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The paper provides a sufficient condition on the oblique derivative. Under this condition, an existence, uniqueness and regularity theorem is proved for the oblique boundary value problem of Monge-Ampère equations in a smoothly bounded strictly convex domain in Eulidean spaces.

1. Introduction and main results.

Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$. We consider the Monge-Ampère equation:

$$\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = f(x, u, \nabla u), \quad x \in \Omega; \tag{1.1}$$

associated to the oblique boundary condition:

$$D_\beta u(x) = \varphi(x, u(x)), \quad x \in \partial\Omega, \tag{1.2}$$

where $\beta(x)$ is a smooth unit vector field on $\overline{\Omega}$ satisfying

$$\beta(x) \cdot \nu(x) > 0, \quad \varphi_u(x, u) \le -\gamma_0(x) < 0. \tag{1.3}$$

Here ν is the unit outer normal to $\partial\Omega$, γ_0 is a positive function on $\partial\Omega$, and $D_\beta = \beta \cdot \nabla$.

The existence and uniqueness of the classical convex solution of the equations (1.1) and (1.2) under some suitiable conditions on f, φ and ν was studied by P.L. Lions, N. Trudinger and J. Urbas [**9**]. They applied the method of continuity plus *a priori* estimates to study the problem. The key and hard analysis in their argument is to obtain *a priori* estimates on the convex solution u up to $C^{2,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. By a result obtained by P.L. Lions and N. Trudinger [**8**], the problem can be reduced to prove the $C^{1,1}(\overline{\Omega})$ *a priori* estimate on u. The $C^1(\overline{\Omega})$ *a priori* estimate of u for a general oblique boundary condition was obtained in [**9**]. A very elegent argument was applied in [**9**] to obtain an *a priori* estimate on the second derivatives of u when $\beta = \nu$. Therefore, they solved the existence and uniqueness of the classical convex solution u for Neumann boundary problems (1.1) and (1.2) (with $\beta = \nu$) under the condition (1.3) and some constructive condition on f. Their technique is highly dependent on the assumption that β is normal.

For the case $n = 2$, the oblique boundary value was solved by J. Urbas [10] and later by X-J. Wang [12] with the condition:

$$\left[-2\delta_i\left(\frac{\beta_j}{\nu\cdot\beta}\right)(x)+\frac{\varphi_z(x,z)}{\nu\cdot\beta(x)}\delta_{ij}\right]\tau_i\tau_j\le -\gamma<0,\quad x\in\partial\Omega \tag{1.4}$$

for all $\tau(x) \in T_x(\partial\Omega) \cap S^{n-1}$, where $\delta_i = (\delta_{ij} - \nu_i\nu_j)\partial_j$. However, the existence of the classical convex solution of oblique boundary problem remains open when dimension $n > 2$. Since $\det(u_{ij})$ becomes much more complicated, there are essential difficulties to be overcome. In fact, V. Pogorelov gave a counterexample indicating that, in general, the oblique boundary value has no smooth solution even if β is strictly oblique and smooth. It is natural to search for a condition on β so that the oblique boundary value has a classical convex solution. It was shown in [9] that the oblique boundary value problems (1.1) and (1.2) has classical convex solution under the condition (1.4) when Ω is the unit ball $B \subset \mathbb{R}^n$. This suggests us to transfer the problem on Ω to a related problem on the unit ball B by using a change of variables. Unfortunately, the Monge-Ampère equation is not invariant under a change of variables. However, this observation is still helpful. In this paper we shall isolate the difficulty and formulate a suitable condition to avoid it in order to obtain our main results (we solve the general oblique boundary value problem when γ_0 is big enough).

Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with smooth boundary. Let ρ be a convex defining function for Ω so that $\nu(y) = \nabla\rho(y)$ for $y \in \partial\Omega$. Let ρ attain its minimum at $y_0 \in \Omega$. Without loss of generality, we may assume that $y_0 = 0$. Then we define

$$t_0(x) = \max\{\lambda > 0 : \lambda x \in \Omega\},\quad x \in \partial B. \tag{1.5}$$

It is easy to show that $t_0 \in C^\infty(\partial B)$ when $\partial\Omega$ is C^∞. We let (for $x \neq 0$)

$$t(x) = |x|^4[t_0(x/|x|) - m] + m,\quad m = \min\{t_0(x) : x \in \partial B\}. \tag{1.6}$$

Let

$$d_{ij}(x) = \frac{1}{t + x\cdot\nabla t}[2t^{-1}\partial_i t\,\partial_j t - \partial_{ij}t]. \tag{1.7}$$

Let $\lambda(x,\xi)$ be the smallest eigenvalue of the matrix $T = [T_{ij}(x,\xi)]$ with

$$T_{ij}(x,\xi) = \sum_{k=1}^n \xi_k|x|\partial_k d_{ij}(x),\qquad \xi \in S^{n-1}.$$

Then we define

$$\Lambda(x) = \min\{\lambda(x,\xi) : \xi \in S^{n-1}\}. \tag{1.8}$$

It is clear that $\Lambda(x) \le 0$. In particular, $\Lambda(x) \equiv 0$ when Ω is the unit ball and $y_0 = 0$. Let $\lambda_0(x)$ and Λ_0 be the smallest and largest eigenvalues of the matrix $[d_{ij}]$ at x, and let $\lambda(x) = \max\{|\lambda_0(x)|, |\Lambda_0(x)|\}$. Let $\phi(x) = t(x)x$

which is a $C^{2,1}$-map from B to Ω. Let $b_1(x)$ be the smallest eigenvalue of $\phi'(x)^t\phi'(x)$ at $x \in \overline{B}$ and let b_0 be the maximum of the largest eigenvalue of $\phi'(x)^t\phi'(x)$ for all x over ∂B. We let $b(x) = b_1(x)/b_0$.

Let $r(x) = \rho(\phi(x))$ for $x \in \overline{B}$. It is easy to see from the chain rule that $\nabla r(x) = \phi'(x)\nabla\rho \circ \phi(x)$, and we let $\eta(x) = \phi'(x)^{-1}\beta \circ \phi(x)$. Then

$$\gamma_1(x) := \nabla r(x) \cdot \eta(x) = \nabla\rho \cdot \beta \circ \phi(x). \tag{1.9}$$

It is easy to see that $\nabla r(x) = |\nabla r(x)|x$ for all $x \in \partial B$. Let

$$\phi'(x)^t H(\rho) \circ \phi(x)\phi'(x) = [h_{ij}(x)] \tag{1.10}$$

where $H(\rho)$ denotes real Hessian matrix of ρ. We let $h(x)$ be the smallest eigenvalue of $[h_{ij}]$ at x. Since $[h_{ij}]$ is positive definite, we have $h(x)b(x) > 0$ on $\overline{B}$. We let

$$K = \max\left\{\frac{2\lambda(x) - \Lambda(x) - \Lambda(x)(1-|x|^2)^2}{h(x)b(x)} : x \in \overline{B}\right\}. \tag{1.11}$$

Finally, for $x \in \partial B$ we let

$$c(x) = \min\left\{\frac{\partial}{\partial x_k}\left[\frac{\eta_p}{x\cdot\eta}\right](x)\xi_k\xi_p : \xi \in S^{n-1} \cap T_x(\partial B)\right\}, \tag{1.12}$$

and for each $x \in \partial B$ we assume

$$\gamma(x) := (x\cdot\eta(x))^2\gamma_0(\phi(x)) + 2(x\cdot\eta)^3 c(x) - K\gamma_1(x)|\eta(x)|^2 > 0. \tag{1.13}$$

We are ready to state our theorems, for simplicity, we state them only for $f(x, u, p) = f(x, u)$, as follows.

