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Let $\{S_i\}_{i=1}^n$ be generators of the Cuntz algebra $\mathcal{O}_n$ and let Φ be the *-endomorphism of $\mathcal{O}_n$ defined by $\Phi(x) = \sum_{i=1}^n S_i x S_i^*$. Then both of Connes–Narnhofer–Thirring's entropy $h_\phi(\Phi)$ and Voiculescu's topological entropy $ht(\Phi)$ are $\log n$, where ϕ is the unique $\log n$-KMS state of $\mathcal{O}_n$. Also Longo's canonical endomorphism for $N \subset M$ have the same entropy $\log n$, where the inclusion $N \subset M$ comes from $\mathcal{O}_n$.

1. Introduction.

Connes-Stϕrmer entropy $H(\cdot)$ extended the entropy invariant of Kolmogorov-Sinai to trace preserving automorphisms of finite von Neumann algebras ([**CS**]). Replacing a finite trace to an invariant state ϕ, Connes-Narnhofer-Thirring entropy $h_\phi(\cdot)$ is defined for automorphisms of C^*-algebras as a generalization of $H(\cdot)$ ([**CNT**]). These entropies depend on an invariant state under a given automorphism.

The first typical interesting example to compute the entropy is the Bernoulli shift β_n on the infinite product space of n-point sets.

In the context of operator algebras (von Neumann algebras or C^*-algebras), the non-commutative Bernoulli shift α_n takes the place of the the Bernoulli shift β_n. It is the shift automorphism on the infinite tensor product $A = \bigotimes_{i=-\infty}^{\infty} A_i$ (where A_i is the $n \times n$-matrix algebra) and $H(\alpha_n) = \log n = h_\tau(\alpha_n)$ ([**CS**], [**CNT**]), where τ is the unique tracial state of A.

Ler γ be an aperiodic automorphism of an algebra B. Then there exists an implimenting unitary operator u for γ in the crossed product $M = B \rtimes_\gamma \mathbb{Z}$. The inner automorphism $Ad_u, (Ad_u(x) = uxu^*)$ of M is an extension of γ to M. In general, the entropy of γ is less than the entropy of Ad_u. Stϕrmer [S] asked if the equality between the entropies of γ and Ad_u holds.

Voiculescu [**V**] defined topological entropy $ht(\cdot)$ for automorphisms of nuclear C^*-algebras (cf. [**Hu**], [**T**]), which does not depend on any state but is based on approximations. As an application, he showed that his topological entropy satisfies the equality for the Bernoulli shift β_n, so that Connes-Narnhofer-Thirring entropy does too.

In this paper, we show the equality for both of the automorphism α_n and the unital *-endomorphism of the type of the non-commutative Bernoulli shift.

In §3, we denote only the fact that

$$H(Ad_u) = h_\tau(Ad_u) = ht(Ad_u) = \log n,$$

where τ is the unique tracial state of the reduced crossed product $A \rtimes_{\alpha_n} \mathbb{Z}$. These are proved by similar method as in §4 and §5.

The definition of Connes-Stϕrmer entropy is available to trace preserving *-endomorphisms on finite von Neumann algebras. Similarly, we can apply the definition of Connes-Narnhofer-Thirring entropy to unital and state preserving *-endomorphisms of C^*-algebras, and also Voiculescu's topological entropy to unital *-endomorphisms of nuclear C^*-algebras. We apply here, in particular, to the unital *-endomorphism which is an extension of the *-endomorphism coming from the non-commutative Bernoulli shift α_n as follows.

