z of a dual space X^* is said to have the *polar property* if for every k, l in $\mathbb{N}$ and every linear function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^l$ we have

$$\overline{f_X\left(B_{\ell_1^k(X)}\right)} = \left({f^*}_{X^*}|_{\ell_\infty^l(Z)}^{-1} B_{\ell_\infty^k(Z)}\right)_\circ .$$

Observe that the inclusion $\overline{f_X\left(B_{\ell_1^k(X)}\right)} \subset \left({f^*}_{X^*}|_{\ell_\infty^l(Z)}^{-1} B_{\ell_\infty^k(Z)}\right)_\circ$ is always true. So, taking into account that the identities $\overline{T(B_X)} = ({T^*}^{-1}B_{X^*})_\circ$, ${T^*}^{-1}B_{X^*} = (TB_X)^\circ$ hold for every operator $T \in \mathcal{B}(X,Y)$, we obtain the following elementary (but useful) characterization of the polar property.

Proposition 3. *A subspace Z of a dual space X^* has the polar property if and only if for every linear function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^l$, we have*

$$ {{f^*}_{X^*}}^{-1}\left(B_{\ell_\infty^k(X^*)}\right) = \overline{{f^*}_{X^*}|_{\ell_\infty^l(Z)}^{-1}\left(B_{\ell_\infty^k(Z)}\right)}^{\sigma(\ell_\infty^l(X^*),\ell_1^l(X))} .$$

The following result shows that the polar property is a useful tool to study finite representability.

Theorem 4. *A subspace Z of a dual space X^* has the polar property if and only if X^* is finitely representable in Z preserving the duality.*

Proof. Assume that X^* is f.d.-r. in Z and Z does not have the polar property. In virtue of Proposition 3, there are $L \in \mathcal{L}(\ell_1^k(X), \ell_1^l(X))$, a $0 < \theta < 1$ and a l-tuple

$$(f_i)_{i=1}^l \in {L^*}^{-1}B_{\ell_\infty^k(X^*)} \Big\backslash \overline{\left(L^*|_{\ell_\infty^l(Z)}^{-1} B_{\ell_\infty^k(Z)}\right)}^{\sigma(\ell_\infty^l(X^*),\ell_1^l(X))}$$

such that $L^*((f_i)_{i=1}^l) = (g_i)_{i=1}^k \in (1-\theta)B_{\ell_\infty^k(X^*)}$.

By the Hahn-Banach Theorem, there exists a number $0 < \varepsilon < 1$ and a l-tuple $(x_i)_{i=1}^l \in \ell_1^l(X)$ such that the $\sigma(\ell_\infty^l(X^*), \ell_1^l(X))$-neighborhood of $(f_i)_{i=1}^l$, given by

$$\mathcal{V} := \left\{(h_i)_{i=1}^l : \left|\sum_{i=1}^l (f_i - h_i)(x_i)\right| < \varepsilon\right\}$$

verifies

$$\mathcal{V} \cap \overline{L^* \mid_{\ell_\infty^l(Z)}^{-1} B_{\ell_\infty^k(Z)}}^{\sigma(\ell_\infty^l(X^*),\ell_1^l(X))} = \emptyset. \tag{1}$$

Since X^* is f.d.-r. in Z, given $\delta > 0$ such that $(1-\theta)(1+\delta) \leq 1$, there are $\widetilde{f}_1, \dots, \widetilde{f}_l$ in Z such that the operator

$$G : \left\langle \widetilde{f}_i : i = 1, \dots, l \right\rangle \longrightarrow \langle f_i : i = 1, \dots, l \rangle$$

given by $G(\widetilde{f}_i) := f_i$ is a δ-isometry and $\widetilde{f}_i(x_j) = f_i(x_j)$ for all i and all j.

Associated to L^*, there is a matrix of numbers $(\lambda_{ij})_{i=1}^{l}{}_{j=1}^{k}$ such that

$$L^*\left((h_i)_{i=1}^l\right) = \left(\sum_{i=1}^l \lambda_{ij} h_i\right)_{j=1}^k.$$

So, every g_j is equal to $\sum_{i=1}^l \lambda_{ij} f_i \in (1-\theta)B_{X^*}$. Since G is a δ-isometry, for every $j = 1, \dots, k$, we have that $\widetilde{g}_j := \sum_{i=1}^l \lambda_{ij} \widetilde{f}_i \in B_Z$, hence $(\widetilde{f}_i)_{i=1}^l \in L^* \mid_{\ell_\infty^l(Z)}^{-1} \left(B_{\ell_\infty^k(Z)}\right)$. On the other hand, the equalities $\widetilde{f}_i(x_j) = f_i(x_j)$ for all $i = 1, \dots, l$ and all $j = 1, \dots, l$ imply that $(\widetilde{f}_i)_{i=1}^l \in \mathcal{V}$, in contradiction with (1).

For the converse implication, let E and F be finite dimensional subspaces of X^* and X, let us denote $n := \dim E$ and $k := \dim F$, and let $0 < \varepsilon < 1$.

The Auerbach Lemma allows us to take $(h_j, y_j)_{j=1}^n$ in $E^* \times E$ such that $\|h_j\| = \|y_j\| = 1$ for all $j = 1, \dots, n$ and $h_i(y_j) = \delta_{ij}$. The identity $id : E \longrightarrow X^*$ is given by $id(e) = \sum_{j=1}^n h_j(e) y_j$. We shall find $z_1, \dots, z_n$ in Z for the wished ε-isometry $L : E \longrightarrow Z$ to be defined as $L(e) := \sum_{j=1}^n h_j(e) z_j$.