Theorem 1.1. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with smooth boundary. Let $f(x, u) \in C^\infty(\overline{\Omega} \times \mathbb{R})$ be positive and non-decreasing in u. Let $\beta \in C^\infty(\overline{\Omega})$ be a unit vector field and let $\varphi \in C^\infty(\partial\Omega \times \mathbb{R})$ satisfy (1.3), and (1.13). Then the oblique boundary value problem of the Monge-Ampère equations (1.1) and (1.2) has a unique convex solution $u \in C^\infty(\overline{\Omega})$.*

Theorem 1.2. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with C^4 boundary. Let f be a nonnegative function on $\overline{\Omega}$ such that $f^{1/n} \in C^{1,1}(\overline{\Omega})$ and $\varphi \in C^3(\partial\Omega \times \mathbb{R})$ satisfying (1.3), and (1.13). Then the oblique boundary value problem of the Monge-Ampère equations (1.1) and (1.2) has a unique convex solution $u \in C^{1,1}(\overline{\Omega})$.*

The paper is organized as follows. In Section 2, we collect some known results and formulate the problems for later sections. We will translate the problems from a convex domain to other problems in the unit ball in Section 3. The main theorems will be proved in Section 4. The statements and proofs of the results for more general equations are also given there.

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Added to proof: The referee pointed out that the related work was also carried out by J. Urbas [**11**] independently using a completely different method.

2. Preliminary.

In this section, we shall outline how to prove our main results, and formulate the problems for later sections.

We consider a more general equation by replacing (1.1) by

$$\det(H(u) - \sigma(x)u) = f(x, u) > 0 \tag{2.1}$$

where $\sigma(x)$ is $n \times n$ symmetric non-negative matrix with smooth entries on $\overline{\Omega}$ with $\sigma_0 \geq 0$. Here $\sigma_0(x)$ be the smallest eigenvalue of $\sigma(x)$ at $x \in \Omega$.

By a standard argument, one can prove the uniqueness of the solution of (2.1) and (1.2) under the assumption that f is non-decreasing in u. For convenience of the readers, we recall a theorem in [**8**] concerned with the existence of a convex solution. By the method of continuity and *a priori* estimate, the following theorem is proved by P. L. Lions and N. S. Trudinger [**8**].

Theorem 2.1. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with C^4 boundary. Let $\varphi \in C^3(\partial\Omega \times \mathbb{R})$ and $\beta \in C^4(\overline{\Omega})$ be unit vector fields satisfying*

$$\gamma_0(x) \geq 0, \quad \gamma_0(x) + \sigma_0(x) > 0, \quad \text{and} \quad \beta(x) \cdot \nu(x) > 0. \tag{2.2}$$

Let $f(x,t) \in C^2(\overline{\Omega} \times \mathbb{R})$ be positive and non-decreasing in t. Then (2.1) and (1.2) has a strictly convex solution $u \in C^{3,\alpha}(\overline{\Omega})$ so that

$$|u|_{C^{3,\alpha}(\overline{\Omega})} \leq C(\alpha, |u|_{C^{1,1}(\overline{\Omega})}) \tag{2.3}$$

for all $0 < \alpha < 1$ provided $\|u\|_{C^{1,1}(\overline{\Omega})}$ is finite, where C is a constant depending only on $|f|_{2,\overline{\Omega}}$, Ω, γ_0, $\gamma_0 + \sigma_0$, α, $|\varphi|_{3,\partial\Omega}$, $\min\{\gamma_1(z) : x \subset \partial B\}$ and $\|u\|_{C^{1,1}(\overline{\Omega})}$. Here $|u|_{k,X}$ denotes the C^k norm on X.

Combining Theorem 2.1 and uniqueness, the proofs of Theorems 1.1 and 1.2 are reduced to proving *a priori* estimates on the convex solution u up to the second order derivatives on $\overline{\Omega}$. We first need the following result.

Lemma 2.2. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with C^3 boundary. Let β be a smooth unit vector field on $\partial\Omega$, let $\varphi \in C^2(\partial\Omega \times \mathbb{R})$ satisfy (2.2), and let $f(x,t) \in C^1(\overline{\Omega} \times \mathbb{R})$ be positive and non-decreasing in t. If $u \in C^3(\overline{\Omega})$ is a strictly convex solution of (2.1) and (1.2), then*

$$|u|_{1,\overline{\Omega}} \leq C, \tag{2.4}$$

where C is a constant depending only on $|f|_{1,\overline{\Omega}}, \Omega, \gamma_0, |\varphi|_{0,\partial\Omega}$ and $\min\{\gamma_1(z) : x \subset \partial B\}$.

Proof. Lemma 2.2 is a special case of Theorems 2.1 and 2.2 in [**9**]. In fact, one can easily estimate the upper bound of u, say, N. Then $g(x) = f(x, N)$ and $h(p) = 1$ satisfy the conditions (2.3) and (2.4) in [**9**]. Thus, Lemma 2.2 follows as a special case of their theorems. □

The main result of this paper is now the following theorem, which will be proved in Section 4.

Theorem 2.3. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with C^4 boundary. Let β be a smooth vector field on $\partial\Omega$ and let $\varphi \in C^3(\partial\Omega \times \mathbb{R})$ satisfy (2.2) and (1.13). Let $f(x,t) \in C^{1,1}(\overline{\Omega} \times \mathbb{R})$ be positive and non-decreasing in t. If $u \in C^4(\overline{\Omega})$ is a convex solution of (2.1) and (1.2), then*

$$|D^2u(x)| \leq C, \text{ for all } x \in \overline{\Omega}, \tag{2.5}$$

where C is a constant depending only on $|f|_{2,\overline{\Omega}}$, Ω, γ_0, $\min\{\beta \cdot \nu(x) : x \in \partial\Omega\}$, $|\varphi|_{3,\partial\Omega}$ and $\min\{\gamma_1(z) : x \subset \partial B\}$.

