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FACTORIZATION OF PROPER HOLOMORPHIC MAPPINGS THROUGH THULLEN DOMAINS

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Given a bounded domain in $\mathbb{C}^2$ and a proper holomorphic mapping onto the unit ball in $\mathbb{C}^2$, we give a criterion for a proper mapping to factor through Thullen domains by its branch locus. As an application, we present a characterization of the Thullen domains among bounded domains admitting a proper holomorphic mapping onto the unit ball.

1. Introduction.

A continuous map is called *proper*, if the pre-image of every compact set is compact. Let $f : D \to G$ be a proper holomorphic mapping between bounded pseudoconvex domains in $\mathbb{C}^n$. Denote by Z_{df} the analytic set defined by the relation that its holomorphic Jacobian determinant is zero. The Proper Mapping Theorem of Remmert and Stein ([**10**]) implies that f is a finite-to-one holomorphic covering map from $D \setminus Z_{df}$ onto its image. Thus, all proper holomorphic mappings between equidimensional bounded domains of holomorphy possess a generic degree.

For every positive integer k, we consider the Thullen domain

$$\mathcal{E}_k = \{(z, w) \in \mathbb{C}^2 \mid |z|^{2k} + |w|^2 < 1\}.$$

For the entire paper, we use the notation

$$\varphi_k(z, w) = (z^k, w) : \mathcal{E}_k \to B^2$$

where B^2 is the open unit ball in $\mathbb{C}^2$.

The primary goal of this article is to establish the factorization theorem presented in the following. This theorem shows in particular that the Thullen domains $\mathcal{E}_k$ together with the map φ_k form a standard model for proper mappings from a bounded domain with a real analytic boundary to the unit ball in $\mathbb{C}^2$, in the sense that a proper mappings with controlled branch locus factors through the map φ_k for an appropriate positive integer k.

Theorem 1. *Let D be a bounded simply connected pseudoconvex domain in $\mathbb{C}^2$ with a real analytic boundary. Assume that $f : D \to B^2$ is a proper holomorphic mapping with generic degree m such that the analytic variety Z_{df} admits an irreducible component V satisfying:*

(1) $f^{-1}(f(V \cap \partial D)) = V \cap \partial D$.

(2) *$V \cap \partial D$ is connected and contains no singular point of the variety Z_{df}. Denote by k the degree of the mapping f in a sufficiently small tubular neighborhood of V. Then m is divisible by k and there exists a generically m/k-to-1 proper holomorphic mapping $g : D \to \mathcal{E}_k$ which extends holomorphically across ∂D such that*

$$f = \alpha \circ \varphi_k \circ g$$

for some holomorphic automorphism α of B^2, where $\varphi_k(z, w) = (z^k, w)$.

By [**9**] (see also [**2**], [**3**], [**7**]) all proper holomorphic mappings between bounded pseudoconvex domains with entirely real analytic smooth boundaries extend holomorphically across the boundaries. Thus, the conditions (1) and (2) in the statement of Theorem 1 (as well as in the statement of Theorem 2 below) are meaningful.

We would like to point out that Theorem 1 treats the factorization of proper holomorphic mappings *into* the ball. This distinguishes our theorem from a result by Rudin ([**11**]) which study the factorization of proper maps *from* the ball.

Notice that Theorem 1 implies in particular the following characterization of the Thullen domain $\mathcal{E}_k$ among the source domains of proper holomorphic mappings onto the unit ball by its branch locus.

Theorem 2. *Let D be a bounded simply connected pseudoconvex domain in $\mathbb{C}^2$ with a real analytic boundary. If D admits a generically k-to-1 proper holomorphic mapping $f : D \to B^2$ such that there exists an irreducible subvariety V of Z_{df} satisfying:*

(1) *f is a local k-to-1 branched covering at every point of $V \cap \partial D$;*

(2) *$V \cap \partial D$ is connected and contains no singular point of the variety Z_{df}; then D is biholomorphic to $\mathcal{E}_k$.*

We would like to remark that the hypothesis in the factorization theorem above is essential in the sense that there are many examples of unfactorizable proper mappings with branch locus violating the conditions in the hypothesis.

Although it is desirable in principle to obtain a generalization of the above theorems to higher complex dimensions, one of the main difficulties lies in the fact that there are no known effective normal forms of Chern-Moser type near the weakly pseudoconvex points in complex dimensions higher than two. Since our methods depend heavily upon the Chern-Moser normal forms and subsequent developments and generalizations by Barletta and Bedford ([**4**]), our two dimensional theorem seems in fact optimal.

Acknowledgments. We are indebted to the referee for valuable suggestions especially concerning Theorem 4.3.

2. Organization of the paper.

The rest of this article is organized as follows. In order to have the exposition as clear as possible, Section 3 presents the precise definitions and standard notation for the concepts such as pointed CR surfaces and associated domains, holomorphic mappings between them, branch points, local automorphisms and weakly spherical normal forms of Barletta and Bedford. Section 4 discusses the rigidity of holomorphic mappings from a normalized weakly spherical surface into a normalized strongly pseudoconvex surface in the sense of Chern-Moser. The first key step (Proposition 4.1) is an involved investigation on the effect of holomorphic mappings to the normal forms. Based upon this result, we then show that the only weakly spherical pointed CR surface that admits a regular branched holomorphic mapping to a pointed Siegel surface is defined by $\mathrm{Re}\, w - |z|^{2k} = 0$. We further show that the holomorphic mapping has to be essentially $(z, w) \mapsto (z^k, w)$. This establishes an important step towards our proof of the factorization theorem. In Section 5, the proofs of main theorems are given. The arguments rely upon the local rigidity established in Section 4 and an analysis of the splitting of holomorphic correspondences.