Let us take $0 < \alpha < \min\{2/5\,,\, (1-\varepsilon)^{-1} - 1\,,\, \varepsilon(1+n/2)^{-1}\}$ and choose

a basis $\{x_j\}_{j=1}^k$ in F,
an $\alpha/4$-net $\{e_j\}_{j=1}^N$ in B_E, and
vectors $\{u_j\}_{j=1}^N$ in B_X

such that $\|e\| \leq (1+\alpha)\sup_{1 \leq j \leq N} |e(u_j)|$ for all $e \in E$. The condition $\alpha < 2/5$ guarantees the existence of the vectors u_j. We write

$$e_j = \sum_{s=1}^n \lambda_s^j y_s, \ j = 1, \dots, N.$$

In order to simplify the notation, we denote

$$\begin{aligned} U &:= \ell_1^n(X), \\ M &:= U \oplus_1 \ell_1^N(X), \\ W &:= \ell_\infty^n(Z), \\ V &:= W \oplus_\infty \ell_\infty^N(Z). \end{aligned}$$

Let $\Lambda^* : U^* \longrightarrow M^* = U^* \oplus_\infty \ell_\infty^N(X^*)$ and $S^* : U^* \longrightarrow \mathbb{R}^{nk \times nN}$ be the conjugate operators given respectively by

$$\Lambda^*((g_s)_{s=1}^n) := \left((g_s)_{s=1}^n \, , \, \left(\sum_{s=1}^n \lambda_s^j g_s \right)_{j=1}^N \right)$$

and

$$S^*((g_s)_{s=1}^n) := (g_r(x_i), g_t(u_j)) \, .$$

Since Z has the polar property, Proposition 3 gives that

$$\Lambda^{*-1}(B_{M^*}) = \overline{\Lambda^* \, |_W^{-1} \, (B_V)}^{\sigma(U^*,U)}.$$

So, the $\sigma(U^*,U)$-continuity of S^* yields

$$S^*(\Lambda^{*-1}(B_{M^*})) \subset \overline{S^* \left(\Lambda^* \, |_W^{-1} \, (B_V) \right)}. \tag{2}$$

Observe that $R(S^*) = R(S^* \, |_W)$ because $\overline{W}^{\sigma(U^*,U)} = U^*$. Thus, by Formula 2, since $(y_j)_{j=1}^n \in \Lambda^{*-1}(B_{M^*})$, given any number ε' such that $0 < \varepsilon' < \alpha \|\Lambda^*\|^{-1}$, we can find $(c_j)_{j=1}^n \in \Lambda^* \, |_W^{-1} \, (B_V)$ and $(b_j)_{j=1}^n \in \varepsilon' B_W$ so that

$$S^*((y_j)_{j=1}^n) = S^*((c_j)_{j=1}^n) + S^*((b_j)_{j=1}^n).$$

Taking $z_j := c_j + b_j$ for each $j = 1, \ldots, n$ in the definition of the operator L, we obtain $(z_j)_{j=1}^n \in (1+\alpha)\Lambda^{*-1}(B_V)$ and $S^*((y_j)_{j=1}^n) = S^*((z_j)_{j=1}^n)$. Evidently, the condition $(Le)(x) = e(x)$ holds for all $e \in E$ and all $x \in F$.

In order to check that L is an ε-isometry, let $e \in B_E$. On the one hand, we have

$$\|Le\| \geq \sup_{1 \leq j \leq N} |(Le)(u_j)| = \sup_{1 \leq j \leq N} |e(u_j)| \geq (1+\alpha)^{-1} \|e\| \geq 1 - \varepsilon.$$

On the other hand, first note that $\|L\| \leq 2n$ because

$$\|Le\| = \left\| \sum_{j=1}^n h_j(e) z_j \right\| \leq \sum_{j=1}^n (1+\alpha) \|h_j\| \leq 2n.$$

Choose now vectors e_k so that $\|e - e_k\| \leq \alpha/4$. Since $\Lambda^*((z_j)_{j=1}^n) \in (1+\alpha) B_V$ we have that $\|Le_k\| = \| \sum_{s=1}^n \lambda_s^k z_s \| \leq 1 + \alpha$. Therefore,

$$\|Le\| \leq \|Le_k\| + \|L(e - e_k)\| \leq 1 + \varepsilon;$$

hence L is an ε-isometry. $\square$

Remark. If we take $f = id : \mathbb{R} \longrightarrow \mathbb{R}$ in Proposition 3, then we obtain that $B_{X^*} = \overline{B_Z}^{\sigma(X^*,X)}$. Thus, if a subspace Z of a dual space X has the polar property, then Z is norming on X [1].

The converse implication is not true. There are norming subspaces of dual Banach spaces which fail the polar property.

For example, let us consider the Rademacher-like sequence (x_n) in ℓ_∞, where $x_1 = (1, -1, 1, -1, 1, -1, \dots)$ and for $n \in \mathbb{N}$ the sequence x_{n+1} consists of successive repetitions of the block

$$1, \overset{(2^n)}{\dots}, 1, -1, \overset{(2^n)}{\dots}, -1.$$

Then (x_n) is 1-equivalent to the unit vector basis of ℓ_1. Now, if we take an enumeration $\{A_n : n \in \mathbb{N}\}$ of all the finite sequences of numbers in $\{1, -1\}$, with $\mathrm{card}(A_m) \le \mathrm{card}(A_n)$ for $m < n$, and modify each x_n in a finite number of coordinates so that the initial segment of x_n coincides with A_n, then (x_n) continues to be 1-equivalent to the unit vector basis of ℓ_1, and it generates a norming subspace $[x_n]$ of $\ell_\infty = \ell_1^*$.

However, since ℓ_∞ is not f.r. in ℓ_1, the subspace $\langle x_n \rangle$ of ℓ_∞ does not have the polar property, by Theorem 4.

The following concept is a generalization of the polar property.

Definition 5. Let $\mathcal{A}$ be a class of operators. A subspace Z of X^* is said to have the *$\mathcal{A}$-polar property* if given a Banach Y, an operator $T \in \mathcal{A}(Y, X)$, integers $p, q \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$, and a couple of matrices of real numbers $(a_{ij})_{i=1}^{q}{}_{j=1}^{p}$, $(b_{ij})_{i=1}^{r}{}_{j=1}^{p}$, we have that the operator $L : \ell_1^q(X) \oplus_1 \ell_1^r(Y) \longrightarrow \ell_1^p(X)$, given by

$$L((x_i)_{i=1}^q, (y_j)_{j=1}^r) := \left(\sum_{i=1}^{q} a_{ij} x_i + \sum_{i=1}^{r} b_{ij} T y_i \right)_{j=1}^{p},$$

satisfies $\overline{L\left(B_{\ell_1^q(X)\oplus_1 \ell_1^r(Y)}\right)} = \left(L^* \mid_{\ell_\infty^p(Z)}^{-1} B_{\ell_\infty^q(Z)\oplus_\infty \ell_\infty^r(Y^*)}\right)_\circ$; equivalently,

$$L^{*-1} B_{\ell_\infty^q(X^*)\oplus_\infty \ell_\infty^r(Y^*)} = \overline{L^* \mid_{\ell_\infty^p(Z)}^{-1} B_{\ell_\infty^q(Z)\oplus_\infty \ell_\infty^r(Y^*)}}^{\sigma(\ell_\infty^p(X^*), \ell_1^p(X))}.$$

Remark. In the case $r = 0$, we assume that the matrix $(b_{ij})_{i=1}^{r}{}_{j=1}^{p}$ appearing in Definition 5 has no entries. Therefore, the $\mathcal{A}$-polar property implies the polar property.

