INTEGRAL SPECIALIZATION
OF FAMILIES OF RATIONAL FUNCTIONS

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Suppose C is an algebraic curve, f is a rational function on C defined over $\mathbb{Q}$, and $\mathcal{A}$ is a fractional ideal of $\mathbb{Q}$. If f is not equivalent to a polynomial, then Siegel's theorem gives a necessary condition for the set $C(\mathbb{Q}) \cap f^{-1}(\mathcal{A})$ to be infinite: C is of genus 0 and the fiber $f^{-1}(\infty)$ consists of two conjugate quadratic real points. We consider a converse. Let $\mathcal{P}$ be a parameter space for a smooth family $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1$ of (degree n) genus 0 curves over $\mathbb{Q}$. That is, the fiber $\mathcal{T}_{\boldsymbol{p}}$ of points of $\mathcal{T}$ over $\boldsymbol{p} \times \mathbb{P}^1$ has genus 0 for $\boldsymbol{p} \in \mathcal{P}$. Assume a Zariski dense set of $\boldsymbol{p} \in \mathcal{P}(\mathbb{Q})$ have fiber $\Phi_{\boldsymbol{p}}^{-1}(\boldsymbol{p} \times \infty)$ over ∞ consisting of two conjugate quadratic real points. The family Φ is then a *Siegel family*. We ask when the conclusion of Siegel's theorem — $\Phi_{\boldsymbol{p}}(\mathbb{Q}) \cap \mathcal{A}$ is infinite — holds for a Zariski dense subset of $\boldsymbol{p} \in \mathcal{P}(\mathbb{Q})$.

We show how braid action on covers and Hurwitz spaces can tackle this. It refines a unirationality criterion for Hurwitz spaces. A particular family, $_{10}\Phi'$, of degree 10 rational functions, illustrates this. It arises as the exceptional case for a general result on Hilbert's Irreducibility Theorem. Fried, 1986 says the only indecomposable polynomials $f(y) \in \mathbb{Q}[y]$ with $f(y) - t$ reducible in $\mathbb{Q}[y]$ for infinitely many $t \in \mathcal{A} \setminus f(\mathbb{Q})$ have degree 5. We show the family $_{10}\Phi'$ satisfies the converse to Siegel's theorem. Thus, exceptional polynomials of degree 5 in Fried, 1986 do exist.

We suspect this result generalizes, thus codifying arithmetic accidents occurring in $_{10}\mathcal{P}'$. To illustrate, we've cast this paper as a collection of elementary group theory tools for extracting from a family of covers special cases with specific arithmetic properties. Examples of *Siegel* and *Néron* families show the efficiency of the tools, though each case leaves a diophantine mystery.

1. Introduction and the basic families.

This paper completes part of [**DF90a**]. It is a tools paper, illustrating with one main (and several support) examples how to analyze a certain style of question. When a diophantine or geometric problem about curves

defines possible exceptional cases, how do you find if the exceptions do exist? In practice the production of illustrative curves occurs with extra data. That data usually includes a reference variable defining the curve as a cover. That is why, in practice, Hurwitz-style families of covers are natural. Our illustrations especially discuss tools for treating the value sets of genus 0 covers. This will show elementary use of Hurwitz families and braid group action as does [**Deb99**], [**Fri90**, **Fri95b**, **Fri95a**], [**MM**] and [**Völ96**]. [**Ser92**] has simpler examples not requiring Hurwitz families and braid groups.

This paper culminates an era in which the *monodromy method* has proved itself innumerable times on many problems of renown. It is time to say these (as delineated here) are the elementary tools. The program *Modular Towers* of [**Fri95c**] has outlined the next stage of problems. It requires a set of tools (like modular representation theory and higher monodromy) that divides it from the tools here. Still, there are many problems around which require neat analysis for which the tools here suffice. We hope the community will use them.

Any cover, $\varphi : X \to \mathbb{P}^1_z$ has an associated group G and attached conjugacy classes $C_1, \ldots, C_r$. Newcomers will find [**Fri95b**] examples an easy introduction to these ideas. As here, they include families of polynomials in practical applications. Group theory in this paper occurs mostly within the groups S_5 and S_{10}. Reader: We act on the *right* of the integers with permutation groups. That means, $(1\,2\,3)$ times $(2\,3\,4)$ is $(1\,3)(2\,4)$, not $(3\,4)(1\,2)$. Also, the braid group B_r acts on the right of Nielsen classes (§1.1).

Notations. Let K be a field and $\bar{K}$ an algebraic closure. A rational function $f \in K(y)$ is *decomposable* over K if f is a functional composition $f_1(f_2(y))$ with $\deg(f_i) > 1$, $f_i(y) \in K(y)$ for $i = 1, 2$. Otherwise it is indecomposable over K. Suppose $f \in K[y]$ and $(\deg(f), \operatorname{char}(K)) = 1$. Then, f is indecomposable over K if and only if it is indecomposable over $\bar{K}$ ([**FM69**] or [**Sch82**, p. 14]). For $g \in S_n$, write $g = \beta_1 \cdots \beta_t$, a product of disjoint cycles (including those of length 1), the $\operatorname{ind}(g) = n - t$. Each element in the conjugacy class of g has the same index; refer to it as the index of the conjugacy class. This notation appears in the Riemann-Hurwitz formula (1.1).

Finally, we use *fiber products* of morphisms $\psi_i : Y_i \to X$, $i = 1, 2$. Basically, these are pairs $\{(y_1, y_2) \in Y_1 \times Y_2 \mid \psi_1(y_1) = \psi_2(y_2)\}$. Often, however, this set theoretic product has singularities, though Y_1 and Y_2 are nonsingular varieties. Example: If ψ_1 and ψ_2 are ramified over a common point, the fiber product has a singular point at the pair of ramified points. So, it is usual to normalize the variety to remove some (all for curves) of this singularity. Even after normalizing the fiber product notation is reminiscent of $Y_1 \times_X Y_2$.

1.1. An exceptional family of covers represented by polynomials.
A family, of genus zero covers (represented by polynomial maps), appears

in [**DF90a**, §2.1]. Each cover in this family has $r = 4$ branch points. Its parameter space $\mathcal{H}_\infty = \mathcal{H}_\infty(S_5, \mathbf{C})$ is a natural closed subvariety of a minimal (absolute) Hurwitz parameter space $\mathcal{H} = \mathcal{H}(S_5, \mathbf{C})$. The word *absolute* here means that two covers $\varphi_1 : Y_1 \to \mathbb{P}^1_z$ and $\varphi_2 : Y_2 \to \mathbb{P}^1_z$, corresponding to distinct points $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathcal{H}$, are *inequivalent*: There exists no continuous map $\psi : Y_1 \to Y_2$ for which $\varphi_2 \circ \psi = \varphi_1$.

Any smooth irreducible family of covers with each cover having r (fixed) branch points has an associated group G and a set of conjugacy classes $\mathbf{C} = (\mathrm{C}_1, \ldots, \mathrm{C}_r)$ of G. In our special case: $G = S_5$, $r = 4$, $\mathrm{C}_2 = \mathrm{C}_3$ is the class of 2-cycles, C_1 the class of products of two disjoint 2-cycles and C_4 that of 5-cycles. A summary of this is in the data for the associated *Nielsen class*:

$$\mathrm{Ni} = \mathrm{Ni}(S_5, \mathbf{C}) = \{\boldsymbol{g} = (g_1, \ldots, g_4) \mid g_1 g_2 g_3 g_4 = 1, \langle \boldsymbol{g} \rangle = S_5 \text{ and } \boldsymbol{g} \in \mathbf{C}\}.$$

The expression $\boldsymbol{g} \in \mathbf{C}$ means these g_i s, $i = 1, 2, 3, 4$, are in the respective conjugacy classes C_i, $i = 1, 2, 3, 4$, in some order.

Notice that S_5 acts on $\mathrm{Ni}(S_5, \mathbf{C})$ by conjugation: For $g \in S_5, \boldsymbol{g} \in \mathrm{Ni}(S_5, \mathbf{C})$, $\boldsymbol{g} \mapsto g\boldsymbol{g}g^{-1}$. Denote the quotient Ni/S_5 by $\mathrm{Ni}^{\mathrm{abs}}$. *Riemann's Existence Theorem* (RET) gives a simple interpretation of the elements of this Nielsen class. Fix the four branch points as $\boldsymbol{z}^0 = (z_1^0, \ldots, z_4^0)$. RET says elements of $\mathrm{Ni}^{\mathrm{abs}}$ correspond one-one with equivalence classes of covers having branch points $\boldsymbol{z}^0$ and associated conjugacy classes and group. We say these covers are in the Nielsen class.

Denote elements $\boldsymbol{g} \in \mathrm{Ni}$ with $g_4 \in \mathrm{C}_4$ by Ni_∞. Let $\varphi : X \to \mathbb{P}^1_z$ be a (degree 5) cover in this Nielsen class. This cover canonically determines a fixed branch point in the family corresponding to the conjugacy class C_4. It is the unique point z' having just one point of X over it. Let λ be any linear fractional transformation taking z' to ∞. The cover $\lambda \circ \varphi : X \to \mathbb{P}^1_z$ is in the same Nielsen class and has ∞ as its branch point associated to C_4. This $\mathrm{SL}_2(\mathbb{C})$ action is a special case of an action §6.2 discusses briefly. In our example family, it normalizes coordinates to use this observation and consider only the sublocus (that is the space $\mathcal{H}_\infty$) in the complete family $\mathcal{H}$ where a designated fourth branch point is ∞.

The reader might wish that we use this normalization and immediately write out coordinates expressing our example as a family of polynomials. This is sensible, reasonable and yet, probably impossible as §1.7 explains. §2.3 uses another parameter space $\mathcal{P}_{\alpha, \beta, c, d}$, pullback from $\mathcal{H}$. This has explicit coordinates that we combine with the following property. §1.5 produces a new family over the parameter space $\mathcal{H}$ (and $\mathcal{H}_\infty$) whose fibers are degree 10 covers of $\mathbb{P}^1_z$. Pull this family back over $\mathcal{P}_{\alpha, \beta, c, d}$ and for any rational point in $\mathcal{P}_{\alpha, \beta, c, d}$, the fiber over this point has infinitely many rational points.

This rational-points-in-fibers property doesn't hold for $\mathcal{H}_\infty$. This is so significant that distinctions between these families is a major subtheme of this paper. The $\mathrm{SL}_2(\mathbb{C})$ observation above, however, implies the $\mathrm{SL}_2(\mathbb{C})$ quotient of $\mathcal{H}$, and an analogous quotient of $\mathcal{H}_\infty$ are the same. This turns a crucial task, showing these particular families have only one irreducible $\mathbb{Q}$-component, into group theory calculation. Subsections 1.2 and 1.3 show how the braid group produces this family from the Nielsen class.

1.2. Enter the braid group. Suppose a cover $\varphi : X \to \mathbb{P}^1_z$ is in a given Nielsen class. So an r-tuple from the Nielsen class is a *branch cycle description* of the cover. The Riemann-Hurwitz formula gives the covering curve genus g as

$$(1.1) \qquad 2(n + g - 1) = \sum_{i=1}^{r} \mathrm{ind}\, C_i, \ \ n = \deg(\varphi).$$

For the particular Nielsen class $\mathrm{Ni}(S_5, \mathbf{C})$ the genus satisfies $2(5 + g - 1) = 2 + 1 + 1 + 4 = 8$, or $g = 0$. Such branch cycle descriptions correspond to polynomial covers (as below).

This example uses Ni_∞ modulo the conjugation action of S_5: $\mathrm{Ni}_\infty^{\mathrm{abs}} = \mathrm{Ni}_\infty/S_5$. All classes, modulo S_5, have a representative with g_4 equal $g_\infty = (1\,2\,3\,4\,5)^{-1}$. Further normalize by conjugating by a power of g_∞. Thus assume the entry that is a product of two disjoint 2-cycles (type $(2)(2)$) fixes 1. Suppose the $(2)(2)$ type element is in the first position. The following lists the first three elements in these normalized entries of Ni_∞.

List A. $\mathrm{Ni}_\infty^{\mathrm{abs}}$ *representatives having* $(2)(2)$ *in first position*

a) $((2\,3)(4\,5), (1\,2), (1\,4))$ \qquad b) $((2\,3)(4\,5), (1\,4), (2\,4))$
c) $((2\,3)(4\,5), (2\,4), (1\,2))$ \qquad d) $((2\,5)(3\,4), (1\,2), (3\,5))$
e) $((2\,5)(3\,4), (3\,5), (1\,2))$.

The *braid group* B_r has generators $Q_1, \dots, Q_{r-1}$ modulo the two relations

$$(1.2) \qquad Q_i Q_j = Q_j Q_i \quad \text{for } |i - j| > 1 \quad \text{and}$$
$$Q_{i+1} Q_i Q_{i+1} = Q_i Q_{i+1} Q_i, \quad 1 \le i \le r-2.$$

There are two other significant words:

(1.3a) $Q_1 \cdots Q_{r-1} Q_{r-1} \cdots Q_1$; and
(1.3b) $(Q_1 \cdots Q_{r-1})^r$.

The *Hurwitz monodromy group* H_r is the quotient of B_r modulo (1.3a). The *mapping class group* M_r is the quotient of H_r modulo (1.3b). Most significant, the braid group acts on a Nielsen class. For $\mathbf{g} \in \mathrm{Ni}(G, \mathbf{C})$,

$$(1.4) \qquad (\mathbf{g})Q_i = (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_r),$$
$$i = 1, \dots, r-1.$$

The word $w \in B_r$ appearing in (1.3a) conjugates $\boldsymbol{g}$ by g_1:

$$\boldsymbol{g} \mapsto (\boldsymbol{g})w = g_1 \boldsymbol{g} g_1^{-1} = (\ldots, g_1 g_i g_1^{-1}, \ldots).$$

A list similar to List A has the $(2)(2)$ type element in the 2nd or in the 3rd position. To get these, apply braid elements Q_1 and $Q_2 Q_1$ to List A.

Riemann's Existence Theorem interprets List A. Polynomials h_1 and h_2 are *affine equivalent* if $h_1(ay + b) = h_2(y)$ for some $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$. Given three distinct values, $z_1, z_2, z_3 \in \mathbb{C}$, up to affine equivalence there are exactly *five* polynomials f of degree 5 with these properties.

(1.5a) $f : \mathbb{P}_y^1 \to \mathbb{P}_z^1$ has z_1, z_2, z_3 as its finite branch points.
(1.5b) The Galois closure of the cover (1.5a) has group S_5.
(1.5c) Inertia group generators for ramification from the cover by f over the three branch points are in C_1, C_2 and C_3, *in that order*.

Thus, there are exactly 15 equivalence classes of polynomials without stipulating the order of attachment of the z_is to the conjugacy classes in (1.5c). We say a polynomial, with a description from List A, or its images under Q_1 or $Q_1 Q_2$, is in the Nielsen class $\mathrm{Ni}(S_5, \mathbf{C})$.

[**DF90a**, §1.3] illustrates braid group action on elements of a Nielsen class. As above, denote equivalence classes of covers of $\mathbb{P}_z^1$ in Ni_∞ by $\mathcal{H}_\infty$. For each $\boldsymbol{p} \in \mathcal{H}_\infty$, there exists $f_{\boldsymbol{p}} : \mathbb{P}^1 \to \mathbb{P}^1$, a polynomial mapping in the equivalence class of $\boldsymbol{p}$.

1.3. $\mathcal{H}_\infty$ and a cover of the λ-line. Let $\mathbb{A}^n$ be affine n-space. Denote the *fat diagonal* of $\mathbb{A}^n$ by Δ_n. This is the set of coordinates with two or more entries equal. Then, S_n also acts on $\mathbb{A}^n \setminus \Delta_n$; the quotient $\mathbb{A}^n \setminus \Delta_n / H$ by any subgroup H of S_n is an affine variety. Denote the quotient $\mathbb{A}^3 \setminus \Delta_3 / \langle (2\,3) \rangle$ by U_∞.

Let f be a degree 5 polynomial in Ni_∞. Then, $D(f) = (z_1, \{z_2, z_3\})$ signifies the unordered pair $\{z_2, z_3\}$ corresponds to the 2-cycle branch points. The $(2)(2)$ type branch point of f is z_1. Identify points of U_∞ with these symbols. Similarly, let U_3 be the quotient of $\mathbb{A}^3 \setminus \Delta_3$ by S_3. Let $D'(f)$ be the unordered triple $\{z_1, z_2, z_3\} \in U_3$ of branch points of f.

This notation applies to any covers in a Nielsen class (no polynomial structure is necessary). Then, $D : \mathcal{H}_\infty \to U_\infty$ is a degree 5 unramified covering map and $D' : \mathcal{H}_\infty \to U_3$ is a degree 15 unramified covering map. Permutation representations of subquotient groups of B_4 on Ni_∞ produce these covers. [**DF90a**, p. 115] introduces H_3^*: the image in H_4 of the group Q_1 and Q_2 generate. Transitivity of H_3^* on Ni_∞ is equivalent to irreducibility of $\mathcal{H}_\infty$ [**DF90a**, Prop. 1.10]. Squares of the Q_is, $i = 1, 2$, generate a subgroup SH_3^* of H_3^*.

Consider the fiber product $\mathcal{H}' = \mathcal{H}_\infty \times_{U_3} \mathbb{A}^3 \setminus \Delta_3$ of $\mathcal{H}_\infty \to U_3$ and $\mathbb{A}^3 \setminus \Delta_3 \to U_3$ (top of §1). This degree 15 cover of $\mathbb{A}^3 \setminus \Delta_3$ has three natural components. One of these is the subspace of $\boldsymbol{p} \in \mathcal{H}'$ having this property: If $\boldsymbol{p}$ lies over $(z_1, z_2, z_3) \in \mathbb{A}^3 \setminus \Delta_3$, then z_1 is the $(2)(2)$ type branch point of $f_{\boldsymbol{p}}$. Call this space $\mathcal{H}^\infty$; it is an irreducible, degree 5 unramified cover of $\mathbb{A}^3 \setminus \Delta_3$, precisely because SH_3^* is transitive on List A [**DF90a**, Prop. 1.11]. Denote restriction of $\mathcal{H}^\infty$ to the set

$$\{(z_1, 0, 1) \mid z_1 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}\}$$

by $\mathcal{H}_{0,1,\infty}$. (Related calculations might replace 0 by z_2^0 and 1 by z_3^0.) Then, $\mathcal{H}_{0,1,\infty}$ is irreducible precisely because $a_{1,2}$ and $a_{1,3}$ below act transitively on List A [**BF82**, §1 Lemma 1.6] (see (1.6) and §6). Compatible with the treatment of §6.2, the variable z_1 is the traditional variable λ from modular functions.

