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A SIMPLE PROOF OF AN INEQUALITY DOMINATING THE AREA INTEGRAL BY THE NONTANGENTIAL MAXIMAL FUNCTION

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In this paper, we give a simple proof for a good-λ inequality which means that nontangential maximal functions controls area integrals.

Let u be a harmonic function on $\mathbf{R}^{n+1}_+$. The nontangential maximal function and the area integral function of f are defined by

$$N_\beta(u)(x) = \sup_{(y,t)\in\Gamma_\beta(x)} |u(y,t)| \qquad (\beta \in \mathbf{R}^1_+),$$

$$A_\alpha(u)(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(y,t)|^2 t^{1-n} dydt\right)^{\frac{1}{2}} \qquad (\alpha \in \mathbf{R}^1_+).$$

The main aim of this paper is to give a simple proof of the inequality

$$(1) \qquad \|A_\alpha(u)\|_p \le C_{n,p,\alpha,\beta} \|N_\beta(u)\|_p \qquad (0 < p < \infty,\ 0 < \alpha,\ \beta < \infty).$$

As we know, this inequality is very important in H^p-theory, it is also a main difficulty in generalizing H^p-theory of one parameter to H^p -theory of several parameters, see [**2**, **6**, **7**, **8**]. The first proof of (1) is probabilistic which was given by Burkholder, Gundy and Silverstein, see [**1**]; Fefferman and Stein first got an analytic proof of (1) by dealing with a kind of Green's formula on $\mathcal{R} = \cup_{x\in E}\Gamma_\alpha(x)$, see [**4**]; a sharpened inequality was obtained in [**5**] by a different approach. In the two-parameter case, Gundy and Stein set up a similar inequality to (1) by dealing with some multi-sub-linear operators like

$$B(u,v)(x) = \left(\int_{\Gamma(x)} |\nabla_1 u|^2 |\nabla_2 v|^2 t_1^{1-n_1} t_2^{1-n_2}\, dx_1 dt_1\, dx_2 dt_2\right)^{\frac{1}{2}},$$

see [**9**]; Merryfield ([**8**]) and author ([**2**]) generalized Gundy-Stein's work to multi-parameter case independently and differently. In our proof ([**2**]), we introduced a kind of Carleson measure technique which does not depend on the dilation and translation structures of $\mathbf{R}^n$ such that the method works on more general case (see Chen and Wang [**3**]). Here, we shall use the idea to give a simple proof of (1).

At first, we notice that for $1 < p < \infty$, the proof of (1) is elementary, and for $0 < p \le 1$, (1) can be followed from

$$|\{x : A_\alpha(u)(x) > \lambda\}| \le C_{n,\alpha,\beta}\Big(|\{x : N_\beta(u)(x) > \lambda\}| + \lambda^{-2} \int_{N_\beta(u)(x)\le\lambda} N_\beta(u)^2(x)dx\Big) \tag{2}$$

where $0 < \alpha, \beta, \lambda < \infty$ (note that, for $0 < \alpha < \beta < \infty$, (2) was set up in [**4**]). Now, we shall prove (2).

By a limitation procedure, we may assume $u(x,t) = \tilde{u}(x, t+\epsilon)$, where $N_\beta(\tilde{u}) \in L^p$. $\forall\lambda > 0$, set $E_\lambda = \{x : N_\beta(u)(x) \le \lambda\}$, $\delta_0 = \delta(n,\beta) = \int_{|x|<\beta} p_1(x)dx \in (0,1)$, where p_t is the Poisson kernel. Take a closed subset F_λ of E_λ such that $|F_\lambda^c| \le C_{n,\alpha,\beta}|E_\lambda^c|$, $p_t * \chi_{E_\lambda} \ge 1 - \frac{1}{2}\delta_0$ (on $\cup_{x\in F_\lambda}\Gamma_\alpha(x)$), which is possible by the definition of p_t and the weak type (1,1)-boundedness of nontangential maximal function operator; then, take $\varphi \in C^2(\mathbf{R}^1) \cap L^\infty(\mathbf{R}^1)$, such that $\varphi|_{(-\infty,1-\delta_o)} = 0$, $\varphi|_{(1-\frac{1}{2}\delta_o,+\infty)} = 1$, $|\varphi'| + |\varphi''| \le c\varphi^{3/4}$ (by using $e^{-t^{-2}}$). Now, set $v = p_t * \chi_{E_\lambda}$, then