The polar property is equivalent to the $\mathcal{I}$-polar property, where $\mathcal{I}$ stands for the class of all identities of Banach spaces.

The difficulty to check the $\mathcal{A}$-polar property depends on the structure of $\mathcal{A}$. Sometimes it is possible to do it easily, as in the next example which will play an important role in Proposition 13.

Example 6. Let $\mathcal{B}^d := \{T^* : T \in \mathcal{B}\}$ denote the class of all conjugate operators. Then every Banach space X has the $\mathcal{B}^d$-polar property as a subspace of X^{**}.

In fact, take an operator $T^* : Y^* \longrightarrow X^*$, and matrices $(a_{ij})_{i=1}^{q}{}_{j=1}^{p}$, $(b_{ij})_{i=1}^{r}{}_{j=1}^{p}$ of real numbers. As in Definition 5, we define $L : \ell_1^q(X^*) \oplus_1 \ell_1^r(Y^*) \longrightarrow \ell_1^p(X^*)$ by

$$L((f_i)_{i=1}^q, (g_j)_{j=1}^r) := \left(\sum_{i=1}^q a_{ij} f_i + \sum_{i=1}^r b_{ij} T^* g_i\right)_{j=1}^p .$$

Let us write $U := \ell_\infty^q(X) \oplus_\infty \ell_\infty^r(Y)$ and $V := \ell_\infty^p(X)$. Clearly, L can be identified with a conjugate operator $S^* : U^* \longrightarrow V^*$, and consequently, we have that $S^*(B_{U^*}) = (S^{-1}B_U)_\circ$; therefore,

$$S^*(B_{U^*}) = \left(S^{**} \mid_V^{-1} B_{\ell_\infty^q(X)\oplus_\infty \ell_\infty^r(Y^{**})}\right)_\circ .$$

We saw in Theorem 4 that a subspace Z of X^* has the polar property if and only if X^* is f.d.-r. in Z. Now we prove a strengthening of the direct implication.

Theorem 7. *Let Z be a subspace of X^*, let E and F be finite dimensional subspaces of X^* and X respectively, and let $\mathcal{A}$ be a class of operators.*

If Z has the $\mathcal{A}$-polar property and $T \in \mathcal{A}(Y, X)$, then for every $0 < \varepsilon < 1$ and $\delta' > 0$ there is an ε-isometry $L : E \longrightarrow Z$ such that $(Le)(x) = e(x)$ for all $e \in E$ and all $x \in F$, and satisfies the following additional condition $\|T^* \mid_{L(E)} \| \leq \|T^* \mid_E \| + \delta'$.

Proof. It is similar to the second part of the proof of Theorem 4. We shall use here the same notations.

Let $\delta := \|T\|$. Without loss of generality, we suppose that $0 < \varepsilon < 2^{-3}\delta^{-1}\delta'$. We choose $0 < \alpha < \min\{2/5, (1-\varepsilon)^{-1} - 1, \varepsilon(1 + n/2)^{-1}\}$ and define the operator Λ^* as in the proof of Theorem 4. Moreover, we write $\widetilde{M}^* := M^* \oplus_\infty \ell_\infty^N(Y^*)$, and define the conjugate operator $\widetilde{\Lambda}^* : U^* \longrightarrow \widetilde{M}^*$ by

$$\widetilde{\Lambda}^*\left((g_s)_{s=1}^n\right) := \left(\Lambda^*((g_s)_{s=1}^n)\, ,\, \left(\delta^{-1}T^*\left(\sum_{s=1}^n \lambda_s^j g_s\right)\right)_{j=1}^N\right).$$

Write $\widetilde{V} := V \oplus_\infty \ell_\infty^N(Y^*)$. Since Z has the $\mathcal{A}$-polar property, we have

$$\overline{\widetilde{\Lambda}^* \mid_W^{-1} \left(B_{\widetilde{V}}\right)}^{\sigma(U^*,U)} = \widetilde{\Lambda}^{*-1}\left(B_{\widetilde{M}^*}\right).$$

Thus, $(y_j)_{j=1}^n \in \widetilde{\Lambda}^{*-1}(B_{\widetilde{M}^*})$, and proceeding as in the proof of Theorem 4, we find

$$(z_j)_{j=1}^n \in (1+\alpha)\widetilde{\Lambda}^*\big|_W^{-1}\left(B_{\widetilde{V}}\right)$$

such that $S^*((y_j)_{j=1}^n) = S^*((z_j)_{j=1}^n)$. Again as in the proof of Theorem 4, the condition $\alpha < \min\{2/5,\, (1-\varepsilon)^{-1}-1,\, \varepsilon(1+n/2)^{-1}\}$ implies that the operator $L : E \longrightarrow Z$ is an ε-isometry and $(Le)(x) = e(x)$ for $e \in E$ and $x \in F$. Let us check that $\|T^* \mid_{L(E)} \| \leq \delta + \delta'$. Note that $(z_s)_{s=1}^n \in (1+\alpha)\widetilde{\Lambda}^{*-1}(B_{\widetilde{V}})$ implies that

$$\|T^*(Le_j)\| = \left\| T^* \left(\sum_{s=1}^{n} \lambda_s^j z_s \right) \right\| \leq \delta(1+\alpha) \quad \text{for} \quad j = 1, \dots, N.$$

Since L is an ε-isometry, if $\beta := 4^{-1}\alpha(1+\varepsilon) + 2\varepsilon$, then $\{Le_j\}_{j=1}^N$ is a β-net in $(1+\varepsilon)B_{L(E)}$, and taking $w_j := (1+\varepsilon)^{-1}Le_j$ we obtain a $(1+\varepsilon)^{-1}\beta$-net $\{w_j\}_{j=1}^N$ in $B_{L(E)}$. Let $w \in B_{L(E)}$ and pick w_j such that $\|w - w_j\| \leq (1+\varepsilon)^{-1}\beta$. We have

$$\|T^*w\| \leq \|T^*(w - w_j)\| + \|T^*w_j\| \leq \frac{\beta}{1+\varepsilon}\|T^*\| + \frac{\delta(1+\alpha)}{1+\varepsilon}.$$

Since $\alpha < \{2/5,\, 2^{-2}\delta^{-1}\delta'\}$ and $\varepsilon < 2^{-3}\delta^{-1}\delta'$, we conclude $\|T^*w\| \leq \delta + \delta'$. □

As a consequence of Theorem 7, we derive a version of the principle of local reflexivity [8].