Label elements of $\mathrm{Ni}_\infty^{\mathrm{abs}}$ from List A as $\{0_a, 0_b, 0_c, 0_d, 0_e\}$. While Q_3 doesn't act on Ni_∞, its square and the following elements $a_{1,2}, a_{1,3}$ and $a_{1,4}$ do. Here are specific actions from (1.4):

(1.6) $\quad a_{1,2} = Q_1^{-2} = (0_a\, 0_d\, 0_b)(0_c\, 0_e), \ a_{1,3} = Q_1 Q_2^{-2} Q_1^{-1} = (0_a\, 0_c\, 0_e)(0_b\, 0_d).$

These generate S_5 itself—a transitive group—acting on List A. Here, where $r = 4$, consider a 3rd element's action on List A:

(1.7) $\qquad\qquad a_{1,4} = Q_1 Q_2 Q_3^{-2} Q_2^{-1} Q_1^{-1} = (0_c\, 0_b\, 0_a).$

From (1.3a), $a_{1,2}a_{1,3}a_{1,4} = (Q_1 Q_2 Q_3^2 Q_2 Q_1)^{-1}$ acts trivially on $\mathrm{Ni}_\infty^{\mathrm{abs}}$. Thus, the actions of $a_{1,2}$, $a_{1,3}$ and $a_{1,4}$ on List A give branch cycles for a degree 5 cover $\bar\varphi : \bar{\mathcal{H}}_{0,1,\infty} \to \mathbb{P}^1_{z_1}$ of the sphere. Three values of z_1 ramify in $\bar\varphi$: $0, 1, \infty$. Formula (1.1) says the genus g of $\bar{\mathcal{H}}_{0,1,\infty}$ is 0: $2(5 + g - 1) = \mathrm{ind}(a_{1,2}) + \mathrm{ind}(a_{1,3}) + \mathrm{ind}(a_{1,4}) = 3 + 3 + 2$. For this and future reference, Prop. 6.5 explains (based on [**BF82**]) how this calculation shows $\mathcal{H}_\infty$ has a function field that is a subfield of a pure transcendental field over $\mathbb{Q}$. That is, $\mathcal{H}_\infty$ is *unirational*, and $\mathcal{H}_\infty(\mathbb{Q})$ is dense.

[**DF90a**] also wrote explicit coordinates for $\mathcal{H}_\infty$ revealing this directly. Usually, however, such computation is too difficult. The proof of Main Theorem 1.2 does rely on these coordinates. We don't know if it is possible to replace these special manipulations with a straightforward group criterion. Group theory provides more efficient (and intrinsic) interpretation than does the shape of the equations for these parameter spaces and fiber products.

There is a total family of representing covers over the space $\mathcal{H}_\infty$. Specifically, there is a smooth, absolutely irreducible, variety $\mathcal{T}_5$ with $\Psi_5 : \mathcal{T}_5 \to \mathcal{H}_\infty \times \mathbb{P}^1_z$ a finite degree 5 map having the following properties.

(1.8a) Ψ_5 and $\mathcal{T}_5$ have field of definition $\mathbb{Q}$.
(1.8b) $\mathcal{T}_5$ is absolutely irreducible.

(1.8c) For $\boldsymbol{p} \in \mathcal{H}_\infty$, the fiber over $\boldsymbol{p}$ is a cover $\Psi_{5,\boldsymbol{p}} : \mathcal{T}_{5,\boldsymbol{p}} \to \mathbb{P}_z^1$ equivalent to a polynomial map $f_{\boldsymbol{p}} : \mathbb{P}_y^1 \to \mathbb{P}_z^1$.

Equivalence in (1.8c) means there exists $\alpha : \mathcal{T}_{5,\boldsymbol{p}} \to \mathbb{P}_y^1$ with $f_{\boldsymbol{p}} \circ \alpha = \Psi_{5,\boldsymbol{p}}$, restriction of Ψ_5 to the fiber over $\boldsymbol{p}$.

Existence (and uniqueness) of Ψ_5 is a consequence of the following property of the absolute Nielsen class. The subgroup S_4 of S_5 defines the degree 5 coset representation giving the monodromy group of the cover. Also, S_4 is *self-normalizing*—consisting exactly of elements in S_5 normalizing S_4 [**Fri77**, §1]. This means covers in the family have no automorphisms. In particular, $\mathcal{H}_\infty$ is an irreducible fine moduli space for equivalence classes of covers in this Nielsen class.

Another group property translates arithmetically (see references above). As all conjugacy classes in S_n are rational, $\mathbf{C}$ consists of rational classes of S_5. This guarantees $\mathcal{H}_\infty$, $\mathcal{H}_{0,1,\infty}$ and the total families over them, have field of definition $\mathbb{Q}$ [**Fri77**, Thm. 5.1].

1.4. HIT motivation from [Fri86] for (1.8)**.** Let $\mathcal{A}$ be a fractional ideal of $\mathbb{Q}$. For $f \in \mathbb{Q}[y]$ denote $\{z_0 \in \mathcal{A} \mid f(y) - z_0 \text{ is reducible}\}$ by $\mathcal{R}_f(\mathcal{A})$. Also, denote $\mathcal{A} \cap f(\mathbb{Q})$ by $\mathcal{V}_f(\mathcal{A})$. Use the notation $f \in S_\mathcal{A}$ if the following holds:
(1.9) $f \in \mathbb{Q}[y]$ with $\mathcal{R}_f(\mathcal{A}) \setminus \mathcal{V}_f(\mathcal{A})$ infinite.

Theorem 1.1 (Explicit HIT Theorem [**Fri86**]). *Suppose* $f \in S_\mathcal{A}$. *Then:*

(1.10a) *Either* $\deg(f) = 5$; *or*
(1.10b) f *is decomposable over* $\mathbb{Q}$ (*top of* §1).

Further, if (1.10b) *doesn't hold, then either* f *is in the family* (1.8c), *or coalescing from* (1.8) *gives* f *as a cover with only two finite branch points* (§3.3).

[**DF90a**] discussed geometric and arithmetic properties of family (1.8). What it didn't show is that family (1.8) genuinely gives nontrivial examples of $f \in S_\mathcal{A}$. For $\boldsymbol{p} \in \mathcal{H}_\infty(\mathbb{Q})$, Main Theorem 1.2 explains when the corresponding polynomial cover $f_{\boldsymbol{p}} : \mathbb{P}_y^1 \to \mathbb{P}_z^1$ is in $S_\mathcal{A}$. An auxiliary cover $h_f : \mathbb{P}_w^1 \to \mathbb{P}_z^1$ plays a role in that result.

Form the 2-times fiber product of $f_{\boldsymbol{p}}$. Natural notation for this is

$$Z_{\boldsymbol{p}} = \mathbb{P}_y^1 \times_{\mathbb{P}_z^1} \mathbb{P}_y^1 = \{(y_1, y_2) \mid f_{\boldsymbol{p}}(y_1) = f_{\boldsymbol{p}}(y_2)\}.$$

This contains the diagonal component, $\Delta_2 = \{(y, y) \mid y \in \mathbb{P}^1\}$. Remove Δ_2 from $Z_{\boldsymbol{p}}$. The group S_2 acts on what remains by mapping (y_1, y_2) to (y_2, y_1). Form the quotient by S_2. Then normalize what you get in its function field. This gives a projective nonsingular degree 10 cover $\Psi_{5,\boldsymbol{p}}^{(2)} : \mathcal{T}_{5,\boldsymbol{p}}^{(2)} \to \mathbb{P}_z^1$. Symmetric functions in (y_1, y_2), subject to $f_{\boldsymbol{p}}(y_1) = f_{\boldsymbol{p}}(y_2)$, generate the function field of $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$.

1.5. Properties of $\Psi_{5,\boldsymbol{p}}^{(2)}$. Unordered pairs (ten) of distinct integers from $\{1,\ldots,5\}$ are letters for a permutation representation $R : S_5 \to S_{10}$. The cover $\Psi_{5,\boldsymbol{p}}^{(2)}$ has *geometric monodromy group* S_5 as a subgroup of S_{10} through this representation. Let $\boldsymbol{g}$ be a branch cycle description of the cover $f_{\boldsymbol{p}}$. Riemann-Hurwitz gives the genus g of $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$ [**DF90a**, p. 109]:

$$(1.11) \qquad\qquad 2(10 + g - 1) = \sum_{i=1}^{4} \mathrm{ind}(R(g_i)).$$

If g_1 and g_2 are 2-cycles from S_5, then $R(g_i)$ has shape $(2)(2)(2)$, $i = 1, 2$. Similarly, if g_3 is the product of two disjoint 2-cycles, then $R(g_3)$ has shape $(2)(2)(2)(2)$. Finally, $R(g_4)$ has shape $(5)(5)$. Thus, the total contribution to the right side of (1.11) is $2 \cdot 3 + 4 + 2 \cdot 4 = 18$: $g = 0$.

Apply the fiber product construction above directly to $\Psi_5 : \mathcal{T}_5 \to \mathcal{H}_\infty \times \mathbb{P}_z^1$. As above, quotient by S_2 and normalize to get

$$\Psi_5^{(2)} : \mathcal{T}_5^{(2)} \to \mathcal{H}_\infty \times \mathbb{P}_z^1.$$

1.6. Explanation of the Main Theorem. Suppose a polynomial function $f = f_{\boldsymbol{p}}$ corresponds to a point $\boldsymbol{p} \in \mathcal{H}_\infty(\mathbb{Q})$. There are two possibilities.

(1.12a) $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$ has a $\mathbb{Q}$ point and is $\mathbb{Q}$ isomorphic to $\mathbb{P}_w^1$.

(1.12b) $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$ has no $\mathbb{Q}$ point and is $\mathbb{Q}$ isomorphic to a conic $C_{\boldsymbol{p}} \subset \mathbb{P}^2$.

For case (1.12a), $\Psi_{5,\boldsymbol{p}}^{(2)} : \mathcal{T}_{5,\boldsymbol{p}}^{(2)} \to \mathbb{P}_z^1$ is equivalent to a cover $h_f : \mathbb{P}_w^1 \to \mathbb{P}_z^1$.

Let $\mathcal{A}$ be any nontrivial fractional ideal of $\mathbb{Q}$ and consider $S_{\mathcal{A}}$ from §1.4. Part I of Th. 1.2 below clarifies the necessary condition for $f \in S_{\mathcal{A}}$ of Th. 1.1. Part II is preliminary to Part III, which is the main motivation of this paper; there are nontrivial examples of $f \in S_{\mathcal{A}}$.

Theorem 1.2 (Main Theorem). I. *Suppose* $f \in S_{\mathcal{A}}$ *is indecomposable* ((1.10b) *does not hold*). *Then:*

(1.13a) *Either* $f = f_{\boldsymbol{p}}$ *for some* $\boldsymbol{p} \in \mathcal{H}_\infty(\mathbb{Q})$ *and* (1.12a) *holds; or*

(1.13b) $f(y) = \beta(v(\alpha(y)))$ *with* v *among an explicit finite set of polynomials,* $\alpha(y) = ay + b$ *and* $\beta(y) = cy + d$ *for some* $a, b, c, d \in \mathbb{Q}$.

II. *Suppose* (1.13a) *holds. Then,* $\mathcal{R}_f(\mathcal{A})$ *is the union of* $(f(\mathbb{Q}) \cup h_f(\mathbb{Q})) \cap \mathcal{A}$ *with a finite set* V *and* $f(\mathbb{Q}) \cap h_f(\mathbb{Q}) \cap \mathcal{A}$ *is finite.*

III. *For each* $\mathcal{A}$, $\{\boldsymbol{p} \in \mathcal{H}_\infty \mid f_{\boldsymbol{p}} \in S_{\mathcal{A}}\}$ *is Zariski dense.*

Much of Parts I and II are in [**Fri86**] ([**Fri99**, §1.4] documents most of this). What remains of Part I is to codify the second possibility of Th. 1.1. That is, to show the entire contribution to (1.13b) comes from coalescing branch points of covers in (1.8). §3.2 shows there are three polynomials v_1, v_2, v_3 giving all the covers to which (1.13b) refers. §3.3 explains how each v_i derives from *coalescing* (specializing branch points of) a general

member of the family over $\mathcal{H}_\infty$. For Part II, that leaves only finiteness of $f(\mathbb{Q}) \cap h_f(\mathbb{Q}) \cap \mathcal{A}$. This is condition (1.14c) of Prop. 1.3. Part III follows from Part II and the two other assertions of Prop. 1.3. §2.3 starts the proof of Prop. 1.3. This completes applying [**DF90a**] to Main Theorem 1.2.

Proposition 1.3 (Main Proposition)**.** *In the notation above, the following hold.*

(1.14a) $\boldsymbol{p} \in \mathcal{H}_\infty(\mathbb{Q})$ *with* (1.12a) *holding are dense.*
(1.14b) *For a dense set of $\boldsymbol{p}$ from* (1.14a)*, $h_{f_{\boldsymbol{p}}}(\mathbb{Q}) \cap \mathcal{A}$ is infinite.*
(1.14c) *For all $\boldsymbol{p} \in \mathcal{H}_\infty(\mathbb{Q})$, $f_{\boldsymbol{p}}(\mathbb{Q}) \cap h_{f_{\boldsymbol{p}}}(\mathbb{Q}) \cap \mathcal{A}$ is finite.*

1.7. Comments on families of rational functions. [**DF90a**] took a pullback of the family of (1.8) to another space $\mathcal{P}$. This pullback gave simple coordinates for $\mathcal{T}_{5,\boldsymbol{P}}^{(2)}$. §2.1 includes explicit explanation of this family. It is a *family of polynomials*. This is more than equivalence of each cover in the family to a polynomial cover.

Definition 1.4. *Family of rational functions.* Call $\Psi : \mathcal{T} \to \mathcal{H} \times \mathbb{P}_z^1$ a *smooth family of genus zero covers* of degree n if the following hold.

(1.15a) Ψ is a finite morphism of smooth varieties.
(1.15b) For $\boldsymbol{p} \in \mathcal{H}$, $\mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}_z^1$ is a degree n cover of (nonsingular) genus 0 curves.

Call Ψ a *family of rational functions* if the following hold. There is an isomorphism $\Gamma : \mathcal{T} \to \mathcal{H} \times \mathbb{P}_y^1$ and a degree n smooth morphism $\Psi' : \mathcal{H} \times \mathbb{P}_y^1 \to \mathcal{H} \times \mathbb{P}_z^1$ with $\Psi' \circ \Gamma = \Psi$. It is a *family of polynomials* if the only points in $\mathcal{H} \times \mathbb{P}_y^1$ over $\mathcal{H} \times \infty \subset \mathcal{H} \times \mathbb{P}_z^1$ are in $\mathcal{H} \times \infty$.

Recall the component $\mathcal{H}^\infty$ of the fiber product $\mathcal{H}_\infty \to U_3$ with $\mathbb{A}^3 \setminus \Delta_3 \to U_3$ (§1.3). Projection from this component induces a map

$$H_{et}^1(\mathcal{H}_\infty, \mathbb{Z}/2) \to H_{et}^1(\mathcal{H}^\infty, \mathbb{Z}/2).$$

[**DF90a**, §2.3] produced $\alpha \in H_{et}^1(\mathcal{H}_\infty, \mathbb{Z}/2)$ with these properties.

(1.16a) α is trivial if and only if $\Psi_5 : \mathcal{T}_5 \to \mathcal{H}_\infty \times \mathbb{P}_z^1$ is a family of rational (or polynomial) functions.
(1.16b) The image $\alpha^* \in H_{et}^1(\mathcal{H}^\infty, \mathbb{Z}/2)$ of α is trivial.

Fibers in a family of rational functions over a $\mathbb{Q}$ point must have many $\mathbb{Q}$ points. [**DF90a**, Th. 4.2] shows there is a dense set of $\boldsymbol{p} \in \mathcal{H}_\infty(\mathbb{Q})$ with no rational point on the fiber $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$ of the family

$$(1.17) \qquad\qquad \Psi_5^{(2)} : \mathcal{T}_5^{(2)} \to \mathcal{H}_\infty \times \mathbb{P}_z^1.$$

In particular, (1.17) can't be a family of rational functions. Yet, the general $\mathbb{Q}$ fiber of the pullback of (1.17) over $\mathcal{H}^\infty$ (see §2.1) does have rational points.

Problem 1.5. Is α nontrivial (so (1.16a) is not a family of rational functions)? Is the pullback of (1.17) over $\mathcal{H}^\infty$ a family of rational functions?

§4.1 describes other Siegel families appropriate for versions of Main Theorem 1.2. §5.1 updates Hilbert-Siegel problems analogous to HIT Theorem 1.1. It also discusses Müller's result [**Mül96**] extending both Theorem 1.1 and results from [**Fri86**] on varying coefficients of polynomial functions other than the constant term.

2. Pullback family and proof of Main Theorem.

[**DF90a**, §2.1] considers (1.17). This is a family of degree 10 covers — by genus 0 curves — with associated Nielsen class $\mathrm{Ni}(S_5, \mathbf{C}, R)$. Here R indicates the permutation representation of S_5 in S_{10} is that of the monodromy group of these degree 10 covers. The point $z_4 = \infty$ ramifies in any fiber $\mathcal{T}_{5,\boldsymbol{p}}^{(2)} \to \mathbb{P}_z^1$ with cycle type $(5)(5)$. The first step to prove the Main Theorem is finding a Zariski dense subset of $\boldsymbol{p} \in \mathcal{H}_\infty(\mathbb{Q})$ where $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$ has one (and therefore, infinitely many) $\mathbb{Q}$ points.

2.1. Family from adjoining branch points. Suppose $\mathcal{P}$ is any $\mathbb{Q}$ variety whose $\mathbb{Q}$ points are Zariski dense. If $\beta : \mathcal{P} \to \mathcal{H}_\infty$ is a dominant $\mathbb{Q}$ birational morphism, then any dense set of $\mathbb{Q}$ points in $\mathcal{P}$ will have Zariski dense image in $\mathcal{H}_\infty$. Thus, to find Zariski dense $\mathbb{Q}$ points on $\mathcal{H}_\infty$, it may pay to pull back to any such variety $\mathcal{P}$. Our choice: Take $\mathcal{P} = \mathcal{H}^\infty$ (§1.3). One valuable ingredient is that §6.1 applied to §1.3 shows $\mathcal{H}^\infty$ is a $\mathbb{Q}$ rational variety.

Denote the pullback of $\mathcal{T}_5^{(2)}$ to $\mathcal{H}^\infty$ by $\bar{\mathcal{T}}_5^{(2)}$. It has the following geometric property [**DF90a**, §2.1].

(2.1)   The natural projection $\bar{\Psi}_5^{(2)} : \bar{\mathcal{T}}_5^{(2)} \to \mathcal{H}^\infty \times \mathbb{P}_z^1$ is a family (smooth and irreducible) of degree 10 covers with fibers in $\mathrm{Ni}(S_5, \mathbf{C}, R)$.

Further, the family (2.1) has these crucial arithmetic properties.

(2.2a) Over a Zariski open subset of $\mathcal{H}^\infty$, $\mathrm{proj}_1 \circ \bar{\Psi}_5^{(2)} : \bar{\mathcal{T}}_5^{(2)} \to \mathcal{H}^\infty$ has a section.

(2.2b) For each $\boldsymbol{p}_0 \in \mathcal{H}^\infty(\mathbb{Q})$, $\bar{\mathcal{T}}_{5,\boldsymbol{p}_0}^{(2)}(\mathbb{Q})$ is nonempty.

Here is how you get the section s in (2.2a).