If we restrict our algebra A to the *half side* infinite C^*-tensor product (or von Neumann tensor product) $B = \bigotimes_{i=0}^{\infty} A_i$ of matrix algebras, then the restriction of α_n to B defines a unit preserving *-endomorphism σ_n of B, which is canonical in the sense of [**Ch2**, **Ch3**]. Then we have the extension algebra $\langle B, \sigma_n \rangle$ of B by σ_n ([**Ch2**, **Ch3**]). In the case of C^* algebras, $\langle B, \sigma_n \rangle$ is the crossed product $B \rtimes_\rho \mathbb{N}$ of B by the corner endomorphism ρ in [**R**, **I2**], which is defined by σ_n using the canonical property of σ_n. Further, the canonical extension $\hat{\sigma}_n$ (in the sense of [**Ch2**, **Ch3**]) of σ_n to $\langle B, \sigma_n \rangle$ is obtained. The *-endomorphism $\hat{\sigma}_n$ of $\langle B, \sigma_n \rangle$ is defined by a modification of the automorphism Ad_u of $A \rtimes_{\alpha_n} \mathbb{Z}$ and has the property like the canonical extension in the sense of [**I1**, **HS**]. In the case of C^*-algebras, the extension algebra $\langle B, \sigma_n \rangle$ is the Cuntz algebra $\mathcal{O}_n$ and $\hat{\sigma}_n$ is nothing but Cuntz's canonical inner endomorphism Φ of $\mathcal{O}_n$ defined by $\Phi(x) = \sum_{i=1}^{n} S_i x S_i^*$, $(x \in \mathcal{O}_n)$ for generators $\{S_1, \dots, S_n\}$ of $\mathcal{O}_n$. In the case of von Neumann algebras, $\langle B, \sigma_n \rangle$ is the unique injective type $\mathrm{III}_{1/n}$ factor and $\hat{\sigma}_n$ is Longo's canonical endomorphism for the subfactor of $\langle B, \sigma_n \rangle$, which appears naturally in the construction of the extension algebra $\langle B, \sigma_n \rangle$ by the canonical *-endomorphism σ_n ([**Ch3**]).

In §4, we show that

$$ht(\Phi) = \log n = ht(\sigma_n).$$

Applying to Connes-Narnhofer-Thirring's entropy $h_\phi(\cdot)$ relative to the unique $\log n$-KMS state ϕ of $\mathcal{O}_n$, we have

$$h_\phi(\Phi) = \log n = h_\phi(\sigma_n).$$

This relation implies the same relation for Longo's canonical endomorphism. Thus the canonical extension of the non-commutative Bernoulli shift has the same entropy with the original one in the case of *-endomorphisms too.

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2. Preliminaries.

2.1. Let H_0 be a Hilbert space of dimension $n < \infty$. Put $H_i = H_0,\ i \in \mathbb{Z}$. For two integers i and j with $i < j$, we put

$$H_{[i,j]} = H_i \otimes H_{i+1} \otimes \cdots \otimes H_j.$$

Let $\{\delta(i) : i = 1, ..., n\}$ be an orthonormal basis of H_0. The emmbedding $H_{[i,j]} \hookrightarrow H_{[i-1,j+1]}$ is given by $\xi \in H_{[i,j]} \to \delta(1) \otimes \xi \otimes \delta(1) \in H_{[i-1,j+1]}$. We denote by $\mathcal{H}_i$ the inductive limit of $\{H_{[i,i+j]} : j = 0, 1, ...\}$ and by $\mathcal{H}$ the inductive limit of the incleasing sequence $\{\mathcal{H}_i : i = 0, -1, ...\}$.

Given $k, l \in \mathbb{Z}\ \ k < l$, let

$$W^n_{[k,l]} = \{\mu = (\mu_k, \dots , \mu_l) : \mu_i \in \{1, \dots , n\},\ (k \leq i \leq l)\}.$$

Let $\mu \in W^n_{[k,l]}$ and $\nu \in W^n_{[l+1,m]}$. We put

$$\mu \cdot \nu = (\mu_k, \dots , \mu_l, \nu_{l+1}, \dots , \nu_m).$$

Further, let

$$W^n_0 = \{0\}, \quad W^n_{[0,\infty]} = \cup_{k=0}^{\infty} W^n_{[0,k]} \quad \text{and} \quad W^n_\infty = \cup_{k=0}^{\infty} W^n_{[-k,k]}.$$

The shift $\alpha : i \in \mathbb{Z} \to i+1$ induces the mapping on W^n_∞, which we denote by the same notation α.

For $\mu \in W^n_{[k,l]}$, we put

$$\delta(\mu) = \delta(\mu_k) \otimes \cdots \otimes \delta(\mu_l) \in H_{[k,l]}.$$

Then $\{\delta(\mu) : \mu \in W^n_{[k,l]}\}$ is an orthonormal basis in $H_{[k,l]}$.