In order to prove Theorem 2.3, we need the following result, also proved in [**9**]. Let $M_0 = |u|_{0,\overline{\Omega}}$ and $M_1 = |u|_{1,\overline{\Omega}}$. Then:

Lemma 2.4. *With the assumptions of Theorem 2.3, we have $|D_\beta D_k u| \leq C$ on $\partial\Omega$, where C is the constant depending only on $|u|_{C^1(\overline{\Omega})}$, $\|f\|_{2,\overline{\Omega}\times[-M_0,M_0]}$ and $|\varphi|_{2,\partial\Omega\times[-M_0,-M_0]}$ and Ω.*

For convenience of the reader, we sketch their proof here. Let

$$z(x) = D_\beta u(x) - \varphi(x, u) + K_1\rho(x).$$

Then we can choose a suitiable K_1 depending on M_1 and given data so that $z(x)$ attains its maximum on $\partial\Omega$, but $z \equiv 0$ on $\partial\Omega$. Then we have $D_{\beta\beta}u(x) \leq C$ on $\partial\Omega$. Moreover, for $\mathbf{e}_k = (0, \ldots, 1, \ldots, 0) = a_k\tau^k + b_k\beta$ with $\tau^k \in S^{n-1}$ and $\langle\tau^k, \nu\rangle = 0$. Thus

$$\begin{aligned}
|D_\beta D_k u| &= |D_\beta(a_k D_{\tau^k} u + b_k D_\beta u)| \\
&= |D_\beta(a_k) D_{\tau^k} u + a_k D_\beta D_{\tau^k} u + (D_\beta b_k) D_\beta u + b_k D_{\beta\beta} u| \\
&\leq CM_1 + |a_k (D_\beta \tau_j^k) D_j u| + |a_k D_{\tau^k} D_\beta u| + C(M_1) \\
&\leq C(M_1) + |a_k D_{\tau^k} \varphi(x, u)| \\
&\leq C(M_1)
\end{aligned}$$

where $C(M_1)$ is a constant depending only on M_1 and the smallest eigenvalue of $H(\rho)$ over $\overline{\Omega}$. Therefore, the proof of Lemma 2.4 is complete. □

3. A translation of the Problem.

In this section, we shall translate our problem from a general convex domain to a more complicated problem in the unit ball $B_n \subset \mathbb{R}^n$.

Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with C^∞ boundary $\partial\Omega$. Let t_0, t and ϕ be given as in Section 1 (see (1.5), (1.6), etc.) From the definition of ϕ, one can see that ϕ is a $(C^{2,1})$ homeomorphism from B_n onto Ω and also from $\overline{B}_n$ onto $\overline{\Omega}$. We shall use the following notation.

$$\phi(x) = (\phi_1(x), \dots, \phi_n(x)) = (x_1 t(x), \dots, x_n t(x)), \tag{3.1}$$

and

$$\phi'(x) = [\phi_{ij}]_{n\times n} = \left[\frac{\partial \phi_i}{\partial x_j}\right]_{n\times n}, \quad [\phi^{ij}] = \phi'(x)^{-1}. \tag{3.2}$$

It is easy to see that

$$\phi_{ki} = \frac{\partial \phi_k}{\partial x_i} = t(x)\delta_{ki} + x_k \partial_i t(x)$$

and

$$\frac{\partial \phi_k}{\partial x_i \partial x_j} = \delta_{ki}\partial_j t(x) + \delta_{kj}\partial_i t(x) + x_k \partial_{ij} t(x).$$

Therefore

$$\phi'(x) = t(x) I_n + x \otimes \nabla t(x) \tag{3.3}$$

and

$$H(\phi_k) = \mathbf{e}_k \otimes \nabla t(x) + \nabla t(x) \otimes \mathbf{e}_k + x_k H(t), \tag{3.4}$$

where $\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots, 0)$.

Let

$$a(x) = t(x)(t(x) + x \cdot \nabla t(x))^{-1}. \tag{3.5}$$

Then we have

$$\phi'(x)^{-1} = \frac{1}{t(x)} I_n - \frac{a(x)}{t(x)^2}(x \otimes \nabla t(x)). \tag{3.6}$$

In fact, if we let $B(x) = t(x)^{-1} I_n - a(x)t(x)^{-2}(x \otimes \nabla t(x))$, we shall prove $B\phi'(x) \equiv 1$. From

$$(x \otimes \nabla t)(x \otimes \nabla t) = (x \cdot \nabla t(x))(x \otimes \nabla t(x))$$

and

$$\nabla t(x) = |x|^4 \nabla t_0 + 4|x|^2 (t_0 - m)x,$$

it follows that

$$x \cdot \nabla t(x) = |x|^4 x \cdot \nabla t_0 + 4|x|^4 (t_0 - m) = 4(t - m) > 0. \tag{3.7}$$

Thus

$$a(x) = \frac{t(x)}{5t(x) - 4m} \in C^{3,1}(\overline{B}_n)$$

and

$$\begin{aligned} B(x)\,\phi'(x) &= I_n + \frac{1}{t(x)} x \otimes \nabla t(x) \\ &\quad - \frac{a(x)}{t(x)} x \otimes \nabla t(x) - \frac{a(x)}{t(x)^2}(x \otimes \nabla t(x))(x \otimes \nabla t(x)) \\ &= I_n + \frac{1}{t(x)} x \otimes \nabla t(x) - \frac{a(x)(t(x) + x \cdot \nabla t(x))}{t(x)^2} x \otimes \nabla t(x) \\ &= I_n. \end{aligned}$$

Therefore, $\phi'(x)^{-1} = B(x)$, the proof of (3.6) is complete.

Let A_k be smooth symmetric matrix-valued function defined as follows:

$$\begin{bmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} = -\phi'(x)^{-1} \begin{bmatrix} H(\phi_1) \\ \cdot \\ \cdot \\ \cdot \\ H(\phi_n) \end{bmatrix}.$$

Then we have the following proposition:

Proposition 3.1. *If $\rho^0 \in C^2(\Omega)$ and $r^0(x) = \rho^0(\phi(x))$, then*

$$H(r^0)(x) + \sum_{k=1}^{n} \frac{\partial r^0}{\partial x_k} A_k = \phi'(x)^t H(\rho^0) \circ \phi(x)\phi'(x). \tag{3.8}$$

Proof. A simple calculation shows that

$$H(r^0)(x) = \phi'(x)^t H(\rho^0) \circ \phi(x)\phi'(x) + \sum_k \frac{\partial \rho^0}{\partial y_k} \circ \phi(x) H(\phi_k). \tag{3.9}$$

By the definition of A_k, we have

$$\sum_k r_k^0(x) A_k(x) = -\sum_k \rho_k^0(\phi(x)) H(\phi_k)(x). \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$H(r^0)(x) = \phi'(x)^t H(\rho^0) \circ \phi(x)\phi'(x) - \sum_k r_k^0(x) A_k(x)$$

which completes the proof of the proposition. □

Let u be a solution of (2.1) and (1.2) with $H(u) \geq \sigma(x)u$. We define

$$v(x) = u(\phi(x)), \qquad x \in \overline{B}, \tag{3.11}$$

and

$$\eta(x) = \phi'(x)^{-1}\beta(\phi(x)). \tag{3.12}$$

Then we have following proposition.