3. Basic terminology and notation.

3.1. Pointed CR hypersurfaces and associated domains. Let (M, p) be a real three dimensional real analytic pointed hypersurface in $\mathbb{C}^2$. It naturally admits the standard CR structure. Furthermore, the Implicit Function Theorem shows that there exists a real-valued real analytic function $\rho_M(z, w)$ defined in an open neighborhood U_M of p in $\mathbb{C}^2$ such that

$$M \cap U_M = \{(z, w) \in U_M \mid \rho_M(z, w) = 0\}$$

and such that the gradient $\nabla \rho_M$ is not zero at any point of $M \cap U_M$. Shrinking U_M sufficiently small, we may assume that $U_M \setminus M$ consists of two domains. We will call such U_M a *neighborhood associated with* (M, p). In case M is pseudoconvex and not Levi flat, only one of the two is a pseudoconvex domain. We assume that

$$\Omega_M \equiv \{(z, w) \in U_M \mid \rho_M(z, w) > 0\}$$

is pseudoconvex and call it a *domain associated with* (M, p). Although the concepts of associated neighborhoods and associated domains contain ambiguities, they can be readily made precise by exploiting the concept of germs. Therefore, in this paper we choose to use the above terminologies without introducing the concept of germs, since there is no danger of confusion in this exposition.

3.2. Holomorphic mappings of pointed CR hypersurfaces. The next concept to consider is the mapping between pointed CR hypersurfaces (or equivalently, between their germs) in $\mathbb{C}^2$. Given a pair of pointed CR hypersurfaces (M,p), (N,q) with associated neighborhoods U_M, U_N and associated domains Ω_M, Ω_N respectively, we say a mapping $F:(M,p)\to(N,p)$ to be *a holomorphic mapping between pointed CR hypersurfaces*, if it extends to a holomorphic mapping $F:U_M\to U_N$ satisfying the conditions

$$F(p)=q, \quad F(M)\subset N, \quad F(\Omega_M)\subset\Omega_N.$$

We also introduce the notation for the branch locus (or equivalently, the Segre variety) of F in the following:

$$Z_{dF}\equiv\{z\in U_M \mid \det JF(z)=0\}.$$

By a *branch point* of the mapping F we mean an element of Z_{dF}. We call a branch point *regular*, if it is a smooth point of Z_{df}.

3.3. Local Automorphisms. Furthermore, for a pointed CR hypersurface (M,p) we denote by

$$\mathrm{Aut}\,(M,p)=\{\varphi:(M,p)\to(M,p)\,|\,\text{1-1 and holomorphic}\}$$

the *local automorphism group for M at p.*

3.4. Weakly spherical hypersurfaces and normal forms. Finally, we introduce an important special class of pointed CR hypersurfaces following [4]. A real analytic CR hypersurface M is called *weakly spherical of order $k_0\geq 1$ at $p\in M$*, if it admits a local holomorphic coordinate system (z,w) at p ($p=(0,0)$ in the new coordinates) in which (M,p) is defined by the following normalized defining function (an analogue of the Chern-Moser normal form introduced in [8])

$$\rho(z,w)=\mathrm{Re}\,w-|z|^{2k_0}-\mathrm{Re}\sum_{\substack{j,h\geq 1\\ j+h\geq 2k_o+1}} a_{jh}(\mathrm{Im}\,w)\,z^j\,\bar{z}^h \tag{3.4.1}$$

where a_{jh} are real analytic functions in the variable $\mathrm{Im}\,w$ satisfying:

$$\begin{aligned} &a_{2k_0\,2k_0}=a_{3k_0\,3k_0}=0\\ &a_{jk_0}=a_{k_0j}=0,\forall j\geq k_0+1. \end{aligned} \tag{3.4.2}$$

If a real analytic CR hypersurface M in $\mathbb{C}^2$ contains the origin $o=(0,0)$, and if a defining function for M in a neighborhood of o takes the form (3.4.1) without a change of coordinates of $\mathbb{C}^2$ and satisfies the condition (3.4.2), then we call the pointed CR hypersurface (M,o) *normalized weakly spherical.*

For a pointed real analytic CR hypersurface (M,o), we say that a defining function is *reduced* if it is of the form $\rho(z,w)=\mathrm{Re}\,w-G(z,\bar{z},\mathrm{Im}\,w)$. Note that in each fixed holomorphic coordinate system, a CR hypersurface admits at most one reduced defining function.

Suppose that there exists a holomorphic mapping F from (M,o) into a smooth strongly pseudoconvex CR hypersurface (N,o) for which o is a regular branch point. Then, according to [4], for every *reduced* defining function ρ_M for (M, o), there exists a local biholomorphism γ at the point o such that $\rho_M \circ \gamma$ is the weakly spherical normal form, which takes the form (3.4.1) and satisfies (3.4.2). See [4] for detailed arguments.

4. Holomorphic mapping into the Siegel hypersurface.

We now investigate the actions of the holomorphic mappings on the normal forms of the CR hypersurfaces. We first present the following technical result on the holomorphic mappings from a normalized weakly spherical CR hypersurface to a normalized strongly pseudoconvex hypersurface.

Proposition 4.1. *Let (M, o) be a normalized pointed weakly spherical CR hypersurface in $\mathbb{C}^2$ of order $k_0 > 2$, and let (N, o) be a normalized CR hypersurface in $\mathbb{C}^2$ which is strongly pseudoconvex at the origin $o = (0, 0)$. Then, for every holomorphic mapping $F : (M, o) \to (N, o)$ with a regular branching point at o, there exist local automorphisms $g_N \in \operatorname{Aut}(N, o)$ and $g_M \in \operatorname{Aut}(M, o)$ such that*

$$\rho_M = \rho_N \circ g_N \circ F \circ g_M$$

where ρ_M and ρ_N are the defining functions in normal form for (M, o) and (N, o), respectively.