Let $A_0 = B(H_0)$ and $\{e(i, j) : i, j = 1, ..., n\}$ be the matrix unit of A_0 with respect to the orthonormal basis $\{\delta(i) : i = 1, ..., n\}$. We denote the trace (1/n)Tr of A_0 by τ_0. Put $A_i = A_0, (i \in \mathbb{Z})$ and $\tau_i = \tau_0$. For two integers $i < j$, let

$$A_{[i,j]} = A_i \otimes A_{i+1} \otimes \cdots \otimes A_j.$$

For $\mu, \nu \in W^n_{[k,l]}$, we put

$$e(\mu, \nu) = e(\mu_k, \nu_k) \otimes \cdots \otimes e(\mu_l, \nu_l) \in A_{[k,l]}.$$

Then $\{e(\mu, \nu) : \mu, \nu \in W^n_{[k,l]}\}$ is a matrix units of $A_{[k,l]}$.

2.2. We apply the entropy of Connes-Narnhofer-Thirring and Voiculescu's topological entropy to both of automorphisms and unital *-endomorphisms on C^*-algebras. To fix notations, we recall the definition of the topological entropy. Let B be a nuclear C^*-algebra with unity. Let $CAP(B)$ be triples (ρ, η, C), where C is a finite dimensional C^*-algebra, and $\rho : B \to C$ and $\eta : C \to B$ are unital completely positive maps. Let Ω be the set of finite subsets of B. For an $\omega \in \Omega$, put

$$rcp(\omega; \delta) = \inf\{\text{rank } C : (\rho, \eta, C) \in CAP(B), \|\eta \cdot \rho(a) - a\| < \delta, a \in B\},$$

where rank C means the dimension of a maximal abelian self-adjoint subalgebra of C. For a unital *-endomorphism β of B, put

$$ht(\beta,\omega\ ;\delta)=\overline{\lim}_{\mathrm{N}\to\infty}\frac{1}{\mathrm{N}}\log rcp\left(\omega\cup\beta(\omega)\cup\cdots\cup\beta^{\mathrm{N}-1}(\omega);\delta\right)$$

and

$$ht(\beta,\omega)=\sup_{\delta>0}ht(\beta,\omega;\delta).$$

Then the topological entropy $ht(\beta)$ of β is defined by

$$ht(\beta)=\sup_{\omega\in\Omega}ht(\beta,\omega).$$

Assume that there exists an increasing sequence $(\omega_j)_{j\in\mathbb{N}}$ of finite subsets of B such that the linear span of $\cup_{j\in\mathbb{N}}\ \omega_j$ is dense in B. Even in the case of *-endomorphisms which are not automorphisms, by the obvious analogoues of [**V**, Proposition 4.3], $ht(\cdot)$ is obtained as the following form which we use later:

$$ht(\beta)=\sup_{j\in\mathbb{N}}ht(\beta,\omega_j).$$

Let ϕ be a state of B with $\phi\cdot\beta=\phi$. The essential relation between $ht(\beta)$ and Connes-Narnhofer-Thirring entropy $h_\phi(\beta)$ is by [**V**, Proposition 4.6]

$$h_\phi(\beta)\le ht(\beta).$$

3. Entropy of Ad_u for non-commutative Bernoulli shift.

In this section, we only state results without proof. We remark that these are proved by similar methods as in §4 and §5.

3.1. Let $n(2\le n<\infty)$ be an integer. Let $A_i,\ \tau_i(i\in\mathbb{Z})$ be as in §2.1 and let A be the infinite C^*-tensor product $A=\bigotimes_{i\in\mathbb{Z}}A_i$. We denote the unique tracial state of A by τ. The non-commutative Bernoulli shift α_n is the automorphism of the C^*-algebra A induced by the shift $\alpha : i(\in\mathbb{Z})\to i+1$. Let u be the implimenting unitary in the reduced C^*-crossed product $A\rtimes_{\alpha_n}\mathbb{Z}$ for α_n. Let E be the conditional expectation of $A\rtimes_{\alpha_n}\mathbb{Z}$ onto A with $E(u^j)=0,(j\ne 0)$. Then $\tau\cdot E$ is a tracial state of $A\rtimes_{\alpha_n}\mathbb{Z}$ which is invariant under Ad_u. We denote by the same notation α_n the extension of α_n to the hyperfinite II_1 factor $\bigotimes_{i\in\mathbb{Z}}(A_i,\tau_i)\rtimes_{\alpha_n}\mathbb{Z}$.