Proposition 3.2. *For any $x \in \partial B_n$, we have*

$$D_\eta v(x) = \varphi(\phi(x), v(x)), \quad \gamma_1(x) := \eta(x)\cdot\nabla r(x) = (\beta\cdot\nabla\rho)\circ\phi(x) > 0. \tag{3.13}$$

Proof. The first identity of (3.13) follows from the chain rule. Since

$$\nabla r(x) = \phi'(x)^t \nabla\rho \circ \phi(x)$$

we have

$$\begin{aligned} \eta(x)\cdot\nabla r(x) &= (\phi'(x)^{-1}\beta\circ\phi(x))\cdot(\phi'(x)^t\nabla\rho\circ\phi(x)) \\ &= (\phi'(x)\phi'(x)^{-1}\beta\circ\phi(x))\cdot(\nabla\rho\circ\phi(x)) \\ &= (\beta\cdot\nabla\rho)\circ\phi(x) > 0. \end{aligned}$$

Therefore, the second identity follows, and the proof of the proposition is complete. □

It is easy to see that (2.1) is equivalent to

$$\det\left(H(v) + \sum_{k=1}^n v_k A_k - \tilde\sigma(x)v\right) = f_0(x, v), \tag{3.14}$$

where

$$\tilde\sigma(x) = \phi'(x)^t\sigma(\phi(x))\phi'(x) \tag{3.15}$$

$$f_0(x, v) = f(\phi(x), v)\det(\phi'(x))^2.$$

If u is a solution of (2.1) and (1.2) with $H(u) \geq \sigma(x)u$ then v is a solution of (3.14) and (3.13) with $H(v) + \sum_{k=1}^n v_k A_k - \tilde\sigma(x)v$ positive definite.

For use later, we shall compute $A_k(x)$ explicitly. By (3.4)–(3.6), we have

$$\begin{aligned} \begin{bmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} &= -t(x)^{-1}\begin{bmatrix} H(\phi_1) \\ \cdot \\ \cdot \\ \cdot \\ H(\phi_n) \end{bmatrix} + \frac{a(x)}{t(x)^2}(x\otimes\nabla t(x))\begin{bmatrix} H(\phi_1) \\ \cdot \\ \cdot \\ \cdot \\ H(\phi_n) \end{bmatrix} \\ &= -t(x)^{-1}\begin{bmatrix} H(\phi_1) \\ \cdot \\ \cdot \\ \cdot \\ H(\phi_n) \end{bmatrix} + \frac{a(x)}{t(x)^2}\begin{bmatrix} x_1\sum_j t_j H(\phi_j) \\ \cdot \\ \cdot \\ \cdot \\ x_n\sum_j t_j H(\phi_j) \end{bmatrix}. \end{aligned}$$

Therefore

$$A_k = -t(x)^{-1}H(\phi_k) + a(x)t(x)^{-2}x_k \sum_{j=1}^{n} \frac{\partial t}{\partial x_j} H(\phi_j). \tag{3.16}$$

By (3.4), we have

$$\begin{aligned}\sum_j \frac{\partial t}{\partial x_j} H(\phi_j) &= \sum_j \frac{\partial t}{\partial x_j} \mathbf{e}_j \otimes \nabla t(x) + \sum_{j=1}^{n} \frac{\partial t}{\partial x_j} \nabla t \otimes \mathbf{e}_j + \sum_j x_j \frac{\partial t}{\partial x_j} H(t) \\ &= 2\nabla t(x) \otimes \nabla t(x) + (x \cdot \nabla t) H(t).\end{aligned}$$

Thus

$$\begin{aligned}& -t^{-1}x_k H(t) + a(x)t^{-2}x_k(x \cdot \nabla t)H(t) \\ &= x_k t^{-2}(-t + a(x)(x \cdot \nabla t))H(t) \\ &= x_k t^{-1}\left[-1 + \frac{x \cdot \nabla t}{t + x \cdot \nabla t}\right] H(t) \\ &= -\frac{x_k}{t + x \cdot \nabla t} H(t).\end{aligned}$$

Therefore

$$\begin{aligned}A_k(x) &= -\frac{1}{t(x)} H(\phi_k) + a(x)t(x)^{-2}x_k[2\nabla t(x) \otimes \nabla t(x) + (x \cdot \nabla t)H(t)] \\ &= -\frac{1}{t(x)}[\mathbf{e}_k \otimes \nabla t(x) + \nabla t(x) \otimes \mathbf{e}_k] \\ &\quad + \frac{2x_k a(x)}{t^2} \nabla t \otimes \nabla t - \frac{x_k}{t} H(t) + \frac{a(x)x_k}{t^2}(x \cdot \nabla t)H(t) \\ &= -t(x)^{-1}[\mathbf{e}_k \otimes \nabla t(x) + \nabla t(x) \otimes \mathbf{e}_k] \\ &\quad + \frac{x_k}{t + x \cdot \nabla t}[2t^{-1}\nabla t(x) \otimes \nabla t - H(t)].\end{aligned}$$

Thus if we let

$$c_{ij}^p = -t(x)^{-1}[\delta_{pi}\partial_j t + \delta_{pj}\partial_i t] \tag{3.17}$$

and

$$d_{ij} = \frac{2t^{-1}\partial_i t\, \partial_j t - \partial_{ij} t}{t + x \cdot \nabla t} = \frac{2t^{-1}\partial_i t\, \partial_j t - \partial_{ij} t}{5t - 4m} \tag{3.18}$$

then

$$a_{ij}^p = c_{ij}^p + x_p d_{ij}. \tag{3.19}$$

As a corollary of Lemmas 2.2 and 2.4, we have:

Proposition 3.3. *With the assumptions of Theorem 2.3, we have*

(i) $|v|_{1,\overline{B}} \le CM_1 \le C$;
(ii) $|D_\eta D_k v| \le C$ *on* ∂B.

Here C is the constant depending only on $|u|_{C^1(\overline{\Omega})}$, $\|f\|_{1,\overline{\Omega}\times[-M_0,M_0]}$ and $|\varphi|_{2,\partial\Omega\times[-M_0,-M_0]}$ and Ω.

4. The proof of Theorem 2.3.

In this section, we shall complete the proof of Theorem 2.3. Let $u \in C^4(\overline{\Omega})$ be a solution of (2.1) and (1.2) so that $H(u) \geq \sigma(x)u$.

Let $W = [w_{ij}] = H(v) + A_p v_p - \tilde{\sigma}(x)v$ and $[F^{ij}] = W^{-1}$. Then we have the following identity (see [**9**] and [**7**]).

$$F^{ij}\partial_{\xi\xi}w_{ij} = \partial_{\xi\xi}g(x,v) + F^{i\ell}F^{jk}\partial_\xi w_{ij}\partial_\xi w_{k\ell} \tag{4.1}$$

for all $\xi \in \partial B_n$, where $\partial_\xi = \sum_i \xi_i \partial_i$ and $g(x) = \log f_0(x)$.

