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In 1981, Nomizu introduced isoparametric hypersurfaces in Lorentzian space forms and studied the Cartan identities. Later Hahn, 1984, generalized Nomizu's work to the pseudo-Riemannian space forms and presented many examples. In general, the shape operator of a hypersurface in a pseudo-Riemannian space form may be not diagonalizable. This makes the isoparametric theory in pseudo-Riemannian space form different from that in Riemannian space forms. In 1985, Megid classified Lorentzian isoparametric hypersurfaces in R_1^{n+1}. He showed that there are three types of Lorentzian isoparametric hypersurfaces in R_1^{n+1}. Type I are exactly cylinders and umblic hypersurfaces while the other two types of hypersurfaces have properties close to cylinders and umblic hypersurfaces. Megid called them generalized cylinders and umblic hypersurfaces. In this paper, the local classification of Lorentzian isoparametric hypersurfaces in H_1^{n+1} is obtained and the properties of them are discussed.

Introduction.

A hypersurface in H_1^{n+1} is called isoparametric if the minimal polynomial of the shape operator is constant. This allows complex or non-simple principal curvatures (eigenvalues of the shape operator). In this paper, we classify Lorentzian isoparametric hypersurfaces in an anti-de Sitte sphere H_1^{n+1}. More precisely, we show that there are four types of such hypersurfaces. Type I hypersurfaces are determined by two orthogonal subspaces of R_2^{n+2} and the principal curvatures; type II and type III hypersurfaces are determined by two 1-parameter orthogonal subspaces of R_2^{n+2} and the principal curvatures; and the type IV hypersurfaces are homogeneous.

The classification theorem we obtain here plays an essential role in the study of isoparametric hypersurfaces in complex hyperbolic spaces CH^n [**9**]. A connected hypersurface in CH^n is called isoparametric if all parallel hypersurfaces M_t for t sufficiently close to zero have constant mean curvatures. In [**9**], we get the complete classification of isoparametric hypersurfaces in

CH^n. In fact, we prove that all isoparametric hypersurfaces are homogenous.

The paper is organized as follows. In Section 1, we recall basic definitions, notations and the structural equations of a Lorentzian hypersurface in H_1^{n+1}. We use a result of Megid [4] to conclude that there are four types of local isoparametric hypersurfaces in H_1^{n+1}. In Section 2, we study the Cartan identities and show that a Lorentzian isoparametric hypersurface has at most a pair of conjugate complex and two real principal curvatures. In Sections 3, 4 and 5, we classify hypersurfaces of type I, II, III and IV, respectively. Combining these results, we get the classification.

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1. Preliminaries.

In this section we recall the basic definitions and the structure equations of a Lorentzian hypersurface in H_1^{n+1}. Then we give the definition of an isoparametric hypersurface and show the forms of the shape operator.

Let R_2^{n+2} be an $n+2$-dimensional real vector space with a bilinear form of signature (2,n) given by

$$\langle x, x\rangle = -\sum_{i=1}^{2} x_i^2 + \sum_{i=3}^{n+2} x_i^2,$$

H_1^{n+1} be the hypersurface

$$\{x \in R_2^{n+2} \mid \langle x, x\rangle = -1\},$$

which is the anti-de Sitte sphere with constant sectional curvature -1. H_1^{n+1} is a non-simply connected Lorentzian space form.

Let V be a vector space with a Lorentzian metric $\langle\ ,\ \rangle$. An orthonomal basis $\{E_1, \dots, E_n\}$ is one satisfying $\langle E_1, E_1\rangle = -1, \langle E_i, E_j\rangle = \delta_{ij}, \langle E_1, E_j\rangle = 0$ for $2 \le i, j \le n$. A pseudo-orthonormal basis is a basis $\{X, Y, E_1, \dots, E_{n-2}\}$ such that $\langle X, X\rangle = 0 = \langle Y, Y\rangle = \langle X, E_i\rangle = \langle Y, E_i\rangle$, $\langle X, Y\rangle = -1$ and $\langle E_i, E_j\rangle = \delta_{ij}$ for $1 \le i, j \le n-2$.

Generally, a hypersurface M in H_1^{n+1} is called a Lorentzian hypersurface if the induced metric has signature $(1, n-1)$. Next, we recall the structure equations of a Lorentzian hypersurface M.

Let X be the position vector of M, i.e., X is the inclusion map from M to R_2^{n+2}, $e_1, \dots, e_{n+1}$ a local frame on $H_1^{n+1} \subset R_2^{n+2}$ such that $e_1, \dots, e_n$ are tangent to M, e_{n+1} normal to M, and $\omega^1, \dots, \omega^{n+1}$ the dual 1-forms.

We can write

$$dX = \sum_{A=1}^{n+1} \omega^A e_A, \tag{1}$$

$$de_A = \sum_{B=1}^{n+1} \omega_A^B e_B + \omega_A X, \tag{2}$$

where $A = 1, \ldots, n+1$, and ω_A^B and ω_A satisfy the first structural equation of H_1^{n+1}:

$$d\omega^A + \sum_{B=1}^{n+1} \omega_B^A \wedge \omega^B = 0, \tag{3}$$

$$dg_{AB} = \sum_{C=1}^{n+1} g_{CB}\omega_A^C + g_{AC}\omega_B^C,$$

$$\omega_A = \sum_{B=1}^{n+1} g_{AB}\omega^B.$$

Especially for an orthonomal frame, $\omega_1^1 = 0, \omega_1^i = \omega_i^1, \omega_i^j + \omega_j^i = 0$, $(2 \leq i, j \leq n+1)$, and for a pseudo-orthonomal frame, $\omega_1^1 + \omega_2^2 = 0, \omega_1^2 = \omega_2^1 = 0, \omega_1^i = \omega_i^2, \omega_2^i = \omega_i^1, \omega_i^j + \omega_j^i = 0$, $(3 \leq i, j \leq n+1)$.

The second structural equation of H_1^{n+1} is:

$$d\omega_A^B - \sum_{C=1}^{n+1} \omega_A^C \wedge \omega_C^B - \omega_A \wedge \omega^B = 0. \tag{4}$$

Restricting these forms to M, we have

$$\omega^{n+1} = 0, \quad \omega_{n+1}^{n+1} = 0. \tag{5}$$

Write

$$\omega_{n+1}^i = \sum_{j=1}^{n} h_j^i \omega^j. \tag{6}$$

Exterior differenting (5), we get

$$\sum_{k=1}^{n} g_{ik} h_j^k = \sum_{k=1}^{n} g_{jk} h_i^k. \tag{7}$$

The shape operator is a linear transformation for any $x \in M$ defined by

$$A : T_xM \to T_xM : e_i \mapsto \sum_{j=1}^{n} h_i^j e_j. \tag{8}$$

A is a symmetric linear transformation on T_xM with Lorenzian product, i.e., for any $X, Y \in T_xM$,

$$\langle AX, Y\rangle = \langle X, AY\rangle = II(X, Y).$$

Here $II(X, Y)$ is the second fundamental form of M. The eigenvalues of A are called principal curvatures of M. Corresponding to every principal curvature λ, we have algebraic multiplicity and geometric multiplicity. Algebraic multiplicity ν is the exponent of $(x-\lambda)$ in the characteristic polynomial and geometric multiplicity μ is the dimension of the eigenspace

$$T_\lambda = \{X \in T_xM \mid AX = \lambda X\}.$$

A principal curvature λ is called diagonalizable if $\nu = \mu$.