Theorem 3.2. *Under the above notations,*

$$ht(\alpha_n)=ht(Ad_u)=h_{\tau\cdot E}(Ad_u)=h_\tau(\alpha_n)=\log n=H(\alpha_n)=H(Ad_u).$$

4. Entropy of Cuntz's canonical endomorphism.

In this section, we apply the definition of Connes-Narnhofer-Thirring entropy and Voiculescu's topological entropy for unital *-endomorphisms of nuclear C^*-algebras. All facts for automorphisms, which we need here, work for unital *-endomorphisms by the analogues of definitions and proofs in [**CNT**] and [**V**].

Let n $(2 \leq n < \infty)$ be an integer. Given n isometries $\{S_i\}$ on a Hilbert space such that $\sum_{i=1}^n S_i S_i^* = 1$, Cuntz defined the Cuntz algebra $\mathcal{O}_n$ as the C^*-algebra generated by $\{S_i\}_i$ ([**Cu1**]). So called Cuntz's canonical endomorphism Φ of $\mathcal{O}_n$ is defined by

$$\Phi(x) = \sum_{i=1}^n S_i x S_i^*, \quad (x \in \mathcal{O}_n).$$

The $\mathcal{O}_n$ has exactly one $\log n$-KMS state ϕ ([**OP**]). In this section we compute Voiculescu's topological entropy of Φ and Connes-Narnhofer-Thiring's entropy $h_\phi(\Phi)$. Applying to the factor generated by $\pi_\phi(\mathcal{O}_n)$, we get the entropy of Longo's canonical endomorphism.

4.1. To compute the entropy of Φ, we recall some of the representation for the Cuntz algebra $\mathcal{O}_n$ as a crossed product in [**Cu1**], (cf., [**Ch2**, **I2**, **P**, **R**]). Let $A_i, \tau_i, (i \in \mathbb{Z})$ and $e(i,j), (i,j \in \mathbb{N})$ be the same as in §2.1. For a $j \in \mathbb{Z}$, $\mathcal{A}_j$ is given as the infinite tensor product:

$$\mathcal{A}_j = \bigotimes_{i=j}^{\infty} A_i.$$

Define embeddings

$$\mathcal{A}_j \hookrightarrow \mathcal{A}_{j-1} \hookrightarrow \mathcal{A}_{j-2} \hookrightarrow \cdots$$

by $x \in \mathcal{A}_j \to e_{j-1}(1,1) \otimes x \in \mathcal{A}_{j-1}$, where $e_{j-1}(i,l)$ is a copy of $e(i,l)$ in A_{j-1}. The inductive limit of this sequence is denoted by $\mathcal{A}$. Since the embedding $\mathcal{A}_j \hookrightarrow \mathcal{A}_{j-1}$ and the embedding $\mathcal{H}_i \hookrightarrow \mathcal{H}_{i-1}$ in §2.1 are compatible, we can consider $\mathcal{A}$ acting faithfully on $\mathcal{H}$.

The automorphism σ of $\mathcal{A}$ is induced by the shift $\alpha : i (\in \mathbb{Z}) \to i+1$.

Then the crossed product $\mathcal{A} \rtimes_\sigma \mathbb{Z}$ acts faithfully on the Hilbert space

$$K = \sum_{i \in \mathbb{Z}} \bigoplus u^i \mathcal{H},$$

where u is the implimenting unitary in $\mathcal{A} \rtimes_\sigma \mathbb{Z}$ for the automorphism σ of $\mathcal{A}$. Let p be the unit of $\mathcal{A}_0 \subset \mathcal{A} \rtimes_\sigma \mathbb{Z}$ and put

$$w = up.$$

We remark $u^j p = w^j$.

Then Cuntz algebra $\mathcal{O}_n$ is reresented as $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$, which is the C^* subalgebra $C^*(\mathcal{A}_0, w)$ of $(\mathcal{A} \rtimes_\sigma \mathbb{Z})$ generated by $\{\mathcal{A}_0,\ w\}$. There exists a conditional expectation E of $C^*(\mathcal{A}_0, w)$ onto $\mathcal{A}_0$ with $E(w^j) = 0$ for all $j = 1, 2, \dots$. The unique tracial state τ of $\mathcal{A}_0$ is extended to the state ϕ of $C^*(\mathcal{A}_0, w)$ by $\phi = \tau \cdot E$. Then ϕ is the unique $\log n$-KMS state of $C^*(\mathcal{A}_0, w)$ ([**OP**]).