Theorem 4.1. *Under the assumptions of Theorem 2.3, let $u \in C^4(\overline{\Omega})$ is a solution of (2.1) and (1.2) with $H(u) \geq \sigma u$. Then $v = u \circ \phi$ satisfies (3.14), (3.13) and*

$$\sum_{k,\ell}(\delta_{k\ell} - x_k x_\ell)\partial_{k\ell}v(x) \leq C \quad x \in \partial B, \tag{4.2}$$

where C is a constant depending only on given data.

In order to prove Theorem 4.1, we need some lemmas.

Lemma 4.2. *If $v = u \circ \phi(x)$ is a solution of (3.14) and (3.13) with $H(v) + A_p v_p - \tilde{\sigma}(x)v$ positive definite on B then*

$$\mathrm{tr}(v_{k\ell})(x) \leq C(M_1) + b(x)^{-1}M_{2,\partial B} \tag{4.3}$$

where

$$M_{2,\partial B} = \max\{\mathrm{tr}(v_{k\ell})(x) : x \in \partial B\}. \tag{4.4}$$

Here $b(x)^{-1}$ the ratio of the maximum largest eigenvalue of $\phi'(x)^t\phi'(x)$ on $\overline{B}$ and the minimum eigenvalue of $\phi'(y)^t\phi'(y)$ for $y \in \partial B$.

Proof. Notice that

$$W(x) = H(v) + \sum_{k=1}^n a_{ij}^k v_k = \phi'(x)^t H(u)\circ\phi(x)\phi'(x).$$

It is easy to verify there is a constant $K_3 = C$ depending only on M_1, f, σ and their first and second derivatives on $\overline{\Omega}$ so that $\mathrm{tr}(u_{k\ell})\circ\phi(x) + K_3 r(x)$ attains its maximum over $\overline{B}$ at some point $x^1 \in \partial B$. Since

$$\mathrm{tr}(u_{k\ell})\circ\phi(x) + K_3 r(x) = \mathrm{tr}\big((\phi'(x)^{-1})^t W \phi'(x)^{-1}\big) + K_3 r(x)$$

attains its maximum over $\overline{B}$ at the point $x^1 \in \partial B$, we have, with the notation of b_1 being minimum of the smallest eigenvalue of $\phi'(x)^t\phi'(x)$ over $\overline{B}$,

$$\mathrm{tr}\big(W\big)(x) + K_3 b_1(x)^{-1}r(x) \leq b(x)^{-1}\mathrm{tr}(W)(x^1)$$

and so

$$\mathrm{tr}\big(W\big)(x) \le b(x)^{-1}\mathrm{tr}\big(W\big)(x^1) - K_3 b_1(x)^{-1} r(x).$$

Therefore (4.3) holds, and the proof of the lemma is complete. □

Lemma 4.3. *If $v = u \circ \phi(x)$ is a solution of (3.14) and (3.13) with $H(v) + A_p v_p - \sigma(x)v$ positive definite on B then*

$$M_{2,\partial B} \le C(M_1) + \sup_{x\in\partial B} \frac{|\eta|^2}{(x\cdot\eta)^2} \left\{ \sum_{k,\ell=1}^{n} (\delta_{k\ell} - x_k x_\ell) v_{k\ell}(x) \right\}. \tag{4.5}$$

Proof. We write

$$\eta = a_1 x + a_2 \tau(x)$$

where $\tau(x) \in T_x(\partial B) \cap S^{n-1}$, and $a_1^2 + a_2^2 = |\eta|^2$ with $a_j \ge 0$. Let

$$N[v](x) = \sum_{k\ell=1}^{n} x_k x_\ell v_{k\ell}(x). \tag{4.6}$$

Then for any $x \in \partial B$ we have

$$\begin{aligned} N[v](x) &\le a_1^{-1} D_\nu D_\eta v(x) - a_1^{-1} a_2 D_\nu D_\tau v(x) + C \\ &\le C(M_1) - a_1^{-1} a_2 D_\nu D_\tau v(x) \\ &\le C(M_1) + \frac{1}{2} N[v] + \frac{1}{2}\frac{a_2^2}{a_1^2} D_{\tau\tau} v(x), \end{aligned}$$

and hence

$$N[v](x) \le 2C + \frac{a_2^2}{a_1^2} D_{\tau\tau} v(x). \tag{4.7}$$

Therefore for $x \in \partial B$, we have

$$\mathrm{tr}(v_{k\ell}) \le C(M_1) + \left(1 + \frac{a_2^2}{a_1^2}\right) \sum_{k,\ell=1}^{n} [\delta_{k\ell} - x_k x_\ell] v_{k\ell}(x).$$

Notices that $1 + a_2^2 a_1^{-2} = |\eta(x)|^2 (x\cdot\eta)^{-2}$. We have (4.5) hold, and the proof of the lemma is complete. □

We now are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let

$$M(x) = \sum_{k\ell} \big(\delta_{k\ell} - x_k x_\ell\big) v_{k\ell} + K_1 |\nabla v|^2 + K_0 r(x) \tag{4.8}$$

and let

$$V(x) = M(x) + K_2 M_{2,\partial B} r(x) \qquad x \in \overline{B} \tag{4.9}$$

where

$$K_2 = K + \frac{1}{2}\min\{\gamma(x)|\eta(x)|^{-2}\gamma_1(x)(x\cdot\eta(x))^{-3} : x \in \partial B\}.$$

Let

$$\mathcal{L} = F^{ij}\partial_{ij} + F^{ij}a^p_{ij}\partial_p. \tag{4.10}$$

We shall compute $\mathcal{L}V(x)$. For convenience, we shall use the following notation: $v_j = \partial_j v$, $v_{ij} = \partial_{ij} v$ and $v_{ijk} = \partial_{ijk} v$, etc.. First, note that

$$F^{ij}\partial_{ij}[\delta_{k\ell} - x_k x_\ell]v_{k\ell} = -4x_k F^{ij} v_{ijk} - 2F^{ij}v_{ij} + F^{ij}[\delta_{k\ell} - x_k x_\ell]v_{k\ell ij}. \tag{4.11}$$

The first term of RHS of (4.11) satisfies the following estimate:

$$\begin{aligned} -4x_k F^{ij} v_{ijk} &= -4x_k F^{ij}\partial_k w_{ij} + 4x_k F^{ij}\partial_k[a^p_{ij}v_p - \tilde{\sigma}_{ij}v] \\ &= -4x_k F^{ij}\partial_k g + 4x_k F^{ij}a^p_{ij}v_{pk} + 4x_k v_p \partial_k a^p_{ij} - 4F^{ij}x_k\partial_k\left[\tilde{\sigma}_{ij}v\right] \\ &\geq -C(M_1)\mathrm{tr}(F^{ij}) + 4x_k v_{pk}F^{ij}a^p_{ij}, \end{aligned}$$

the second term of RHS of (4.11) satisfies the following estimate:

$$-2F^{ij}v_{ij} = -2F^{ij}w_{ij} + 2F^{ij}[a^p_{ij}v_p - \tilde{\sigma}_{ij}v] \geq -C[M_1]\mathrm{tr}(F^{ij}).$$