Remark 4.2. A direct application of Hopf's lemma implies only that $\rho_N \circ F$ is also a defining function of (M, o), which in turn yields at best that $h \cdot \rho_M = \rho_N \circ F$ for some positive real valued function h. The first merit of the proposition above is that we have limited the ambiguity to a significant degree, by showing that h can be replaced by a composition by two local automorphisms of the surfaces. Even with this fact alone, it seems providing an interesting aspect for a further investigation, which should be of a separate interest.

Proof of Proposition 4.1. Let W_M be the set of weakly pseudoconvex points on $M \cap U_M$. It is given by

$$W_M = \{(z, w) \in M \cap U_M \,|\, z = 0\}$$

since M is normalized. Since F maps holomorphically M into N and Ω_M into Ω_N respectively, Hopf's lemma implies that

$$W_M = \{(z, w) \in M \cap U_M \,|\, \det(JF)(z, w) = 0\}.$$

Since o is a regular branch point for F, the Weierstrass Preparation Theorem implies that, by shrinking U_M if necessary, the mapping $F : U_M \to F(U_M) \subset U_N$ is a branched covering such that $F(Z_{df} \cap U_M)$ is a smooth analytic variety. Furthermore, we may choose a change of holomorphic local coordinates at

o by a local biholomorphism ψ so that

$$(\psi \circ F)(Z_{df} \cap U_M) \subset \{(0, w) \in \mathbb{C}^2 \mid w \in \mathbb{C}\}$$
$$(\psi \circ F)(0, w) = (0, w).$$

Hence,

$$(\psi \circ F)(z, w) = (z^m \cdot k(z, w), w + z \cdot g(z, w))$$

where $k(0, w)$ is not identically zero. Since the map $(z, w) \mapsto (z, w + z \cdot g(z, w))$ is invertible near the origin $o = (0, 0)$, we define a local biholomorphism ϕ_1 in a neighborhood of the origin by

$$\phi_1^{-1}(z, w) = (z, w + z \cdot g(z, w)).$$

Thus we have

$$(\psi \circ F \circ \phi_1)(z, w) = (z^m h(z, w), w),$$

where $h(z, w) = (k \circ \phi_1)(z, w)$. Now we show that $h(0, 0) \neq 0$. Since

$$\det J(\psi \circ F \circ \phi_1) = z^{m-1} \left(mh + z \frac{\partial h}{\partial z} \right) \tag{4.1}$$

and since the origin o is a smooth point of the variety $Z_{d(\psi \circ F \circ \phi_1)}$, the equation $\det J(\psi \circ F \circ \phi_1) = 0$ must determine a smooth variety near the origin. From (4.1) above, we must then have either $h(0, 0) \neq 0$ or $mh + z\frac{\partial h}{\partial z}$ vanish identically on $\{(0, w) \in \mathbb{C}^2\}$. However the latter implies that $h(0, w)$ is identically zero. Since $h(0, w) = k(0, w)$, this contradicts to the fact that $k(0, w)$ is not identically zero. Hence $h(0, 0) \neq 0$. (This paragraph is a modification of an argument in Barletta-Bedford [**4**].)

Therefore, in an open neighborhood of the origin we may choose a local biholomorphism ϕ_2 given by

$$\phi_2^{-1}(z, w) = (z(h(z, w))^{1/m}, w)$$

by taking any branch. Let

$$\phi = \phi_1 \circ \phi_2.$$

Then we have

$$\tilde{F}(z, w) \equiv (\psi \circ F \circ \phi)(z, w) = (z^m, w) \tag{4.2}$$

and

$$\begin{aligned} F^{\#}(z, w) \equiv (\psi \circ F)(z, w) &= (\tilde{F} \circ \phi^{-1})(z, w) \\ &= (z^m k(z, w), w + z \cdot g(z, w)) \end{aligned} \tag{4.3}$$

in a neighborhood of the origin o.

We now consider the pointed CR hypersurfaces (N', o) and (M', o) defined by

$$N' \equiv \psi(N \cap U_N), \qquad M' \equiv \phi^{-1}(M \cap U_M) = \tilde{F}^{-1}(N' \cap \psi(U_N)),$$

where U_M and U_N are taken sufficiently small so that ψ and ϕ are well-defined.

To conclude the proof, we prove the following:

Claim. There exist two reduced defining functions $\rho_{N'}$ and $\rho_{M'}$ for (N', o) and (M', o) such that

$$\rho_{M'} = \rho_{N'} \circ \tilde{F}.$$

Suppose for the moment that the claim is proved. Since ρ_N and ρ_M are two defining functions in normal form, according to [**4**] and [**8**] there exist two local biholomorphic mappings $\gamma : (N', o) \to (N, o)$ and $\chi : (M', o) \to (M, o)$ such that

$$\rho_N = \rho_{N'} \circ \gamma^{-1}, \qquad \rho_M = \rho_{M'} \circ \chi^{-1}.$$

This implies that

$$\begin{aligned} \rho_M &= \rho_N \circ \gamma \circ \tilde{F} \circ \chi^{-1} \\ &= \rho_N \circ (\gamma \circ \psi) \circ F \circ (\phi \circ \chi^{-1}), \end{aligned}$$

where it is obvious that $(\gamma \circ \psi) \in \operatorname{Aut}(N, o)$ and $(\phi \circ \chi^{-1}) \in \operatorname{Aut}(M, o)$. Thus the assertion of the proposition is obtained.

Therefore it remains only to prove the claim above. Let us now look for a *reduced* defining function $\rho_{N'}$. If we denote by

$$\delta = \rho_N \circ \psi^{-1}$$

then, by shrinking U_M if necessary, we have

$$F^{\#}(\Omega_M) = \{(z, w) \in U_M \mid \delta(z, w) < 0\} \tag{4.4}$$

and

$$F^{\#}(M \cap U_M) = \{(z, w) \in U_M \mid \delta(z, w) = 0\}. \tag{4.5}$$

By Hopf's Lemma, $\delta \circ F^{\#}$ is also a defining function for (M, o, Ω_M). Therefore,

$$\delta \circ F^{\#}(z, w) = \alpha(z, w)\, \rho_M(z, w)$$

for some positive real analytic function $\alpha : U_M \to \mathbb{R}$.