4.2. Since

$$\sigma^j(p)(\mathcal{H}) = \mathcal{H}_j, \quad j \in \mathbb{Z},$$

the algebra $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$ is acting faithfully on

$$pK = \sum_{i\in\mathbb{Z}} \bigoplus u^i \mathcal{H}_{-i}.$$

The restriction $\sigma|_{\mathcal{A}_0}$ of σ to $\mathcal{A}_0$ is the one sided non commutative Bernoulli shift. Cuntz's canonical inner endomorphism Φ of $\mathcal{O}_n$ is nothing but the extension of $\sigma|_{\mathcal{A}_0}$ to the Cuntz algebra $C^*(\mathcal{A}_0, w)$ which maps

$$a \to \sigma(a),\ (a \in \mathcal{A}_0), \quad \text{and} \quad w \to vw,$$

where

$$v = \sum_{j=1}^{n} e((j,1),(1,j)) \in A_{[0,1]},$$

([**Cu2**], cf. [**Ch2**]).

4.3. Let $k, m \in \mathbb{N}$. We define

$$K(k,m) = \sum_{l=-k}^{k} \bigoplus u^l H_{[-l,-l+m]}$$

and we denote the orthogonal projection of K onto $K(k,m)$ by $Q(k,m)$. The set $\{u^j\delta(\mu) : -k \le j \le k,\ \mu \in W^n_{[-j,-j+m]}\}$ is an orthonomal basis of $K(k,m)$. We denote by $E((j,\mu),(l,\nu))$ the partial isometry in $B(K(k,m))$ such that

$$E((j,\mu),(l,\nu)) : u^l\delta(\nu) \to u^j\delta(\mu), \quad \left(\mu \in W^n_{[-j,-j+m]},\ \nu \in W^n_{[-l,-l+m]}\right).$$

Then the set

$$\mathcal{E}(k,m) = \left\{E((j,\mu),(l,\nu)) : -k \le j, l \le k,\ \mu \in W^n_{[-j,-j+m]},\ \nu \in W^n_{[-l,-l+m]}\right\}$$

is a matrix units of $B(K(k,m))$.

4.4. Let $k, m \in \mathbb{N}$. We define the completely positive unital linear map

$$\varphi_{k,m} : p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p \to B(K(k,m))$$

by

$$\varphi_{k,m}(x) = Q(k,m)xQ(k,m)|_{K(k,m)}, \quad x \in p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p.$$

We remark that if $e(\mu,\nu)w^j \neq 0$ for ν in $W^n_{[0,b]}$, $(b \geq j)$, then

$$\nu = (1, \dots, 1, \nu_j, \dots, \nu_b) \quad \text{and} \quad \delta(\nu) \in H_{[j,b]}.$$

For two integers a and b with $a < b$, we let

$$\omega_{a,b} = \left\{ e(\mu,\nu)w^j : 0 \leq j \leq a \quad \text{and} \quad \mu, \nu \in W^n_{[0,b]} \right\}.$$

Let $e(\mu,\nu)w^j \in \omega_{a,b}$ for $a, b \in \mathbb{N}$, $(a < b)$ and $e(\mu,\nu)w^j \neq 0$. Since $\sigma^{-l}(p)\delta(\mu) = \delta(\mu)$ for $u^l\delta(\mu) \in K(k,m)$, we have that if $k \geq a$ and $m \geq b$ then

$$\varphi_{k,m}(e(\mu,\nu)w^j) = \sum_{l=-k}^{k-j} E((j+l,\ \alpha^{-(j+l)}(\mu)\cdot\beta_l),\ (l,\ \alpha^{-(j+l)}(\nu)\cdot\gamma_l)),$$

where

$$\beta_l = (1, \dots, 1) \in W^n_{[-(j+l)+b+1, -(j+l)+m]},$$
$$\gamma_l = (1, \dots, 1) \in W^n_{[-l+b+1, -l+m]}.$$

We remark that

$$\delta(\alpha^{-(j+l)}(\nu)) \in H_{[-l, -l+b+1]},$$

so that $E((j+l,\ \alpha^{-(j+l)}(\mu)\cdot\beta_l),\ (l,\ \alpha^{-(j+l)}(\nu)\cdot\gamma_l)) \in \mathcal{E}(k,m)$.