Notice that $a^p_{ij} \in C^2(\overline{B})$, we have $F^{ij}\partial_{k\ell}a_{ij} \geq -C\mathrm{tr}(F^{ij})$. Thus the third term of RHS of (4.11) satisfies that

$$\begin{aligned} &F^{ij}[\delta_{k\ell} - x_k x_\ell]v_{k\ell ij} \\ &\quad = F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_{k\ell}w_{ij} - F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_{k\ell}(a^p_{ij}v_p - \tilde{\sigma}_{ij}v) \\ &\quad \geq [\delta_{k\ell} - x_k x_\ell]\partial_{k\ell}g - F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_{k\ell}(a^p_{ij}v_p - \tilde{\sigma}_{ij}v) \\ &\quad \geq -C(M_1)(\mathrm{tr}(F^{ij}) + \mathrm{tr}(v_{k\ell})) + F^{ij}\tilde{\sigma}_{ij}M(x) - F^{ij}a^p_{ij}\partial_p[(\delta_{k\ell} - x_k x_\ell)v_{k\ell}] \\ &\qquad -F^{ij}[\delta_{k\ell} - x_k x_\ell][\partial_k a^p_{ij}v_{\ell p} + \partial_\ell a^p_{ij}v_{kp}] - 2F^{ij}x_k v_{kp}a^p_{ij}. \end{aligned}$$

Moreover, we have the following two inequalities:

$$\begin{aligned} F^{ij}\partial_{ij}|\nabla v|^2 &= 2v_k F^{ij}v_{ijk} + 2F^{ij}v_{ki}v_{jk} \\ &= 2v_k F^{ij}\partial_k w_{ij} - 2v_k F^{ij}\partial_k[a^p_{ij}v_p - \tilde{\sigma}_{ij}v] + 2F^{ij}v_{ki}v_{jk} \\ &\geq 2v_k\partial_k g - F^{ij}a^p_{ij}\partial_p|\nabla v|^2 - C(M_1)\mathrm{tr}(F^{ij}) + 2F^{ij}v_{ki}v_{jk}, \end{aligned}$$

and

$$\mathcal{L}r(x) = F^{ij}h_{ij} + F^{ij}a^p_{ij}\partial_p r \geq h(x)\mathrm{tr}(F^{ij}) - C\mathrm{tr}(F^{ij}).$$

By combining the above five estimates and (4.11), we have

$$\begin{aligned} &\mathcal{L}M(x) \\ &\geq -C(M_1)[(1+K_1)\mathrm{tr}(F^{ij}) + \mathrm{tr}(v_{k\ell})] + F^{ij}\tilde{\sigma}_{ij}M(x) \\ &- 2F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_k a^p_{ij}v_{\ell p} + 2x_k v_{kp}F^{ij}a^p_{ij} + K_1F^{ij}v_{ik}v_{jk} + K_0h(x)\mathrm{tr}(F^{ij}) \\ &\geq [K_0h(x) - C(M_1)(K_1+1)]\mathrm{tr}(F^{ij}) + [K_1 - C(M_1)]\mathrm{tr}(v_{k\ell}) + F^{ij}\tilde{\sigma}_{ij}M(x) \\ &- 2F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_k a^p_{ij}v_{\ell p} + 2x_k v_{kp}F^{ij}a^p_{ij}. \end{aligned}$$

We write

$$I[v] = -2F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_k c_{ij}^p\, v_{\ell p} + 2x_k v_{kp} F^{ij} c_{ij}^p \tag{4.12}$$

and

$$J[v] = -2F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_k (x_p d_{ij}) v_{\ell p} + 2x_k x_p v_{kp} F^{ij} d_{ij}. \tag{4.13}$$

We claim that

$$I[v] \geq -C[M_1]\mathrm{tr}(F^{ij}). \tag{4.14}$$

In fact,

$$\begin{aligned}
&I[v] \\
&= 2t(x)^{-1}F^{ij}[\delta_{k\ell} - x_k x_\ell](\partial_k t_j v_{\ell i} + \partial_k t_i v_{j\ell}) \\
&\quad - 2t(x)^{-2}[\delta_{k\ell} - x_k x_\ell]\partial_k t F^{ij}(\partial_j t\, v_{i\ell} + \partial_i t\, v_{j\ell}) - 2t^{-1}x_k F^{ij}(v_{ki}t_j + v_{kj}t_i) \\
&= \frac{4}{t}[\delta_{k\ell} - x_k x_\ell]F^{ij}t_{kj}v_{\ell i} - \frac{4}{t^2}[\delta_{k\ell} - x_k x_\ell]\partial_k t F^{ij}\partial_j t\, v_{i\ell} - \frac{4}{t}x_k F^{ij}v_{ki}t_j \\
&\geq -C(M_1)\mathrm{tr}(F^{ij}) + \frac{4}{t}\big([\delta_{k\ell} - x_k x_\ell]F^{ij}w_{\ell i}(t_{kj} - t(x)^{-1}t_k t_j) - x_k F^{ij}w_{ki}t_j\big) \\
&\geq -C(M_1)\mathrm{tr}(F^{ij})
\end{aligned}$$

and the proof of (4.14) is complete.

Next we consider $J[v]$. It is obvious that

$$\begin{aligned}
J[v] &= -2F^{ij}[\delta_{k\ell} - x_k x_\ell]\partial_k(x_p d_{ij})v_{p\ell} + 2x_k v_{pk} x_p F^{ij} d_{ij} \\
&= -2F^{ij}[\delta_{k\ell} - x_k x_\ell][x_p v_{p\ell}\partial_k d_{ij} + \delta_{pk} d_{ij} v_{p\ell}] + 2x_k v_{pk} x_p F^{ij} d_{ij} \\
&= 2F^{ij}d_{ij}(-\mathrm{tr}(v_{k\ell}) + 2N[v]) - 2F^{ij}[\delta_{k\ell} - x_k x_\ell]x_p v_{p\ell}\partial_k d_{ij}.
\end{aligned}$$

Now we consider

$$\begin{aligned}
|x|^2 N_1[v]^2 := \sum_{k=1}^n \left(\sum_{p,\ell=1}^n [\delta_{k\ell} - x_k x_\ell]x_p v_{p\ell}\right)^2
&= \sum_{k=1}^n \left(\sum_{p=1}^n x_p v_{pk} - x_k N[v]\right)^2 \\
&= \sum_{k=1}^n \left(\sum_{p=1}^n x_p v_{pk}\right)^2 - 2\sum_{k=1}^n x_k N[v]\sum_{p=1}^n x_p v_{pk} + \sum_{k=1}^n x_k^2 N[v]^2 \\
&= \sum_{k=1}^n \left(\sum_{p=1}^n x_p v_{pk}\right)^2 - 2N[v]^2 + |x|^2 N[v]^2.
\end{aligned}$$