As in §1.3, z_2 is a branch point of the cover corresponding to the shape (2) conjugacy class in S_5. In the representation R, the conjugacy class has shape $(2)(2)(2)$ (§1.5). Let $D_{\boldsymbol{p}}$ be the divisor on $\bar{\mathcal{T}}_{5,\boldsymbol{p}}^{(2)}$ consisting of the three ramified points lying over z_2 in the map $\bar{\Psi}_{5,\boldsymbol{p}}^{(2)} : \bar{\mathcal{T}}_{5,\boldsymbol{p}}^{(2)} \to \mathbb{P}_z^1$. Then, let $D'_{\boldsymbol{p}}$ be the divisor of the *differential* of the function $\bar{\Psi}_{5,\boldsymbol{p}}^{(2)}$. Since $\bar{\mathcal{T}}_{5,\boldsymbol{p}}^{(2)}$ has genus 0, $D'_{\boldsymbol{p}}$ has degree -2. On a genus zero curve, all divisors of a given degree are linearly equivalent. Further, on any (nonsingular projective) curve over K, functions over K generate any linear system whose divisor has field of

definition K. Thus, the linear equivalence class of $D_{\boldsymbol{p}}+D'_{\boldsymbol{p}}$ contains a positive (degree 1) $\mathbb{Q}(\boldsymbol{p})$ divisor. This is therefore a point $s(\boldsymbol{p})$ on $\bar{\mathcal{T}}^{(2)}_{5,\boldsymbol{p}}$. Apply this to a generic point $\boldsymbol{p}$ of $\mathcal{H}^{\infty}$. This defines a map $\boldsymbol{p}_0 \mapsto s(\boldsymbol{p}_0)$ for $\boldsymbol{p}_0$ in a Zariski open subset of $\mathcal{H}^{\infty}$ [**DF90a**, Ex. 3.5]. The argument applies to each point of $\mathcal{H}^{\infty}(\mathbb{Q})$ providing a rational point on the fiber of that point. Note: $s(\boldsymbol{p})$ is not unique. This gives (2.2b).

2.2. Real conjugate points over ∞. Consider $\boldsymbol{p} \in \mathcal{H}^{\infty}$. Let $\boldsymbol{z}$ be the image of $\boldsymbol{p}$ in U_3 as in §2.1. Consider the behavior over ∞ of the cover

$$(2.3) \qquad\qquad \bar{\Psi}^{(2)}_{5,\boldsymbol{p}} : \bar{\mathcal{T}}^{(2)}_{5,\boldsymbol{p}} \to \mathbb{P}^1_z.$$

Two points, $x_{1,\infty}, x_{2,\infty}$, on $\bar{\mathcal{T}}^{(2)}_{5,\boldsymbol{p}}$ lie over $z=\infty$. These correspond to the two disjoint 5-cycles of $R(g_4)$. (Recall: g_4 is the last member of a branch cycle description of the cover.) [**DF90b**] gave a general test for points in the fiber of a cover $\psi : X \to \mathbb{P}^1_z$ to be real. Suppose $\mathbb{R}$ contains a field of definition of (X, ψ). Then, the branch points are permuted by complex conjugation. That is, as a set they are in $\mathbb{R}$. The most interesting case is when the fiber X_{z_0} is that over a branch point z_0. Our explicit description of this family allows checking this less abstractly. Still, the efficient group theoretic approach works on examples like those of §4.2 (specifically see §4.3.1 and §4.3.2) without having their explicit equations.

For $G \leq S_n$ denote the normalizer of G in S_n by $N_{S_n}(G)$. In the general case, a given branch point z_i is either in $\mathbb{R}$, or it is complex conjugate to another branch point $z_{i'}$. Let $\bar{C}_i$ be the conjugacy class in the group $N_{S_n}(G)$ containing C_i. If z_i and $z_{i'}$ are conjugate, then the *branch cycle argument* of, say, [**Fri95a**] gives an involution in $N_{S_n}(G)$ conjugating $\bar{C}_i$ to $\bar{C}_{i'}$. For $\boldsymbol{p} \in \mathcal{H}_{\infty}(\mathbb{R})$, it is possible z_2 and z_3 are complex conjugate. A cover, however, associated to $\boldsymbol{p} \in \mathcal{H}^{\infty}(\mathbb{R})$ must have real branch points. Since our covers will come from points of $\boldsymbol{p} \in \mathcal{H}^{\infty}(\mathbb{R})$, the next lemma suffices. (There is, however, a version for the case where pairs of branch points are complex conjugate.) It illustrates how [**Fri95c**, App.C] detects real points on Hurwitz spaces.

Lemma 2.1 (Real Point Lemma [**DF90b**]). *Let* $z_1 < z_2 \cdots < z_{r-1} \in \mathbb{A}^1$ *be real points. Suppose* $\mathbf{C} = (C_1, \ldots, C_r)$ *are conjugacy classes of* G*; classes* $C_1, \ldots, C_{r-1}$ *correspond to the branch points* $z_1, \ldots, z_{r-1}$*. Let* $\varphi_i : X_i \to \mathbb{P}^1$*,* $i = 1, \ldots, t$*, be a complete set of representatives of equivalence classes of covers in the absolute Nielsen class* $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}_{\infty}$ *having* $z_1 < z_2 \cdots < z_{r-1}$ *and* ∞ *(corresponding to* C_r*) as branch points. Let* M *be those from* $\{\varphi_i : X_i \to \mathbb{P}^1\}^t_{i=1}$ *equivalent to a cover with field of definition* $\mathbb{R}$*. There is a one-one correspondence between elements of* M *and* $\boldsymbol{g} = (g_1, \ldots, g_r) \in$ $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}_{\infty}$ *with the following property. There is an involution* $G' \in N_{S_n}(G)$ *with*

$$G' g_i G'^{-1} = g_1 \cdots g_{i-1} g_i^{-1} g_{i-1}^{-1} \cdots g_1^{-1}, \ i = 1, \ldots, r.$$

If no elements of S_n centralize G, then $C = C_{\boldsymbol{g}}$ is unique. Call C the complex conjugation operator associated to a cover with branch cycles $\boldsymbol{g}$ (see statement on the association below).

Let g_r be the branch cycle corresponding to ∞; assume it nontrivial. Since $g_1 \cdots g_r = 1$, this says $C\,g_r C = g_r^{-1}$. In particular, $C = C_{\boldsymbol{g}}$ conjugates any disjoint cycle β of g_r to the inverse of another disjoint cycle in g_r. If there are exactly two such disjoint cycles, then either $C\,\beta C = \beta^{-1}$, or $C\,\beta C \neq \beta^{-1}$. Suppose $\varphi' : X' \to \mathbb{P}^1$ is a cover from M. Let C' be the complex conjugation operator corresponding to it by hypothesis. Then, real points of X' over ∞ correspond exactly to disjoint cycles β with $C'\beta C' = \beta^{-1}$.

The Real Point Lemma 2.1 correspondence comes from a specific bouquet of paths on the punctured sphere $\mathbb{P}^1 \setminus \{\boldsymbol{z}\}$ [**DF90b**, §2]. Compute branch cycles of $\varphi_i : X_i \to \mathbb{P}^1$, $i = 1, \ldots, t$, in Lemma 2.1 from this set of paths to get the correspondence in one direction. Cor. 2.2 requires letters for the representation R:

$$s'_i = \{1, i+1\}, \ i = 1, 2, 3, 4, \quad s'_{4+i} = \{2, i+2\}, \ i = 1, 2, 3,$$
$$s'_8 = \{3, 4\}, \quad s'_9 = \{3, 5\}, \quad s'_{10} = \{4, 5\}.$$

If $\boldsymbol{p} \in \mathcal{H}^\infty(\mathbb{R})$, Cor. 2.2 shows the cover $(2.3_{\boldsymbol{p}})$ passes the *necessary* Siegel Theorem condition for a positive conclusion to Main Proposition 1.3: (1.14b) can't hold unless the two points of $h_{\boldsymbol{p}}^{-1}(\infty)$ are real conjugates.

Corollary 2.2. *Consider the fifteen equivalence classes of covers having branch cycles $\boldsymbol{g}$ in $\mathrm{Ni}(S_5, \mathbf{C}, R)$ and three finite branch points $(z_1, z_2, z_3) = \boldsymbol{z} \in \mathbb{Q}^3$. From those five with $g_1 \in \mathrm{C}_1$ (and $g_4 \in \mathrm{C}_4$), exactly two have field of definition $\mathbb{R}$. Suppose one of these, $\varphi : X \to \mathbb{P}^1$, has field of definition $\mathbb{Q}$. Let $x_{1,\infty}$ and $x_{2,\infty}$ be the geometric points of $\varphi^{-1}(\infty)$. These are conjugate over $\mathbb{Q}(\sqrt{5})$.*

Proof. Apply representation R to elements in List A to get branch cycles for the covers of the corollary (see Lemma 2.1). [**DF90a**, Lemma 2.1] shows S_5 is self-normalizing in S_{10}. The information that some $\alpha \in S_5$ takes g_1 (product of two disjoint 2-cycles) to g_1^{-1} and g_4 (5-cycle) to its inverse determines $C = R(\alpha)$. For example, apply R to entry c), $((2\,3)(4\,5), (2\,4), (1\,2))$ from List A. A calculation shows $C = R((2\,5)(3\,4))$ satisfies the criterion of Lemma 2.1:

$$C\,R((2\,4))C = R((2\,3)(4\,5)(2\,4)(2\,3)(4\,5)) = R((3\,5)).$$

Also, if some C works for List A items a) and b), it must be $R(\alpha)$. Note, however,

$$C\,R((1\,2))C \neq R((2\,3)(4\,5)(1\,2)(2\,3)(4\,5)) = R((1\,3)) \quad \text{and}$$
$$C\,R((1\,4))C \neq R((2\,3)(4\,5)(1\,4)(2\,3)(4\,5)) = R((1\,5)).$$

Similarly, for List A items d) and e): α works for e); nothing works for d).

Now check the Lemma 2.1 criterion for the field of definition of the points over ∞ for the covers corresponding to c) and e). Use i in place of s_i' to simplify notation. The two disjoint cycles of $R(g_4)$ are then $\beta_1 = (1\,5\,8\,10\,4)^{-1}$ and $\beta_2 = (2\,6\,9\,3\,7)^{-1}$. Apply Lemma 2.1. That our choice of C works and is unique is easy. Now check that conjugation by C takes β_1 to $(8\,5\,1\,4\,10)^{-1} = \beta_1^{-1}$. This shows $x_{1,\infty}$ and $x_{2,\infty}$ have coordinates in $\mathbb{R}$.

Finally, suppose one of these covers, $\varphi : X \to \mathbb{P}_z^1$, has field of definition $\mathbb{Q}$. We show $x_1 = x_{1,\infty}$ and $x_2 = x_{2,\infty}$ aren't $\mathbb{Q}$ points. Let $\zeta_5 = e^{\frac{2\pi i}{5}}$. Consider $f : \mathbb{P}_y^1 \to \mathbb{P}_z^1$, the corresponding degree 5 polynomial cover. It totally ramifies over ∞. So, all Puiseux expansions for local functions on $\mathbb{P}_y^1$ over $z = \infty$ embed in $\mathbb{Q}(\zeta_5)\{\{z^{-1/5}\}\}$: Laurent series in $z^{-1/5}$ over $\mathbb{Q}(\zeta_5)$. Thus, the Galois closure Ω of the extension $\mathbb{Q}(y)/\mathbb{Q}(z)$ embeds in $\mathbb{Q}(\zeta_5)\{\{z^{-1/5}\}\}$. Since the Galois closure of the field $\mathbb{Q}(X)/\mathbb{Q}(z)$ is also Ω, $K = G(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ acts on coefficients of Puiseux expansions around x_1 and x_2. A generator γ of K acting on these expansions has order four. The square of $R(\gamma)$ gives the effect of C. So, $\gamma = (2\,3\,5\,4)$ or its inverse. Conjugation by $R(\gamma) = (1\,2\,4\,3)(7\,8)(5\,9\,10\,6)$ takes β_1 to $(\beta_2)^{-2}$: x_1 and x_2 are conjugate over the real quadratic subfield of $\mathbb{Q}(\zeta_5)$. $\qquad\square$

Remark 2.3. Q_1 *and* Q_1Q_2 *applied to List A.* For given z_1, z_2, z_3, as in Cor. 2.2, apply Q_1 and Q_1Q_2 to List A to get 10 other covers in the Nielsen class with the same branch points. Some of these covers will also have field of definition $\mathbb{R}$ and some won't. For example, apply Q_1 to a) of List A to get $((1\,3),(2\,3)(4\,5),(1\,4))$. This has $C = R((1\,3)(4\,5))$ as complex conjugation operator.

2.3. Proof of Main Prop. 1.3. We must prove (1.14a-c). We follow these points for the rest of §2.

> **(P$_1$):** For $\boldsymbol{p} \in \mathcal{H}^\infty(\mathbb{Q})$, apply [**DF90a**, §4.4] to identify $\bar{\mathcal{T}}_{5,\boldsymbol{p}}^{(2)}$ with a conic $\mathcal{C}_{\boldsymbol{p}}$.
>
> **(P$_2$):** Find a rational point on $\mathcal{C}_{\boldsymbol{p}}$.
>
> **(P$_3$):** Use Euler's parametrization to identify $\bar{\mathcal{T}}_{5,\boldsymbol{p}}^{(2)}$ with $\mathbb{P}_w^1$.
>
> **(P$_4$):** Present the covering map $\bar{\Psi}_{5,\boldsymbol{p}}^{(2)} : \bar{\mathcal{T}}_{5,\boldsymbol{p}}^{(2)} \to \mathbb{P}_z^1$ as a function $h_{\boldsymbol{p}}(w)$ in w.
>
> **(P$_5$):** Study values of $h_{\boldsymbol{p}}$.

A further pullback of family (2.1) seems necessary for task (P$_4$) for (as in §1.7) we need to replace (2.1) by a family of rational functions. [**DF90a**, Theorem 4.2] (see §2.4) gives $\bar{\mathcal{T}}_{5,\boldsymbol{p}}^{(2)}$ as a cover of $\mathbb{P}_z^1$ by a conic:

$$(2.4) \quad \left\{ (U,V) \mid U^2 + V^2 - 3UV - \frac{5}{4}(c+d)U + \frac{5}{2}(c+d)V - 5cd = 0 \right\}.$$

For later, refer to the parameter space of this family by the notation $\mathcal{P}_{\alpha,\beta,c,d}$. Its points correspond to the parameters (α, β, c, d) that appear below. Here c and d are in $\mathbb{Q}$. They are coordinates for the unique ramified points on $\mathcal{T}_{5,\boldsymbol{p}}$ above z_2 and z_3, the 2-cycle branch points. [**DF90a**, Prop. 2.5] pulls back the family of (1.8) over $\mathbb{G} \times \mathbb{A}^2 \times \mathbb{G} \overset{\text{def}}{=} \mathcal{P}$ with $\mathbb{G} = \mathbb{A}^1 \setminus \{0\}$. This gives

$$f_{\alpha,\beta,s,t}(y) = \alpha(y^5/5 - sy^4/4 + 2ty^3 - 5sty^2/2 + 5t^2y) + \beta.$$

Here $s = c + d, t = cd$. With it is associated an allied parameter space $\mathcal{P}'_{\alpha,\beta,s,t} = \mathcal{P}'$. Later we use the map $\varphi^* : \mathcal{P} \to \mathcal{H}_\infty$ sending (α, β, c, d) to the equivalence class of the cover $f_{\alpha,\beta,s,t} : \mathbb{P}^1_y \to \mathbb{P}^1_z$. Apply the fiber product construction of §1.4 directly to $f_{\alpha,\beta,s,t}$ to produce the pullback of (2.4) to $\mathcal{P}$. Denote this by

$$(2.5) \qquad \qquad \bar{\Psi}_{\alpha,\beta,s,t} : \overline{\mathcal{C}}_{\alpha,\beta,s,t} \to \mathbb{P}^1_z$$

as in [**DF90a**, §4.4]. Equation (2.4) is an affine portion of (2.5).

2.4. Rational parametrization of (2.5)**.** Use the coordinates y_1 and y_2 at the end of §1.4. [**DF90a**, Lemma 3.18] produces a rational point $\boldsymbol{m} \in \overline{\mathcal{C}}_{\alpha,\beta,s,t}$ over z_2: $(y_1 + y_2)(\boldsymbol{m}) = 2c$ and $(y_1y_2)(\boldsymbol{m}) = c^2$. Equivalently,

$$U(\boldsymbol{m}) = (y_1 + y_2)(\boldsymbol{m}) = U_0 = 2c \text{ and}$$

$$V(\boldsymbol{m}) = (y_1y_2 - 5cd)/(y_1 + y_2)(\boldsymbol{m}) = V_0 = \frac{c^2 - 5cd}{2c} = \frac{c - 5d}{2}.$$

Here is how $\boldsymbol{m}$ appears in the fiber product construction of §1.4.

Regard $\overline{\mathcal{C}}_{\alpha,\beta,s,t}$ as the normalization of one of the fiber products $Z_{\boldsymbol{p}} = \mathbb{P}^1_{y_1} \times_{\mathbb{P}^1_z} \mathbb{P}^1_{y_2}$ modulo the S_2 action. Let y' be the unique ramified point over z_2 in the degree 5 cover $f_{\boldsymbol{p}} : \mathbb{P}^1_y \to \mathbb{P}^1_z$. Then, $\boldsymbol{m}$ is the preimage of (y', y').

Let w be the parameter for tangent directions in $\mathbb{P}^2$ at the point $\boldsymbol{m}$. Euler's parametrization writes $U = wt + U_0$, $V = t + V_0$. Solve for the value of $t \neq 0$ giving an intersection different from (U_0, V_0) with the conic if there is one. This displays coordinates for the conic as a function of w:

(2.6)
$$U(w) = \frac{8cw^2 + (-14c + 10d)w + 3c - 25d}{4(w^2 - 3w + 1)}, \qquad \qquad \text{and}$$

$$V(w) = \frac{12cw^2 + (-11c + 5d)w + 2(c - 5d)}{4(w^2 - 3w + 1)} \qquad \text{where } w = \frac{V - \frac{c-5d}{2}}{U - 2c}.$$

Start with z as a function of U and V [**DF90a**, proof of Prop. 4.1] and rewrite $\bar{\Psi}_{\alpha,\beta,s,t}$ using w :

$$(2.7) \qquad z = \frac{\alpha}{2}\left[(U^5/5 - U^4V + U^3V^2) - \frac{c+d}{4}(U^4 - 4U^3V + 2U^2V^2)\right.$$

$$\left. + cd(-3U^3 + 4U^2V) + \frac{5}{2}cd(c+d)U^2 + \frac{25}{2}(cd)^2(c+d)\right] + \beta.$$

2.5. Values of $z(w)$ (from $h_{\boldsymbol{p}}(w)$ in (P_4)). Fix a fractional ideal $\mathcal{A} = \delta\mathbb{Z}$ of $\mathbb{Q}$. Make the following further assumption:

(2.8) $\alpha, \beta, c, d, \frac{\alpha}{10\delta}$ and $\frac{\beta}{\delta}$ are integers.

The next two results complete the proof of Main Prop. 1.3 (and Main Theorem 1.2). Substitute (2.6) in (2.7) to complete computing $z(w)$.