Write as $z = x + iy$ and $w = u + iv$. Since $F^{\#}(z, w) = (z^m k(z, w), w + z \cdot g(z, w))$, we have

$$\left.\frac{\partial(\delta \circ F^{\#})}{\partial u}\right|_o = \left.\frac{\partial \delta}{\partial u}\right|_o. \tag{4.6}$$

On the other hand,

$$\left.\frac{\partial(\delta \circ F^{\#})}{\partial u}\right|_o = \left.\frac{\partial(\alpha \cdot \rho_M)}{\partial u}\right|_o = \alpha(0,0) \cdot \left.\frac{\partial \rho_M}{\partial u}\right|_o = \alpha(0,0). \tag{4.7}$$

Therefore,

$$\left.\frac{\partial \delta}{\partial u}\right|_o = \alpha(0,0) > 0. \tag{4.8}$$

The Implicit Function Theorem now implies that the domain $F^{\#}(\Omega_M)$ admits a defining function $\rho_{N'}$ near o given by

$$\rho_{N'}(z,w) = \operatorname{Re} w - G_{N'}(z,\bar{z},v)$$

where

$$G_{N'}(z,\bar{z},v) = \sum_{j,h\geq 1} b_{jh}(v) z^j \bar{z}^h$$

is real analytic.

Now, applying Hopf's Lemma, we obtain that $\rho_{N'} \circ \tilde{F}$ is a defining function of $\Omega_{M'}$ near o, and more importantly that it is in a *reduced* form. Thus, by letting

$$\rho_{M'} \equiv \rho_{N'} \circ \tilde{F},$$

the assertion of the claim above is verified. Consequently the proof of Proposition 4.1 is complete. □

Now we are ready to present the following main result of this section, which shows the local rigidity of Siegel hypersurfaces among the target hypersurfaces of proper holomorphic mappings from normalized weakly spherical hypersurfaces. From now on, we denote by

$$\varphi_k(z,w) = (z^k, w).$$

Theorem 4.3. *Let (M,o) be a real analytic normalized weakly spherical pointed CR hypersurface in $\mathbb{C}^2$ of order $k_0 > 1$. Let (Σ, o) be the pointed Siegel hypersurface given by the defining equation*

$$\rho_\Sigma(z,w) \equiv \operatorname{Re} w - |z|^2 = 0.$$

If there is a holomorphic mapping $F : (M,o) \to (\Sigma,o)$ for which o is a regular branch point, then:

(1) *(M,o) is defined by the equation $\operatorname{Re} w - |z|^{2k_0} = 0$, and*
(2) *$F(z,w) = (g_\Sigma \circ \varphi_{k_0} \circ g_M)(z,w)$, for some $g_\Sigma \in \operatorname{Aut}(\Sigma,o)$ and $g_M \in \operatorname{Aut}(M,o)$.*

Proof. By the preceding proposition, we may choose two elements $g_\Sigma \in \operatorname{Aut}(\Sigma,o)$ and $g_M \in \operatorname{Aut}(M,o)$ such that

$$\rho_M = \rho_\Sigma \circ g_\Sigma \circ F \circ g_M$$

where ρ_M is the weakly spherical normal form of (M,o). We put

$$(g_\Sigma \circ F \circ g_M)(z,w) = f(z,w) = (f_1(z,w), f_2(z,w)).$$

Then we have

$$\operatorname{Re} f_2(z,w) - |f_1(z,w)|^2 = \operatorname{Re} w - |z|^{2k_0} - \sum_{\substack{j,h\geq 1\\ j+h\geq 2k_0+1}} a_{jh}(\operatorname{Im} w) z^j \bar{z}^h \tag{4.9}$$

where the coefficients a_{jh} satisfy the vanishing conditions specified in (3.4.2).

Since $f(0,0) = (0,0)$, the Taylor expansions of f_1 and f_2 take the form

$$f_1(z,w) = b_{10}z + b_{01}w + R_{1,2}(z,w)$$
$$f_2(z,w) = \beta_{10}z + \beta_{01}w + R_{2,2}(z,w)$$

where $R_{k,m}(z,w)$ denotes the remainder term of order m or higher in the Taylor expansion of $f_k(z,w)$. Thus, (4.9) becomes

$$\begin{aligned} &\operatorname{Re}(\beta_{10}z + \beta_{01}w) + \operatorname{Re} R_{2,2}(z,w) - |b_{10}z + b_{01}w + R_{1,2}(z,w)|^2 \\ &= \operatorname{Re} w - |z|^{2k_0} - \sum a_{jh}(\operatorname{Im} w)\; z^j \bar{z}^h. \end{aligned} \tag{4.10}$$

Let $w = 0$ in (4.10), then we get $\beta_{10} = 0$. Then put $z = 0$ in turn to get $\beta_{01} = 1$. Since $k_0 > 1$, we must have $\det Jf(0,0) = 0$. This then implies $b_{10} = 0$. In summary, we have

$$b_{10} = \beta_{10} = 0, \quad \beta_{01} = 1$$

and

$$\operatorname{Re}(R_{2,2}(z,w)) - |b_{01}w + R_{1,2}(z,w)|^2 = -|z|^{2k_0} - \sum a_{jh}(\operatorname{Im} w) z^j \bar{z}^h. \tag{4.11}$$

Letting $z = 0$ in (4.11), we obtain

$$\operatorname{Re} R_{2,2}(0,w) = |f_1(0,w)|^2.$$

Since the left hand side is a harmonic function whereas the right hand side attains its minimum at $w = 0$, we deduce that

$$\operatorname{Re} R_{2,2}(0,w) = f_1(0,w) = 0,$$

for all values of w in a neighborhood of $w = 0$. Thus, in their Taylor expansions, neither f_1 nor f_2 has any nonzero pure terms in w^h with $h > 1$. Moreover, it holds that

$$b_{01} = 0.$$

By setting now $w = 0$, we get

$$\begin{aligned} &\operatorname{Re} f_2(z,0) - |b_{20}z^2 + \ldots + b_{k_0\,0}z^{k_0} + R_{1,k_0+1}(z,0)|^2 \\ &= -|z|^{2k_0} - \sum a_{jh}(0) z^j \bar{z}^h \end{aligned} \tag{4.12}$$

where $R_{1,k_0+1}(z,0) = O(|z|^{k_0+1})$.