Corollary 8. *Given an operator $T \in \mathcal{B}(X, Y)$, a pair of finite dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and numbers $0 < \varepsilon < 1$ and $\delta > 0$, there exists an ε-isometry $L : E \longrightarrow X$ such that we have $f(Le) = e(f)$ for $f \in F$ and $e \in E$, and $\|T \mid_{L(E)} \| < \|T^{**} \mid_E \| + \delta$.*

Proof. We have seen in Example 6 that X has the $\mathcal{B}^d$-polar property as a subspace of X^{**}. Thus, X plays here the role of Z in Theorem 7 as a subspace of the dual space $(X^*)^*$, and the result holds. □

3. Applications.

Here we apply the previous results to study the duality theory for ultrapowers of operators and some semigroups of operators related with the super-reflexivity and the finite representability of c_0 and ℓ_1 in a Banach space.

First, we show in Theorem 10 that for every operator T, the kernel $\ker(T^{**}{}_{\mathfrak{U}})$ is f.r. in $\ker(T_{\mathfrak{U}})$. In order to do that, we need the following technical Lemma.

Lemma 9. *Given an ultrafilter $\mathfrak{U}$ on I and a sequence $(B_n) \subset \mathfrak{U}$ such that $B_{n+1} \subset B_n$ for every n, there is a sequence $(C_n) \subset \mathfrak{U}$ verifying $C_{n+1} \subset C_n \subset B_n$, $C_n \setminus C_{n+1} \neq \emptyset$ for every $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} C_n = \emptyset$.*

Proof. Let $\{I_n : n \in \mathbb{N}\}$ be a partition of I such that $\emptyset \neq I_n \notin \mathfrak{U}$ for every $n \in \mathbb{N}$. We build the sets C_n inductively. We take $C_1 := B_1$. Assume we already have the sets $(C_n)_{n=1}^k \subset \mathfrak{U}$ verifying $C_{n+1} \subset C_n \subset B_n$ and $C_{n+1} \neq C_n$ for $n = 1, \dots, k-1$. Let us denote $n_k := \min\{n \geq k : I_n \cap C_k \neq \emptyset\}$ and

$$C_{k+1} := \bigcup_{n=n_k+1}^{\infty} (I_n \cap B_{k+1}).$$

It is easy to check that the sequence $(C_n)_n$ satisfies the desired conditions. □

Theorem 10. *Let $T \in \mathcal{B}(X,Y)$ be an operator and let $\mathfrak{U}$ be an ultrafilter on I. Then the kernel $\ker({T^{**}}_{\mathfrak{U}})$ is finitely representable in $\ker(T_{\mathfrak{U}})$.*

Proof. Let E be a finite dimensional subspace of $\ker({T^{**}}_{\mathfrak{U}})$ and let $\{\mathbf{F}_k = [F_k^i]_i : k = 1, \dots, n\}$ be a basis of E. Let us denote $E_i := \langle F_k^i : k = 1, \dots, n\rangle$ and let $T_i : E \longrightarrow E_i$ be the operator given by $T_i(\mathbf{F}_k) := F_k^i$.

Taking into account that $\lim_{\mathfrak{U}(i)} T^{**} F_k^i = 0$, a result of Heinrich [**5**, Proposition 6.1] allows us to find, for each positive integer m, a set $J_m \in \mathfrak{U}$ such that for every $i \in J_m$, the operators T_i are $1/m$-isometries and $\|T^{**}F_k^i\| < 1/m$ for $k = 1, \dots, n$. By Lemma 9, we can take a decreasing sequence $(C_n)_n$ in $\mathfrak{U}$ such that $\cap_{n=1}^{\infty} C_n = \emptyset$, $C_n \neq C_{n+1}$ and $C_n \subset J_n$ for all $n \in \mathbb{N}$; add the set $C_0 := I$ to this sequence. For every $i \in I$, we denote by m_i the unique element of $\mathbb{N} \cup \{0\}$ so that $i \in C_{m_i} \setminus C_{m_i+1}$.

For every $i \in C_1$, since $i \in C_{m_i} \setminus C_{m_i+1}$, by Corollary 8 we can choose $x_1^i, \dots, x_n^i$ in X such that the operator $L_i : E_i \longrightarrow F_i := \langle x_1^i, \dots, x_n^i\rangle$ given by $L_i(F_k^i) := x_k^i$ is a $1/m_i$-isometry and $\|Tx_k^i\| \leq 2/m_i$ for $k = 1, \dots, n$.

Define $\mathbf{x}_k := [x_k^i]$ and write $F := \langle \mathbf{x}_k : k = 1, \dots, n\rangle$. Thus we have that every $\mathbf{x}_k$ belongs to $\ker(T_{\mathfrak{U}})$ and F is isometric to E. In fact, on the one hand $C_m = \{i : m_i \geq m\} \in \mathfrak{U}$ for each $m \in \mathbb{N}$. So $\|Tx_k^i\| \leq 2/m$ for all $i \in C_m$. Therefore $\lim_{\mathfrak{U}(i)} Tx_k^i = 0$ and $\mathbf{x}_k \in \ker(T_{\mathfrak{U}})$ for $k = 1, \dots, n$.