The structural equations of M are

$$d\omega^i + \sum_{j=1}^{n} \omega_j^i \wedge \omega^j = 0, \tag{9}$$

$$dg_{ij} = \sum_{k=1}^{n} (g_{ik}\omega_j^k + g_{kj}\omega_i^k),$$

$$\omega_i = \sum_{j=1}^{n} g_{ij}\omega^j,$$

$$g_{ij}\omega_{n+1}^j + \omega_i^{n+1} = 0, \tag{10}$$

$$d\omega_i^j - \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^j = \omega_i^{n+1} \wedge \omega_{n+1}^j + \omega_i \wedge \omega^j, \tag{11}$$

$$d\omega_{n+1}^i = \sum_{j=1}^{n} \omega_{n+1}^j \wedge \omega_j^i. \tag{12}$$

Among these equations, (11) and (12) are called Gauss equation and Codazzi equation of M, respectively.

Define

$$\sum_{k=1}^{n} h_{j,k}^i \omega^k = dh_j^i - \sum_{k=1}^{n} h_k^i \omega_j^k + \sum_{k=1}^{n} h_j^k \omega_k^i. \tag{13}$$

Then Codazzi equation becomes

$$h_{j,k}^i = h_{k,j}^i. \tag{14}$$

A hypersurface is called *isoparametric* if the minimal polynomial of shape operator is constant. In this paper we only consider Lorentzian isoparametric hypersurfaces. In [4], Megid showed that such a hypersurface has constant principal curvatures and the shape operator A can be put into exactly one of the canonical forms I, II, III or IV.

I.

$$A = \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{pmatrix}.$$

II.

$$A = \begin{pmatrix} a_0 & 0 & & & \\ 1 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{pmatrix}.$$

III.

$$A = \begin{pmatrix} a_0 & 0 & 0 & & & \\ 0 & a_0 & 1 & & & \\ -1 & 0 & a_0 & & & \\ & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_{n-3} \end{pmatrix}.$$

IV.

$$A = \begin{pmatrix} a_0 & b_0 & & & \\ -b_0 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{pmatrix}.$$

Here b_0 is assumed to be non-zero. In cases I and IV A is represented with respect to an orthonomal basis while in cases II and III the basis is a pseudo-orthonomal basis. In cases I, II and III the eigenvalues are real, while $a_0 \pm ib_0$ are eigenvalues in case IV. Throughout this paper, a Lorentzian isoparametric hypersurface in H_1^{n+1} is called a type I, II, III or IV hypersurface according to the form of the shape operator A.

2. Cartan identities.

In this section, we use Hahn's result on Cartan identities to study the possible number of principal curvatures of Lorentzian isoparametric hypersurfaces in H_1^{n+1}, and prove the following theorem.

Theorem 2.1. *A type* I, II *or* III *Lorentzian isoparametric hypersurface has at most two real principal curvatures, and a type* IV *Lorentzian isoparametric hypersurface has a pair of conjugate complex principal curvatures and at most two real principal curvatures.*

We need a couple of Lemmas to prove the theorem. The first one is proved by Hahn:

Lemma 2.2 ([2]). *Let M be a Lorentzian isoparametric hypersurface in H_1^{n+1}, and $\lambda_1, \dots, \lambda_p$ all distinct principal curvatures of M with algebraic multiplicities $\nu_1, \dots, \nu_p$. If λ_i is a (real) diagonalizable principal curvature, then we have Cartan identity*

$$\sum_{j=1, j\neq i}^{p} \nu_j \frac{-1+\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0.$$

Lemma 2.3. *Let M be a type* I, II *or* III *hypersurfaces. Then $p \leq 2$. Moreover, if $p = 2$, then $\lambda_1\lambda_2 = 1$.*

Proof. If M is type I, then all principal curvatures of M are real and diagonalizable. By Lemma 2.2, we have

$$\sum_{j=1, j\neq i}^{p} \nu_j \frac{-1+\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0 \tag{15}$$

for any i in $\{1, \dots, p\}$. Without loss of generalities, we may assume $\lambda_1 < \lambda_2 < \dots < \lambda_p$, and $\lambda_p \geq 0$. Choose the largest nonnegative λ_i such that $\lambda_i\lambda_{i-1} \leq 1$. Then

$$\frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} \leq 0$$

for any $j \neq 0$. Hence $\lambda_i\lambda_j = 1$ if $i \neq j$. Therefore $p \leq 2$.

If M is type II or type III, then only one principal curvature of M is not diagonalizable. Without loss of generalities, we may assume that λ_1 is not diagonalizable. By Lemma 2.2, we have

$$\sum_{j=1, j\neq i}^{p} \nu_j = \frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} = 0 \tag{16}$$

for any i in $\{2, \dots, p\}$.

Note that

$$\begin{aligned} &\sum_{i,j=1, i\neq j}^{p} \nu_i\nu_j \frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} \\ &= \sum_{i<j}^{p} \nu_i\nu_j(\lambda_i\lambda_j - 1)\left(\frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_j - \lambda_i}\right) \\ &= 0. \end{aligned} \tag{17}$$

Combining (16) and (17), we have

$$\sum_{j=1, j\neq i}^{p} \nu_j \frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} = 0 \tag{18}$$

for any i in $\{1, \dots, p\}$, which is exactly the equation (15). Hence we know that $p \leq 2$. □

Lemma 2.4. *Let M be a type* IV *hypersurface, and $\lambda_1, \dots, \lambda_p$ all distinct principal curvatures of M. Then $p \leq 4$.*

Proof. If M is a type IV, then M has a pair of conjugate complex principal curvatures with algebraic multiplicities 1. We may assume $\lambda_1 = a_0 + ib_0$, $\lambda_2 = a_0 - ib_0$, $b_0 \neq 0$, $\nu_1 = \nu_2 = 1$, and $\lambda_3 < \lambda_4 < \cdots < \lambda_p$. By Lemma 2.2, $\lambda_3, \dots, \lambda_p$ satisfy

$$\frac{2a_0(1+\lambda_i^2) - 2\lambda_i(a_0^2+b_0^2+1)}{\lambda_i^2 - 2a_0\lambda_i + (a_0^2+b_0^2)} + \sum_{j=3, j\neq i} \nu_j = \frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} = 0 \tag{19}$$

for any i in $\{3, \dots, p\}$.

First we claim that if $a_0 \leq 0$, then $\lambda_i \leq 0$ for any i in $\{3, \dots, p\}$.