Proposition 2.4. *As w runs over $\mathbb{Q}$, $z(w)$ takes on infinitely many values in $\mathcal{A}$. Thus, the subset of $\mathcal{H}_{\infty}(\mathbb{Q})$ satisfying the conclusion of Prop. 1.3 is Zariski dense.*

Proof. Write w as m/n with $m, n \in \mathbb{Z}$. Then (2.6) becomes

$$U = \frac{8cm^2 + (-14c + 10d)mn + (3c - 25d)n^2}{4(m^2 - 3mn + n^2)}, \qquad \text{and}$$

$$V = \frac{12cm^2 + (-11c + 5d)mn + 2(c - 5d)n^2}{4(m^2 - 3mn + n^2)}.$$

Rewrite $m^2 - 3mn + n^2$ as $(m - \frac{3n}{2})^2 - \frac{5n^2}{4}$. The fundamental unit of $\mathbb{Q}(\sqrt{5})$ is $(1 + \sqrt{5})/2$. Its 6th power is $\theta = 9 + 4\sqrt{5}$. The components (u, v) of powers of θ give integer solutions of $u^2 - 5v^2 = 1$, with $v \equiv 0 \bmod 4$. Each such solution (u, v) gives a solution $(m, n) \in \mathbb{Z}^2$ of $m^2 - 3mn + n^2 = 1$ with $n \equiv 0 \bmod 4$. Further, with this holding, there are infinitely many (m, n) for which (2.8) forces $U(m/n)$ to be a multiple of 2, $V(m/n)$ to be an integer and $z(m/n) \in \delta\mathbb{Z}$.

Use the map $\varphi^* : \mathcal{P} \to \mathcal{H}_{\infty}$ from §2.3. Let $\mathcal{O}'$ be the open subset of $\mathcal{P}$ consisting of 4-tuples (α, β, c, d) mapping by φ^* to $(\alpha, \beta, c + d, cd) \in \mathcal{H}_{\infty}$ [**DF90a**, §4.1]. Further, let Ω' be the subset of $\mathcal{O}'$ satisfying (2.8).

Then Ω' is Zariski dense in $\mathcal{O}'$. Further, points of Ω' give a cover $\overline{\mathcal{C}}_{\alpha,\beta,s,t} \to \mathbb{P}^1_z$ for which the conclusion of the Main Theorem holds. The image Ω' in $\mathcal{H}_{\infty}$ is clearly Zariski dense. To finish only requires showing (1.14c) holds. The next proposition does exactly that. $\qquad\square$

Proposition 2.5. *Suppose $f = f_{\boldsymbol{p}}$ corresponds to $\boldsymbol{p} \in \mathcal{P}(\mathbb{Q})$ as above. Then, $h_f(\mathbb{Q}) \cap f(\mathbb{Q}) \cap \mathcal{A}$ is finite.*

Proof. We show $h_f(\mathbb{Q}) \cap f(\mathbb{Q}) \cap \mathcal{A}$ is finite. This means showing the normalization W of the fiber product $\mathbb{P}^1_y \times_{\mathbb{P}^1_z} \mathbb{P}^1_w$ has only finitely many $\mathbb{Q}$ points over values of z in $\mathcal{A}$. Siegel's own result shows this if each absolutely irreducible component of the fiber product has genus at least 1. The remainder

of the proof shows there are two such components, one of genus 1, the other
of genus 2.

Since W covers $\mathbb{P}_y^1$, components of W correspond to classes of conjugate
points on W over a generic point y on $\mathbb{P}_y^1$. Points on W over y have the
form, (y, w_i), $i = 1, \ldots, 10$. Let H be the stabilizer of 1 in the representa-
tion $R : S_5 \to S_{10}$ of S_5. Galois theory corresponds components of W to
orbits of H in the representation R. There are two orbits: $\{s_1', \ldots, s_4'\}$; and
the complement of this set of symbols from §2.2. Label the corresponding
components as W_1 and W_2.

Abhyankar's Lemma in §7 lets us list points from $\mathbb{P}_y^1$ ramified in W:

(2.9a) The unique value y_1 unramified over z_1;

(2.9b) the three points $y_{2,j}$, $j = 1, 2, 3$, unramified over z_2; and

(2.9c) the analogous three points $y_{3,k}$, $k = 1, 2, 3$, unramified over z_3.

Branch cycles for these points are conjugates, fixing 1 in the standard repre-
sentation of S_5, of branch cycles for the corresponding points z_i, $i = 1, 2, 3$.
This information suffices to apply the Riemann-Hurwitz formula to $W_i/\mathbb{P}_y^1$.

So, $2(4 + g(W_1) - 1) = e_1 + e_2 + e_3$: e_1, the contribution from y_1, is 2; e_2,
from the three points over z_2, is 3; and so is e_3. Thus, $g(W_1)$ is 1.

Here is the cycle type of the action of the branch cycles for W_2 on the
symbols, s_j', $j = 5, \ldots, 10$. (Use only their subscripts to identify them.)

(2.10a) $(2\,3)(4\,5)$ acts on the last six symbols as $(6\,9)(7\,8)$.

(2.10b) $(2\,3)$ acts as $(6\,8)(7\,9)$.

Similar to above: $2(6 + g(W_2) - 1) = 2 + 3 \cdot 2 + 3 \cdot 2 = 14$, or $g = 2$.   □

3. Completion of the Main Theorem with comments.

§3.1 analyzes the totally arithmetic aspects of the proof of Thm. 1.2. §3.2
has exceptional degree 5 polynomials giving covers of $\mathbb{P}^1$ with only two finite
branch points. Treating these in detail completes a missing item from the
Main Theorem.

3.1. Comments on the proof of Thm. 1.2. Preliminaries to Thm. 1.2
used branch cycle description and the group theory of Nielsen classes. Re-
ducing computations to group theory can be more efficient than manipu-
lating equations. Hurwitz type families support translation to group theory
because they have a natural moduli definition. Thus, they include *all* appro-
priate covers even when it would be prohibitive to write equations for these.
[**DF90a**] (and related papers) suggest how to detect diophantine properties
of members of these families from group theory alone. You never see the
coefficients of the equations, so their complexity is irrelevant. Thm. 1.2 in-
cludes a diophantine question we haven't yet reduced to group theory. The
following remarks discuss related points in the proof.

Remark 3.1. The highest homogeneous part of conic (2.4) (in U and V) is constant in c and d. [**DF90a**, p. 135] finds U and V as generators of the linear system of the divisor $\boldsymbol{m}_1 + \boldsymbol{m}_2$ on $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$ over $z = \infty$. The fiber product construction and Abhyankar's Lemma ($\S 7$) trace this to a property of the Puiseux expansions for the cover $f_{\boldsymbol{p}} : \mathbb{P}_y^1 \to \mathbb{P}_z^1$ around $z = \infty$. It is that a constant field extension, $\mathbb{Q}(\zeta_5)$, over the coordinates for $\boldsymbol{p}$ contains the coefficients of these expansions.

Remark 3.2. Eulerian substitution in (2.4) replaces U by $wt + U_0$ and V by $t + V_0$. Then solve (2.4) for $t \neq 0$ giving the other intersection of this line through (U_0, V_0) with the conic. Since (U_0, V_0) are polynomials in c and d, Remark 3.1 implies c and d appear only in the numerators of $U(w)$ and $V(w)$ from (2.6). The functions $U = y_1 + y_2$ and $V = (y_1 y_2 - 5cd)/(y_1 + y_2)$ generate the function field of $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$. Their polar divisors have support in the divisor on $\mathcal{T}_{5,\boldsymbol{p}}^{(2)}$ over $z = \infty$. Thus, z in (2.7) is a polynomial in all nonconstant variables excluding w. So, the denominator of z is a power of the denominator of U (and V). These conclusions therefore come from Remark 3.1 and that (U_0, V_0) ($\S 2.4$) are polynomials in c and d. Will this hold generally when *ramification provides rational points* (the major theme of [**DF90a**])?

Remark 3.3. Expression $4(w^2 - 3w + 1)$ appears in the denominator for z. Congruence conditions (2.8) cancel the appearance of 4. Further, when we set $w = m/n$ and homogenize, there are integers m, n making the denominator of z equal to 1. In similar situations expect these to hold by adding congruence conditions like those of (2.8). This assumes Remark 3.1 and 3.2 conditions generalize.

3.2. Main Theorem 1.2 three branch point specializations. This section produces branch cycles for polynomial covers $v : \mathbb{P}_y^1 \to \mathbb{P}_z^1$ that satisfy (1.13b) in Main Theorem 1.2. The argument for Theorem 1.1 applies, except these covers have only *two* finite branch points, instead of three. As in $\S 1.5$, $R : S_5 \to S_{10}$ is the degree 10 representation of S_5. As before, $z = \infty$ totally ramifies and branch cycles $(g_1, g_2, g_3) = \boldsymbol{g}$ satisfy the following.

(3.1a) g_3 is a 5-cycle corresponding to total ramification over ∞.

(3.1b) $g_1 g_2 g_3 = 1$ and $\langle g_1, g_2, g_3 \rangle = A_5$ or S_5.

(3.1c) With $\boldsymbol{g}' = R(\boldsymbol{g})$, Riemann-Hurwitz shows $\boldsymbol{g}$ and $\boldsymbol{g}'$ give genus zero covers.

The 3-tuples $\boldsymbol{g}'$ are branch cycles for the cover we previously were calling h_v. As in (1.11), (3.1c) means $\text{ind}(g_1) + \text{ind}(g_2) = 4$ and $\text{ind}(g_1') + \text{ind}(g_2') = 10$. For example, if $\text{ind}(g_i) = 2$, then g_i is either a 3-cycle or a product of two disjoint 2-cycles, $i = 1, 2$. With $g_3 = (5\,4\,3\,2\,1)$, possible branch cycles for

(g_1, g_2) are

(3.2) $$((1\,2\,3),(1\,4\,5)),\ ((1\,3\,5),(2\,3)(4\,5)).$$

Reject another apparent possibility: $((1\,2)(3\,5),(1\,3)(4\,5))$. As both g_1 and g_2 are involutions, they generate the dihedral group of degree 5, not A_5 or S_5.

The (3.2) examples gives (g_1, g_2) generating A_5. The next requirement is

(3.3) $$\operatorname{ind}(g_1') + \operatorname{ind}(g_2') = 10.$$

This interprets that h_v (notation as in §1.6 for h_f) is of genus 0. As in §1.5, for g a 2-cycle, $R(g)$ has shape $(2)(2)(2)$; for g a product of disjoint 2-cycles, $R(g)$ has shape $(2)(2)(2)(2)$; and for g a 3-cycle, $R(g)$ has shape $(3)(3)(3)$. Only the second from (3.2) satisfies (3.3). That is, suppose v is a polynomial cover with branch cycles from the first entry in (3.2). Then, instead of giving h_v as in Main Prop. 1.3, it produces a genus 1 cover. (Note: This is a coalescing from List B of §4.4.)

The other possibility from $\operatorname{ind}(g_1) + \operatorname{ind}(g_2) = 4$ is that one of $\operatorname{ind}(g_i)$ is 1, the other 3, $i = 1, 2$. We might have g_1 a 2-cycle and g_2 a 4-cycle; or g_1 a 2-cycle and g_2 a product of disjoint 2 and 3-cycles. Here are examples:

$$((1\,2),(1\,3\,4\,5)),\ ((1\,3),(2\,3)(1\,4\,5)).$$

Both branch cycle descriptions generate S_5. If g is a 4-cycle, then $R(g)$ has shape $(2)(4)(4)$, and if g has shape $(2)(3)$, then $R(g)$ has shape $(6)(3)$. Both pass the genus 0 condition for $R(\boldsymbol{g})$. The next result amends Main Th. 1.2 by showing these branch cycles have a corresponding (v, h_v) over $\mathbb{Q}$.

Lemma 3.4 (Completion Lemma)**.** *The polynomials in* (1.13b) *of Main Theorem* 1.2 *consist of* $v_1, v_2, v_3 \in \mathbb{Q}[y]$ *(with explicit branch cycle descriptions in the proof below). That is, suppose* $f \in S_A$ *for some fractional* $\mathbb{Q}$ *ideal,* (1.13a) *does not hold, and* $f : \mathbb{P}^1_y \to \mathbb{P}^1_z$. *Then, there are affine functions* α, β *for which* $f(y) = \beta(v_i(\alpha(y)))$ *for some value of* i *from* $\{1, 2, 3\}$.

Proof. From the proof of the Main Proposition, it suffices to find three polynomials v_1, v_2, v_3 giving each f satisfying (3.1) by the affine change (from the statement of the lemma). Suppose the finite branch points of f are z_1 and z_2. The above discussion shows, beyond (3.1), the major properties of any branch cycle description for f. In each case the finite branch cycles are in distinct conjugacy classes of $N_{S_5}(A_5) = S_5$. So, $z_1, z_2 \in \mathbb{Q}$. (For this and non-Galois cover application of rigidity, see the *branch cycle argument* in [**Fri95a**, §3] or the source of that in [**Fri77**, start of §5].) If its finite branch cycles are (g_1, g_2) — g_i corresponding to z_i — then, one of these hold. Either: g_1 has shape (3) and g_2 shape (2)(2); g_1 has shape (2) and g_2 shape (4); g_1 has shape (2) and g_2 shape (2)(3); or switching g_1 and g_2 gives these shapes.

We now produce polynomials v_1, v_2, v_3 in y. These are unique up to affine change of y and composition with one of the functions $z \mapsto z$ or $z \mapsto 1 - z$ given the following conditions. They have branch cycle shapes at the points 0 and 1 as one of the first three above. As previously, take the branch cycle for $z = \infty$ to be $g_3 = (1\,2\,3\,4\,5)^{-1}$. Riemann's existence theorem produces corresponding covers, up to absolute equivalence, if the branch cycles equal one of

$$(3.4) \quad a)\ ((1\,3\,5), (2\,3)(4\,5)),\ b)\ ((1\,2), (1\,3\,4\,5))\ \text{or}\ c)\ ((1\,3), (2\,3)(1\,4\,5)).$$

This is a simple combinatorial computation.

Polynomials give the three covers because they are of genus 0, the covering space has a rational point (the ramified point over 0) and $z = \infty$ totally ramifies (g_3 is a 5-cycle). Polynomials with coefficients in $\mathbb{Q}$ represent these covers because their equivalence class of branch cycles uniquely determines them. This is from the non-Galois cover application of rigidity [**Fri77**, Theorem 5.1].

Cases (3.4a) *and* (3.4b). For (3.4a), g_2 is a product of two disjoint 2-cycles. Conjugate by a power of g_3, to assume up to absolute equivalence, the support of g_2 excludes 1. We claim, $g_1 g_2 = (1\,2\,3\,4\,5)$ determines g_1 and g_2. Write $g_1 = (1\,k\,j)$ and $g_2 = (k\,k')(j\,j')$. Then, the claim follows from $g_1 g_2 = (1\,k'\,k\,j'\,j)$. For Case (3.4b), g_2 is a 4-cycle. This is similar to (3.4a).

Case (3.4c). Here g_2 has shape (2)(3). The support of g_1 has common support in both the 2-cycle and the 3-cycle of g_2. The branch cycles (g_1, g_2) therefore look like this: $((1\,i), (1\,j)(i\,k\,l))$, and $g_1 g_2 = (1\,k\,l\,i\,j)$. As in (3.4a), $g_1 g_2 = (1\,2\,3\,4\,5)$ determines g_1 and g_2. $\qquad\square$

3.3. Arithmetic of the polynomials in Completion Lemma 3.4.

It is easy to find a rational point on the degree 10 cover $\mathcal{T}_{(i)} \to \mathbb{P}^1$ arising from v_i (of Lemma 3.4) as in §1.5, $i = 1, 2, 3$. For example, the divisor of the fiber of $\mathcal{T}_{(1)} \to \mathbb{P}^1_z$ over 0 has shape (3)(3)(3). Therefore, $\mathbb{Q}$ is a field of definition of the unramified point in this fiber. For $j = 2, 3$, the divisor of the fiber of $\mathcal{T}_{(j)} \to \mathbb{P}^1_z$ over 0 has shape (2)(2)(2). Thus, the set of 3 ramified points is a $\mathbb{Q}$ divisor. As in §2.1, this produces a $\mathbb{Q}$ point from ramification. Conclude the following.

Proposition 3.5. *There is a $\mathbb{Q}$ rational function $h_{v_i} : \mathbb{P}^1_w \to \mathbb{P}^1_z$ giving a cover equivalent to $\mathcal{T}_{(i)} \to \mathbb{P}^1_z$, $i = 1, 2, 3$.*

Now we explain the meaning that these v_is are coalescings from the polynomials of Thm. 1.1. Algebraically it is that v_i is a specialization of functions at the generic point of $\mathcal{H}_\infty$. This, however, is equivalent to saying a *coalescing of branch cycles* from Ni_∞ gives branch cycles for these covers. Combinatorially: Each v_i has a branch cycle description from one of $(g_1 g_2, g_3, g_4)$,

(g_1, g_2g_3, g_4) or (g_1, g_2, g_3g_4) where $(g_1, g_2, g_3, g_4) \in \text{Ni}_\infty$. (§4.3 discusses a general definition for *coalescing of branch cycles*.) Here are examples.

Coalesce the first and second branch cycles in $((2\,3)(4\,5), (1\,2), (1\,4))$. This gives $((1\,2\,3)(4\,5), (1\,4))$. These are the finite branch cycles of a polynomial cover. Let $\alpha(z) = 1 - z$. This cover is equivalent to $\alpha \circ v_3 : \mathbb{P}^1_y \to \mathbb{P}^1_z$. Similarly, coalesce the second and third branch cycles in $((2\,3)(4\,5), (1\,2), (1\,4))$ to get $((2\,3)(4\,5), (1\,2\,4))$ for finite branch cycles of a cover equivalent to $\alpha \circ v_1 : \mathbb{P}^1_y \to \mathbb{P}^1_z$.

Finally, coalesce the first and second branch cycles in $((2\,5)(3\,4), (3\,5), (1\,2))$ to get $((2\,3\,4\,5), (1\,2))$, finite branch cycles for a cover equivalent to $\alpha \circ v_2 : \mathbb{P}^1_y \to \mathbb{P}^1_z$. Theoretically it requires complicated base change of coordinates to see this specialization. An argument for this is in [**Fri95c**, §III.F]: The discussion of *specialization sequences*. Here, however, coordinates for the space $\mathcal{P}$ of §2.3 suffice and it is easy to see the coalescing corresponds to specialization of variables for $\mathcal{P}$.

There is a way to view coalescing as coming from paths. Suppose a description of branch cycles (g_1, g_2, g_3, g_4) comes from a choice of paths P_1, P_2, P_3, P_4 on the sphere punctured at four points. Then, there is a natural set of paths to attach to coalescing these branch cycles to (g_1g_2, g_3, g_4): $P_1 \cdot P_2, P_3, P_4$. That is, you replace the first two paths by their product. Do this and then choose a homotopy of the product $P_1 \cdot P_2$ to a loop P' enclosing the branch points z_1 and z_2. Keep P' fixed and let the branch points come together complex analytically. This choice, for example, preserves the complex conjugation operator of Real Point Lemma 2.1.

We haven't ruled out that one from the v_is may be in $S_\mathcal{A}$ for *each* fractional ideal $\mathcal{A}$. Suppose for example, $v = f(x)/u(x)^t$ with $f, u \in \mathbb{Z}[x]$ monic polynomials, $\deg(u) = 2$ has real roots α_1, α_2 conjugate over $\mathbb{Q}$ and $\deg(f) = 2t$. Assume $\epsilon = r_1 - \alpha_1 r_2$ is the fundamental unit of $\mathbb{Q}(\alpha_1)$ with $s_1, s_2 \in \mathbb{Q}$, and $\epsilon^s = m_s - \alpha_1 n_s$. An arbitrary such v might satisfy the following: $v(x) \equiv 0 \bmod p^e$ has a solution for each prime p and each positive integer e. This happens if $f(x)$ has a linear factor over $\mathbb{Q}$. There are other cases too.