On the other hand, let $\sum_{k=1}^n \lambda_k \mathbf{F}_k$ be a norm-one element in E. Let M be an upper bound for $\{\sum_{k=1}^n \lambda_k F_k^i : i \in I\}$. Take any $m \in \mathbb{N}$. For every $i \in \{i : m_i \geq m\}$, we have that L_i is a $1/m$-isometry; thus

$$C_m \subset \left\{ i : \left| \left\| \sum_{k=1}^n \lambda_k F_k^i \right\| - \left\| \sum_{k=1}^n \lambda_k x_k^i \right\| \right| \leq M/m \right\} \in \mathfrak{U}.$$

In this way we obtain that $\lim_{\mathfrak{U}(i)} \left\|\sum_{k=1}^n \lambda_k F_k^i\right\| - \left\|\sum_{k=1}^n \lambda_k x_k^i\right\| = 0$. Since

$$\left\| \sum_{k=1}^n \lambda_k \mathbf{x}_k \right\| = \lim_{\mathfrak{U}(i)} \left\| \sum_{k=1}^n \lambda_k x_k^i \right\| \quad \text{and} \quad \lim_{\mathfrak{U}(i)} \left\| \sum_{k=1}^n \lambda_k F_k^i \right\| = \left\| \sum_{k=1}^n \mathbf{F}_k \right\|,$$

we get $\|\sum_{k=1}^n \lambda_k \mathbf{x}_k\| = \|\sum_{k=1}^n \mathbf{F}_k\|$. Hence, E is isometric to F and therefore $\ker({T^{**}}_{\mathfrak{U}})$ is f.r. in $\ker(T_{\mathfrak{U}})$. □

A second application of Theorem 4: Heinrich [5] showed that ${X_{\mathfrak{U}}}^*$ is f.d.-r. in ${X^*}_{\mathfrak{U}}$. This is a consequence of our Theorem 11 below, applied to the zero operator. But our Theorem 4 is stronger than the result in [5]. Indeed, it will allow us to prove in Theorem 11 that $\ker({T_{\mathfrak{U}}}^*)$ is f.d.-r. in $\ker({T^*}_{\mathfrak{U}})$ for any operator $T \in \mathcal{B}(X,Y)$. Note that $\ker({T_{\mathfrak{U}}}^*)$ can be identified with the dual space $\operatorname{coker}(T_{\mathfrak{U}})^*$, but is not necessarily the second dual of any space. Moreover, in general $\operatorname{coker}(T_{\mathfrak{U}})$ is not an ultrapower [3]. Thus neither the principle of local reflexivity nor the result of Heinrich are applicable.

Theorem 11. *Let $T : X \longrightarrow Y$ be an operator and $\mathfrak{U}$ and ultrafilter on I. Then the kernel $\ker({T_{\mathfrak{U}}}^*)$ is finitely representable in $\ker({T^*}_{\mathfrak{U}})$ preserving the duality.*

Proof. By Theorem 4, it is enough to prove that $\ker({T^*}_{\mathfrak{U}})$ has the polar property as a subspace of $\ker({T_{\mathfrak{U}}}^*)$. We identify isometrically the kernel $\ker({T_{\mathfrak{U}}}^*)$ with the dual of the cokernel $\operatorname{coker}(T_{\mathfrak{U}})$. Along this proof, when we say that an operator $A : U_1 \longrightarrow U_2$ is identified with $B : V_1 \longrightarrow V_2$ we mean that there are isometries onto $J_1 : U_1 \longrightarrow V_1$ and $J_2 : U_2 \longrightarrow V_2$ so that $A = J_2^{-1} B J_1$.

Pick a linear function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^l$ and write

$$f_{\operatorname{coker}(T_{\mathfrak{U}})} : \ell_1^k(\operatorname{coker}(T_{\mathfrak{U}})\,) \longrightarrow \ell_1^l(\operatorname{coker}(T_{\mathfrak{U}})\,).$$

We only have to check the inclusion

$$\left((f_{\operatorname{coker}(T_{\mathfrak{U}})})^* \,|^{-1}_{\ell_\infty^l(\ker({T^*}_{\mathfrak{U}}))}\, B_{\ell_\infty^k(\ker({T^*}_{\mathfrak{U}}))}\right)_{\circ} \subset \overline{f_{\operatorname{coker}(T_{\mathfrak{U}})} B_{\ell_1^k(\operatorname{coker}(T_{\mathfrak{U}}))}}. \tag{3}$$

The proof is divided into three cases. The main one is the case $k = l$. The cases $k < l$ and $k > l$ will be obtained as a consequence of the main case.

Case a) $k = l$. We consider the operator $T^k : \ell_1^k(X) \longrightarrow \ell_1^k(Y)$, defined by $T^k((x_j)_{j=1}^k) := (Tx_j)_{j=1}^k$, and write $U := T^k$, $V := \ell_1^k(X)$ and $W := \ell_1^k(Y)$. We can identify $U_{\mathfrak{U}} : V_{\mathfrak{U}} \longrightarrow W_{\mathfrak{U}}$ with

$$(T_{\mathfrak{U}})^k : \ell_1^k(X_{\mathfrak{U}}) \longrightarrow \ell_1^k(Y_{\mathfrak{U}}).$$

The operator $\varphi : W_{\mathfrak{U}} \longrightarrow \ell_1^k(\operatorname{coker}(T_{\mathfrak{U}}))$ which sends $[(y_i^j)_{j=1}^k]_i$ to $([y_i^j]_i + \overline{R(T_{\mathfrak{U}})})_{j=1}^k$ is surjective, $\ker(\varphi) = \overline{R(U_{\mathfrak{U}})}$ and it is easy to check that the induced operator

$$\widetilde{\varphi} : \operatorname{coker}(U_{\mathfrak{U}}) \longrightarrow \ell_1^k(\operatorname{coker}(T_{\mathfrak{U}}))$$

is an isometry. Now, since $(f_Y)_{\mathfrak{U}} : W_{\mathfrak{U}} \longrightarrow W_{\mathfrak{U}}$ satisfies $(f_Y)_{\mathfrak{U}}(R(U_{\mathfrak{U}})) \subset R(U_{\mathfrak{U}})$, it induces an operator $L : \operatorname{coker}(U_{\mathfrak{U}}) \to \operatorname{coker}(U_{\mathfrak{U}})$ given by

$$L\left([y_i] + \overline{R(U_{\mathfrak{U}})}\right) := (f_Y)_{\mathfrak{U}}([y_i]) + \overline{R(U_{\mathfrak{U}})}.$$