Suppose the claim is false. Then $\lambda_p > 0$. We choose the largest positive λ_i such that $\lambda_i\lambda_{i-1} \leq 1$. Then

$$\frac{2a_0(1+\lambda_i^2) - 2\lambda_i(a_0^2+b_0^2+1)}{\lambda_i^2 - 2a_0\lambda_i + (a_0^2+b_0^2)} \leq 0, \tag{20}$$

$$\frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} \leq 0 \tag{21}$$

for any j in $\{3, \dots, p\} - \{i\}$. From (19), (20) and (21), it follows that

$$2a_0(1+\lambda_i^2) - 2\lambda_i(a_0^2+b_0^2+1) = 0. \tag{22}$$

This is a contradiction since $a_0 \leq 0$ and $\lambda_i > 0$. Therefore $\lambda_i \leq 0$ for any i in $\{3, \dots, p\}$. Similarly, we can prove that $\lambda_i \geq 0$ for any i in $\{3, \dots, p\}$ if $a_0 \geq 0$.

Note that

$$\begin{aligned} &\frac{2a_0(1+\lambda_i^2) - 2\lambda_i(a_0^2+b_0^2+1)}{\lambda_i^2 - 2a_0\lambda_i + (a_0^2+b_0^2)} \\ &= \frac{2a_0(t)(1+\lambda_i^2(t)) - 2\lambda_i(t)(a_0^2(t)+b_0^2(t)+1)}{\lambda_i^2(t) - 2a_0(t)\lambda_i(t) + (a_0^2(t)+b_0^2(t))}, \\ &\frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} = \frac{\lambda_i(t)\lambda_j(t) - 1}{\lambda_i(t) - \lambda_j(t)}. \end{aligned} \tag{23}$$

Here

$$\begin{aligned} a_0(t) &= \frac{(a_0^2+b_0^2+1)\sinh t\cosh t + a_0(\cosh^2 t + \sinh^2 t)}{\cosh^2 t + 2a_0\cosh t\sinh t + (a_0^2+b_0^2)\sinh^2 t}, \\ b_0(t) &= \frac{b_0}{\cosh^2 t + 2a_0\cosh t\sinh t + (a_0^2+b_0^2)\sinh^2 t}, \\ \lambda_i(t) &= \frac{\sinh t + \lambda_i\cosh t}{\cosh t + \lambda_i\sinh t} \end{aligned} \tag{24}$$

and t is any real number satisfying $\cosh t + \lambda_i\sinh t \neq 0$ for any i in $\{3, \dots, p\}$.

Let $\lambda_1(t) = a_0(t) + ib_0(t)$, $\lambda_2(t) = a_0(t) - ib_0(t)$. Then $\lambda_1(t), \lambda_2(t), \dots, \lambda_p(t)$ are p distinct numbers satisfying the equation system (19) for any t satisfying $\cosh t + \lambda_i \sinh t \neq 0$. Hence if $a_0(t) \leq 0$ then $\lambda_i(t) \leq 0$ for any i in $\{3, \dots, p\}$ and if $a_0(t) \geq 0$ then $\lambda_i(t) \geq 0$ for any i in $\{3, \dots, p\}$.

If $a_0 = 0$, then $\lambda_i = 0$ for any i in $\{3, \dots, p\}$ which implies $p = 3$. Note that $b_0 \neq 0$. If $a_0 \neq 0$, then we can choose $t_0 \in R - \{0\}$ such that

$$a_0 + \frac{1}{a_0} + \frac{b_0^2}{a_0} = -\tanh\, t_0 - \frac{1}{\tanh\, t_0},$$

which implies $a_0(t_0) = 0$. We claim that $\lim_{t\to t_0} \lambda_i(t) = 0$ or ∞ for all $3 \leq i \leq p$. Suppose the claim is false. Then $\lim_{t\to t_0} \lambda_k(t) > 0$ or $\lim_{t\to t_0} \lambda_k(t) < 0$ for some k in $\{3, \dots, p\}$. Without loss of generalities, we may assume that $\lim_{t\to t_0} \lambda_k(t) > 0$. Hence we can choose a real t_1 satisfying that $cosht_1 + \lambda_i \sinh t_1 \neq 0$ for any i in $\{3, \dots, p\}$ such that $a_0(t_1) < 0$ and $\lambda_k(t_1) > 0$. This is a contradiction. So $\lambda_i = -\tanh\, t_0$ or $-\coth\, t_0$ for any i in $\{3, \dots, p\}$. Hence $p \leq 4$ and λ_3, λ_4 satisfy the following equation

$$2a_0(1 + \lambda_i^2) - 2\lambda_i(a_0^2 + b_0^2 + 1) = 0.$$

□

As a consequence of Lemma 2.3 and 2.4, we obtain Theorem 2.1.

3. Type I hypersurfaces.

The main result of this section is the following result.

Theorem 3.1. *Let M be a Lorentzian isoparametric hypersurfaces in H_1^{n+1}. Then M is type* I *if and only if M is congruent to an open part of one of the following hypersurfaces:*

i) $H_1^m\left(\sqrt{1-\lambda^2}\right) \times S^{n-m}\left(\sqrt{\frac{1-\lambda^2}{\lambda^2}}\right)$, *where* $-1 < \lambda < 1$;

ii) $S_1^m\left(\sqrt{\lambda^2-1}\right) \times H^{n-m}\left(\sqrt{\frac{\lambda^2-1}{\lambda^2}}\right)$, *where λ is real and* $\lambda^2 > 1$;

iii) $\{x \in H_1^{n+1} \mid \langle x, C\rangle = \lambda\}$, *where λ is real, and C is a constant vector with* $\langle C, C\rangle = 1 - \lambda^2$.

Proof. We shall arrange the index as follows: $1 \leq i \leq m$, $m+1 \leq \alpha \leq n$.

Let M be a type I hypersurface. By Lemma 2.3, we can choose a local orthonormal frame $e_1, \dots, e_m, e_{m+1}, \dots, e_n, e_{n+1}$ such that e_{n+1} is normal to M, $\omega_{n+1}^i = \lambda\omega^i$ for $1 \leq i \leq m$ and $\omega_{n+1}^\alpha = \frac{1}{\lambda}\omega^\alpha$ for $m+1 \leq \alpha \leq n$. Note that $m = n$ if $\lambda = 0$ or ± 1.

Consider the Codazzi equation

$$d\omega_{n+1}^i = \sum_{A=1}^n \omega_{n+1}^A \wedge \omega_A^i = \lambda \sum_{j=1}^m \omega^j \wedge \omega_j^i + \frac{1}{\lambda} \sum_{\alpha=m+1}^n \omega^\alpha \wedge \omega_\alpha^i. \tag{25}$$

On the other hand

$$d\omega_{n+1}^i = \lambda(d\omega^i) = \lambda \sum_{j=1}^{m} \omega^j \wedge \omega_j^i + \lambda \sum_{\alpha=m+1}^{n} \omega^\alpha \wedge \omega_\alpha^i. \tag{26}$$

Hence

$$\sum_{\alpha=m+1}^{n} \omega^\alpha \wedge \omega_\alpha^i = 0. \tag{27}$$

By Cartan's Lemma, ω_α^i is a linear combination of $\omega^{m+1}, \dots, \omega^n$. Similarly we can prove that ω_i^α is a linear combination of $\omega^1, \dots, \omega^m$.