For example, consider $f(x) = h(x)(x^2 - 2)(x^2 - 3)(x^2 - 6)$ with $h(x)$ having a 2-adic and a 3-adic zero, neither in $\mathbb{Q}$. For such an f, the Chinese Remainder Theorem produces for $f(x) \equiv 0 \bmod w$ a solution for each integer w. Here, however, assuring the denominator equals ± 1, requires for each prime power p^e there is an s with $n_s^{2t} f(m_s/n_s) \equiv 0 \bmod p^e$ having a solution.

Problem 3.6. Suppose $h : \mathbb{P}^1_w \to \mathbb{P}^1_z$ is a $\mathbb{Q}$ rational function with exactly two real conjugate (geometric) points over $z = \infty$. Is there a nontrivial fractional ideal $\mathcal{A}$ with $h \notin S_\mathcal{A}$ (§1.4)?

4. Siegel and Néron families.

§4.1 defines *Siegel families* and §4.2 lists many examples. §4.4 comments briefly by example on what we call Néron families. This is an analog of Siegel families of covers, except the covers $\mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ (like those of §1.4) for $\boldsymbol{p} \in \mathcal{P}$ are by curves of genus 1. The arithmetic assumption is that $\mathcal{P}(\mathbb{Q})$ is Zariski dense in $\mathcal{P}$. The arithmetic question is whether $\mathcal{T}_{\boldsymbol{p}}(\mathbb{Q})$ is infinite for a dense set of $\boldsymbol{p} \in \mathcal{P}(\mathbb{Q})$. The principal example of §4.4 resembles that of Main Theorem 1.2: Does some Néron family of covers contribute to $\mathcal{R}_f(\mathbb{Q}) \setminus \mathcal{V}_f(\mathbb{Q})$ being infinite.

4.1. Definition of Siegel families. Let $\mathrm{Ni}(G, \mathbf{C}, R)$, $\mathbf{C} = (\mathrm{C}_1, \dots, \mathrm{C}_r)$, be a Nielsen class (as in §1.1) with $R : G \to S_m$ a degree m faithful representation of G. Assume for $g \in \mathrm{C}_r$, $R(g)$ is a product of two disjoint $m/2$-cycles. Consider a smooth irreducible family $\Psi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ of curve covers over $\mathbb{Q}$ with degree m fibers $\Psi_{\boldsymbol{p}} : \mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ in $\mathrm{Ni}(G, \mathbf{C})$ with geometric monodromy group G and associated representation R. If the following properties hold, we say Ψ is a *Siegel family* (in the Nielsen class $\mathrm{Ni}(G, \mathbf{C})$).

(4.1a) For each $\boldsymbol{p} \in \mathcal{P}$, $z = \infty$ in $\Psi_{\boldsymbol{p}} : \mathcal{T}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ has associated branch cycle in C_r.

(4.1b) For each $\boldsymbol{p} \in \mathcal{P}$, $\mathcal{T}_{\boldsymbol{p}}$ has genus 0.

(4.1c) For a Zariski dense subset of $\boldsymbol{p} \in \mathcal{P}(\mathbb{Q})$, the (geometric) support points $y_{1,\boldsymbol{p}}$, $y_{2,\boldsymbol{p}}$ of $\Psi_{\boldsymbol{p}}^{-1}(\infty)$ are real and conjugate over $\mathbb{Q}$.

The Hurwitz space $\mathcal{H}(G, \mathbf{C}, R)^{\mathrm{abs}}$ is always available. Denote the quotient of $(\mathbb{P}^1)^r \setminus \Delta_r$ by the stabilizer of r in S_r (§6.2) by $U_{S_{r-1}}$. Label the subset of $U_{S_{r-1}}$ with ∞ in the rth coordinate as $U_{S_{r-1},\infty}$. Then, points of the fiber product

$$\mathcal{H}^{\mathrm{abs}}_\infty = \mathcal{H}(G, \mathbf{C}, R)^{\mathrm{abs}} \times_{\mathbb{P}^r \setminus D_r} U_{S_{r-1},\infty}$$

represent covers of $\mathbb{P}^1_z$ in the Nielsen class $\mathrm{Ni}(G, \mathbf{C})$. These automatically inherit properties (4.1a) and (4.1b). The shape of C_r guarantees covers of $\mathbb{P}^1_z$ representing $\boldsymbol{p}' \in \mathcal{H}^{\mathrm{abs}}_\infty$ have two points $y_{1,\boldsymbol{p}'}$, $y_{2,\boldsymbol{p}'}$ of ramification index $m/2$ lying over ∞. Call a Siegel family *complete* if the natural map $\mathcal{P} \to \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}_\infty$ is birationally surjective on an absolutely irreducible component of $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}_\infty$. Then, condition (4.1c) produces a dense set of $\boldsymbol{p}' \in \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}_\infty(\mathbb{Q})$ for which $y_{1,\boldsymbol{p}'}$ and $y_{2,\boldsymbol{p}'}$ are real and conjugate over $\mathbb{Q}$. Let $G(1)$ be the subgroup of elements $g \in G$ for which $R(g)$ fixes the integer 1 of the representation. Recall: §1.3 notes $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}_\infty$ supports a unique total family $\mathcal{T}$ of representing covers if and only if $G(1)$ is self-normalizing in G. Still, even without this assumption, working with complete families returns us to properties of $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}_\infty$.

4.2. Siegel families with r large. There are probably too many Siegel families to classify them. Further, it is difficult — though as we are showing, sometimes possible — in practice to know if $\mathcal{P}$ has a Zariski dense set

of $\mathbb{Q}$ points (property (4.1c)). This subsection gives several examples of Siegel families. For any Siegel family, one can ask if the conclusion of Main Theorem 1.2 holds: Is

$$\{\boldsymbol{p} \in \mathcal{P}(\mathbb{Q}) \mid |\Psi_{\boldsymbol{p}}(\mathcal{T}_{\boldsymbol{p}}(\mathbb{Q})) \cap \mathcal{A}| = \infty\}$$

Zariski dense? Also, is this a family of rational functions (§1.7)?

The easiest complete Siegel family has $G = S_m$, R the standard representation and $m = 2k$. Take $\mathcal{H}_\infty(G, \mathbf{C})_\infty^{\mathrm{abs}}$ where $r = m+1$, $\mathbf{C} = (C_1, \ldots, C_{m+1})$ has C_{m+1} with cycle type $(k)(k)$ and C_i, $i = 1, \ldots, m$, is the conjugacy class of 2-cycles. Pullback of $\mathcal{H}_\infty(G, \mathbf{C})_\infty^{\mathrm{abs}}$ to an appropriate analog of $\mathcal{P}$ as in §2.3 produces a family of rational functions of form $f(w) = M(w)/(w^2+cw+d)^k$. Here, $M(w)$ is a polynomial of degree at most $2k$, and the zeros of $M(w)$ and of $w^2 + cw + d$ are all distinct. It simplifies to assume — a choice we can make canonically — that $\deg(M(w)) \leq 2k - 1$. An additional condition assures f gives a cover in the Nielsen class:

(4.2) $\frac{dM}{dw}(w^2 + cw + d) - M(w)(2w + c) = 0$ has distinct zeros and these have $2k$ distinct images under f.

This includes the case $M(w)$ is a general polynomial of degree $m - 1$ and c, d are also general. The field of definition of the points w_1 and w_2 over ∞ varies with $\boldsymbol{p} \in \mathcal{P}$. This is because zeros of $w^2 + cw + d$ vary with c and d and these give coordinates for the w_is. It is easy to see condition (4.1c) holds. There is a Zariski dense set of c and d in $\mathbb{Q}$ with $w^2 + cw + d$ having real conjugate zeros.

Here is a more serious example. The parameter space is $\mathcal{H}_\infty^{\mathrm{abs}} = \mathcal{H}(G, \mathbf{C}')_\infty^{\mathrm{abs}}$ where $G = A_n$ for some integer $n = 2k \geq 6$ and $r = k+1$. Specific conjugacy classes are C_r' with cycle type $(k)(k)$ and C_i', $i = 1, \ldots, r-1$, the class of 3-cycles. §4.3 gives representative elements of $\mathrm{Ni}(A_{2k}, \mathbf{C}')$. Some experience with this example comes from [**Fri96**]. This lists components of $\mathcal{H}(A_n, \mathbf{C}_{3^r})$ for all r and all $n \geq 5$: $\mathbf{C}_{3^r}$ is r repetitions of the conjugacy class of 3-cycles. When $r = n - 1$, there is one absolutely irreducible component (defined over $\mathbb{Q}$).

[**Mes90**] uses rational functions with square discriminant. For n odd, this produces a unirational family of rational function covers over $\mathbb{Q}$ having $r = n - 1$ 3-cycles as branch cycles. As [**Fri96**] gives only one component in this case, $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})$ is unirational. Here, though, we are in the case n is even. In the next remark, we produce elements of $\mathrm{Ni}(A_{2k}, \mathbf{C}')$ satisfying the conditions above. Yet, the Hurwitz spaces may not give Siegel families, because we don't know if they have a dense set of $\mathbb{Q}$ points.

4.3. Producing elements of $\mathrm{Ni}(A_{2k}, \mathbf{C}')$ by coalescing Nielsen classes. Let $\mathrm{Ni}(G, \mathbf{C})$ and $\mathrm{Ni}(G^*, \mathbf{C}^*)$ be Nielsen classes with $\mathbf{C}$ an r-tuple and $\mathbf{C}^*$ an r^*-tuple of conjugacy classes from respective groups G and G^*. Assume P partitions $\{1, \ldots, r^*\}$ into r sets, each consisting of contiguous integers. Example: If $r^* = 5$ and $r = 4$, $P = \{1\} \cup \{2\} \cup \{3\} \cup \{4, 5\}$ is such

a partition. Given P, there is a natural map

$$\mathrm{Ni}(G^*, \mathbf{C}^*) \to \{\boldsymbol{g} \in (G^*)^r \mid g_1 \cdots g_r = 1\}$$

by $\boldsymbol{g}^* \mapsto (g_1, \ldots, g_r)$; g_i is the product in order of entries whose subscripts are in the ith term of the partition P. Use P also to denote this map. We say, $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ is *in the P boundary* of $\mathrm{Ni}(G^*, \mathbf{C}^*)$ if there exists $\boldsymbol{g}^* \in \mathrm{Ni}(G^*, \mathbf{C}^*)$ with $P(\boldsymbol{g}^*) = \boldsymbol{g}$. Write $\mathrm{Ni}(G, \mathbf{C})$ as a union $O_1 \cup O_2 \cup \cdots \cup O_t$ of braid orbits (§1.2). If $\boldsymbol{g} \in O_u$, then say O_u is in the P boundary of $\mathrm{Ni}(G^*, \mathbf{C}^*)$.

4.3.1. Case: $k = 3$, $r = 5$. From [**Fri96**],

$$\boldsymbol{g} = ((1\,2\,3), (1\,3\,4), (1\,4\,2), (1\,5\,6), (1\,5\,6)^{-1})$$

is in the unique braid orbit on $\mathrm{Ni}(A_6, \mathbf{C}_{3^5})$. The first three entries of $\boldsymbol{g}$ are transitive on $\{1, 2, 3, 4\}$; the last two have product 1. So, the Blocks Lemma [**Fri96**, §1.3] produces a braid conjugating the last two entries by $(1\,4)(2\,3)$ (fixed on the other entries). Then, braid to $((4\,6\,5), (1\,2\,3), (1\,3\,4), (1\,4\,2), (4\,5\,6))$. Finally, braid $(1\,2\,3)$ unchanged past the two 3-cycles to its right. This gives

$$((4\,6\,5), (1\,4\,2), (2\,4\,3), (1\,2\,3)(4\,5\,6)):$$

A representative of $\mathcal{H}(A_6, \mathbf{C}')$ for this case. Here $G = (2\,3)(4\,5)$ in the Real Point Lemma 2.1. This gives a cover where the two points over ∞ are real.

4.3.2. Case: $k = 4$ $r = 7$. The unique braid orbit of $\mathrm{Ni}(A_8, \mathbf{C}_{3^7})$ contains

$$((1\,2\,3), (1\,3\,4), (1\,4\,2), (1\,5\,6), (6\,5\,1), (1\,7\,8), (1\,8\,7)).$$

If there is an analog of the case $k = 3$, there is a product of three 3-cycles giving cycle type $(4)(4)$ over the special branch point. Yet, no element of $\mathrm{Ni}(A_8, \mathbf{C}_{3^7})$ coalesces to an element of $\mathcal{H}(A_8, \mathbf{C}')$. Argument: Suppose a product of three 3-cycles has shape $(4)(4)$. Then, all eight integers appear among the nine places of support in the three 3-cycles. Only one integer (say, 1) can appear twice in the supports of the 3-cycles. Up to reordering integers, the 3-cycles must be $((1\,2\,3), (1\,4\,5), (6\,7\,8))$. Yet, this product has shape $(5)(3)$.

Here, however, is an element of $\mathrm{Ni}(A_8, \mathbf{C}')$:

$$(4.3) \qquad ((1\,2\,3), (1\,4\,5), (6\,7\,8), (1\,6\,5), (4\,3\,2\,1)(8\,7\,6\,5)).$$

Apply Real Point Lemma 2.1 to see if it gives a cover with real points over ∞. (Some element of this Nielsen class must have this property for this to be a Siegel family.) The complex conjugation operator takes (4.3) to

$$(4.4) \qquad ((1\,3\,2), (3\,5\,4), (6\,8\,7), (4\,8\,5), (1\,2\,3\,4)(5\,6\,7\,8)).$$

So, if G exists, it conjugates (4.3) to (4.4). The 1st, 3rd and 5th entries force G to be $(1\,3)(6\,8)$. This doesn't, however, conjugate the 2nd and

fourth entries of (4.3) to the corresponding entries of (4.4). That leaves a programming question.

Problem 4.1. Does $\mathrm{Ni}(A_8, \mathbf{C}')$ contain branch cycles for a cover (corresponding to Lemma 2.1) with two real points over ∞?

4.3.3. Case: general $n = 2k$, $k \geq 5$ odd. Here is a representative of $\mathrm{Ni}(A_{2k}, \mathbf{C}_{3^{2k-1}})$:

$$(4.5) \qquad \begin{aligned} ((1\,2\,3), (1\,3\,4), (1\,4\,2), (1\,5\,6), (1\,5\,6)^{-1}, (1\,7\,8), (1\,7\,8)^{-1}, \; \ldots, \\ (1\,2k{-}1\,2k), (1\,2k{-}1\,2k)^{-1}). \end{aligned}$$

As with $n = 6$, a braid conjugates

$$((1\,k{+}2\,k{+}3), (1\,k{+}2\,k{+}3)^{-1}, \ldots, (1\,2k{-}1\,2k), (1\,2k{-}1\,2k)^{-1})$$

by $(1\,4)(2\,3)$ without changing anything else. This changes these $(k-1)/2$ pairs to $((4\,k{+}2\,k{+}3), (4\,k{+}2\,k{+}3)^{-1}, \ldots, (4\,2k{-}1, 2k), (4\,2k{-}1\,2k)^{-1})$. Braid the 1st, 3rd, 5th, etc. of these together to form $((4\,k{+}2\,k{+}3), \ldots, (4\,2k{-}1\,2k), *)$. Coalesce entries up to *. This gives a k-cycle on $\{4, k{+}2, k{+}3, \ldots, 2k{-}1, 2k\}$. From the remaining 3-cycles, $(1\,2\,3), (1\,5\,6), \ldots, (1\,k\,k{+}1)$ appear from left to right in (4.5). Braid these together to the exact left of the previous k-cycle. Then, coalesce them to a k-cycle on $\{1, 2, 3, 5, \ldots, k{+}1\}$. The two k-cycles have disjoint supports. Coalescing leaves an element of shape $(k)(k)$ and 3-cycles. This shows how to coalesce an element of $\mathrm{Ni}(A_{2k}, \mathbf{C}_{3^{k+1}})$ to an element of $\mathrm{Ni}(A_{2k}, \mathbf{C}')$.

4.4. Néron version of the problem. §5.1 has an update on results extending Main Theorem 1.2. This includes variants of the original problem. Example: Drop the restriction that z_0, with $f(y) - z_0$ reducible, must lie in a given fractional ideal. That is, replace the condition $f \in S_{\mathcal{A}}$ in §1.4 with $f \in S_{\mathbb{Q}}$:

(4.6) $f \in \mathbb{Q}[y]$ with $\mathcal{R}_f(\mathbb{Q}) \setminus \mathcal{V}_f(\mathbb{Q})$ infinite.

As in Thm. 1.2, $f \in S_{\mathbb{Q}}$ comes from a simple geometry setup. Let $\hat{\varphi} : \hat{X} \to \mathbb{P}^1_z$ be the minimal Galois closure of $f : \mathbb{P}^1_y \to \mathbb{P}^1_z$. The cover f produces a standard faithful permutation representation $T_f : G(\hat{X}/\mathbb{P}^1_z) = \hat{G} \to S_n$ with $n = \deg(f)$. Call a subgroup of $\hat{G}$ *intransitive* if its image in S_n is an intransitive subgroup.

For each subgroup H of $\hat{G}$ consider the quotient cover $\varphi_H : \hat{X}/H \to \mathbb{P}^1_z$. Let $\mathbb{H}$ run over maximal proper subgroups H satisfying the following.

(4.7a) $\hat{X}/H$ has ∞-ly many $\mathbb{Q}$ points.
(4.7b) H is intransitive.

One element of $\mathbb{H}$ is G_1, the subgroup defining the cover f. Remove from $\mathbb{H}$ groups containing conjugates of G_1. Denote the result by $\mathbb{H}'$. The next result is in [**Fri86**] and [**Fri77**, Intro.].

Proposition 4.2. *Suppose $f \in S_{\mathbb{Q}}$ and f is indecomposable. Then $\mathcal{R}_f(\mathbb{Q}) \setminus \mathcal{V}_f(\mathbb{Q})$ is the union of a finite set and*

$$\bigcup_{H \in \mathbb{H}'} \varphi_H \left((\hat{X}/H)(\mathbb{Q}) \right).$$

Remark 4.3 (Comments on Prop. 4.2). Suppose $f \in K[y]$ has degree prime to the characteristic of K. Then f is indecomposable over K if and only if it is indecomposable over $\bar{K}$ (see top of §1). From Faltings' Theorem, $f \in S_{\mathbb{Q}}$ requires there exists $H \in \mathbb{H}'$ with $\hat{X}/H$ of genus 0 or 1. [**Ser92**, §5.4] discusses a similar situation using Faltings' Theorem. Branch cycle descriptions of covers allow checking the genus of $\hat{X}/H$ by applying Riemann-Hurwitz. That is, let G be the geometric monodromy group of the cover, $\boldsymbol{g}$ a branch cycle description of $f : \mathbb{P}^1_y \to \mathbb{P}^1_z$ and $T_H : G \to S_{(G:H)}$ the permutation representation on cosets of H. The genus of $\hat{X}/H$ is g_H in

$$2((G : H) + g_H - 1) = \sum_{i=1}^{r} \operatorname{ind}(T_H(g_i)).$$

Müller's Theorems (§5.1) handle occurrences when $\hat{X}/H$ has genus 0 with one extra condition; it has at most two places over $z = \infty$. §5.1 discusses where this condition appears in [**Mül96**]. There are, however, likely further cases from those he cites where $\hat{X}/H$ has genus 0. Excluding the cases of degree 5 of this paper, and the *Davenport polynomials* from [**Fri86**], his examples are polynomial covers of $\mathbb{P}^1_z$ with two finite branch points. To complete considering $\mathcal{R}_f(K) \setminus \mathcal{V}_f(K)$ leaves knowing, when the genus is 0, if corresponding families have fibers with rational points. This was the concern of [**DF90a**]. When $\hat{X}/H$ has genus 1, there is no general technique to decide if these Nielsen classes are arithmetically exceptional.