Comparing the terms in the power series expansions of both sides, we get

$$b_{20} = \cdots = b_{(k_0-1)\,0} = 0, \quad |b_{k_0\,0}| = 1. \tag{4.13}$$

Moreover, we obtain $f_2(z,0) \equiv 0$ since the monomials in the expansion of $\operatorname{Re} f_2(z,0)$ are types of either z^ℓ or $\bar{z}^\ell$ which are not found from any other terms of (4.12). Therefore we now have

$$b_{k_0\,0}z^{k_0}\,\overline{R_{1,k_0+1}(z,0)} + \overline{b_{k_0\,0}z^{k_0}}\,R_{1,k_0+1}(z,0) + |R_{1,k_0+1}(z,0)|^2 \tag{4.14}$$
$$= \sum_{\substack{j,h\geq 1\\ j+h\geq 2k_0+1}} a_{jh}(0) z^j \bar{z}^h$$

where $|b_{k_0\,0}| = 1$.

Now we use the vanishing condition (3.4.2) for a_{jh} that is

$$a_{k_0\,h} = 0 = a_{j\,k_0},\ \forall j, h \geq k_0 + 1.$$

Applying this to (4.14) we obtain

$$R_{1,k_0+1}(z,0) \equiv 0.$$

As a consequence, we also have

$$b_{j0} = 0,\ \forall j \geq k_0 + 1 \tag{4.15}$$

and

$$a_{jh}(0) = 0,\ \forall j, h \geq 1 \text{ with } j + h \geq 2k_0 + 1. \tag{4.16}$$

Thus, it follows that

$$f_1(z,w) = b_{k_0\,0}\,z^{k_0} + r_1(z,w)$$
$$f_2(z,w) = w + r_2(z,w)$$

where both $r_1(z,w)$ and $r_2(z,w)$ are holomorphic functions of the class $O(|zw|)$.

Now, we let $w = \epsilon \in \mathbb{R}$, with ϵ sufficiently small but positive. Using (4.16), the identity (4.10) now becomes

$$\epsilon + \operatorname{Re} r_2(z,\epsilon) - |b_{k_0\,0}z^{k_0} + r_1(z,\epsilon)|^2 = \epsilon - |z|^{2k_0} \tag{4.17a}$$

or, equivalently

$$\operatorname{Re} r_2(z,\epsilon) - b_{k_0\,0}z^{k_0}\,\overline{r_1(z,\epsilon)} - \overline{b_{k_0\,0}z^{k_0}}\,r_1(z,\epsilon) = |r_1(z,\epsilon)|^2. \tag{4.17b}$$

Here, we want to show first that $r_2(z,\epsilon) \equiv 0$. For each fixed $\epsilon > 0$ that is sufficiently small, we expand each term in the identity (4.17b) using the Taylor expansion of r_1 and r_2 in the variable z at the origin. The monomial terms in the expansion of $\operatorname{Re} r_2(z,\epsilon)$ consist solely of the form z^ℓ and $\bar{z}^\ell$, whereas the remaining terms contain only the mixed power terms of z and $\bar{z}$. Thus the comparison of coefficients immediately implies that $r_2(z,\epsilon) = 0$ for every z in an open neighborhood of the origin and for every sufficiently small $\epsilon > 0$. Since r_2 is holomorphic in z, w, this implies immediately that $r_2(z,w)$ is indeed identically zero.

Due to the preceding arguments, (4.17a) now becomes

$$|b_{k_0\,0}z^{k_0} + r_1(z,\epsilon)|^2 = |z|^{2k_0}$$

or equivalently

$$\left| b_{k_0\,0} + \frac{r_1(z,\epsilon)}{z^{k_0}} \right| = 1$$

for all z in a neighborhood of the origin in $\mathbb{C}$ and for all sufficiently small values of $\epsilon > 0$. It then follows immediately that

$$b_{k_0,0} z^{k_0} + r_1(z,\epsilon) = e^{ig(\epsilon)} z^{k_0}$$

for some real-valued real-analytic function g defined in an open neighborhood of the origin in $\mathbb{R}$. Thus, we conclude that $r_1(z,w) = z^{k_0}\hat{r}(w)$, where $\hat{r}(w)$ is holomorphic in the complex variable w.

Therefore, we now arrive at that our mapping f is of the form

$$f(z,w) = \left(z^{k_0}(b_{k_0,0} + \hat{r}(w)), w \right).$$

We apply this information to the identity (4.9) and we immediately observe the following:

(1) All coefficients a_{ij} vanish identically, and
(2) $|b_{k_0,0} + \hat{r}(w)| \equiv 1$, and hence $b_{k_0,0} + \hat{r}(w) \equiv \lambda$, for some constant $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Thus, we obtain

$$(g_\Sigma \circ F \circ g_M)(z,w) = f(z,w) = (\lambda z^{k_0}, w),$$

which proves the assertion. □

5. Factorization of proper maps through a complex ellipsoid.

Denote by

$$E_m \equiv \{(z,w) \in \mathbb{C}^2 \,|\, |z|^{2m} + |w|^2 = 1\}$$

and by

$$\Omega_{E_m} \equiv \{(z,w) \in \mathbb{C}^2 \,|\, |z|^{2m} + |w|^2 < 1\}$$

the *complex ellipsoid* and the *Thullen domain of order* m, respectively. Similarly, denote by

$$\Sigma_m \equiv \{(z,w) \in \mathbb{C}^2 \,|\, \operatorname{Re} w - |z|^{2m} = 0\}$$

and

$$\Omega_{\Sigma_m} \equiv \{(z,w) \in \mathbb{C}^2 \,|\, \operatorname{Re} w - |z|^{2m} > 0\}$$

the *normalized Siegel hypersurface* and the *Siegel upper-half space* of order m. Recall that, for any positive integer m, the map

$$\mu_m(z,w) = \left(\frac{z}{(1+w)^{1/m}}, \frac{1-w}{1+w} \right)$$

sends *biholomorphically* the pointed CR hypersurface $(E_m, (0,1))$ to the pointed CR hypersurface $(\Sigma_m, (0,0))$.