From (3),we know that $\omega_\alpha^i = \omega_i^\alpha$ (if $i = 1$) or $-\omega_i^\alpha$ (if $i > 1$) since $e_1, \dots, e_n$ is an orthonormal frame. Therefore

$$\omega_\alpha^i = 0. \tag{28}$$

From (1), (2), (3) and (28), we get

$$d(X - \lambda e_{n+1}) = (1 - \lambda^2) \sum_{i=1}^{m} \omega^i e_i, \tag{29}$$

$$de_i = \left(\sum_{j=1, j \neq i}^{m} \omega_i^j e_j \right) + \omega_i (X - \lambda e_{n+1}),$$

which imply that

$$d(e_1 \wedge e_2 \wedge \cdots e_m \wedge (X - \lambda e_{n+1})) = 0. \tag{30}$$

Similarly, we can prove that

$$d(e_{m+1} \wedge e_{m+2} \wedge \cdots \wedge e_n \wedge (\lambda X - e_{n+1})) = 0. \tag{31}$$

Let $W_1(x)$ be the linear span of $\{e_1(x), e_2(x), \dots, e_m(x), X - \lambda e_{n+1}(x)\}$, and $W_2(x)$ the linear span of $\{e_{m+1}(x), \dots, e_n(x), \lambda X - e_{n+1}(x)\}$. From (30) and (31) we know that $W_1(x)$ and $W_2(x)$ are fixed subspaces in R_2^{2n+2}. Denote them by W_1 and W_2, respectively.

If $\lambda \neq \pm 1$, then

$$R_2^{2n+2} = W_1 + W_2$$

is a direct sum of subspaces. Write

$$X = X_1 + X_2,$$

where X is the position vector field of M, $X_1 \in W_1$ and $X_2 \in W_2$. Since $X - \lambda e_{n+1} \in W_1$, $\lambda X - e_{n+1} \in W_2$, we know that

$$\lambda e_{n+1} = \lambda^2 X_1 + X_2.$$

Since $\langle X, X \rangle = -1$, $\langle e_{n+1}, e_{n+1} \rangle = 1$ and $\langle X_1, X_2 \rangle = 0$, we have

$$\langle X_1, X_1 \rangle = \frac{1}{\lambda^2 - 1}, \quad \langle X_2, X_2 \rangle = \frac{\lambda^2}{1 - \lambda^2}. \tag{32}$$

If $\lambda = \pm 1$, then from (29) we have $d(e_{n+1} - \lambda X) = 0$. Hence $C = e_{n+1} - \lambda X$ is a fixed vector in R_2^{n+2} and $\langle X, C\rangle = \lambda$.

Therefore M can be represented as in Theorem 3.1. □

4. Type II and type III hypersurfaces.

In this section, we classify the type II and type III hypersurfaces. We state the classification as a couple of theorems.

4.1. Type II hypersurfaces.

In this subsection, we arrange the index as follows: $3 \le i, j \le m$, $m+1 \le \alpha, \beta \le n$. By a direct calculation, we have:

Theorem 4.1. *Let $\gamma(s)$ be a null curve in $H_1^{n+1} \subset R_2^{n+2}$, and $\{\dot\gamma(s), Y(s), U_3(s), \dots, U_m(s), V_{m+1}(s), \dots, V_n(s), \xi(s)\}$ a pseudo-orthonormal basis of $T_{\gamma(s)}H_1^{n+1}$ such that*

$$\dot V_\alpha(s) \in \operatorname{span}\{Y(s), V_{m+1}(s), \dots, V_n(s)\},$$

$$\dot\xi(s) = \lambda\dot\gamma(s) + B(s)Y(s)$$

for some nonzero $B(s)$. If M is one of the following parametrized hypersurfaces in $H_1^{n+1} \subset R_2^{n+2}$:

(1) $\lambda^2 \ne 0$ *or* 1,

$$\begin{aligned} &f(s, y, a_3, \dots, a_m, b_{m+1}, \dots, b_n)\\ &= \epsilon_1(\lambda)\sqrt{\frac{1}{(\lambda^2-1)^2} - \sum_{i=3}^m \frac{a_i^2}{\lambda^2-1}}(\gamma(s) - \lambda\xi(s))\\ &\quad + \epsilon_2(\lambda)\sqrt{\frac{\lambda^2}{(1-\lambda^2)^2} - \sum_{\alpha=m+1}^n \frac{b_\alpha^2}{1-\lambda^2}}(\xi(s) - \lambda\gamma(s))\\ &\quad + yY(s) + \sum_{i=3}^m a_iU_i(s) + \sum_{\alpha=m+1}^n b_{\alpha=}V_\alpha(s), \end{aligned}$$

where

$$\epsilon_1(\lambda) = \begin{cases} -1, & \text{if } \lambda^2 > 1\\ 1, & \text{if } \lambda^2 < 1, \end{cases}$$

$$\epsilon_2(\lambda) = \begin{cases} -1, & \text{if } \lambda(\lambda^2-1) > 0\\ 1, & \text{if } \lambda(\lambda^2-1) < 0. \end{cases}$$

(2) $\lambda = 0$,

$$f(s,y,a_3,\dots,a_n) = \sqrt{1+\sum_{i=3}^{n} a_i^2\gamma(s)+yY(s)+\sum_{i=3}^{n} a_iU_i(s)}.$$

(3) $\lambda^2=1$,

$$f(s,y,a_3,\dots,a_n) = \left(1+\frac{1}{2}\sum_{i=3}^{n} a_i^2\right)\gamma(s)-\lambda\left(\frac{1}{2}\sum_{i=3}^{n} a_i^2\right)\xi(s)+yY(s)+\sum_{i=3}^{n} a_iU_i(s),$$

then M is type II.

Conversely, we have:

Theorem 4.2. *Let M be a type* II *hypersurface in H_1^{n+1}. Then for any $p\in M$, there is a neighborhood U_p of p in M such that U_p is exactly one of the parametrized hypersurfaces in Theorem 4.1.*

Before proceeding to give the proof, we separate off the following lemma.

Lemma 4.3. *Let M be a type* II *hypersurface, $e_1,e_2,\dots,e_{n+1}$ a local pseudo-orthonormal frame such that e_{n+1} is normal to M, $\omega^1_{n+1}=\lambda\omega^1,\omega^2_{n+2}=\lambda\omega^2+\omega^1,\omega^i_{n+1}=\lambda\omega^i$, $\omega^\alpha_{n+1}=\frac{1}{\lambda}\omega^\alpha$, and T_λ, $T_{\frac{1}{\lambda}}$ the distributions defined as follows:*

$$T_\lambda=\ker(A-\lambda)=\operatorname{span}\{e_2,e_3,\dots,e_m\},$$

and

$$T_{\frac{1}{\lambda}}=\ker(A-\frac{1}{\lambda})=\operatorname{span}\{e_{m+1},\dots,e_n\}.$$

Then all distributions T_λ, $T_{\frac{1}{\lambda}}$ and $T_\lambda+T_{\frac{1}{\lambda}}$ are integrable, and $\omega^1_\alpha=\omega^\alpha_2=\omega^i_\alpha=0$, $\omega^i_2\wedge\omega^1=0$ and $\omega^2_\alpha\wedge\omega^1=0$.