4.5. Example of a family of genus 1 covers akin to Main Prop. 1.3. As above, use description by branch cycles. Here $_+\mathrm{C}_1$ and $_+\mathrm{C}_3$ are the conjugacy classes of 2-cycles in S_5, $_+\mathrm{C}_2$ is the conjugacy class of a 3-cycle and $_+\mathrm{C}_4$ is the conjugacy class of a 5-cycle.

The corresponding Nielsen class is $_+\mathrm{Ni}_\infty = \mathrm{Ni}_\infty(S_5, {}_+\mathbf{C})$:

$$\{\boldsymbol{g} = (g_1, \dots, g_4) \mid g_1 g_2 g_3 g_4 = 1, \langle \boldsymbol{g} \rangle = S_5 \text{ and } \boldsymbol{g} \in {}_+\mathbf{C}\}.$$

As with List A of §1.2, there is a moduli space $\mathcal{H}(S_5, {}_+\mathbf{C})_\infty = {}_+\mathcal{H}_\infty$ of such polynomials. We compute as in §1.3 to see $_+\mathcal{H}_\infty$ is a parameter space for a family of degree 10 covers. Here, however, $_+\mathcal{H}_\infty$ parameterizes genus 1 covers of $\mathbb{P}^1_z$. Then, as above, we check that $_+\mathcal{H}_\infty$ is irreducible and its $\mathbb{Q}$ points are Zariski dense.

4.5.1. Genus of the corresponding degree 10 covers. Let $\boldsymbol{g}$ be a branch cycle description of the cover from $_+\mathrm{Ni}_\infty$. Compute the genus g of $_+\mathcal{T}_{\boldsymbol{p}}^{(2)}$ from Riemann-Hurwitz:

$$(4.8) \qquad 2(10 + g - 1) = \sum_{i=1}^{4} \mathrm{ind}(R(g_i)).$$

Suppose g_1 and g_3 are 2-cycles from S_5. Then, $R(g_i)$ has shape $(2)(2)(2)$ in the representation R, $i = 1, 3$. Similarly, if g_2 is a 3-cycle, $R(g_3)$ has shape $(3)(3)(3)$. Finally, $R(g_4)$ has shape $(5)(5)$. Thus, the total contribution to the right side of (4.8) is $2 \cdot 3 + 6 + 2 \cdot 4 = 20$ and $g = 1$.

Next: Compute $_+\mathrm{Ni}_\infty$ modulo conjugation by S_5. Choose S_5 representatives with g_4 equal $g_\infty = (1\,2\,3\,4\,5)^{-1}$. Divide $_+\mathrm{Ni}_\infty$ into two sets T_1 and T_2: $\boldsymbol{g} \in T_1$ has g_1 and g_2 with no integers of common support, and $\boldsymbol{g} \in T_2$ has g_1 and g_2 with one integer of common support. Conjugate by a power of $\boldsymbol{g}_\infty$ to assure elements of T_1 have $g_1 = (1\,j)$ with $j = 2$ or 3. Similarly, elements in T_2 have 1 as common support of g_1 and g_2. From this, list $_+\mathrm{Ni}_\infty{}^{\mathrm{abs}}$.

4.5.2. List B. $_+\mathrm{Ni}_\infty{}^{\mathrm{abs}}$ *representatives with 2nd position a 3-cycle.*

  a) $((1\,2), (3\,4\,5), (1\,3))$ b) $((1\,2), (1\,4\,5), (3\,4))$ c) $((1\,2), (1\,3\,5), (4\,5))$
  d) $((1\,2), (1\,3\,4), (1\,5))$ e) $((1\,3), (1\,4\,5), (2\,3))$.

Similar lists have the type (3) element in the 1st or in the 3rd positions. Apply braid elements Q_1 (resp., Q_2) to List B to produce these. Finally: Draw conclusions exactly as in §1.3. Start by computing the $a_{1,j}$s:

$$a_{1,2} = Q_1^{-2} = (0_b\, 0_e\, 0_d\, 0_c), \ a_{1,3} = Q_1 Q_2^{-2} Q_1^{-1} = (0_a\, 0_d\, 0_e).$$

These elements generate S_5 itself — a transitive group — in their action on List B. As previously, consider a 3rd element's action on List B:

$$a_{1,4} = Q_1 Q_2 Q_3^{-2} Q_2 Q_1^{-1} = (0_a\, 0_b\, 0_c\, 0_d).$$

Expression (1.3a) shows the product

$$a_{1,2} a_{1,3} a_{1,4} = (Q_1 Q_2 Q_3^2 Q_2 Q_1)^{-1}$$

acts trivially on $_+\mathrm{Ni}_\infty{}^{\mathrm{abs}}$. Thus, $(a_{1,2}, a_{1,3}, a_{1,4})$ are branch cycles for a degree 5 cover $_+\varphi : {}_+\mathcal{H}_{\infty 0,1,\infty} \to \mathbb{P}^1$ of the sphere ramifying at 3-places: $0, 1, \infty$. Riemann-Hurwitz gives the genus of this cover as 0. Apply §6.1 to see $_+\mathcal{H}_\infty$ has a dense set of rational places.

Proposition 4.4. *The family* $_+\Psi : {}_+\mathcal{T}^{(2)} \to {}_+\mathcal{H}_\infty \times \mathbb{P}_z^1$ *has a section making it a family of elliptic curves. Suppose* $\boldsymbol{p} \in {}_+\mathcal{H}_\infty(\mathbb{Q})$ *has fiber* $_+\mathcal{T}_{\boldsymbol{p}}^{(2)}$. *Then,* $|{}_+\Psi_{\boldsymbol{p}}({}_+\mathcal{T}_{\boldsymbol{p}}^{(2)}(\mathbb{Q})) \cap f_{\boldsymbol{p}}(\mathbb{Q})| < \infty$.

Proof. Consider any cover $_+\varphi^{(2)} : {}_+\mathcal{T}_{\boldsymbol{p}}^{(2)} \to \mathbb{P}_z^1$ in the family. The unique unramified point in $_+\varphi^{(2)}$ over the branch point of the 3-cycle has a $\mathbb{Q}(\boldsymbol{p})$ rational point. This point serves as the fiber origin on the family of genus 1 curves.

The last sentence is the analog of Prop. 2.5. This follows if the normalization $_+W$ of the fiber product $W = \mathbb{P}_y^1 \times_{\mathbb{P}_z^1} {}_+\mathcal{T}_{\boldsymbol{p}}^{(2)}$ has but finitely many $\mathbb{Q}$ points lying over $\mathbb{Q}$ values of z. Faltings' Theorem gives this if each component of the fiber product has genus at least 2.

Prop. 2.5's proof shows components of W correspond to $S_5(1)$ orbits in the representation R. There are two such orbits: $\{s_1', \ldots, s_4'\}$; and the complement of this set of symbols from §2.2. Label corresponding components W_1 and W_2.

Abhyankar's Lemma §7 lets us list points from $\mathbb{P}_y^1$ ramified in W:

(4.9a) The three values $y_{1,j}$ unramified over z_1, $k = 1, 2, 3$;
(4.9b) the two points $y_{2,k}$, $k = 1, 2$, unramified over z_2; and
(4.9c) three points $y_{3,l}$, $l = 1, 2, 3$, unramified over z_3.

Again, take branch cycles as conjugates, fixing 1 in the standard representation of S_5, of branch cycles for the corresponding points z_i, $i = 1, 2, 3$.

Apply Riemann-Hurwitz. So, $2(4 + g(W_1) - 1) = e_1 + e_2 + e_3$ where e_1 (resp., e_3), the contribution from $y_{1,k}$ s (resp., $y_{3,l}$ s), is 3 and e_2, from the $y_{2,k}$ s is 4. Thus, $g(W_1)$ is 2. Similarly, $2(6 + g(W_2) - 1) = 12 + 8 = 20$ or $g(W_2) = 5$. □

Problem 4.5. Is there a Zariski dense set of $\boldsymbol{p} \in {}_+\mathcal{H}_\infty(\mathbb{Q})$ for which $_+\mathcal{T}_{\boldsymbol{p}}^{(2)}$ has an infinite set of $\mathbb{Q}$ points.

5. Müller's additions to [Fri86].

§5.1 reviews Müller's (and [Fri86]'s) applications to Hilbert's Irreducibility Theorem of variation of a coefficient, other than the constant term, in a fixed polynomial. Müller goes beyond [Fri86] at two places. He treats variation of the penultimate coefficient (next to the constant term, instead of the constant term). He does this by describing all geometric monodromy groups from covers of the form $f(y)/y$ with (§5.2). He also rediscusses over an arbitrary number field the [Fri86] results for varying the y^i coefficient for $2 \le i \le n - 2$ over $\mathbb{Q}$. [Fri99, §7-§8] contains much of what we quote from [Fri86], including a complete description of the Davenport polynomials of degree 13, in the style of examples here.

5.1. Variation of coefficients other than the constant term. [Mül96] reviews known results from [Fri86] and [Fri74]. Beyond Explicit HIT Theorem 1.1, a more general result over any number field considers variation of the constant coefficient of a fixed polynomial. Use previous notation, except replace $\mathbb{Q}$ with a general number field K. As before, $\mathcal{A}$ is a fractional ideal

of K. (Results still hold if $\mathcal{A}$ is a subring of K, finitely generated over $\mathcal{O}_K$.) The following is in [**Fri86**, Th. 7.2].

Result 5.1 (HIT). *Suppose $f \in K[y]$ and $\mathcal{R}_f(\mathcal{A}) \setminus \mathcal{V}_f(\mathcal{A})$ is infinite. If f is indecomposable (over K) then, either $\deg(f) = 5$ or $\deg(f) \in \{7, 11, 13, 15, 21, 31\}$.*

The Hurwitz space approach to the degree 5 example allows immediate extension over any number field. So, exceptions to finiteness of $\mathcal{R}_f(\mathcal{A}) \setminus \mathcal{V}_f(\mathcal{A})$ over *any* number field K for *each* fractional ideal $\mathcal{A}$ come from the degree 5 case. [**Fri86**] completely discusses exceptional degrees $\{7, 11, 13, 15, 21, 31\}$. The Hurwitz space analysis for this is more demanding. It was the first foray (from 1968) in practical use of Hurwitz spaces. Yet, the arithmetic analysis is simpler than for §2.3-§2.5. This is because polynomials in these families (of degrees $\{7, 11, 13, 15, 21, 31\}$) have no contribution to $\mathcal{R}_f(\mathcal{A})$ from non-polynomial rational functions. Still, we explain why $m = 7$ and $m = 13$ are especially interesting.

Let m be an integer from $\{7, 11, 13, 15, 21, 31\}$. For each m there is a least field of definition $K_m \neq \mathbb{Q}$ for polynomials of that degree providing exceptions to HIT Result 5.1. The extension $K_m/\mathbb{Q}$ has degree equal the number of conjugacy classes in $N_{S_m}(G)$ of m-cycles from G, the monodromy group of the polynomial. For example, $K_7 = \mathbb{Q}(\sqrt{-7})$, and $K_{13} = \mathbb{Q}(\zeta_{13} + \zeta_{13}^3 + \zeta_{13}^9)$, a degree 4 extension of $\mathbb{Q}$. For each m and m-cycle conjugacy class, there is one corresponding absolutely irreducible Hurwitz space of polynomials, defined over K_m. For each case the Hurwitz parameter space is a rational variety. (It was exactly the techniques of this paper that gave this.) This is most compelling for $m = 7$ and 13 where exceptions form families of polynomials having three (finite) branch points. In the other cases, the polynomials have only two finite branch points, altogether three branch points, counting ∞.

Now we explain the possibilities for variation of other coefficients of a fixed polynomial $f \in K[y]$ with K a number field. Consider the ith coefficient of f, with $1 \leq i \leq n - 1$. For varying this, define $\mathcal{V}_{f,i}(\mathcal{A})$ to be

$$\{z_0 \in \mathcal{A} \mid \exists y_0 \in K \text{ with } f(y_0) - z_0 y_0^i = 0\}.$$

Define $\mathcal{R}_{f,i}(\mathcal{A})$ similarly. Assume the following:

(5.1)   a) $f(0) \neq 0$; b) $\gcd(i, n) = 1$; and   c) $2 \leq i \leq n - 2$.

[**Fri86**, Thm. 7.4] contains the following (Müller comments on this in [**Mül96**, §10]).

Result 5.2 (HIT i-Result, $2 \leq i \leq n - 2$). *Assume i and f satisfy (5.1). Then $\mathcal{R}_{f,i}(\mathcal{A}) \setminus \mathcal{V}_{f,i}(\mathcal{A})$ is finite.*

Remark 5.3 (Classification Comments). Result 5.1 uses the classification of finite simple groups. This comes by directly or indirectly quoting results

from [**Kan80**] that list doubly transitive groups. Still, Result 5.2 uses little more than a combination of the ingredients of Result 5.1 and Marggraf's old theorem (see §5.2). The outcome gives much to a classification of variables separated rational functions $f(x) - g(y)$. Specifically it does classify those where g is either a polynomial or its denominator is a power of a linear or quadric polynomial. When $K = \mathbb{Q}$ and g is a polynomial, the result is in [**Fri74**] and it is classification free.

5.2. Group theory and the HIT 1-Result. Conditions (5.1a) and (5.1b) allow assuming indecomposability of the rational function $f(y)/y^i$. There should be more precise results even removing these conditions. Still, the case $i = 1$ (or equivalently, $i = n - 1$), posed by [**Fri86**], gives serious new phenomena [**Mül96**, Th. 1.3].

Result 5.4 (HIT 1-Result). *Assume* (5.1a). *If* $K = \mathbb{Q}$ *then* $\mathcal{R}_{f,1}$ *is finite. If* $K \neq \mathbb{Q}$ *and* $\mathcal{R}_{f,1}(\mathcal{A}) \setminus \mathcal{V}_{f,1}(\mathcal{A})$ *is infinite, then* $n \in \{4, 6, 8, 9, 12, 16\}$. *Further, suppose* n *is one of these exceptional integers. Then, there are number fields* K *and* $f \in K[y]$ *for which* $\mathcal{R}_{f,1}(\mathcal{A}) \setminus \mathcal{V}_{f,1}(\mathcal{A})$ *is infinite for some fractional ideals* $\mathcal{A}$.

As above, the classification of finite simple groups appears when using a list of 2-transitive groups of degree n. [**Mül96**] proceeds by listing those containing an $(n-1)$-cycle. Recall the group theory statements that appear in Comments on Prop. 4.2. [**Mül96**] lists geometric monodromy groups of rational functions $f(y)/y$ with $f \in K[y]$ and $\mathcal{R}_{f,i}(\mathcal{A}) \setminus \mathcal{V}_{f,i}(\mathcal{A})$ infinite for some fractional ideal $\mathcal{A}$. The remainder of the section describes the exact conditions on a group G for there to exist such a triple $(f, K, \mathcal{A})$. Analogous to §4.4, a necessary condition is that G have a subgroup H for which $\hat{X}/H$ has genus 0, producing a rational function h_f (below).

5.2.1. [**Mül96**] **starting assumptions.** Since G is doubly transitive, its minimal normal subgroup is a simple (nonabelian) group or it is abelian and we take it to be $A = \mathbb{F}_p^m$ (the *affine* case). In the affine case, G is transitive on A (since A is transitive on A). So, G is isomorphic to $\mathbb{F}_p^m \times^s U$ with U a subgroup of $\mathrm{GL}_m(\mathbb{F}_p)$ acting faithfully on A.

5.2.2. Quotation of [**Kan80**]. [**Kan80**, Lemma 6.3] assumes G is an affine group of degree n containing an $n-1$-cycle. Then, $n = q^e$ for some prime power q and $G = \mathbb{F}_q^e \times^s U$ with $\mathrm{GL}_e(q) \leq U \leq \Gamma L_e(q)$. Interpret the *singer cycle* as a generator of the multiplicative group of nonzero elements of $\mathbb{F}_{q^e} \cong \mathbb{F}_q^e$ (as a vector space over $\mathbb{F}_q$). This gives an $n-1$-cycle in identifying the integers of the representation with $\mathbb{F}_q^e$.

5.2.3. [**Mül96**] **Main Theorem.** Call a group *projective* if $\mathrm{PSL}_m(q) \leq G \leq P\Gamma L_m(q)$ where G acts on points of the projective space of dimension $m - 1$. Let M_n, $n \in \{11, 12, 22, 23, 24\}$ be the Mathieu groups. (The most

interesting case in [**Mül96**] is M_{11}, with its 3-transitive representation of degree 12.) The next theorem requires the classification of finite simple groups. Without it, [**Mül96**, Th. 6.1] proves only that G is affine, or $\mathrm{PSL}_2(p)$ with $p \geq 5$, or 3-transitive.

Theorem 5.5. *If G is transitive and contains an $n-1$-cycle, then G is one of the following: Affine, A_n (n even), S_n, $\mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ with $p \geq 5$, M_{11} (of degree 12), M_{12} or M_{24}.*

[**Mül96**, Prop. 7.1] describes those from this list having an intransitive subgroup V (choose it maximal among intransitive subgroups) and

(5.2) an $n-1$-cycle g having at most two orbits on the coset space G/V. As [**Fri86**] and [**Mül96**, Prop. 4.1 (4)] note, the representation on G/V is automatically faithful. Given f, condition (5.2) translates to h_f having exactly two points over $z = \infty$.

5.2.4. Fractional values. Statement (5.2) is necessary (from Siegel's Theorem over general number fields) for h_f to have infinitely many values in some fractional ideal. [**Fri86**, Thm. 7.2] (and [**Mül96**, Prop. 5.2]) also note the following: Orbits of g in (5.2) have lengths k and ℓ where $\mathrm{lcm}(k, \ell) = n-1$ and $k + \ell \geq n$. The latter condition follows from a 2-transitive group having no intransitive coset representation of degree smaller than n. Further, if the degree is n, and the subgroup is intransitive, the (group) characters of the two permutation representations are the same. From this conclude $k = n-1$ and $\ell | n - 1$.

Further, even in excluding the case $2 \leq i \leq n - 2$, [**Mül96**, Prop. 4.2] and [**Fri86**] use the two-orbit condition. This occurs in applying Marggraf's Theorem [**Wie64**, p. 13.8] to g^{n-i} being an i-cycle and concluding $G = A_n$ or S_n. The following result is a [**Mül96**, Prop. 7.1] conclusion.