For simplicity, in the following we will often write just $E = \partial B^2$ instead of E_1, Σ instead of Σ_1 and μ instead of μ_1. We also continue using the notation $\varphi_m(z,w) = (z^m, w)$. Observe that, for any positive integer m

$$\varphi_m \circ \mu_m = \mu \circ \varphi_m. \tag{5.1}$$

We now present a key lemma.

Lemma 5.1. *Let U_1 and V_1 be open neighborhoods of the point $(0,1)$ in $\mathbb{C}^2$ and let $F : U_1 \to V_1$ be a local biholomorphism of the pointed CR hypersurface $(E_m, (0,1))$ into itself. Then F extends uniquely to a global automorphism of the Thullen domain Ω_{E_m}.*

Proof. Let μ_m and φ_m be as above. Furthermore, let us denote by

$$\hat{F} = \mu_m \circ F \circ \mu_m^{-1} : \Sigma_m \to \Sigma_m$$

and

$$G = \varphi_m \circ \hat{F} : \Sigma_m \to \Sigma.$$

An important step toward the proof is the following:

Claim.

There exists a local automorphism ψ of (Σ, o) such that

$$\rho_{\Sigma_m} = \rho_\Sigma \circ \psi \circ G$$

where ρ_{Σ_m} and ρ_Σ are the defining functions in normal form for Σ_m and Σ respectively.

Proof of the claim. Notice that $\hat{F}$ preserves the set $\{z = 0\} \cap (\Sigma_m, (0,0))$, which is the set of weakly pseudoconvex points of Σ_m. Hence, it follows that

$$\hat{F}(z,w) = (zk(z,w), f(w) + zg(z,w))$$

for some holomorphic functions k, f and g. We rewrite $g(z,w)$ in its MacLaurin expansion:

$$g(z,w) = \sum g_{r,s} z^r w^s.$$

We want to show that $g \equiv 0$ first. Let us restrict $\hat{F}$ to the set of points of the type $(z, |z|^{2m} + ic)$, for sufficiently small positive real values of c. Since $\hat{F}$ is a local automorphism of Σ_m preserving $o = (0,0)$, we have

$$\operatorname{Re}(zg(z, |z|^{2m} + ic)) = |z|^{2m} |k(z, |z|^{2m} + ic)|^{2m} - \operatorname{Re} f(|z|^{2m} + ic). \tag{5.2}$$

Now, we apply the operator $\left.\frac{\partial^{r+1}}{\partial z^{r+1}}\right|_{z=0}$ to both sides of (5.2). Then we deduce that there exists $\eta > 0$ such that

$$\sum_s (r+1)!(ic)^s g_{rs} \equiv 0$$

for all $0 < c < \eta$. This implies $g_{rs} = 0$ for all r, s. Hence, $g \equiv 0$.

Now we have

$$\hat{F}(z, w) = (zk(z, w), f(w)).$$

Rewriting the identity (5.2) with g identically zero, we get

$$\operatorname{Re} f(|z|^{2m}) = |z|^{2m} |k(z, |z|^{2m})|^{2m}, \quad \forall z. \tag{5.3}$$

Since $\hat{F}(0, 0) = (0, 0)$, it follows that $f(0) = 0$. Hence, when we rewrite f in its MacLaurin expansion, we get

$$f(w) = \sum_{j \geq 1} f_j w^j.$$

It follows by (5.3) that

$$0 \equiv \frac{1}{2} \sum_{j \geq 1} (f_j + \overline{f}_j) |z|^{2m(j-1)} - |k(z, |z|^{2m})|^{2m}. \tag{5.4}$$

From this identity, we may deduce in particular that

$$0 = \frac{\partial^h}{\partial z^h} |k(z, |z|^{2m})|^{2m} \bigg|_{z=0}$$

for any positive integer h. Notice also that

$$\frac{\partial^h}{\partial z^h} |k(z, |z|^{2m})|^{2m} = \frac{\partial^h}{\partial t^h} |k(t, |z|^{2m})|^{2m} \bigg|_{t=z} + O(|z|). \tag{5.5}$$

Since $\hat{F}$ is a local automorphism at $(0, 0)$, $k(0, 0) \neq 0$. Altogether, it follows by (5.5) that $\frac{\partial^h}{\partial t^h} k(t, 0) = 0$ for any positive integer h. This implies that $k(z, w) = k(w)$. In conclusion, the map $\hat{F}$ is of the form

$$\hat{F}(z, w) = (zk(w), f(w))$$

with $k(0) \neq 0$ and $\frac{\partial f}{\partial w}(0) \neq 0$. Thus,

$$G(z, w) = (z^m k(w)^m, f(w)).$$

Let us consider the local biholomorphism ψ in a neighborhood of the origin in $\mathbb{C}^2$ defined by

$$\psi^{-1}(z, w) = (zk^m(w), f(w)).$$

Also denote by

$$G^{\#}(z, w) = \psi \circ G(z, w).$$

Then

$$G^{\#}(z, w) = (z^m, w) = \varphi_m(z, w). \tag{5.6}$$

Notice that $G^{\#}(\Sigma_m \cap U_{\Sigma_m}) = \varphi_m(\Sigma_m \cap U_{\Sigma_m}) \subset \Sigma$ and that $\rho_\Sigma \circ G^{\#} = \rho_\Sigma \circ \varphi_m = \rho_{\Sigma_m}$. Moreover, it is easy to see that ψ is a local automorphism of Σ, because the images of G and $G^{\#} = \psi \circ G$ are both contained in Σ.