The operator L is identified with

$$f_{\operatorname{coker}(T_{\mathfrak{U}})} : \ell_1^k(\operatorname{coker}(T_{\mathfrak{U}})) \longrightarrow \ell_1^k(\operatorname{coker}(T_{\mathfrak{U}}))$$

because $L = \widetilde{\varphi}^{-1} f_{\mathrm{coker}(T_{\mathfrak{U}})} \widetilde{\varphi}$. Since $\ell_1^k(\mathrm{coker}(T_{\mathfrak{U}}))^* = \ell_\infty^k(\ker(T_{\mathfrak{U}}{}^*))$, we can identify

$$\left(f_{\mathrm{coker}(T_{\mathfrak{U}})}\right)^* : \ell_\infty^k(\ker(T_{\mathfrak{U}}{}^*)) \to \ell_\infty^k(\ker(T_{\mathfrak{U}}{}^*))$$

with $L^* : \ker(U_{\mathfrak{U}}{}^*) \longrightarrow \ker(U_{\mathfrak{U}}{}^*)$. Under the previous identification we have that $\ker(U^*{}_{\mathfrak{U}}) = \ell_\infty^k(\ker(T^*{}_{\mathfrak{U}}))$. So the result that we want to prove (Formula 3) is equivalent to the inclusion

$$\left(L^* \mid_{\ker(U^*{}_{\mathfrak{U}})}^{-1} B_{\ker(U^*{}_{\mathfrak{U}})}\right)_\circ \subset \overline{LB_{\mathrm{coker}(U_{\mathfrak{U}})}}.$$

Suppose that this inclusion is false. Then there exists $\mathbf{h} = [h_i] \in W_{\mathfrak{U}}$ such that

$$\mathbf{h} + \overline{R(U_{\mathfrak{U}})} \in (L^{*-1} B_{\ker(U^*{}_{\mathfrak{U}})})_\circ \setminus \overline{LB_{\mathrm{coker}(U_{\mathfrak{U}})}}.$$

We can assume that $\|h_i\| = K > 0$ for all i. Write $S := f_Y$, $A := S(B_W)$ and $\mathbf{A} := S_{\mathfrak{U}}(B_{W_{\mathfrak{U}}}) = \{[a_i] : a_i \in A,\ i \in I\}$. Take $\varepsilon > 0$ such that

$$\mathrm{dist}\,(\mathbf{h} - \mathbf{A},\ R(U_{\mathfrak{U}})) > \varepsilon > 0.$$

Thus, for every $m \in \mathbb{N}$, we have $\mathrm{dist}\,(\mathbf{h} - \mathbf{A},\ mU_{\mathfrak{U}} B_{V_{\mathfrak{U}}}) > \varepsilon$, hence

$$D_m := \{i \in I : \mathrm{dist}\,(h_i - A, mUB_V) > \varepsilon\} \in \mathfrak{U}.$$

By Lemma 9 we can take a sequence $(C_n)_n$ in $\mathfrak{U}$ such that $C_{n+1} \subset C_n \subset D_n$ and $C_n \neq C_{n+1}$ for every $n \in \mathbb{N}$, and $\cap_{n=1}^\infty C_n = \emptyset$.

For every $i \in I$, let m_i be the unique positive integer for which $i \in C_{m_i} \setminus C_{m_i+1}$, and denote $K_i := h_i - A - m_i UB_V$. Since K_i is convex and $K_i \cap \varepsilon B_W = \emptyset$, by the Hahn-Banach Theorem there is a vector $f_i \in S_{W^*}$ such that $\inf f_i(K_i) \geq \varepsilon$. Hence

$$|f_i(Ux)| \leq \frac{K + \|S\| + \varepsilon}{m_i} \quad \text{for all } \ x \in B_V;$$

equivalently, $\|U^* f_i\| \leq m_i{}^{-1}(K + \|S\| + \varepsilon)$. Since the chain $(C_n)_n$ has empty intersection we have $\lim_{\mathfrak{U}} m_i{}^{-1} = 0$, and therefore $\mathbf{f} := [f_i] \in \ker(U^*{}_{\mathfrak{U}})$.

On the other hand, $\inf f_i(K_i) \geq \varepsilon$ implies $\inf f_i(h_i - A) \geq \varepsilon$. Thus we get $0 < \varepsilon \leq \varepsilon_i := f_i(h_i) \leq K$ for all i and $|f_i(a)| \leq \varepsilon_i - \varepsilon$ for all $a \in A$. Hence, letting $g_i := \varepsilon_i^{-1} f_i$, we obtain that

$$\alpha := \sup[g_i](LB_{\mathrm{coker}(U_{\mathfrak{U}})}) \leq 1 - \varepsilon K^{-1} < 1 = [g_i]([h_i]).$$

Take $y_i := \alpha^{-1} g_i$. We have that $\mathbf{y} := [y_i] = \alpha^{-1}(\lim_{\mathfrak{U}} \varepsilon_i^{-1})[f_i] \in \ker(U^*{}_{\mathfrak{U}})$ and

$$\sup \mathbf{y}(LB_{\mathrm{coker}(U_{\mathfrak{U}})}) \leq 1 < \mathbf{y}(\mathbf{h});$$

thus $\mathbf{y} \in L^{*-1} B_{\ker(U_{\mathfrak{U}}^*)}$ and $\mathbf{h} + \overline{R(U_{\mathfrak{U}})} \notin (L^{*-1} B_{\ker(U^*{}_{\mathfrak{U}})})_\circ \setminus \overline{LB_{\mathrm{coker}(U_{\mathfrak{U}})}}$, a contradiction.

For the cases b) and c), we adopt the notations $Z := \mathrm{coker}(T_{\mathfrak{U}})$ and $H := \ker(T^*{}_{\mathfrak{U}})$, so that $Z^* = \ker(T_{\mathfrak{U}}{}^*)$.

Case b) $k < l$. We consider the operator $\widetilde{L} : \ell_1^k(Z) \oplus_1 \ell_1^{l-k}(Z) \longrightarrow \ell_1^l(Z)$, defined by $\widetilde{L}(a,b) := La$. By case a) we have

$$\widetilde{L}B_{\ell_1^k(Z)\oplus_1\ell_1^{l-k}(Z)} = \left(\widetilde{L}^*\Big|_{\ell_\infty^l(H)}^{-1} B_{\ell_\infty^k(H)\oplus_\infty\ell_\infty^{l-k}(H)}\right)_\circ .$$

Since $LB_{\ell_1^k(Z)} = \widetilde{L}B_{\ell_1^k(Z)\oplus_1\ell_1^{l-k}(Z)}$ and

$$L^*\big|_{\ell_\infty^l(H)}^{-1}\left(B_{\ell_\infty^k(H)}\right) = \widetilde{L}^*\Big|_{\ell_\infty^l(H)}^{-1}\left(B_{\ell_\infty^k(H)\oplus_\infty\ell_\infty^{l-k}(H)}\right),$$

we have that $\overline{LB_{\ell_1^k(Z)}} = \left(L^* \,|_{\ell_\infty^l(H)}^{-1} B_{\ell_\infty^k(H)}\right)_\circ$.