Theorem 5.6. *Cycle lengths of g on G/V are $n - 1$ and ℓ with G in the following.*

(5.3a) $G = A \times^s U$ *with* $A \equiv \mathbb{F}_p^m, U \leq GL_m(p)$ *and* $n = p^m$. *With* p^r *the index of* $V \cap A$ *in* A, $\ell = (p^m - 1)/(p^r - 1)$. *Let* V' *be a conjugate of* V *with* $|V' \cap U|$ *maximal. Then* $V' = N_U(V' \cap A)(V' \cap A)$.

(5.3b) *If G is not affine, then either G is a group from Thm. 5.5 (V is a point stabilizer) or G is among the following:*

- $n = 6$, $\ell = 5$, $G = \mathrm{PSL}_2(5)$ *and V is isomorphic to D_3;*
- $n = 8$, $\ell = 7$, $G = \mathrm{PSL}_2(7)$; *and $V = A_4$; or*
- $n = 12$, $\ell = 11$, $G = M_{11}$ *and $V = A_6$.*

[**Mül96**, Prop. 7.2] shows the affine cases produce examples satisfying all the hypotheses of [**Mül96**, Th. 6.2] and (5.2), the two-orbit condition. The genus 0 condition, from applying Riemann-Hurwitz to generators of G for both representations, excludes all but finitely many examples. For example, [**Mül96**] uses [**GT90**] and [**Neu93**] to handle the affine case. In particular,

with (5.2) in force, [**Mül96**, Lemma 8.4] discusses how a **GAP** program cuts the list of possibilities — details from Müller's thesis. Suppose we drop (5.2) while retaining that a branch cycle description for the representation of G on G/V gives genus 0 or 1. This would likely produce more exceptional examples like those of §4.4.

6. Rational point criterion for Hurwitz spaces.

We give a practical if and only if criterion for Hurwitz spaces of four branch-point covers to be birational images of some projective space. This $\mathbb{Q}$-*unirationality* test often shows when spaces like $\mathcal{H}_\infty$ have a dense set of rational points. The test depends only on braid group action. This phrasing is for full Hurwitz families: Parameterizing all covers in a given Nielsen class $\mathrm{Ni}(G, \mathbf{C})$, with $\mathbf{C}$ consisting of $r \geq 4$ conjugacy classes of G and G a transitive subgroup of S_n.

The test for $r = 4$ of Prop. 6.5 is the best, most likely and easiest to apply criterion. For $r = 4$, the $\mathrm{SL}_2(\mathbb{C})$ quotient action presents each Hurwitz space component as a canonical cover of the classical j-line. The immediate application to $\mathcal{H}_\infty$ follows from §1.1. The criterion computes a branch cycle description for this cover. The test is that the genus computed from this is 0. Though it is less likely, when the test of §6.1 holds there are extra diophantine corollaries. In particular, it implies the Hurwitz space *with* data for the branch points of the covers in the family is unirational. Further, we need the computation that gives us this in §6.4.3 for the proof of Prop. 6.5.

6.1. Tesselating a Hurwitz space with curves. Denote $\mathbb{P}^r \setminus D_r$ by U_r. Hurwitz monodromy action on $\mathrm{Ni}(G, \mathbf{C})/N_{S_n}(G) = \mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ produces an unramified cover $\Psi_{G,\mathbf{C}} : \mathcal{H}(G, \mathbf{C}) \to U_r$ (§1.2). As in §1.3, pull this cover back to $U^r = (\mathbb{P}^1)^r \setminus \Delta_r$ by forming the fiber product $\mathcal{H}(G, \mathbf{C})' = \mathcal{H}(G, \mathbf{C}) \times_{U_r} U^r$. This may have several components as [**DF90a**] takes pains to explain. These components correspond exactly to orbits of the straight braid group (kernel of the natural map of the braid group to S_r) acting on the *straight Nielsen class* $\mathrm{SNi}(G, \mathbf{C})$. Define straight Nielsen classes exactly as Nielsen class (§1.1), except r-tuples $\boldsymbol{g} = (g_1, \ldots, g_r)$ have entries in $\mathbf{C}$ in a *fixed* order: g_i is in the conjugacy class C_i, $i = 1, \ldots, r$. Example: There are (at least) $r!$ components if all conjugacy classes in $\mathbf{C}$ are distinct when extended to conjugacy classes in $N_{S_n}(G)$. Then, each component is a copy of a component of $\mathcal{H}(G, \mathbf{C})$; an especially easy case.

Even if $\mathcal{H}(G, \mathbf{C})$ is absolutely irreducible with $\mathbb{Q}$ as field of definition, absolutely irreducible components of $\mathcal{H}(G, \mathbf{C})'$ may not have field of definition $\mathbb{Q}$. For example, suppose $\mathbf{C}$ satisfies the following.

(6.1a) $\mathbf{C}$ is a rational union of conjugacy classes, and

(6.1b) a conjugacy class (say C_1) is not a rational conjugacy class even in its extension to $N_{S_n}(G)$.

Now, assume $\sigma \in G_{\mathbb{Q}}$ maps to $k \in G(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$ and has the following properties.

(6.2a) k is relatively prime to the order of the elements of C_1.

(6.2b) The extension of conjugacy classes C_1^k and C_1 in $N_{S_n}(G)$ aren't equal.

Then, σ acts nontrivially on the absolutely irreducible components of $\mathcal{H}(G, \mathbf{C})'$. This is a consequence of the *branch cycle argument* in [**Fri95a**, §3] or [**Fri77**, §5]. The goal of this section is to give a criterion for $\mathcal{H}(G, \mathbf{C})$ having a $\mathbb{Q}$ point. For this we need an extra hypothesis. As in §1.3, consider these elements of SH_r:

$$a_{1,2} = Q_1^{-2}, \ a_{1,3} = Q_1 Q_2^{-2} Q_1^{-1}, \ a_{1,4} = Q_1 Q_2 Q_3^{-2} Q_2^{-1} Q_1^{-1}, \dots,$$
$$a_{1,r} = Q_1 \cdots Q_{r-2} Q_{r-1}^{-2} Q_{r-2}^{-1} \cdots Q_1^{-1}.$$

These elements have indices in their action on $\mathrm{SNi}(G, \mathbf{C})$. The following hypothesis appears in Prop. 6.2.

(6.3) $\mathcal{H}(G, \mathbf{C})'$ has an absolutely irreducible $\mathbb{Q}$-component.

Here is a criterion for this [**Fri77**, Thm. 5.1].

Proposition 6.1. *Hypothesis* (6.3) *holds if the following holds.*

(6.4a) SH_r *is transitive on* $\mathrm{SNi}(G, \mathbf{C})$, *and*

(6.4b) *the extension of each conjugacy class* C_i *to* $N_{S_n}(G)$ *is rational in* $N_{S_n}(G)$.

These hypotheses hold in the cases of this paper. That is, we explicitly display the (6.4a) transitivity in List A and List B. The group $N_{S_n}(G)$ here is S_n ($n = 5$), a group with all conjugacy classes rational. Here is a general statement.

Proposition 6.2. *Assume* (6.3) *holds and* $O' \subset \mathrm{SNi}(G, \mathbf{C})/SN_{S_n}(G)$ *is an* SH_r *orbit. Let* $\mathcal{H}_{O'}$ *be the absolutely irreducible* $\mathbb{Q}$-component *of* $\mathcal{H}(G, \mathbf{C})'$ *corresponding to* O'. *Suppose* $|O'| = N$ *and* $\langle a_{1,2}, \dots, a_{1,r} \rangle$ *is transitive on* O' *(automatic when* $r = 4$ [**BF82**, Lemma 1.6]). *Compute* $g_{O'}$ *from*

$$(6.5) \qquad\qquad 2(N + g_{O'} - 1) = \sum_{i=2}^{r} \mathrm{ind}(a_{1,i}).$$

Then, $g_{O'}$ *is the genus of the nonsingular completion* $\bar{\mathcal{H}}_{O',L}$ *of* $\mathcal{H}_{O',L}$, *the restriction of* $\mathcal{H}_{O'}$ *over* $L = \{(z, z_2^0, \dots, z_r^0) \mid z \in \mathbb{P}^1\} \subset U^r$. *Suppose the following:*

(6.6a) $z_2^0, \dots, z_r^0$ *are distinct* $\mathbb{Q}$ *points in* $\mathbb{P}_x^1$.

(6.6b) $g_{O'} = 0$ *and* $\bar{\mathcal{H}}_{O',L}$ *has a* $\mathbb{Q}$ *point.*

Then, $\mathcal{H}_{O'}$ *is unirational and has a dense set of rational points.*

Proof. Identify the line L with $\mathbb{P}^1_z \setminus \{z_2^0, \ldots, z_r^0\}$. [**BF82**, (4.5) and following discussion] identifies $a_{1,2}, \ldots, a_{1,r}$ with generators of the fundamental group of L modulo the relation $a_{1,2} \cdots a_{1,r} = 1$ in H_r. Further, by restriction of $\mathcal{H}_{O'}$ over L, the corresponding permutation representation is restriction from SH_r acting on O'. Thus, formula (6.5) is just Riemann-Hurwitz for the cover $\bar{\mathcal{H}}_{O',L} \to \mathbb{P}^1$ computed from the action of branch cycles.

Finally, suppose $z_2^0, \ldots, z_r^0$ are distinct $\mathbb{Q}$ points. Then $\bar{\mathcal{H}}_{O',L} \to \mathbb{P}^1_z$ is a $\mathbb{Q}$ cover, and $\mathcal{H}_{O'}$ is birational to $W = \bar{\mathcal{H}}_{O',L} \times (\mathbb{P}^1)^{r-1}$. Clearly W is unirational (over $\mathbb{Q}$) if and only if $\bar{\mathcal{H}}_{O',L} = X$ is unirational. Since X is a nonsingular curve, this is equivalent to $\bar{\mathcal{H}}_{O',L}$ being of genus 0 and having a $\mathbb{Q}$ point. $\qquad\square$

6.2. $\mathrm{SL}_2(\mathbb{C})$ **action on Hurwitz spaces and covers of** Λ_r **and** J_r**.** Much of [**DF90a**] is on criteria guaranteeing rational points on families of genus 0 curves. In particular, the results there apply to consider how to satisfy (6.6b). If N is odd, then $\bar{\mathcal{H}}_{O',L}$ has a rational point. That is the easiest criterion when it holds, and it does in the main cases of this paper. The most refined criterion comes from the action of H_r on *pointed* Nielsen classes as in [**DF90a**]. We won't review that. Rather, when $r = 4$, Prop. 6.5 gives the best practical criterion for a component of $\mathcal{H}(G, \mathbf{C})$ to be unirational. This was the criterion, with fewer details, [**Fri90**] applied to degree 5 covers of $\mathbb{P}^1$ with four conjugacy classes of 3-cycles.

The group S_r acts on the space $(\mathbb{P}^1)^r$ by permutation of its coordinates. As with action of B_r, put this action on the right. This gives a natural map $\Psi_r : (\mathbb{P}^1)^r \to \mathbb{P}^r$. Consider $\mathbf{z} = (z_1, \ldots, z_r)$ with none of the coordinates equal ∞. The image of $\mathbf{z}$ under Ψ_r has as representative in projective space the point whose coordinates are the coefficients of the polynomial $\prod_{i=1}^r (x - z_i)$ in z. (If $z_i = \infty$, replace the factor $x - z_i$ by 1.) Also, Ψ_r takes the fat diagonal Δ_r to D_r. This interprets $U_r = \mathbb{P}^r \setminus D_r$ as the space of r distinct unordered points in $\mathbb{P}^1$. Thus, $\Psi_r : U^r \to U_r$ is an unramified Galois cover with group S_r.

Consider $\mathrm{PSL}_2(\mathbb{C})$ as linear fractional transformations acting diagonally on r copies of $\mathbb{P}^1_z$. For $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ and $\mathbf{z} \in U^r$, $\alpha(\mathbf{z}) \mapsto (\alpha(z_1), \ldots, \alpha(z_r))$. Put the action of $\mathrm{PSL}_2(\mathbb{C})$ on the left, since it commutes with the coordinate permutation action of S_r. The quotient $\mathrm{PSL}_2(\mathbb{C}) \backslash U^r = \Lambda_r$ generalizes the λ-line minus the points $0, 1, \infty$ from the theory of modular curves. Also, $\mathrm{PSL}_2(\mathbb{C}) \backslash U_r = J_r$ generalizes the j-line minus the point at ∞. It is a space of complex dimension $r - 3$. Take $r = 4$. Then, four unordered distinct points correspond to the unique unramified degree two cover of $\mathbb{P}^1$ ramified at these four points. This then corresponds to the classical j-invariant of the associated elliptic curve. In summary we have the following.

Proposition 6.3 (J_4). *The quotient of the (unramified) cover* $\Psi_4 : U^4 \to U_4$ *by* $\mathrm{PSL}_2(\mathbb{C})$ *produces the classical (ramified) map* $\psi : \mathbb{P}^1_\lambda \setminus \{0, 1, \infty\} \to$

$\mathbb{P}^1_j \setminus \{\infty\}$. *Though ψ is a Galois cover with group S_3, this copy of S_3 naturally identifies with the quotient of S_4 by the Klein 4-group, rather than with the stabilizer of 1 in S_4 appearing above.*

Proof. Consider $\mathbf{z} \in \mathbb{P}^4 \setminus D_4$. Then, $\mathbf{z}$ corresponds uniquely to a degree 2 cover $Y \to \mathbb{P}^1$ with associated branch locus $\mathbf{z}$. By the Riemann-Hurwitz formula, the genus of Y is one. Thus, regard $\mathbb{P}^4 \setminus D_4$ as a Hurwitz space $\mathcal{H}$, and consider the $\mathrm{PSL}_2(\mathbb{C})$ action from §1.1 on it. Two points $\mathbf{z}, \mathbf{z}' \in \mathbb{P}^4 \setminus D_4$ give genus one covers $\varphi : E \to \mathbb{P}^1_z$ (ramified at $\mathbf{z}$) and $\varphi' : E' \to \mathbb{P}^1_z$ (ramified at $\mathbf{z}'$). Then, E and E' are isomorphic if and only if there exists $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ for which the following statement holds:

(6.7) $\quad \alpha(\mathbf{z}) = \mathbf{z}'$, or equivalently $\alpha \circ \varphi = \varphi'$.

Therefore, J_4 is the j-line, missing only the point at ∞; this doesn't correspond to a genus one curve. □

We extend the $\mathrm{PSL}_2(\mathbb{C})$ action of $\mathbb{P}^r \setminus D_r$ to Hurwitz spaces. Suppose $\mathcal{H}$ is an absolute (or inner) Hurwitz (moduli) space of r-branch point covers of $\mathbb{P}^1_x$ in a given absolute (or inner) Nielsen class $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ (or $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$). We explain further.

There is a map $\beta_r : \mathcal{H} \to U_r$: If $\mathbf{p} \in \mathcal{H}$ has a representative cover $\varphi_{\mathbf{p}} : X_{\mathbf{p}} \to \mathbb{P}^1_z$, then the collection of branch points $\mathbf{z}$ of $\varphi_{\mathbf{p}}$ determines a point in U_r. Call this the *branch point map*. To compute the Nielsen class of the cover choose a standard set of continuous, piece-wise differentiable paths $\mathcal{P}_1, \ldots, \mathcal{P}_r$ in $\mathbb{P}^1_z \setminus \mathbf{z}$ based at a point z_0 (not in $\mathbf{z}$). (For details see [**Fri77**], [**MM**], [**Ser92**] or [**Völ96**].) The salient points are as follows. For some ordering $(z_1, \ldots, z_r)$ of the entries of $\mathbf{z}$:

(6.8a) Each $\mathcal{P}_i = \gamma_i \delta_i \gamma_i^{-1}$, with δ_i a clockwise path around z_i bounding a closed disc D_i with center z_i;

(6.8b) the γ_is meet only at z_0;

(6.8c) γ_i meets D_i only at the beginning point of δ_i, and it doesn't meet D_j if $j \neq i$; and

(6.8d) for a suitably small disk D around z_0, the initial meeting point m_i of γ_i with the boundary of D appear in the clockwise order $m_1, \ldots, m_r$.

For any element $\alpha \in \mathrm{PSL}_2(\mathbb{C})$, compose $\varphi_{\mathbf{p}}$ with α to give

$$\alpha \circ \varphi_{\mathbf{p}} : X_{\mathbf{p}} \to \mathbb{P}^1.$$

As α is conformal, you get paths for computing the Nielsen class for $\alpha \circ \varphi_{\mathbf{p}}$ from $\alpha^{-1}(\mathcal{P}_1), \ldots, \alpha^{-1}(\mathcal{P}_r)$ based at $\alpha^{-1}(x_0)$. These paths go around the respective branch points $\alpha^{-1}(x_1), \ldots, \alpha(x_r)$. From this, the cover $\alpha \circ \varphi_{\mathbf{p}}$ is in the same Nielsen class as is $\varphi_{\mathbf{p}}$. Further, suppose $\varphi_{\mathbf{p}}$ is a Galois cover and $\mu_{\mathbf{p}}$ is a fixed isomorphism between $G(X_{\mathbf{p}}/\mathbb{P}^1_x)$ and G. So composing $\varphi_{\mathbf{p}}$ with α doesn't affect the isomorphism $\mu_{\mathbf{p}}$. Thus, $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ induces an isomorphism on both absolute and inner Hurwitz spaces, commuting with

the branch point map β_r. Further, $\mathrm{PSL}_2(\mathbb{C})$ is a connected space (containing the identity map). So, any α maps *each* absolutely irreducible component of $\mathcal{H}$ into itself.

Further, for any Hurwitz space $\mathcal{H}$, consider the space $\mathcal{H}'$ of pairs $(\boldsymbol{p}, (z_1, ..., z_r))$ with $\boldsymbol{p} \in \mathcal{H}$ and an ordering $(z_1, \dots, z_r)$ of $\beta_r(\boldsymbol{p}) = \boldsymbol{z}$. The space $\mathcal{H}'$ is the fiber product $\mathcal{H} \times_{U_r} U^r$. Denote the projection of $\mathcal{H}' \to \mathcal{H}$ off the first factor by pr_1. Elements of $\mathrm{PSL}_2(\mathbb{C})$ map components of $\mathcal{H}'$ isomorphically onto themselves. Finally, both $\mathcal{H}$ and $\mathcal{H}'$ are finite covers of the affine space U_r. Thus, they are affine algebraic varieties. In particular, applying the *Reynolds operator* argument of [**MF80**, Th. 1.1] shows the $\mathrm{PSL}_2(\mathbb{C})$ quotients of these varieties are also affine varieties. Conclude the next result about the following *strong* equivalence relation between covers.

(6.9) $\varphi_i : X_i \to \mathbb{P}^1$, $i = 1, 2$, are equivalent if there exists $\alpha : X_1 \to X_2$ and $\beta : \mathbb{P}^1 \to \mathbb{P}^1$ with $\varphi_2 \circ \alpha = \beta \circ \varphi_1$.