4.5. We define the linear map

$$\psi_{k,m} : B(K(k,m)) \to p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$$

by

$$\psi_{k,m}(E((j,\mu),(l,\nu))) = \frac{1}{2k+1} p u^j e(\mu,\nu) u^{*l} p,$$

for $E((j,\mu),(l,\nu)) \in \mathcal{E}(k,m)$.

Let $T_j, (j \in \mathbb{Z})$ be the unitary operator on K defined by

$$T_j(u^i\delta(\mu)) = u^{i+j}\delta(\alpha^{-j}(\mu)), \quad i \in \mathbb{Z},\ \mu \in W^n_\infty.$$

Then we have

$$w - \lim_{r \to \omega} \sum_{i=-r}^{r} T_i E((j,\mu),(l,\nu)) T_i^* = u^j e(\mu,\nu) u^{*l}$$

for any $E((j,\mu),(l,\nu)) \in B(K(k,m))$. Here ω is a nontrivial ultrafilter on $\mathbb{N}$. Hence $\psi_{k,m}$ is a unital completely positive map from $B(K(k,m))$ to $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$. Since $u^j p = w^j$, we have

$$\psi_{k,m} \cdot \varphi_{k,m}(e(\mu,\nu)w^j) = \frac{2k-j+1}{2k+1} e(\mu,\nu)w^j,$$

for all $e(\mu,\nu)w^j \in \omega_{a,b}$, $a \le k$ and $b \le m$.

Theorem 4.6. *Let Φ be Cuntz's canonical inner endomorphism of $\mathcal{O}_n$. Then*

$$ht(\Phi) = \log n.$$

Proof. Let $e(\mu,\nu)w^j \in \omega_{a,b}$. Then we have, for $a \le k$ and $b \le m$, by §4.5

$$\|\psi_{k,m} \cdot \varphi_{k,m}(e(\mu,\nu)w^j) - e(\mu,\nu)w^j\| = \frac{j}{2k+1}\|e(\mu,\nu)w^j\| \le \frac{a}{2k+1}$$

and we have for an $i \in \mathbb{N}$

$$\begin{aligned}\Phi^i(e(\mu,\nu)w^j) &= \sigma^i(e(\mu,\nu)) \sum_{s=1}^{n} e(\beta_s,\gamma_s)w^j \\ &= \sum_{s=1}^{n} e(\bar{\beta}_s \cdot \alpha^i(\mu), \gamma_s \cdot \nu_j)w^j.\end{aligned}$$

Here

$$\begin{aligned}\beta_s &= (1,\ldots,1,\underset{i-1}{s},1,\ldots,1) \in W^n_{[0,j+i-1]}, \\ \gamma_s &= (1,\ldots,1,s) \in W^n_{[0,j+i-1]}\end{aligned}$$

and

$$\bar{\beta}_s = (1,\ldots,1,\underset{i-1}{s}) \in W^n_{[0,i-1]}, \quad \nu_j = (\nu_{j+i},\ldots,\nu_b) \in W^n_{[j+i,b+i]}.$$

Hence for $k \ge a$ and $m \ge b+i$ we have

$$\|\psi_{k,m} \cdot \varphi_{k,m}(\Phi^i(e(\mu,\nu)w^j)) - \Phi^i(e(\mu,\nu)w^j)\| \le \frac{an}{2k+1}.$$

Therefore, we have for $\mathrm{N} \in \mathbb{N}$

$$\begin{aligned}rcp\left(\bigcup_{i=0}^{\mathrm{N}} \Phi^i\left(\omega_{a,b} \cup (\omega_{a,b})^* : \frac{an}{2k+1}\right)\right) &\le \text{rank } B(K(k,\mathrm{N}+b+1)) \\ &= (2k+1)n^{\mathrm{N}+b+1},\end{aligned}$$

where $(\omega_{a,b})^* = \{x^*; x \in \omega_{a,b}\}$. This implies that for all integers a, b with $a < b$,

$$ht\left(\Phi, \omega_{a,b} \cup (\omega_{a,b})^*; \frac{an}{2k+1}\right) \le \overline{\lim}_{\mathrm{N}\to\infty} \frac{1}{\mathrm{N}} \log\left((2k+1)n^{\mathrm{N}+b+1}\right) = \log n.$$

