Pacific Journal of Mathematics

LINEARLY UNRELATED SEQUENCES

Jaroslav Hančl

Volume 190 No. 2 October 1999

PACIFIC JOURNAL OF MATHEMATICS
Vol. 190, No. 2, 1999

LINEARLY UNRELATED SEQUENCES

Jaroslav Hančl

The paper deals with the so-called linearly unrelated sequences. The criterion and the application for irrational sequences and series is included too.

1. Introduction.

There are not many new results concerning the linear independence of numbers. Exceptions in the last decade are, e.g., the result of Sorokin [8] which proves the linear independence of logarithmus of special rational numbers, or that of Bezivin [2] which proves linear independence of roots of special functional equations.

The algebraic independence of numbers can be considered as a generalization of linear independence. One can find many results of this nature. For instance, in [4] Bundschuh proves that if the special series of rational numbers converges to infinity very fast then they are algebraically independent. In [7] I prove a similar result for continued fractions. In that paper the so-called continued fractional algebraic independence of sequences was also defined.

If we consider irrationality as a special case of linear independence then we can obtain many results. For instance, in [1] Apery proves the irrationality of $\zeta(3)$ and in [3] Borwein proves the irrationality of the sum $\sum_{n=1}^{\infty} 1/(q^n + r)$, where q and r are integers such that $q > 1$ and $r \neq 0$.

In 1975 Erdös defined the so-called irrationality of sequences in [5] (we will consider a generalization of this definition in Section 3) and in the same paper he proves the irrationality of the sequence $\{2^{2^n}\}$. In 1993 in [6] I proved:

Theorem. *Let $\{r_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of positive real numbers such that $\lim_{n\to\infty} r_n = \infty$, let B be a positive integer, and let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers such that*

$$b_{n+1} \leq r_n^B$$

and

$$a_n \geq r_n^{2^n}$$

holds for every large n. Then the series

$$A = \sum_{n=1}^{\infty} b_n/a_n$$

and the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ are irrational.

2. Linearly Unrelated Sequences.

Definition 2.1. Let $\{a_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K)$ be sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the numbers $\sum_{n=1}^{\infty} 1/(a_{1,n}c_n)$, $\dots$, $\sum_{n=1}^{\infty} 1/(a_{K,n}c_n)$, and 1 are linearly independent, then the sequences $\{a_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K)$ are linearly unrelated.

Theorem 2.1. *Let $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K-1)$ be sequences of positive integers and $\epsilon > 0$ such that*

$$(1) \qquad \frac{a_{1,n+1}}{a_{1,n}} \geq 2^{K^{n-1}}, a_{1,n}|a_{1,n+1} \quad (a_{1,n} \ \textit{divides} \ a_{1,n+1})$$

$$(2) \qquad b_{i,n} < 2^{K^{n-(\sqrt{2}+\epsilon)\sqrt{n}}}, \quad i = 1, \dots, K-1$$

$$(3) \qquad \lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0, \quad \textit{for all} \ j, i \in \{1, \dots, K-1\}, i > j$$

$$(4) \qquad a_{i,n}2^{-K^{n-(\sqrt{2}+\epsilon)\sqrt{n}}} < a_{1,n} < a_{i,n}2^{K^{n-(\sqrt{2}+\epsilon)\sqrt{n}}}, i = 1, \dots, K-1$$

hold for every sufficiently large natural number n. Then the sequences $\{a_{i,n}/b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K-1)$ are linearly unrelated.

Proof. We will prove that for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers and for every $(K-1)$-tuple of integers $\alpha_1, \dots, \alpha_{K-1}$ (not all equal to zero) the sum

$$A = \sum_{j=1}^{K-1} \alpha_j \sum_{n=1}^{\infty} \frac{b_{j,n}}{a_{j,n}c_n}$$

is an irrational number. Suppose that A is a rational number. Let R be a maximal index such that $\alpha_R \neq 0$. Then we have

$$A = \sum_{j=1}^{K-1} \alpha_j \sum_{n=1}^{\infty} \frac{b_{j,n}}{a_{j,n}c_n} = \sum_{n=1}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{b_{j,n}}{a_{j,n}c_n}$$

$$= \sum_{n=1}^{\infty} \frac{b_{R,n}}{a_{R,n}c_n} \left(\sum_{j=1}^{R-1} \alpha_j \frac{b_{j,n}a_{R,n}}{a_{j,n}b_{R,n}} + \alpha_R \right).$$