Proof. Let $\widetilde{\omega}^1_{n+1}=\omega^1_{n+1}-\lambda\omega^1$, $\widetilde{\omega}^2_{n+1}=\omega^2_{n+1}-\lambda\omega^2$, $\widetilde{\omega}^i_{n+1}=\omega^i_{n+1}-\lambda\omega^i$, and $\widetilde{\omega}^\alpha_{n+1}=\omega^\alpha_{n+1}-\lambda\omega^\alpha$. Then

$$\widetilde{\omega}^1_{n+1}=0,\quad \widetilde{\omega}^2_{n+1}=\omega^1,\quad \widetilde{\omega}^i_{n+1}=0,\quad \widetilde{\omega}^\alpha_{n+1}=\left(\frac{1}{\lambda}-\lambda\right)\omega^\alpha. \tag{33}$$

By (3), we have

$$d\widetilde{\omega}^A_{n+1}=\sum_{B=1}^{n}\widetilde{\omega}^B_{n+1}\wedge\omega^A_B, \tag{34}$$

where $A = 1, 2, \dots, n$. (33) and (34) are exactly the Codazzi equation, which Megid discussed in [4]. Following his results, we have Lemma 4.3. $\square$

We are now in a position to give a:

Proof of Theorem 4.2. Let M be a type II hypersurface, x_0 a point of M. By Lemma 2.3, there is a local pseudo-orthonormal frame $e_1, e_2, \dots, e_n$, e_{n+1} defined in a neighborhhod of x_0 such that e_{n+1} is normal to M, and $\omega^1_{n+1} = \lambda\omega^1, \omega^2_{n+2} = \lambda\omega^2 + \omega^1, \omega^i_{n+1} = \lambda\omega^i$, $\omega^\alpha_{n+1} = \frac{1}{\lambda}\omega^\alpha$. Let $\gamma(s)$ be the integral curve of e_1 through x_0, and $N(s)$ the integral manifold of $T_\lambda + T_{\frac{1}{\lambda}}$ through $\gamma(s)$. Fixing s and restricting the forms to $N(s)$, we have $\omega^1 = 0$.

From Lemma 4.3 and (3), we have

$$de_2 = \omega^2_2 e_2. \tag{35}$$

Denote $Y(s) = e_2(\gamma(s))$, $U_i(s) = e_i(\gamma(s))$ and $V_\alpha(s) = e_\alpha(\gamma(s))$. Then $\forall x \in N(s)$, $e_2(x) = \lambda(x)Y(s)$ for some function λ. So the integral curve of e_2 is a straight line in R^{n+2}_2.

Define $W_1(s) = \text{span}\,\{Y(s)\}$. Then $W_1(s) = \text{span}\,\{= e_2(x) \mid \forall x \in N(s)\}$. From (1) and (2), we have

$$dX = \omega^2 e_2 + \sum_{i=3}^{m} \omega^i e_i + \sum_{\alpha=m+1}^{n} \omega^\alpha e_\alpha, \tag{36}$$

$$de_{n+1} = \lambda\omega^2 e_2 + \lambda\sum_{i=3}^{m} \omega^i e_i + \frac{1}{\lambda}\sum_{\alpha=m+1}^{n} \omega^\alpha e_\alpha. \tag{37}$$

Computing (37)$-\lambda$(36), we get

$$d(e_{n+1} - \lambda X) = \left(\frac{1}{\lambda} - \lambda\right)\sum_{\alpha=m+1}^{n} \omega^\alpha e_\alpha. \tag{38}$$

From (3) and Lemma 4.3, we have

$$de_\alpha = \sum_{\beta=m+1,\beta\neq\alpha}^{n} \omega^\beta_\alpha e_\beta - \frac{1}{\lambda}\omega^\alpha e_{n+1} + \omega^\alpha X. \tag{39}$$

It follows from (38) and (39) that

$$d(e_{m+1} \wedge \cdots \wedge e_n \wedge (e_n - \lambda X)) = 0. \tag{40}$$

Denote $W_2(s) = \mathit{span}\{e_{m+1}(\gamma(s)), \dots, e_n(\gamma(s)), e_{n+1}(\gamma(s)) - \lambda\gamma(s)\}$. Then $W_2(s) = \{e_{m+1}(x), \dots, e_n(x),\ e_{n+1}(x) - \lambda x \mid \forall x \in N(s)\}$. Define $W_3(s) = \{e_2(x), e_3(x), \dots, e_m(x), x - \lambda e_{n+1}(x)\}$. Note that $\langle Z(x), e_2(x)\rangle = 0$ for any $Z(x)$ in $T_\lambda + T_{\frac{1}{\lambda}}$. Hence $\forall x \in N(s)$, we have

$$\langle x, Y(s)\rangle = 0. \tag{41}$$

If $m = n$, then it follows from (38) that

$$e_{n+1}(x) - \lambda x = e_{n+1}(\gamma(s)) - \lambda\gamma(s).$$

So $\forall x \in N(s)$, we get

$$\langle x, e_{n+1}(\gamma(s)) - \lambda\gamma(s)\rangle = \lambda.$$

This includes the cases $\lambda = 0$ or ± 1.

Now suppose $\lambda \neq 0, \pm 1$. By (38) and (39) and $e_{n+1}(x) - \lambda x \in W_2(s)$, $x - \lambda e_{n+1}(x) \in W_3(s)$, we can write

$$\begin{aligned} x &= X_1 + X_2, \\ e_{n+1} &= \frac{1}{\lambda}X_1 + \lambda X_2, \end{aligned} \tag{42}$$

where $X_1 \in W_2(s)$ and $X_2 \in W_3(s)$. Since $\langle X_1, X_2\rangle = 0$, we get

$$\langle X_1, X_1\rangle = \frac{\lambda^2}{1-\lambda^2}, \quad \langle X_2, X_2\rangle = \frac{1}{\lambda^2 - 1}. \tag{43}$$

Hence M can be locally represented as a parametrized hypersurface in Theorem 4.1. $\square$

4.2. Type III hypersurfaces.

In this subsection, we arrange the index as follows: $4 \leq i, j \leq m$, $m+1 \leq \alpha, \beta \leq n$. By a direct calculation, we have:

Theorem 4.4. *Let $\gamma(s)$ be a null curve in $H_1^{n+1} \subset R_2^{n+2}$, and $\{\dot\gamma(s), Y(s), U_3(s), \ldots, U_m(s), V_{m+1}(s), \ldots, V_n(s), \xi(s)\}$ a pseudo-orthonormal basis of $T_{\gamma(s)}H_1^{n+1}$ such that*

$$\dot V_\alpha(s) \in \mathrm{span}\{Y(s), V_{m+1}(s), \ldots, V_n(s)\},$$
$$\dot\xi(s) = \lambda\dot\gamma(s) + B(s)U_3(s)$$

for some nonzero $B(s)$. If M is one of the following parametrized hypersurfaces in $H_1^{n+1} \subset R_2^{n+2}$:

(1) $\lambda^2 \neq 0$ *or* 1,

$$\begin{aligned} &f(s, y, a_3, \ldots, a_m, b_{m+1}, \ldots, b_n) \\ &= \epsilon_1(\lambda)\sqrt{\frac{1}{(\lambda^2-1)^2} - \sum_{i=3}^m \frac{a_i^2}{\lambda^2-1}}(\gamma(s) - \lambda\xi(s)) \\ &+ \epsilon_2(\lambda)\sqrt{\frac{\lambda^2}{(1-\lambda^2)^2} - \sum_{\alpha=m+1}^n \frac{b_\alpha^2}{1-\lambda^2}}(\xi(s) - \lambda\gamma(s)) \\ &+ yY(s) + \sum_{i=3}^m a_iU_i(s) + \sum_{\alpha=m+1}^n b_\alpha V_\alpha(s), \end{aligned}$$

where

$$\epsilon_1(\lambda) = \begin{cases} -1, & \text{if } \lambda^2 > 1 \\ 1, & \text{if } \lambda^2 < 1, \end{cases}$$

$$\epsilon_2(\lambda) = \begin{cases} -1, & \text{if } \lambda(\lambda^2 - 1) > 0 \\ 1, & \text{if } \lambda(\lambda^2 - 1) < 0. \end{cases}$$

(2) $\lambda = 0$,

$$\begin{aligned} & f(s, y, a_3, \dots, a_n) \\ =& \sqrt{1 + \sum_{i=3}^{n} a_i^2 \gamma(s) + yY(s) + \sum_{i=3}^{n} a_i U_i(s)}. \end{aligned}$$

(3) $\lambda^2 = 1$,

$$\begin{aligned} & f(s, y, a_3, \dots, a_n) \\ =& \left(1 + \frac{1}{2}\sum_{i=3}^{n} a_i^2\right)\gamma(s) - \lambda\left(\frac{1}{2}\sum_{i=3}^{n} a_i^2\right)\xi(s) + yY(s) + \sum_{i=3}^{n} a_i U_i(s), \end{aligned}$$

then M is type III.

Conversely, we have:

Theorem 4.5. *Let M be a type* III *hypersurface in H_1^{n+1}. Then for any $p \in M$, there is a neighborhood U_p of p in M such that U_p is exactly one of the parametrized hypersurfaces in Theorem 4.4.*

Before proving Theorem 4.5, we need the following Lemma.

Lemma 4.6. *Let M be a type* III *hypersurface, and $e_1, e_2, \dots, e_m, e_{m+1}, ...,$ e_n, e_{n+1} a local pseudo-orthonormal frame such that e_{n+1} is normal to M, $\omega_{n+1}^1 = \lambda\omega^1, \omega_{n+2}^2 = \lambda\omega^2 + \omega^3$, $\omega_{n+1}^3 = \lambda\omega^3 - \omega^1$, $\omega_{n+1}^i = \lambda\omega^i$, $\omega_{n+1}^\alpha \frac{1}{\lambda}\omega^\alpha$, and T_λ, T_λ^2 $T_{\frac{1}{\lambda}}$ the distributions defined as follows:*

$$\begin{aligned} T_\lambda &= \ker(A - \lambda) = \operatorname{span}\{e_2, e_4, \dots, e_m\}, \\ T_\lambda^2 &= \ker(A - \lambda)^2 = \operatorname{span}\{e_2, e_3, \dots, e_m\}, \\ T_{\frac{1}{\lambda}} &= \ker\left(A - \frac{1}{\lambda}\right) = \operatorname{span}\{e_{m+1}, \dots, e_n\}. \end{aligned}$$

Then the distribution $T_\lambda^2 + T_{\frac{1}{\lambda}}$ is integrable, and $\omega_\alpha^1 = \omega_2^\alpha = \omega_\alpha^i = 0$, $\omega_2^i \wedge \omega^1 = 0$ and $\omega_\alpha^2 \wedge \omega^1 = 0$.

Proof. Let $\widetilde{\omega}^1_{n+1} = \omega^1_{n+1} - \lambda\omega^1$, $\widetilde{\omega}^2_{n+1} = \omega^2_{n+1} - \lambda\omega^2$, $\widetilde{\omega}^3_{n+1} = \omega^3_{n+1} - \lambda\omega^3$, $\widetilde{\omega}^i_{n+1} = \omega^i_{n+1} - \lambda\omega^i$, and $\widetilde{\omega}^\alpha_{n+1} = \omega^\alpha_{n+1} - \lambda\omega^\alpha$. Then

$$\widetilde{\omega}^1_{n+1} = 0, \quad \widetilde{\omega}^2_{n+1} = \omega^3, \quad \widetilde{\omega}^3_{n+1} = -\omega^1, \quad \widetilde{\omega}^i_{n+1} = 0, \quad \widetilde{\omega}^\alpha_{n+1} = \left(\frac{1}{\lambda} - \lambda\right)\omega^\alpha.$$

By (3), we have

$$d\widetilde{\omega}^A_{n+1} = \sum_{B=1}^{n} \widetilde{\omega}^B_{n+1} \wedge \omega^A_B,$$

where $A = 1, 2, \dots, n$. The above equations are also the Codazzi equation, which Megid discussed in [**4**]. Hence the Lemma holds. □

We can proceed to the:

Proof of Theorem 4.5. Since the proof is similar to that of Theorem 4.2, we give a sketch here.

Let M be a type III hypersurface, x_0 a point of M. By Lemma 2.3, there is a local pseudo-orthonormal frame $e_1, e_2, \dots, e_n$, e_{n+1} defined in a neighborhhod of x_0 such that e_{n+1} is normal to M, and $\omega^1_{n+1} = \lambda\omega^1, \omega^2_{n+2} = \lambda\omega^2 + \omega^3$, $\omega^3_{n+1} = \lambda\omega^3 - \omega^1$, $\omega^i_{n+1} = \lambda\omega^i$, $\omega^\alpha_{n+1} = \frac{1}{\lambda}\omega^\alpha$ Let $\gamma(s)$ be the integral curve of e_1 and $N(s)$ be the integral manifold of $T^2_\lambda + T_{\frac{1}{\lambda}}$ through $\gamma(s)$. By the similar discussion for type II, we have

$$\begin{aligned} &de_2 \wedge e_2 = 0, \\ &d(e_{m+1} \wedge \cdots \wedge e_n \wedge (e_{n+1} - \lambda X)) \wedge e_2 = 0. \end{aligned} \tag{44}$$

Let $W_1(s)$ be the linear span of $\{e_2(\gamma(s))\}$, $W_2(s)$ the span of $\{e_2(\gamma(s)),$ $e_{m+1}(\gamma(s)), \dots,$ $e_n(\gamma(s))$, $e_{n+1}(\gamma(s)) - \lambda\gamma(s)\}$, and $W_3(s)$ the span of $\{e_2(\gamma(s)), e_3(\gamma(s)), \dots, e_m(\gamma(s)),\ \gamma(s) -\ \lambda e_{n+1}(\gamma(s))\}$.