Increasing k, we have $ht(\Phi, \omega_{a,b} \cup (\omega_{a,b})^*) \leq \log n$, for all $a, b \in \mathbb{N}$ with $a < b$. Put $\omega_a = \omega_{a,2a} \cup (\omega_{a,2a})^*$, for $a \in \mathbb{N}$. Then the set $\{\omega_a : a \in \mathbb{N}\}$ is an increasing sequence of finite subsets of $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$ and the linear span of $\bigcup\{\omega_a : a \in \mathbb{N}\}$ is dense in $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$. Hence

$$ht(\Phi) = \sup_{a \in \mathbb{N}} ht(\Phi, \omega_a) \leq \log n.$$

On the other hand, the restriction $\Phi|\mathcal{A}_0$ of Φ to $\mathcal{A}_0$ is $\sigma|_{\mathcal{A}_0} = \alpha_n|_{\mathcal{A}_0}$ and $h_\tau(\alpha_n|_{\mathcal{A}_0}) = h_\tau(\alpha_n) = \log n$. Since there exists a conditional expectation of $\mathcal{O}_n$ onto $\mathcal{A}_0$ and $\tau \cdot \alpha_n|_{\mathcal{A}_0} = \tau$,

$$\log n = h_\tau(\alpha_n|_{\mathcal{A}_0}) \leq ht(\Phi|_{\mathcal{A}_0}) \leq ht(\Phi) \leq \log n$$

by the version for unital *-endomorphisms of [**V**, Proposition 4.4]. Therefore, $ht(\Phi) = \log n$. □

Corollary 4.7. *Let ϕ be the unique $\log n$-KMS state of $\mathcal{O}_n$. Then*

$$h_\phi(\Phi) = \log n.$$

Proof. Let τ be the unique tracial state of $\mathcal{A}_0$ and E be the conditional expectation of $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$ onto $\mathcal{A}_0$, then $\phi = \tau \cdot E$. Hence $\phi \cdot \Phi = \phi$. This relation implies, by the endomorphism version of [**V**, Proppsition 4.6],

$$\log n = h_\tau(\sigma|\mathcal{A}_0) \leq h_\phi(\Phi) \leq ht(\Phi) = \log n.$$

Therefore $h_\phi(\Phi) = \log n$. □

5. Entropy of Longo's canonical endomorphism.

In this section we apply the result in §4 to Longo's canonical endomorphism. We use the same notations as in §4.

5.1. Let τ_i be the tracial state of A_i, for $i \in \mathbb{N}$ and let

$$\tilde{A} = \bigotimes_{i=0,}^{\infty} (A_i, \tau_i).$$

The $\tilde{A}$ has the canonical trace $\bigotimes_{i=0,}^{\infty} \tau_i$, which we denote by τ. The shift $\sigma|\mathcal{A}_0$ is extended to the *-endomorphism γ of the hyperfinite II_1 factor $\tilde{A}$. The γ is canonical in the sense of [**Ch3**]. Hence we have the extension algebra $\tilde{M} = \langle \tilde{A}, \sigma \rangle$, which is the injective type $\mathrm{III}_{1/n}$ factor generated by $\tilde{A}$ and an isometry W. Then γ is extended to the canonical *-endomorphism Γ of $\tilde{M}$ and

$$\Gamma(a) = \gamma(a), a \in \tilde{A}, \quad \text{and} \quad \Gamma(W) = \pi_\phi(vW).$$

The Γ is Longo's canonical endomorphism for the inclusion $\tilde{N} \subset \tilde{M}$ [**Ch3**, Theorem 6.10]. Here the subfactor $\tilde{N}$ is obtained naturally in the step of constructing $\tilde{M}$. The factor $\tilde{M}$ is the von Neumann algebra generated by $\pi_\phi(\langle \mathcal{A}_0, \sigma|_{\mathcal{A}_0}\rangle)$ and the C^*-algebra $\langle \mathcal{A}_0, \sigma|_{\mathcal{A}_0}\rangle$ is $\mathcal{O}_n$. Hence Γ is the extension of Φ to $\tilde{M}$. Since Φ is ϕ-preserving, as an application of 4.7 Corollary, we have the following by [**CNT**, Theorem VII.2]:

Corollary 5.2. *Let M be the von Neumann algera generated by $\pi_\phi(\mathcal{O}_n)$ and let Γ be the extension of Cuntz's canonical endomorphism Φ of $\mathcal{O}_n$ to M. Then Γ is Longo's canonical endomorphism and*

$$h_\phi(\Gamma) = \log n.$$

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