Without loss of generality, by rotation, we may assume that $x = (|x|, 0, \dots, 0)$ as a vector. Since $H(v) - A_p v_p - \tilde{\sigma} v$ is positive definite and we may assume $\operatorname{tr}(v_{ij}) \ge C$, we have

$$\begin{aligned}\sum_{k=1}^n \left(\sum_{p=1}^n x_p v_{pk}\right)^2 &= \sum_{k=1}^n |x|^2 v_{1k}^2 \le |x|^2 (v_{11} + C)(\operatorname{tr}(v_{k\ell})) \\ &= (N[v] + C|x|^2)(\operatorname{tr}(v_{k\ell})).\end{aligned}$$

Thus

$$\begin{aligned}|x|^2 N_1[v]^2 &\le (N[v] + C|x|^2)\operatorname{tr}(v_{k\ell}) - 2N[v]^2 + |x|^2 N[v]^2 \\ &= N[v](\operatorname{tr}(v_{k\ell}) - 2N[v] + |x|^2 N[v]) + C|x|^2 \operatorname{tr}(v_{k\ell}).\end{aligned}$$

Thus

$$N_1[v]^2 \le \frac{N[v]}{|x|^2}(\operatorname{tr}(v_{k\ell}) - 2N[v] + |x|^2 N[v] + C) + C\operatorname{tr}(v_{k\ell}).$$

Therefore, by Cauchy-Schwarz's inequality, we have

$$\begin{aligned}N_1[v] &\le (1/2)\left(\operatorname{tr}(v_{k\ell}) - 2N[v] + |x|^2 N[v] + |x|^{-2} N[v] + C\sqrt{\operatorname{tr}(v_{k\ell})}\right) \\ &= (1/2)[\operatorname{tr}(v_{k\ell}) - (1 - |x|^2)N[v] + |x|^{-2}N[v](1 - |x|^2)] + C\sqrt{\operatorname{tr}(v_{k\ell})} \\ &= (1/2)[\operatorname{tr}(v_{k\ell}) + |x|^{-2}N[v](1 - |x|^2)^2] + C\sqrt{\operatorname{tr}(v_{k\ell})} \\ &\le (1/2)[1 + (1 - |x|^2)^2]\operatorname{tr}(v_{k\ell}) + C\sqrt{\operatorname{tr}(v_{k\ell})}.\end{aligned}$$

Therefore (since $\Lambda \le 0$)

$$\begin{aligned}&-2F^{ij}[\delta_{k\ell} - x_k x_\ell] x_p v_{p\ell} \partial_k d_{ij} \\ &\quad\ge 2\Lambda N_1[v] \\ &\quad\ge \Lambda \operatorname{tr}(F^{ij})[1 + (1 - |x|^2)^2]\operatorname{tr}(v_{k\ell}) + 2C\Lambda \operatorname{tr}(F^{ij})\sqrt{\operatorname{tr}(v_{ij})}.\end{aligned}$$

It is obvious that

$$-2F^{ij} d_{ij}[\operatorname{tr}(v_{k\ell}) - 2N[v]) \ge -2\lambda(x)\operatorname{tr}(F^{ij})\operatorname{tr}(v_{k\ell}).$$

Thus

$$J[v] \ge [\Lambda(x)(1+(1-|x|^2)^2) - 2\lambda]\operatorname{tr}(F^{ij})\operatorname{tr}(v_{k\ell}) + 2C\Lambda \operatorname{tr}(F^{ij})\sqrt{\operatorname{tr}(v_{ij})}. \tag{4.15}$$

Now we assume that $V(x)$ attains its maximum over $\overline{B}$ at $x^0 \in B$. By Lemmas 4.2 and 4.3, we have (at $x = x^0$)

$$\begin{aligned}M(x^0) &\ge -C - K_2 r(x^0) M_{2,\partial B} + \max\{M(x) : x \in \partial B\} \\ &\ge -C(M_1) + \left[-K_2 r(x^0) + \frac{a_1(x^1)^2}{|\eta(x^1)|^2}\right] M_{2,\partial B} \\ &\ge -C(M_1) + \gamma_2 b(x^0)\operatorname{tr}(v_{k\ell}(x^0))\end{aligned}$$

where

$$\gamma_2 := \min\left\{a_1(x)^2|\eta(x)|^{-2} : x \in \partial B\right\}. \tag{4.16}$$

Therefore, by choosing $K_1 \geq 4C(M_1)(1 + h(x) + b(x)^{-1})$ and $K_0 \geq (K_1 + 1)C(M_1)h(x)^{-1}$ for all $x \in B$, and (4.3), we have

$$\begin{aligned}
&\mathcal{L}M(x)\\
&\geq \frac{K_1}{2}\mathrm{tr}(F^{ij}) + \sigma_0\mathrm{tr}(F^{ij})M(x) + I[v] + J[v] + 2C(M_1)(h(x) + b^{-1})\mathrm{tr}(F^{ij})\\
&> \sigma_0\mathrm{tr}(F^{ij})\gamma_2 b(x)\mathrm{tr}(v_{k\ell})(x) + 2C(M_1)(h(x) + b^{-1}(x))\mathrm{tr}(F^{ij})\\
&\ + [\Lambda(x)(1 + (1 - |x|^2)^2) - 2\lambda]\mathrm{tr}(F^{ij})\mathrm{tr}(v_{k\ell}) + 2C\Lambda\mathrm{tr}(F^{ij})\sqrt{\mathrm{tr}(v_{ij})}\\
&= [\sigma_0\gamma_2 b(x) + \Lambda(x)(1 + (1 - |x|^2)^2) - 2\lambda]\mathrm{tr}(F^{ij})\mathrm{tr}(v_{k\ell})\\
&\ + 2C(M_1)(h(x) + b^{-1}(x))\mathrm{tr}(F^{ij}) + 2C\Lambda\mathrm{tr}(F^{ij})\sqrt{\mathrm{tr}(v_{ij})}\\
&\geq -Kb(x)h(x)\mathrm{tr}(F^{ij})\mathrm{tr}(v_{k\ell}) + 2C\Lambda\mathrm{tr}(F^{ij})\sqrt{\mathrm{tr}(v_{ij})}\\
&\geq -Kh(x)\mathrm{tr}(F^{ij})M_{2,\partial B} + 2C\Lambda\mathrm{tr}(F^{ij})\sqrt{M_{2,\partial B}b(x)^{-1}}
\end{aligned}$$

where $K = \max\{K^0, 0\}$ and

$$K^0 = \min\left\{-\frac{\sigma_0(x)\gamma_2}{h(x)} + \frac{2\lambda - \Lambda - \Lambda(x)(1 - |x|^2)^2}{h(x)b(x)} : x \in \overline{B}\right\}. \tag{4.17}$$