Consequently, we arrive at

$$\rho_{\Sigma_m} = \rho_\Sigma \circ \psi \circ G$$

and the claim is proved.

Now we want to complete the proof of Lemma 5.1.

By [**1**], ψ extends uniquely to a global automorphism, say Ψ, of Σ fixing o. By (5.6) we have

$$\Psi \circ G = \varphi_m.$$

This, together with (5.1), leads us to the identity

$$\varphi_m \circ F = \mu^{-1} \circ \Psi^{-1} \circ \mu \circ \varphi_m = \Phi \circ \varphi_m \tag{5.7}$$

where $\Phi = \mu^{-1} \circ \Psi^{-1} \circ \mu$ is a global automorphism of B^2. Moreover,

$$\Phi(\{(0, it) \,|\, t \in \mathbb{R}\}) \subset \{(0, it) \,|\, t \in \mathbb{R}\}.$$

Appealing to an explicit formula for the automorphism Φ, we are immediately able to deduce that each branch of $\varphi_m^{-1} \circ \Phi \circ \varphi_m$ defines a biholomorphism of whole Σ_m. So the proof of the lemma is complete by letting F be one of the branches. □

Now we present the main theorem of this paper.

Theorem 5.2. *Let D be a bounded simply connected pseudoconvex domain in $\mathbb{C}^2$ with a real analytic boundary. Assume that $f : D \to B^2$ is a proper holomorphic mapping with generic degree m such that the analytic variety Z_{df} admits an irreducible component V satisfying:*

(1) *$f^{-1}(f(V \cap \partial D)) = V \cap \partial D$;*
(2) *$V \cap \partial D$ is connected and contains no singular point of the variety V.*

Then, there exists a proper holomorphic mapping $g : D \to \Omega_{E_k}$ from D to the Thullen domain Ω_{E_k}, where k is the generic degree of f in a sufficiently small tubular neighborhood of $V \cap \partial D$, which extends holomorphically across ∂D such that

$$f = \beta \circ \varphi_k \circ g$$

for some holomorphic automorphism β of B^2.

Notice that this immediately implies:

Theorem 5.3. *Let D be a bounded simply connected pseudoconvex domain in $\mathbb{C}^2$ with a real analytic boundary. If D admits a generically k-to-1 proper holomorphic mapping $f : D \to B^2$ such that there exists an irreducible subvariety V of Z_{df} satisfying:*

(1) *f is a local k-to-1 branched covering at every point of $V \cap \partial D$;*

(2) *$V \cap \partial D$ is connected and contains no singular point of the variety Z_{df}; then D is biholomorphic to Ω_{E_k}.*

The rest of the section is devoted to the proof of Theorem 5.2. We first consider the concept of holomorphic correspondences.

A *holomorphic correspondence* from a domain Ω in $\mathbb{C}^m$ to Ω' in $\mathbb{C}^n$ is a complex analytic set S in $\Omega \times \Omega'$ such that $\pi_\Omega(S) = \Omega$, where $\pi_\Omega : \Omega \times \Omega' \to \Omega$ is the standard projection onto Ω. We denote the correspondence by

$$S : \Omega \multimap \Omega'.$$

For a holomorphic correspondence $S : \Omega \multimap \Omega'$, we denote by

$$S^{-1}(G) = \pi_\Omega(\pi_{\Omega'}^{-1}(G) \cap S), \text{ for } G \subset \Omega'.$$

Furthermore, we call S *proper* if $S^{-1}(K)$ is compact for every compact subset K of Ω'.

An important concept associated with the holomorphic correspondences concerns whether it is actually realized by a union of the graphs of holomorphic mappings. Precisely speaking, a proper holomorphic correspondence $G : \Omega \multimap \Omega'$ between bounded domains in $\mathbb{C}^n$, which extends holomorphically across $\partial\Omega$ is said to *split locally at* $p_0 \in \overline{\Omega}$, if there exist a neighborhood U_0 of p_0 in $\mathbb{C}^n$ and m holomorphic maps $f_j : U_0 \to f_j(U_0)$, $(j = 1, \ldots, m)$ such that

$$G|_{U_0} = \bigcup_{j=1}^{m} \{(z, f_j(z)) \mid z \in U_0\}.$$

A holomorphic correspondence is said to *split globally*, if it is the union of the graphs of globally defined holomorphic mappings. In our proof here, we use the following fact that was observed in Lemmata 3.6 and 3.7 by Bedford-Bell ([**5**]):

Let Ω be a bounded simply connected domain, and let Ω' be a bounded domain in $\mathbb{C}^n$. Let $G : \Omega \multimap \Omega'$ be a proper holomorphic correspondence that extends holomorphically across the boundary of Ω. Then G splits globally if and only if it splits locally at every boundary point of Ω.

Now, let $f : D \to B^2$ be a proper holomorphic mapping given in the hypothesis of Theorem 5.2.

By the extension theorem of Diederich-Fornæss [**9**], the mapping f extends holomorphically to an open neighborhood of the closure $\overline{D}$ of D onto a neighborhood of B^2 mapping ∂D onto ∂B^2. We consider the holomorphic correspondence

$$G \equiv \alpha^{-1} \circ \varphi_k^{-1} \circ \beta \circ u_p \circ f : \overline{D} \multimap \overline{\Omega}_{E_k}$$

for some appropriate $\alpha \in \operatorname{Aut}\Omega_{E_k}$ and $\beta, u_p \in \operatorname{Aut} B^2$, which are to be determined later. In order to verify the assertion of Theorem 5.2, it is enough to show that the proper holomorphic correspondence G above splits at every boundary point of D. Then, G will consist of k graphs of holomorphic mappings, each of which will provide the desired factorization map.