$$\begin{aligned}
&\text{the left side of (2)} \\
&\le |F_\lambda^c| + |F_\lambda \cap \{x : A_\alpha(u)(x) > \lambda\}| \\
&\le C_{n,\alpha,\beta}\left\{|E_\lambda^c| + \lambda^{-2}\int_{F_\lambda}\int_{\Gamma_\alpha(x)} \varphi(v)\,|\nabla u(w,t)|^2\,t^{1-n}dwdtdx\right\} \\
&\le C_{n,\alpha,\beta}\left\{|E_\lambda^c| + \lambda^{-2}\int\int_{\mathbf{R}_+^{n+1}} \varphi(v)\,|\nabla u|^2\,tdwdt\right\}.
\end{aligned} \tag{3}$$

Note that

$$\varphi(v)\,|\nabla u|^2 = -u\varphi'(v)\nabla v \cdot \nabla u - \frac{1}{2}u^2\Delta(\varphi(v)) + \frac{1}{2}\Delta(\varphi(v)u^2);$$

and, $\|\varphi(v)u\|_\infty \le C_\varphi\lambda$ for $v \le 1 - \delta_0$ on $(\cup_{x\in E_\lambda}\Gamma_\beta(x))^c$; in addition, it is not difficult to show that for a fixed $\psi \in C_c^\infty(\mathbf{R}^n)$ satisfying $\psi(|x| \le 1) = 1$, $\psi(|x| \ge 2) = 0$, we have (where $\psi_r(w) := \psi(w/r)$)

$$\begin{aligned}
\int\int_{\mathbf{R}_+^{n+1}} \Delta(\varphi(v)u^2)tdwdt &= \lim_{r\to\infty}\int\int_{\mathbf{R}^n\times(0,r)} \psi_r(w)\Delta(\varphi(v)u^2)tdwdt \\
&= \int_{\mathbf{R}^n} \varphi(v(x,0))u^2(x,0)dx
\end{aligned}$$

by Green's formula, because $N_\beta(\tilde{u}) \in L^p$, and

$$\left\|t^{k+n/p}\nabla^k u\right\|_\infty + \left\|t^k\nabla^k v\right\|_\infty \le C_{\epsilon,n,p,k}(\widetilde{u}) < \infty$$

for $k = 0, 1, 2, \cdots$. Therefore, by Hölder's inequality, we get

$$\begin{aligned}
&\int\int_{\mathbf{R}_+^{n+1}} \varphi(v) \left|\nabla u\right|^2 t dw dt \\
&\le C_\varphi \lambda \left(\int\int_{\mathbf{R}_+^{n+1}} \left|\nabla v\right|^2 t dw dt\right)^{\frac{1}{2}} \left(\int\int_{\mathbf{R}_+^{n+1}} \varphi(v) \left|\nabla u\right|^2 t dw dt\right)^{\frac{1}{2}} \\
&\quad + C_\varphi \lambda^2 \int\int_{\mathbf{R}_+^{n+1}} \left|\nabla v\right|^2 t dw dt + \frac{1}{2}\int_{\mathbf{R}^n} \varphi(v(x,0)) u^2(x,0) dx \\
&\le C_{\varphi,n} (\lambda^2 \left|E_\lambda^c\right|)^{\frac{1}{2}} \left(\int\int_{\mathbf{R}_+^{n+1}} \varphi(v) \left|\nabla u\right|^2 t dw dt\right)^{\frac{1}{2}} \\
&\quad + C_{\varphi,n} \left(\lambda^2 \left|E_\lambda^c\right| + \int_{E_\lambda} N_\beta(u)^2(x) dx\right).
\end{aligned}$$

Thus, by an elementary argument, we get

$$\int\int_{\mathbf{R}_+^{n+1}} \varphi(v) \left|\nabla u\right|^2 t dw dt \le C_{\varphi,n} \left(\lambda^2 \left|E_\lambda^c\right| + \int_{E_\lambda} N_\beta(u)^2(x) dx\right). \tag{4}$$

(3) and (4) give (2).

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