For any x in $N(s)$,

$$\langle x, Y(s)\rangle = 0.$$

Here $Y(s) = e_2(\gamma(s))$. If $\lambda \neq 0, \pm 1$, by (44) and $e_{n+1} - \lambda x \in W_2(s)$, $x - \lambda e_{n+1} \in W_3(s)$, we can write

$$x = X_1 + X_2,$$

$$e_{n+1} = \frac{1}{\lambda}X_1 + \lambda X_2,$$

where $X_1 \in W_2(s)$, $X_2 \in W_3(s)$. Note that $\langle X_1, X_2\rangle = \langle X_1, Y(s)\rangle = \langle X_2, Y(s)\rangle = \langle Y(s), Y(s)\rangle = 0$. So we have

$$\langle X_1, X_1\rangle = \frac{\lambda^2}{1-\lambda^2}, \quad \langle X_2, X_2\rangle = \frac{1}{\lambda^2 - 1}.$$

If $m = n$, then

$$\langle x, e_{n+1}(\gamma(s)) - \lambda\gamma(s)\rangle = \lambda.$$

This completes the proof of Theorem 4.5. □

5. Type IV hypersurfaces.

The number p of distinct principal curvatures of type IV hypersurfaces is 2, 3 or 4. In this section we classify the type IV hypersurfaces. The classification is based on the following theorems.

Let M be a type IV hypersurface, x a point of M. By Lemma 2.4, there is a local orthonormal frame $e_1, e_2, \dots, e_m, e_{m+1}, \dots, e_n$, e_{n+1} defined in a neighborhhod of x such that e_{n+1} is normal to M, $\omega^1_{n+1} = a_0\omega^1 + b_0\omega^2$, $\omega^2_{n+1} = -b_0\omega^1 + a_0\omega^2$ ($b_0 \neq 0$), $\omega^i_{n+1} = \lambda\omega^i$ for $3 \leq i \leq m$, and $\omega^\alpha_{n+1} = \frac{1}{\lambda}\omega^\alpha$ for $m+1 \leq \alpha \leq n$.

Theorem 5.1. *Let M be a type* IV *hypersurface with $p = 2$. Then M is congruent to an open part of a principal orbit of $G \subset O(2,2)$ in H^3_1, where*

$$G = \left\{ \begin{pmatrix} \cos s & \sin s & 0 & 0 \\ -\sin s & \cos s & 0 & 0 \\ 0 & 0 & \cos s & \sin s \\ 0 & 0 & -\sin s & \cos s \end{pmatrix} \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & \cosh t & 0 & \sinh t \\ \sinh t & 0 & \cosh t & 0 \\ 0 & \sinh t & 0 & \cosh t \end{pmatrix} \middle| \; s, t \in R \right\}.$$

Proof. In this case, $n = 2$ and

$$\omega^1_3 = a_0\omega^1 + b_0\omega^2, \omega^2_3 = -b_0\omega^1 + a_0\omega^2. \tag{45}$$

Note that $e_1, \dots, e_n$ is an orthonormal frame. It follows from (3) that

$$\omega^2_1 = \omega^1_2.$$

Exterior differenting (45), we get

$$\omega^1 \wedge \omega^1_2 = 0, \tag{46}$$

$$\omega^2 \wedge \omega^1_2 = 0. \tag{47}$$

From (46) and (47), we arrive at

$$\omega^1_2 = 0. \tag{48}$$

Substituting (45) and (48) to Gauss equation (11), we have

$$a_0^2 + b_0^2 = 1.$$

From the theory of moving frame, (45) and (48) imply that M is locally homogeneous. In fact M is congruent to an open part of a principal orbit of G defined in Theorem 5.1. $\square$

Theorem 5.2. *Let M be a type* IV *hypersurface with $p = 3$. Then M is congruent to an open part of a orbit of $G \subset O(2,3)$ in H_1^4, where the Lie algebra of G is generated by*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \sqrt{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{3} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. In this case, $\omega_{n+1}^1 = a_0\omega^1 + b_0\omega^2$, $\omega_{n+1}^2 = -b_0\omega^1 + a_0\omega^2$ $(b_0 \neq 0)$, $\omega_{n+1}^i = \lambda\omega^i$ for $3 \leq i \leq n$. From the Codazzi equation (12), we know that

$$\omega_1^2 = \omega_2^1 = c\omega^3 \tag{49}$$

for some function c and

$$\omega_1^3 = \frac{2cb_0}{(a_0 - \lambda)^2 + b_0^2}[(a_0 - \lambda)\omega^1 + b_0\omega^2], \tag{50}$$

$$\omega_2^3 = \frac{2cb_0}{(a_0 - \lambda)^2 + b_0^2}[(a_0 - \lambda)\omega^2 - b_0\omega^1], \tag{51}$$

$$\omega_1^i = \omega_2^i = 0 \tag{52}$$

for $i > 3$.

Substituting (50), (51) and (52) to Gauss equation (11), we have

$$\begin{aligned} \frac{4c^2b_0^2}{(a_0 - \lambda)^2 + b_0^2} &= 1 - a_0\lambda, \\ \frac{4c^2(a_0 - \lambda)}{(a_0 - \lambda)^2 + b_0^2} &= \lambda, \\ \frac{8c^2b_0^2}{(a_0 - \lambda)^2 + b_0^2} &= a_0^2 + b_0^2 - 1, \end{aligned} \tag{53}$$

and

$$\begin{aligned} -\omega_1^3 \wedge \omega_3^i &= (a_0\lambda - 1)\omega^1 \wedge \omega^i + b_0\lambda\omega_2 \wedge \omega_i, \\ -\omega_2^3 \wedge \omega_3^i &= (1 - a_0\lambda)\omega^2 \wedge \omega^i + b_0\lambda\omega_1 \wedge \omega_i, \end{aligned} \tag{54}$$

where $i > 3$. But (50), (51) and (54) have no solution for any $i > 3$. This implies $n = 3$. From (49), (50), (51) and (53), we know that M is locally homogeneous. In fact M is congruent to an open part of a principal orbit of G defined in Theorem 5.2. □

Note. $G \cong SL(3, R)$.

Theorem 5.3. *Let M be a type* IV *hypersurface with $p = 4$. Then M is congruent to an open part of the hypersurface*

$$\{(x^1, \dots, x^{2n+2}) \in H_1^{n+1} :| -(x^1 + ix^2)^2 + (x^3 + ix^4)^2 + \cdots + (x^{2n+1} + ix^{2n+2})^2 |= t\}$$

where $t > 1$.

To prove the theorem, we need the following simple Lemma.

Lemma 5.4. *Let $A_1, \dots, A_p$ be $m \times m$ matrices in $o(1, n)$. If rank $(\sum_{j=1}^p a_j A_j) = 2$ and its enginvalues are $\pm i\sqrt{\sum_{j=1}^p a_j^2}$, 0 for any $a_1, ..., a_p \in R$, and $\sum_{j=1}^p a_j^2 \neq 0$. Then $m \geq p + 1$ and there is a invertible matrix P such that $PA_jP^{-1} = e_{1\ j+1} - e_{j+1\ 1}$ for all $1 \leq j \leq p$, where $e_{ij} \in gl(n+1)$ whose ij-th entry is 1 and all other entries are 0.*

Proof. Since we only need Linear algebra, we give an outline of the proof.