Therefore, we have either

$$\mathcal{L}V(x^0) > h(x^0)\mathrm{tr}(F^{ij})M_{2,\partial B}(-K + K_2) + 2C\Lambda\mathrm{tr}(F^{ij})\sqrt{M_{2,\partial B}b(x)^{-1}} > 0$$

or $M_{2,\partial B} \leq C(K_2 - K)^{-2}$. In other words, we have either $x^0 \in \partial B$ or $M_{2,\partial B} \leq C(K_2 - K)^{-2}$. If $M_{2,\partial B} \leq C(K_2 - K)^{-2}$ then the proof of Theorem 4.1 is complete. Without loss of generality, we may assume $x^0 \in \partial B$. Let

$$\eta^0 = (\eta \cdot x)^{-1}\eta, \quad \gamma_3 = \gamma_1(x \cdot \eta)^{-1}(x^0).$$

Now applying Proposition 3.3, at $x = x^0$ with $\delta_k = \sum_{j=1}^n (\delta_{kj} - x_k^0 x_j^0)\partial_j$, we have

$$
\begin{aligned}
0 &\le D_{\eta^0} V(x^0) \\
&\le D_{\eta^0} M(x^0) + 2K_1 v_k(x^0) D_{\eta^0}\partial_k v(x^0) + K_0\gamma_3 + K_2\gamma_3 M_{2,\partial B} \\
&\le C(M_1) + D_{\eta^0} M(x^0) + K_2\gamma_3 M_{2,\partial B} \\
&= C(M_1) + D_{\eta^0} \sum_{k,\ell} (\delta_{k\ell} - x_k^0 x_\ell^0) v_{k\ell}(x^0) + K_2\gamma_3 M_{2,\partial B} \\
&= C(M_1) + D_{\eta^0} \sum_{k=1}^n \left(\partial_k - x_k^0 \sum_\ell x_\ell^0 \partial_\ell\right)\left(\partial_k - x_k^0 \sum_{\ell=1}^n x_\ell^0 \partial_\ell\right) v(x^0) \\
&\quad + K_2\gamma_3\, M_{2,\partial B} \\
&\le C(M_1) + \sum_{k=1}^n \left(\partial_k - x_k^0 \sum_\ell x_\ell^0 \partial_\ell\right)\left(\partial_k - x_k^0 \sum_{\ell=1}^n x_\ell^0 \partial_\ell\right) D_{\eta^0} v(x^0) \\
&\quad - 2\sum_{k=1}^n \sum_p \delta_k \eta_p^0 \delta_k v_p(x^0) + K_2\gamma_3 M_{2,\partial B} \\
&\le C(M_1) + (x^0\cdot\eta)(x^0)^{-1}\varphi_v(\phi(x^0), v(x^0)) M(x^0) - 2(\delta_k\eta_p^0)\delta_k\partial_p v(x^0) \\
&\quad + K_2\gamma_3 M_{2,\partial B} \\
&\le -[(x^0\cdot\eta)(x^0)^{-1}\gamma_0(\phi(x^0)) - K_2\gamma_3|\eta(x^0)|^2 (x^0\cdot\eta(x^0))^{-2}] M(x^0) \\
&\quad + C(M_1) - 2(\delta_k\eta_p^0)\delta_k\partial_k v(x^0).
\end{aligned}
$$

Now we consider $-2(\delta_k\eta_p^0)\delta_k\partial_p v(x^0)$. Without loss of generality, we may assume that $x^0 = (0, \dots, 1)$. Thus $\delta_j = \partial_j$ for $1 \le j < n$ and $\delta_n = 0$. Let

$$
c(x^0) = \min\left\{ \sum_{k,p=1}^{n-1} \frac{\partial \eta_p^0}{\partial x_k}(x^0)\xi_k^0\xi_p^0 : \xi \in S^{n-1} \right\}. \tag{4.18}
$$

Notice that $\eta_n^0(x^0) = 1$, we have

$$
\begin{aligned}
-2(\delta_k\eta_p^0)\delta_k\partial_k v(x^0) &= -2\sum_{p=1}^n \sum_{k=1}^{n-1} \partial_k\eta_p^0\, v_{pk}(x^0) \\
&= -2\sum_{p,k=1}^{n-1} \partial_k\eta_p^0\, v_{pk}(x^0) - 2\sum_{k=1}^{n-1} \frac{\partial\eta_n^0}{\partial x_k} v_{nk}(x^0) \\
&\le -2c(x^0)M(x^0) + C(M_1).
\end{aligned}
$$

Therefore

$$\begin{aligned} 0 &\leq C(M_1) - \left[\frac{\gamma_0(x^0)}{(x\cdot\eta)(x^0)} - 2c(x^0) - K_2\gamma_3|\eta(x^0)|^2(x^0\cdot\eta(x^0))^{-2}\right]M(x^0) \\ &\leq C(M_1) - \left[\frac{\gamma_0(x^0)}{(x\cdot\eta)(x^0)} - 2c(x^0) - K\gamma_3|\eta(x^0)|^2(x^0\cdot\eta(x^0))^{-2}\right]M(x^0) \\ &\qquad + \frac{1}{2}\gamma(x^0)(x^0\cdot\eta(x^0))^{-3}M(x^0) \\ &= C(M_1) - \frac{1}{2}\gamma(x^0)(x\cdot\eta)(x^0)^{-3}M(x^0). \end{aligned}$$

Since $\gamma(x) > 0$ for all $x \in \partial B$. We have $M(x^0) \leq C(M_1)/\gamma(x^0) \leq C(M_1)$, and so

$$M_{2,\partial B} \leq C(M_1). \tag{4.19}$$

Therefore, by combining Lemma 4.2 and (4.19), the proof of Theorem 4.1 is complete, and so is that of Theorem 2.3. □

Combining Theorem 2.1 and Theorem 2.3, we have the following theorem.

Theorem 4.4. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ with C^∞ boundary. Let $f(x,t) \in C^\infty(\overline{\Omega}\times\mathbb{R})$ be positive and non-decreasing in t. Let β be a smooth vector field on $\partial\Omega$ and let $\varphi \in C^\infty(\partial\Omega\times\mathbb{R})$ satisfy (2.2). If K is given by (4.17) and satisfies*

$$(x\cdot\eta)(x)^2\gamma_0(\phi(x)) + 2(x\cdot\eta)(x)^3c(x) - K\gamma_1(x)|\eta(x)|^2 > 0, \quad x \in \partial B, \tag{4.20}$$

then there is a unique solution $u \in C^\infty(\overline{\Omega})$ of (2.1) and (1.2) with $H(u) \geq \sigma(x)u(x)$ on Ω. In paricular, when $\sigma(x) = 0$, Theorem 1.1 holds.

Proof of Theorem 1.2. The proof of Theorem 1.2 can be obtained by combining Theorem 1.1 and the argument of proof of Theorem 4.4 as well as some treatments in [**7**] and [**9**]. We omit the details here. □

References

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