We will make an appropriate choice of $\alpha \in \operatorname{Aut}\Omega_{E_k}$ and $\beta, u_p \in \operatorname{Aut} B^2$ as well as check the local splitting of G at every point of $V \cap \partial D$. Then we will check the local splitting of G at the points not in $V \cap \partial D$.

Step 1. Choice of α, β and u_p.

Pick a point $p \in V \cap \partial D$ and choose a unitary map $u_p \in \operatorname{Aut} B^2$ so that $u_p \circ f(p) = (0,1) \in \partial B^2$. Then, using the notation introduced at the beginning of this section, consider the biholomorphic mapping $\mu : B^2 \to \Omega_\Sigma$. Now, choose a sufficiently small open neighborhood U_p of p in $\mathbb{C}^2$ so that the normalization by Barletta-Bedford [**4**] can be applied. Namely, the boundary ∂D in U_p is weakly spherical and there exists a local biholomorphic mapping ψ_p from U_p onto an open neighborhood of the origin such that $\psi_p(\partial D)$ is now represented by the normal form

$$\operatorname{Re} w = |z|^{2k} + \text{higher order terms}$$

where the higher order terms satisfy the conditions specified in (3.4.1) and (3.4.2). Composing these maps, we arrive at the holomorphic mapping

$$F \equiv \mu \circ u_p \circ f \circ \psi_p^{-1} : (\psi_p(\partial D), o) \to (\Sigma, o)$$

from a normalized weakly spherical pointed CR surface to the normalized Siegel pointed CR surface.

By Theorem 4.3, we deduce that $\psi_p(\partial D \cap U_p)$ is in fact a neighborhood of the origin in the hypersurface Σ_k and there exist $\hat{\beta} \in \operatorname{Aut}(\Sigma, o)$ and $\hat{\alpha} \in \operatorname{Aut}(\Sigma_k, o)$ such that

$$F(z,w) = (\hat{\beta}^{-1} \circ \varphi_k \circ \hat{\alpha})(z,w). \tag{5.8}$$

By [**1**] and Lemma 5.1 the mappings $\mu^{-1} \circ \hat{\beta} \circ \mu$ and $\mu_k^{-1} \circ \hat{\alpha} \circ \mu_k$ extend, respectively, to $\beta \in \operatorname{Aut}(B^2)$ and $\alpha \in \operatorname{Aut}(\Omega_{E_k})$. This implies, in particular, that $\hat{\alpha}$ and $\hat{\beta}$ also extend to global automorphisms of Ω_Σ and Ω_{Σ_k}, respectively.

Define a global holomorphic correspondence

$$G = \alpha^{-1} \circ \varphi_k^{-1} \circ \beta \circ u_p \circ f. \tag{5.9}$$

Note that in U_p, G has a local expression

$$G = \alpha^{-1} \circ \varphi_k^{-1} \circ \beta \circ \mu^{-1} \circ F \circ \psi_p.$$

Using (5.8) and the definition of β, we get

$$G = \alpha^{-1} \circ \varphi_k^{-1} \circ \mu^{-1} \circ \varphi_k \circ \hat{\alpha} \circ \psi_p.$$

It follows by (5.1) that

$$G = \alpha^{-1} \circ \varphi_k^{-1} \circ \varphi_k \circ \alpha \circ \mu_k^{-1} \circ \psi_p. \tag{5.10}$$

Since the correspondence $\varphi_k^{-1} \circ \varphi_k$ splits, and since α, μ_k and ψ_p are local biholomorphisms, it follows by (5.10) that the correspondence G splits in a small neighborhood of p where ψ_p is defined. However, we have yet to see if G splits at every boundary point. We will see this in the next two and final steps.

Step 2. Splitting of G at points of $\partial D \cap V$.

Let $q \in \partial D \cap V$. Assume for a moment that the neighborhood U_q of q satisfies the following additional condition

$$U_q \cap U_p \neq \emptyset.$$

In $U_p \cap U_q$, (5.10) is valid and we have

$$G = \alpha^{-1} \circ \varphi_k^{-1} \circ \varphi_k \circ \alpha \circ \hat{\psi}_p \circ \hat{\psi}_q^{-1} \circ \hat{\psi}_q \tag{5.11}$$

where $\hat{\psi}_p = \mu_k^{-1} \circ \psi_p$ and $\hat{\psi}_q = \mu_k^{-1} \circ \psi_q$, and where ψ_q is the Barletta-Bedford normalization map for $\partial D \cap U_q$. Since $\hat{\psi}_p \circ \hat{\psi}_q^{-1}$ extends to an automorphism of Ω_{E_k} by Lemma 5.1, the expression (5.11) is well-defined on U_q. This shows that G splits at every point of $U_p \cup U_q$.

For an arbitrary point of $\partial D \cap V$, an inductive repetition of this argument yields the desired conclusion for the current step, because $\partial D \cap V$ is a compact connected set.

Step 3. Splitting of G at points of $\partial D \setminus V$.

Let $q \in \partial D \setminus V$. Recall

$$G = \alpha^{-1} \circ \varphi_k^{-1} \circ \tilde{f}$$

where α is a biholomorphism of the Thullen domain Ω_{E_k} and $\tilde{f} = \beta \circ u_p \circ f$ is a proper holomorphic map branching at every point of V. Thus, for $q \in \partial D \setminus V$, we have that $\tilde{f}(q)$ is a strictly pseudoconvex point that does not belong to $\tilde{f}(V \cap \partial D) \subset \{z = 0\}$, by (1) in the hypothesis of the theorem. Thus, $G(q)$ consists only of strictly pseudoconvex points. The splitting of G is then guaranteed by the proof of Theorem 3 of [**5**]. □

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