Since rank$A_1 = 2$ and A_1 has eigenvalues $i, -i$ and 0, we can choose P_1 such that

$$P_1 A_1 P_1^{-1} = \begin{pmatrix} 0 & x_{21} & \cdots & x_{m1} \\ -x_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -x_{m1} & 0 & \cdots & 0 \end{pmatrix}.$$

Since rank $\left(\sum_{j=1}^p a_j A_j\right) = 2$ for any $a_1, \dots, a_p$ satisfying $\sum_{j=1}^p a_j^2 \neq 0$, the $P_1 A_i P_1^{-1}$ take the form

$$P_1 A_i P_1^{-1} = \begin{pmatrix} 0 & x_{2i} & \cdots & x_{mi} \\ -x_{2i} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -x_{mi} & 0 & \cdots & 0 \end{pmatrix}.$$

From the fact that the eigenvalues of $\sum_{j=1}^p a_j A_j$ are $\pm i\sqrt{\sum_{j+1}^p a_j^2}$ and 0, it follows that

$$\sum_{k=2}^m x_{ki} x_{kj} = \delta_{ij}.$$

Hence $m - 1 \geq p$ and the Lemma holds. $\square$

Proof of Theorem 5.3. We consider the focal manifold N of M. Choose a local orthonormal frame $e_1, e_2, \dots, e_m, e_{m+1}, \dots, e_n, e_{n+1}$ such that $e_1, e_2, \dots, e_m$ are tangent to N and $e_{m+1}, \dots, e_n, e_{n+1}$ are normal to N. Note that M is type IV. From Lemma 2.4, we know that for any (local) unit normal field ν of N, its principal curvatures are ib, $-ib$, 0 for some b, and the multiplicity of principal curvature 0 is $m - 2$. Here $i^2 = -1$. From Lemma 5.4, we have

$m \geq n+2-m$, i.e., $2m \geq n+2$. Hence we can choose an orthonormal frame $e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n, e_{n+1}$ such that

$$\omega_{m+l}^1 = b\omega^{l+1}, \quad \omega_{m+l}^{l+1} = -b\omega^1, \quad \omega_{m+l}^i = 0 \tag{55}$$

for any l, and $1 \leq l \leq n-m+1$. Here $1 \leq i \leq m$, and $i \neq 1, l+1$. Note that $n+1-m \geq 2$. Using Coddazzi equation, we get

$$\omega_1^j = 0, \qquad (j = 1, \ldots, m), \tag{56}$$

and

$$\omega_{m+l}^{m+k} = \omega_{l+1}^{k+1}, \quad (1 \leq k, l \leq n-m+1). \tag{57}$$

Substituting (56) and (57) to Gauss equation (11), we have $b^2 = 1$ and $\omega^1 \wedge \omega^j = 0$ for $n-m+3 \leq j \leq m$. This implies $m \leq n-m+2$, i.e., $2m \leq n+2$. Therefore $2m = n+2$.

Now we prove the existence and uniqueness of N.

Let N^m be the focal manifold of a type IV hypersurface $M^n \subset H_1^{n+1} \subset R_2^{n+2}$. Then $n+2 = 2m$ and for any $x \in N$, there is a local orthonormal frame $e_1, e_2, \ldots, e_{n+1}$ such that $e_1, e_2, \ldots, e_m$ are tangent to M, e_{m+1}, $\ldots, e_n, e_{n+1}$ are normal to M,

$$\omega_1^i = 0, \quad (i = 1, \ldots, m), \tag{58}$$

and

$$\omega_{m+i}^{m+j} = \omega_{i+1}^{j+1}, \quad (i, j = 1, \ldots, m-1), \tag{59}$$

$$\omega_{m+i}^j = \delta_1^j \omega^{i+1} - \delta_{i+1}^j \omega^1 \quad (i = 1, \ldots, m-1, j = 1, \ldots, m). \tag{60}$$

Here $\delta_i^j = 1$ if $i = j$, and $\delta_i^j = 0$ if $i \neq j$.

Let $\gamma(s)$ be an integral curve of e_1. Then

$$de_1 = \sum_{i=1}^{m} \omega_1^i e_i + \sum_{j=1}^{m-1} \omega_{m+j}^1 - \omega^1 X, \tag{61}$$

where X is the position vector field of N. From (58) and (60),

$$de_1 = \sum_{i=1}^{m-1} \omega^{i+1} e_{m+i} - \omega^1 X. \tag{62}$$

From (62), we know that e_1 is parallel along γ in H_1^{n+1}. This means that γ is a geodesics of H_1^{n+1}. Hence

$$\gamma(s) = \cos s\gamma(0) + \sin s e_1(\gamma(0)), \tag{63}$$

and $\omega_1 = ds$. Choose the normal vector fields $e_{m+1}, \ldots, e_n, e_{n+1}$ such that each of them is parallel along $\gamma(s)$, i.e., $\omega_{m+i}^{m+j} = 0$, $(i, j = 1, \ldots, m-1)$ on

γ. So along γ, we have

$$de_{i+1} = \omega^1 e_{m+i}, \quad i = 1, \dots, m-1, \tag{64}$$
$$de_{m+i} = \omega^1 e_{i+1}, \quad i = 1, \dots, m-1,$$

which implies that

$$e_{i+1}(\gamma(s)) = \cos s e_{i+1}(\gamma(0)) + \sin s e_{m+i}(\gamma(0)) \tag{65}$$

for i in $\{1, \dots, m-1\}$. Now consider the distribution $E = \text{span}\ \{e_2, \dots, e_m\}$. From the structural equation (3), we have

$$d\omega^1 = 0, \quad d\omega^{m+i} = \omega^{i+1} \wedge \omega^1$$

for i in $\{1, \dots, m-1\}$, which implies that E is an integrable distribution. Denote the integral manifold through $\gamma(s)$ by $P(s)$. Then $e_1, e_{m+1}, \dots, e_{n+1}$ are normal vector fields of $P(s)$. On $P(s)$,

$$de_1 = \sum_{i=1}^{m-1} \omega^{i+1} e_{m+i}, \quad de_{m+i} = \omega^{i+1} e_1, \quad i = 1, \dots, m-1.$$

Hence $P(s)$ is a totally geodesic submanifold of H_1^{n+1} for every s.

Summarizing the arguments above, N is determined uniquely by $e_1(\gamma(0))$, $e_2(\gamma(0))$, $\dots, e_m(\gamma(0)), e_{m+1}(\gamma(0)), \dots, e_{n+1}(\gamma(0))$. In fact N is congruent to an open part of the submanifold:

$$\{(x^1, \dots, x^{2n+2}) \in H_1^{n+1} :| -(x^1 + ix^2)^2 + (x^3 + ix^4)^2 + \cdots + (x^{2n+1} + ix^{2n+2})^2 |= 1\}.$$

The second fundamental form of N is $II(e_1, e_1) = 0$, $II(e_1, e_i) = 0$, $II(e_1, e_i) = e_{n+i}$ for $2 \leq i, j \leq n$. Since M is a tube of N, we obtain Theorem 5.3. $\square$

Hence we have finished the classification of Lorentzian isoparametric hypersurfaces in H_1^{n+1} by combining Theorems in Sections 3, 4 and 5.

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