Case c) $k > l$. We consider the operator $\widetilde{L} : \ell_1^k(Z) \longrightarrow \ell_1^l(Z) \oplus_1 \ell_1^{k-l}(Z)$, defined by $\widetilde{L}(a) := (La, 0)$. By case a) we have

$$\widetilde{L}B_{\ell_1^k(Z)} = \left(\widetilde{L}^*\Big|_{\ell_\infty^l(H)\oplus_\infty\ell_\infty^{k-l}(H)}^{-1} B_{\ell_\infty^k(H)}\right)_\circ .$$

The definition of $\widetilde{L}$ yields $L\left(B_{\ell_1^k(Z)}\right) \times 0_{\ell_1^{k-l}(Z)} = \widetilde{L}\left(B_{\ell_1^k(Z)}\right)$ and

$$L^*\big|_{\ell_\infty^l(H)}^{-1}\left(B_{\ell_\infty^k(H)}\right) \times \ell_\infty^{k-l}(Z^*) = \widetilde{L}^*\Big|_{\ell_\infty^l(H)\oplus_\infty\ell_\infty^{k-l}(H)}^{-1}\left(B_{\ell_\infty^k(H)}\right).$$

So we obtain

$$\left(L^*\big|_{\ell_\infty^l(H)}^{-1}\left(B_{\ell_\infty^k(H)}\right)\right)_\circ \times 0_{\ell_1^{k-l}(Z)} = \left(\widetilde{L}^*\Big|_{\ell_\infty^l(H)\oplus_\infty\ell_\infty^{k-l}(H)}^{-1}\left(B_{\ell_\infty^k(H)}\right)\right)_\circ .$$

Consequently, we get $\overline{LB_{\ell_1^k(Z)}} = \left(L^* \,|_{\ell_\infty^l(H)}^{-1}\left(B_{\ell_\infty^k(H)}\right)\right)_\circ$. □

Corollary 12 ([**9**])**.** *Let* $T : X \longrightarrow Y$ *be an operator and let* $\mathfrak{U}$ *be an ultrafilter on* I*. Then* $\ker(T^*{}_{\mathfrak{U}})$ *is a norming subspace of* $\ker(T_{\mathfrak{U}}{}^*) = (\mathrm{coker}(T_{\mathfrak{U}}))^*$*.*

A class of operators $\mathcal{A}$ is said to be a *semigroup* if the composition of two operators of $\mathcal{A}$ belongs to $\mathcal{A}$ and if for every Banach space X, the identity operator I_X on X belongs to $\mathcal{A}$. As well-known examples of semigroups, we mention the classical semi-Fredholm operators, and the tauberian operators, introduced by Kalton and Wilansky [**7**]. An operator $T \in \mathcal{B}(X,Y)$ is *tauberian* if $T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y$.

Now we apply the previous results to study the dual properties of the semigroups $\mathcal{W}^{up}{}_+$, introduced in [**12**], and $\mathcal{U}^{up}{}_+$ and $\mathcal{R}^{up}{}_+$, introduced in [**4**]. Recall that $S_X = \{x \in X : \|x\| = 1\}$.

An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{W}^{up}{}_+$ if for every $0 < r < 1$ there exists $n \in \mathbb{N}$ for which there are no finite sets $\{x_1, \ldots, x_n\} \subset S_X$ and $\{f_1, \ldots, f_n\} \subset S_{X^*}$ for which $f_i(x_j) > r$ for $i \le j$, $f_i(x_j) = 0$ for $j < i$ and $\|Tx_k\| < 1/k$, for $k = 1, \ldots, n$.

An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{U}^{up}{}_+$ if for every $C \geq 1$ there are $\delta > 0$ and $n \in \mathbb{N}$ for which there is no finite set $\{x_1, \dots, x_n\} \subset S_X$ which is C-equivalent to the unit basis of ℓ_∞^n and satisfies $\|Tx_k\| < \delta$ for $k = 1, \dots, n$.

An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{R}^{up}{}_+$ if for every $C \geq 1$ there are $\delta > 0$ and $n \in \mathbb{N}$ for which there is no finite set $\{x_1, \dots, x_n\} \subset S_X$ which is C-equivalent to the unit basis of ℓ_1^n and satisfies $\|Tx_k\| < \delta$ for $k = 1, \dots, n$.

Note that $T \in \mathcal{W}^{up}{}_+$ if and only if all the ultrapowers $T_\mathfrak{U}$ are tauberian operators [**2**]. Tacon [**13**] proved that $T \in \mathcal{W}^{up}{}_+$ implies $T^{**} \in \mathcal{W}^{up}{}_+$. Nevertheless, his proof does not seem to be applicable to the semigroups $\mathcal{U}^{up}{}_+$ and $\mathcal{R}^{up}{}_+$. Next we show that Theorem 10 allows us to give a unified proof of the result for the three semigroups $\mathcal{W}^{up}{}_+$, $\mathcal{R}^{up}{}_+$ and $\mathcal{U}^{up}{}_+$.

Proposition 13. *Let $\mathcal{A}_+$ be any of the semigroups $\mathcal{W}^{up}{}_+$, $\mathcal{R}^{up}{}_+$ or $\mathcal{U}^{up}{}_+$ and let $T \in \mathcal{B}(X,Y)$. Then $T \in \mathcal{A}_+$ if and only if $T^{**} \in \mathcal{A}_+$.*

Proof. Let $\mathfrak{U}$ be an ultrafilter. It was proved in [**2**] (for $\mathcal{W}^{up}{}_+$) and [**4**] (for $\mathcal{R}^{up}{}_+$ and $\mathcal{U}^{up}{}_+$) that T belongs to $\mathcal{A}_+$ if and only if the zero operator $0_{\ker(T_\mathfrak{U})}$ on $\ker(T_\mathfrak{U})$ belongs to $\mathcal{A}_+$. In any of the three cases, the condition $0_X \in \mathcal{A}_+$ defines a superproperty; i.e., Y f.r. in X and $0_X \in \mathcal{A}_+$ implies $0_Y \in \mathcal{A}_+$.