Proposition 6.4 ($\mathrm{PSL}_2(\mathbb{C})$-quotient). *The quotient spaces $\mathcal{H}/\mathrm{PSL}_2(\mathbb{C}) = \mathcal{H}^{rd}$ and $\mathcal{H}'/\mathrm{PSL}_2(\mathbb{C})$ exist as moduli spaces for the equivalence* (6.9) *of covers in a Nielsen class. Further, they are affine algebraic varieties and there is a natural commutative diagram of induced maps:*

$$(6.10) \qquad \begin{array}{ccc} \mathcal{H}'/\mathrm{PSL}_2(\mathbb{C}) & \xrightarrow{\ \mathrm{pr}_1\ } & \mathcal{H}/\mathrm{PSL}_2(\mathbb{C}) \\ \Big\downarrow{\scriptstyle \beta_r \times \mathrm{Id}} & & \Big\downarrow{\scriptstyle \beta_r} \\ \Lambda_r & \longrightarrow & J_r \end{array} \quad .$$

6.3. $\mathrm{SL}_2(\mathbb{C})$ action j-line covers when $r = 4$. Let $\mathcal{H}_{O'}$ correspond to an H_4 orbit O' in its action on $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}$ (resp. on $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$). Thus, $\mathcal{H}_{O'}$ is an absolutely irreducible component of a Hurwitz space $\mathcal{H}(G, \mathbf{C})$ (over some number field K) of equivalence classes of $r = 4$ branch point covers. Then, Prop. 6.4 produces a finite cover $\beta_4^{rd}(O') : \mathcal{H}_{O'}/PSL_2(\mathbb{C}) = \mathcal{H}_{O'}^{rd} \to \mathbb{P}_j^1 \setminus \{\infty\}$. Complete this to a cover $\bar{\beta}^{rd}(O') : \bar{\mathcal{H}}_{O'}^{rd} \to \mathbb{P}_z^1$.

Let q_1, q_2, q_3 be the images of Q_1, Q_2, Q_3 in the mapping class group M_4 (1.3b). Form one further equivalence on $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ (or on $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$). Recall: For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, $g_1 g_2 g_3 g_4 = 1$. For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, $(Q_1 Q_3^{-1})^2$ has this effect:

$$(6.11) \qquad \boldsymbol{g} \mapsto (g_1 g_2 g_1 g_2^{-1} g_1^{-1}, g_1 g_2 g_1^{-1}, g_4^{-1} g_3 g_4, g_4^{-1} g_3^{-1} g_4 g_3 g_4) \equiv^G \boldsymbol{g}.$$

Form $\mathrm{Ni}(G, \mathbf{C})/\langle Q_1 Q_3^{-1}\rangle$. Denote this $\mathrm{MQ}(G, \mathbf{C})$ for *mapping quotient* classes. Action of H_4 on $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ induces action of $H_4/\langle Q_1 Q_3^{-1}\rangle = \bar{H}_4$ on $\mathrm{MQ}(G, \mathbf{C})$. As for $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ (resp. $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$) there is the quotient set $\mathrm{MQ}(G, \mathbf{C})^{\mathrm{abs}}$ (resp. $\mathrm{MQ}(G, \mathbf{C})^{\mathrm{in}}$). The next result computes branch cycles for the cover $\bar{\beta}^{rd}(O')$.

Proposition 6.5 (j-Line Branch Cycles). *The group $\bar{H}_4$ has generators $\alpha = q_1 q_2$ and $\gamma = q_1 q_2 q_1$ subject to the relations $\alpha^3 = \gamma^2 = 1$: $\bar{H}_4 \equiv \mathrm{PSL}_2(\mathbb{Z})$.*

Orbits of $\bar{H}_4$ on $\mathrm{MQ}(G, \mathbf{C})^{\mathrm{abs}}$ (*resp.* $\mathrm{MQ}(G, \mathbf{C})^{\mathrm{in}}$) *correspond one-one to orbits of H_4 on* $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ (*resp.* $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$). *In particular, consider the orbit O' in the discussion before this proposition. Let α' and γ' be respective actions of α and γ on the image of O' in* $\mathrm{MQ}(G, \mathbf{C})^{\mathrm{abs}}$. *Then (α', γ', q_2') is a description of the branch cycles of the cover $\bar{\beta}^{rd}(O')$.*

Finally, $\mathcal{H}_{O'}$ is unirational over K if and only if the genus of $\bar{\mathcal{H}}_{O'}^{rd}$ is 0 and

(6.12)   *there is a K rational divisor of odd degree on $\bar{\mathcal{H}}_{O'}^{rd}$.*

§6.4 completes the proof of Prop. 6.5. This subsection concludes with comments on practical use of (6.12).

Remark 6.6. *Testing* (6.12). The cover $\bar{\beta}^{rd}(O') : \bar{\mathcal{H}}_{O'}^{rd} \to \mathbb{P}_j^1$ has branch points 0, 1 and ∞. If $\bar{\beta}^{rd}(O')$ has odd degree, then a general K fiber of the cover has odd degree. Often, however, we can do better than that.

Suppose $\varphi : X \to \mathbb{P}_z^1$ is any cover with a branch point z' in K. Then the disjoint cycles of the branch cycle $g_{z'}$ provide one quick test for a K rational divisor of odd degree. Let e be an integer for which $g_{z'}$ has a disjoint cycle of length e. Let n_e be the number of disjoint cycles of $g_{z'}$ of length e. Then, the points of the fiber $X_{z'}$ ramified of index e over z' correspond to the respective disjoint cycles of length e. These form a K rational divisor on $\bar{\mathcal{H}}_{O'}^{rd}$. Reason: The action of G_K on the fiber over z' permutes the points of ramification e among themselves, so it leaves this set of points invariant.

6.4. Proof of Prop. 6.5. Consider the group $\bar{H}_4$ of §6.3. The first subsection presents this group by generators and relations. Denote the element of $\mathrm{PSL}_2(\mathbb{C})$ taking z to $1/z$ by μ. The second subsection shows how quotient action of $\mathcal{H}^{\mathrm{abs}}$ by μ interprets that the cover $\bar{\beta}^{rd}(O') : \bar{\mathcal{H}}_{O'}^{rd} \to \mathbb{P}_j^1$ has monodromy group a quotient of $\bar{H}_4$. The remainder of the proof follows quickly from these observations.

6.4.1. Presentation of $\bar{H}_4$. Relations (1.3a) and $Q_1 Q_3^{-1}$ show that $q_1 q_2 q_1 q_1 q_2 q_1$ in $\bar{H}_4$. From $Q_1 Q_2 Q_1 = Q_2 Q_1 Q_2$ in H_4,

(6.13) $q_1 q_2 q_1 q_1 q_2 q_1 = (q_1 q_2)^3 = (q_1 q_2 q_1)^2.$

This shows the identities $\alpha^3 = \gamma^2 = \alpha \gamma q_2$ from Prop. 6.5. All act trivially on $\mathrm{MQ}(G, \mathbf{C})^{\mathrm{in}}$, since they equal the action of $Q_1 Q_2 Q_3 Q_3 Q_2 Q_1$. So, $\alpha' \gamma' q_2'$ is the identity. It is a candidate for a branch cycle description.

6.4.2. The effect of μ on $\mathcal{H}^{\mathrm{in}}$ or $\mathcal{H}^{\mathrm{abs}}$. First consider the effect of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{H}^{\mathrm{in}}$ (or similarly, on $\mathcal{H}^{\mathrm{abs}}$). For this argument, refer to either of these choices as $\mathcal{H}$. Suppose $\boldsymbol{p} \in \mathcal{H}$ lies over $\boldsymbol{z} = \{z_1, z_2, z_3, z_4\} \in U_4$. Then, any other point $\boldsymbol{p'} \in \mathcal{H}$ to another point $\boldsymbol{p''} \in \mathcal{H}$ lying over $\boldsymbol{z}$. See the comments of §6.5 for the fiber product formulation of this. Prop. 6.8 reminds of an

explicit formula for the j-invariant of the genus one curve ramified over the z-line at $\boldsymbol{z}$. If $z_4 = \infty$, then [**Rob73**, p. I.61] gives the classical formula:

$$(6.14) \qquad j = -4(z_1 z_2 + z_2 z_3 + z_3 z_1)^3 / (z_1 - z_2)^2 (z_2 - z_3)^2 (z_1 - z_3)^2.$$

If $z_4 \neq \infty$, compute j by applying the transformation $z \mapsto 1/(z - z_4)$ to each element of $\boldsymbol{z}$. Let $\bar{\boldsymbol{z}} = j'$ denote the image of $\boldsymbol{z}$ in J_4 (the j-line).

In particular, points of $\mathcal{H}^{rd}$ over $j' \in J_4$ are in orbit for the stabilizer of $\boldsymbol{z}$ in $\mathrm{PSL}_2(\mathbb{C})$. To see this action explicitly, it suffices to consider a specific choice of $\boldsymbol{z}$. Our choice is this: $\boldsymbol{z} = \{0, \infty, +1, -1\}$. Then $z_0 = i$ is a convenient choice of base point for the next calculation. Let $\mathcal{L} = (\lambda_0, \lambda_\infty, \lambda_{+1}, \lambda_{-1})$ be an allowable set of generating paths for $\pi_1(\mathbb{P}^1 \setminus \boldsymbol{z}, z_0)$ (as in §6.2). The subscripts allude to respective points $z' \in \boldsymbol{z}$. These continuous, semisimplicial paths satisfy the following.

(6.15a) For $z' \in \boldsymbol{z}$, $\lambda_{z'}$ is homotopic to a path of the form $\tau_{z'} \delta_{z'} \tau_{z'}^{-1}$ with $\delta_{z'}$ the boundary of a clockwise disk around z'.

(6.15b) The product $\lambda_0 \lambda_\infty \lambda_{-1} \lambda_1$ is homotopic to 1.

(6.15c) For $z' \neq z''$ paths $\lambda_{z'}$ and $\lambda_{z''}$ meet only at their initial and end points z_0.

Applying μ^{-1} to these paths takes them to a new set of paths starting and ending at the same base point. Suppose $\varphi : X \to \mathbb{P}^1_z$ is a cover with $\boldsymbol{z}$ as branch points. Compute branch cycles $\boldsymbol{g}$ for the cover φ relative to the paths $\mathcal{L}$. Let $\mu^{-1}(\mathcal{L})$ be the image paths of $\mathcal{L}$ under μ^{-1}. Then, the cover $\mu \circ \varphi$ has $\boldsymbol{g}$ as branch cycle description relative to $\mu^{-1}(\mathcal{L})$.

Suppose $\boldsymbol{g}'$ is a description of the branch cycles of $\mu \circ \varphi$ relative to $\mathcal{L}$. Further suppose $Q \in H_4$ satisfies $Q(\boldsymbol{g}) = \boldsymbol{g}'$ (Q independent of $\boldsymbol{g}$). Then, the covering group for $\bar{\mathcal{H}}^{rd} \to \mathbb{P}^1_j$ is a quotient of $H_4 / \langle Q \rangle$. Here is a description of the paths we desire. Let C be the great circle on the Riemann sphere going through $0, i, \infty$. Then λ_0 is the path going from i toward 0 along C, down to a neighborhood of 0 where it goes clockwise on a small circle around 0, and then back up along C to i. Similarly, let λ_∞ be the path on C going from i toward ∞, up to a neighborhood of ∞ where it goes clockwise on a small circle around ∞, and then back along C to i. Now, take λ_{+1} along a great circle containing i and $+1$, confined to the right hemisphere (as C divides it) around $+1$ appropriate for (6.15). Finally, let λ_{-1} be *any* path around -1 allowed in (6.15) so λ_{-1} emanates from i between λ_{+1} and λ_0.

Express this choice of $\mathcal{L}$ as words in $\mu^{-1}(\mathcal{L})$. Here is what you get:

(6.16) $\mu^{-1}(\mathcal{L}) = (\mathcal{L}) Q' Q_1^{-1} Q_3$ where $Q' = Q_1 Q_2 Q_3 Q_3 Q_2 Q_1$ has the effect of conjugating $\boldsymbol{g}$ by g_1.

Since Q' is already trivial in H_4, the action of H_4 on the Nielsen class determines an action of $\bar{H}_4$ on $\mathrm{MQ}(G, \mathbf{C})$.

6.4.3. Precise branch cycles. Prop. 6.2 gives precise branch cycles for the pullback cover of $\bar{\mathcal{H}}^{rd} \to \mathbb{P}^1_j$ to the λ-line. Consider the elements

$a_{1,2}, a_{1,3}, a_{1,4}$ from §6.1. Their images in $\bar{H}_4$,

$$a'_{1,2} = (q'_1)^{-2}, \ a'_{1,3} = q'_1(q'_2)^{-2}(q'_1)^{-1}, \ a'_{1,4} = q'_1 q'_2(q'_1)^{-2}(q'_2)^{-1}(q'_1)^{-1},$$

generate the fundamental group $\Gamma(2)$ of the λ-line minus $\{0, 1, \infty\}$. It is easy to list six distinct cosets of $\Gamma(2)$ in $\bar{H}_4$. We don't know why the branch cycles fell so neatly from the relations, though we are happy they did. There is only one ingredient left to the proof: The relation between $\mathcal{H}_{O'}$ being unirational over K and properties of $\bar{\mathcal{H}}_{O'}^{rd}$. Consider the easy direction.

Suppose $\mathcal{H}_{O'}$ is unirational over K. Then there is a copy of $\mathbb{P}_t^1$ for some variable t that projects onto $\bar{\mathcal{H}}_{O'}^{rd}$. One may take the maps here over K. So, by Luroth's theorem, $\bar{\mathcal{H}}_{O'}^{rd}$ has genus 0. Being the image of $\mathbb{P}_t^1$ it contains many K points. These are K divisors of degree 1. Now, consider the converse.

Suppose $\bar{\mathcal{H}}_{O'}^{rd}$ has genus 0 and a K divisor of odd degree. The argument of §2.2 shows $\bar{\mathcal{H}}_{O'}^{rd}$ is isomorphic to $\mathbb{P}_u^1$ for some function u. Choose u to be a generic point on $\bar{\mathcal{H}}_{O'}^{rd}$ and let V_u be the fiber of $\mathcal{H}_{O'} \to \bar{\mathcal{H}}_{O'}^{rd}$ over u. Then, V_u is an open subset of a variety V' isomorphic to $\mathbb{P}^3$ over the algebraic closure of $K(u)$. Further, the argument of §6.5 shows, because $r = 4$, V_u has a $K(u)$ rational point. Thus, V' is a trivial dimension three *Brauer-Severi* variety. So, it is birational to $\mathbb{P}^3$ over $K(u)$, and $\mathcal{H}_{O'}$ is unirational. This concludes the proof of Prop. 6.5.

6.5. Comments on sections to $\mathcal{H} \to \mathcal{H}^{rd}$. These comments apply to the spaces $\mathcal{H} = \mathcal{H}^{\mathrm{abs}}(G, \mathbf{C})$ or to $\mathcal{H}^{\mathrm{in}}(G, \mathbf{C})$. Consider that $\mathcal{H}$ is just the fiber product $\mathcal{H}^{rd} \times_{J_r} U_r$ in §6.1 notion. The key question that arises in applications is this:

Question 6.7. Suppose for some field K, $\bar{\boldsymbol{p}} \in \mathcal{H}(K)^{rd}$. Then, under what conditions will there exist $\boldsymbol{p} \in \mathcal{H}(K)$ lying over $\bar{\boldsymbol{p}}$?

The fiber product formulation is helpful. Suppose $\bar{\boldsymbol{p}}$ lies over $\bar{\boldsymbol{z}} \in J_r(K)$. Then, the answer to the question is this:
(6.17) $\boldsymbol{p} \in \mathcal{H}(K)$ over $\bar{\boldsymbol{p}}$ exists if and only if there exists $\boldsymbol{z} \in U_r(K)$ lying over $\bar{\boldsymbol{z}}$.

To see this just take $\boldsymbol{p} = (\bar{\boldsymbol{p}}, \boldsymbol{z})$ in the fiber product. When $r = 4$ the answer to Question 6.7 is always (at least in characteristic 0).

Proposition 6.8 ($r = 4$-Section). *Suppose $j_0 \in K$. Then, there exists $\boldsymbol{z}$ over K with j-invariant equal to j_0.*

Proof. From [**Rob73**, I.50], $j_0 = g_2^3/(g_2^3 - 27g_3^2)$ with $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3 = 0$ a homogeneous equation for an elliptic curve having j_0 as invariant. So, take $g_2 = g_3 = 27j_0/(j_0 - 1)$ to get a curve over K with this invariant. The K point of $\mathbb{P}^4 \setminus D_4$ is the unordered collection $\boldsymbol{z}$ of branch points for this Weierstrass model. □

For all values of r, there are applications of the existence of such sections as in Prop. 6.8. For example, Serre has noted the analog for larger values of r does not hold (see [**Fri99**]).

7. Appendix on Abhyankar's Lemma.

Here is a convenient form of Abhyankar's Lemma for its use in §2.5 and §4.4.

Lemma 7.1 (Fiber Product Lemma). *Let $\psi_X : X \to Z$ and $\psi_Y : Y \to Z$ be nonsingular projective curve covers tamely ramifying over $z_0 \in Z$. Further, let $(x_1, \ldots, x_s)$ be all points of X ramified over z_0 with respective ramification indices $(e_1, \ldots, e_s)$. Similarly, $(y_1, \ldots, y_t)$ are all points of Y ramified over z_0 with respective ramification indices $(f_1, \ldots, f_t)$. Let W be the normalization of $X \times_Z Y$; $\psi_X^* : W \to X$ is the natural projection onto the first factor. For each pair (x_i, y_j) there are $\gcd(e_i, f_j)$ points of W above x_i, each with ramification index $f_j / \gcd(e_i, f_j)$. For fixed i as j runs from 1 to t, this completely lists points of W over x_i.*

Proof. Everything is local. So the general case reduces to this: $\psi_X : X \to Z$ is $\mathbb{P}^1_x \to \mathbb{P}^1_z$, ψ_X is the e-power map, $\psi_Y : Y \to Z$ is $\mathbb{P}^1_y \to \mathbb{P}^1_z$, ψ_Y is the f-power map and $z_0 = 0$. Thus, W is $\{(x,y)|x^e - y^f = 0\}$ normalized around $(0,0)$. Let m be the gcd and n the lcm of e and f. Assume $\bar{K}$ is algebraically closed and, by the tame ramification assumption, of characteristic prime to e and f. Let R be $\bar{K}[x,y]/(x^e - y^f)$ localized at $(0,0)$. Further, let ζ_m be a primitive mth root of 1. Denote $\bar{K}[x]$ localized at $x = 0$ by $\bar{K}[x]_0$.

The remainder explicitly finds the normalization S of R and lists primes of S over $x = 0$ in $\bar{K}[x]_0$. Actually, S is $\gcd(e, f)$ copies of $\bar{K}[w]_0$ as a $\bar{K}[x]_0$ algebra by the embedding $x \mapsto w^{n/m}$. Let $e' = e/m$ and $f' = f/m$. Euclid's algorithm finds integers a and b with $af' + be' = 1$: $a/e' + b/f' = 1/e'f'$. Consider

$$R_1 = \bar{K}[u,v]_{(0,0)}/(u^m - v^m) = \oplus_j \bar{K}[u,v]_{(0,0)}/(w - \zeta_m^j u).$$

The normalization of R_1 is the direct sum of m rings isomorphic to $\bar{K}[u]_0$. This reduces everything to the case $m = 1$. For that, embed R in $S = \bar{K}[w]_0$ by $x \mapsto w^f$ and $y \mapsto w^e$, so $x^a y^b = w$. □

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