By Theorem 10, $\ker(T^{**}{}_\mathfrak{U})$ is f.r. in $\ker(T_\mathfrak{U})$. Since $\ker(T^{**}{}_\mathfrak{U})$ contains $\ker(T_\mathfrak{U})$, we conclude that $T \in \mathcal{A}_+$ if and only if $T^{**} \in \mathcal{A}_+$. □

Proposition 14. *Let $\mathcal{A}_+$ be any of the semigroups $\mathcal{W}^{up}{}_+$, $\mathcal{R}^{up}{}_+$ or $\mathcal{U}^{up}{}_+$ and let $T \in \mathcal{B}(X,Y)$. Then $T_\mathfrak{U}{}^* \in \mathcal{A}_+$ if and only if $T^*{}_\mathfrak{U} \in \mathcal{A}_+$.*

Proof. The direct implication is clear since $T_\mathfrak{U}{}^*$ is an extension of $T^*{}_\mathfrak{U}$. For the converse implication, observe that Theorem 11 says that $\ker(T_\mathfrak{U}{}^*)$ is finitely representable in $\ker(T^*{}_\mathfrak{U})$. Now, it is enough to observe that $T^* \in \mathcal{W}^{up}{}_+$ if and only if $\ker(T^*{}_\mathfrak{U})$ is superreflexive [**2**], and T^* belongs to $\mathcal{R}^{up}{}_+$ (resp. $\mathcal{U}^{up}{}_+$) if and only if ℓ_1 (resp. c_0) is not finitely representable in $\ker(T^*{}_\mathfrak{U})$ [**4**]. □

The following result was proved by Tacon using nonstandard analysis. Here we give a more transparent proof.

Proposition 15 ([**13**, Theorem 3])**.** *Given an operator $T \in \mathcal{B}(X,Y)$, we have $T^* \in \mathcal{W}^{up}{}_+$ if and only if* $\ker(T^*{}_\mathfrak{U}) = \ker(T_\mathfrak{U}{}^*)$.

Proof. By Corollary 12 we have that $\ker(T^*{}_\mathfrak{U})$ is w^*-dense in $\ker(T_\mathfrak{U}{}^*)$. Moreover, it is shown in [**2**, Theorem 9] that T^* belongs to $\mathcal{W}^{up}{}_+$ if and only if $\ker(T^*{}_\mathfrak{U})$ is reflexive. Hence the direct implication is clear.

Conversely, assume that $T^* \notin \mathcal{W}^{up}{}_+$; equivalently, $\ker(T^*{}_\mathfrak{U})$ is not reflexive. If $\ker(T_\mathfrak{U}{}^*) = \ker(T^*{}_\mathfrak{U})$, since $\ker(T_\mathfrak{U}{}^*) = \operatorname{coker}(T_\mathfrak{U})^*$, the triangular arrays in non-reflexive spaces, discovered in [6] and [11], give normalized sequences $(\mathbf{y}_n + \overline{R(T_{\mathfrak{U}})})_n$ in $\operatorname{coker}(T_{\mathfrak{U}})$ and $(\mathbf{f}_n)$ in $\ker(T^*{}_{\mathfrak{U}})$, and $0 < \varepsilon < 1$ so that

$$\mathbf{f}_k(\mathbf{y}_l) = \begin{cases} > \varepsilon, & \text{if } 1 \leq k \leq l < \infty \\ = 0, & \text{if } 1 \leq l < k < \infty. \end{cases} \tag{4}$$

Write $\mathbf{y}_n = [y_n^i]_i$ and $\mathbf{f}_n = [f_n^i]_i$. Let $\mathbf{g} = [g_i]$ be a w^*-cluster point of $\{\mathbf{f}_n : n \in \mathbb{N}\}$ in $\ker(T^*{}_{\mathfrak{U}})$. Formula 4 yields $\mathbf{g}(\mathbf{y}_n) = 0$ for all $n \in \mathbb{N}$, and allows us to build the sequence $(A_n)_n$ of elements of $\mathfrak{U}$ given inductively by

$$A_1 := \{i \in I : g_i(y_1^i) < \varepsilon/2,\ f_1^i(y_1^i) > \varepsilon\} \in \mathfrak{U}$$

$$A_n := A_{n-1} \cap \{i \in I : g_i(y_n^i) < \varepsilon/2,\ f_k^i(y_n^i) > \varepsilon,\ 1 \leq k \leq n\} \in \mathfrak{U}.$$

Since $(A_n)_n$ is decreasing, by Lemma 9 we find another decreasing sequence $(C_n)_n \subset \mathfrak{U}$ such that $C_n \subset A_n$ and $C_n \setminus C_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} C_n = \emptyset$. Thus, for every $k \in \mathbb{N}$, we take $s_i := y_k^i$ for every $i \in C_k \setminus C_{k+1}$, and define $\mathbf{s} := [s_i]$.

On the one hand, we have that $\mathbf{g}(\mathbf{s}) \geq \varepsilon$. Indeed, let k be a positive integer. For every $l > k$ and every $i \in C_l \setminus C_{l+1}$ we have $f_k^i(s_i) = f_k^i(y_l^i) \geq \varepsilon$. Since $\bigcup_{l=k}^{\infty} (C_l \setminus C_{l+1}) \in \mathfrak{U}$, we have $[f_k^i]([s_i]) \geq \varepsilon$; therefore, $\mathbf{f}_k(\mathbf{s}) \geq \varepsilon$ for each $k \in \mathbb{N}$. Besides that, as $\mathbf{g}$ is a w^*-cluster point of $\{\mathbf{f}_n : n \in \mathbb{N}\}$, we have $\mathbf{g}(\mathbf{s}) \geq \varepsilon$.

On the other hand, for every $n \in \mathbb{N}$ and $i \in C_n \setminus C_{n+1} \subset A_n$ we have that $g_i(s_i) = g_i(y_n^i) < \varepsilon/2$. Thus $\bigcup_{n=1}^{\infty}(C_n \setminus C_{n+1}) \in \mathfrak{U}$ leads to $\mathbf{g}(\mathbf{s}) = [g_i]([s_i]) \leq \varepsilon/2$, and we get a contradiction. $\square$

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