Because of (3), there is a natural number N such that for every $n \geq N$ the number

$$\sum_{j=1}^{R-1} \alpha_j \frac{b_{j,n}a_{R,n}}{a_{j,n}b_{R,n}} + \alpha_R$$

and the number α_R have the same sign. Without loss of generality we may assume $\alpha_R > 0$ and (1)-(4) hold for every $n \geq N$. Thus, there are positive integers p and q such that

$$B = \frac{p}{q} = \sum_{n=N}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{b_{j,n}}{a_{j,n} c_n}.$$

We reorder the sequences $\{a_{j,n} c_n\}_{n=N}^{\infty}$ to obtain the sequences $\{c_{j,n}\}_{n=N}^{\infty}$ $(j = 1, \dots, R)$ so that $c_{1,N} \leq c_{1,N+1} \leq c_{1,N+2} \leq \dots$. Thus, there is a map ϕ:$\{n \geq N\} \to \{n \geq N\}$, such that $c_{1,n} = a_{1,\phi(n)} c_{\phi(n)}$ for $n \geq N$. It follows that

$$B = \frac{p}{q} = \sum_{n=N}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}}, \tag{5}$$

where $d_{j,n} = b_{j,\phi(n)}$ for every $j = 1, \dots, K-1$, $n = N, N+1, \dots$. We will consider two cases.

1. First we assume that

$$\limsup_{n\to\infty} c_{1,n}^{1/K^n} = 2^V. \tag{6}$$

Then (1), (6), and the definition of the sequence $\{c_{1,n}\}_{n=1}^{\infty}$ imply that

$$V > 0.$$

Also, (6) implies that for every $\delta > 0$ there is a $n(\delta)$ such that for every $j > n(\delta)$

$$c_{1,j} < 2^{(V+\delta)K^j}, \tag{7}$$

and there are infinitely many M such that

$$c_{1,M} > 2^{(V-\delta)K^M}. \tag{8}$$

From $c_{1,n} = a_{1,\phi(n)} c_{\phi(n)} \leq 2^{(V+\delta)K^n}$, we get $a_{1,\phi(n)} \leq 2^{(V+\delta)K^n}$. Now, condition (1) gives

$$a_{1,\phi(n)} \geq a_{1,1} 2^{\frac{K^{\phi(n)-1}-1}{K-1}} \geq 2^{\frac{K^{\phi(n)-1}-1}{K-1}}.$$

Thus, $K^{\phi(n)-1} \leq 1 + (K-1)(V+\delta)K^n$ for all sufficiently large n. Hence,

$$\phi(n) - 1 \leq n + \frac{\log(V+\delta) + \log(K-1) + \log\left(1 + \frac{1}{(K-1)(V+\delta)K^n}\right)}{\log K},$$

and, $\phi(n) \leq n + \frac{\log(V+\delta)}{\log K} + 2$ for n sufficiently large. From the latter inequality, it follows from the fact that $x \to x - (\sqrt{2}+\epsilon)\sqrt{x}$ is increasing that

$$d_{j,n} < 2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}}}, j = 1, \dots, R, \tag{9}$$

holds for every $n \geq N_1$, where $\gamma = \frac{\log(V+\delta)}{\log K} + 2$. For the same reason, and with the help of (4), we also obtain that

$$c_{j,n} 2^{-K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}}} < c_{1,n} < c_{j,n} 2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}}}, j = 1, \dots, R \tag{10}$$

holds for every $n \geq N_2$. Now, (9) and (10) imply that

$$\sum_{n=M}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}} \leq \sum_{n=M}^{\infty} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{c_{1,n}} \tag{11}$$

for every sufficiently large M. Let $h \in N$ such that $\gamma + 1 \geq h > \gamma$. Now we will prove

$$T_M = \sum_{n=M}^{\infty} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{c_{1,n}} \leq \frac{2^{K^{M+\beta-(\sqrt{2}+\epsilon)\sqrt{M}+4}}}{c_{1,M}} \tag{12}$$

for every sufficiently large M where $\beta = \gamma + h$. Also (1) yields $a_{1,n} \geq 2^{K^{n-2}}$. Thus $c_{1,n} \geq 2^{K^{n-2}}$. From this and (7) we have

$$\begin{aligned} T_M &= \sum_{n=M}^{\infty} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{c_{1,n}} \\ &= \sum_{n=M}^{M+h} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{c_{1,n}} + \sum_{n=M+h+1}^{\infty} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{c_{1,n}} \\ &\leq (h+1) \frac{2^{K^{M+\gamma-(\sqrt{2}+\epsilon)\sqrt{M}+3+h}}}{c_{1,M}} + \sum_{n=M+h+1}^{\infty} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{c_{1,n}} \end{aligned}$$

because $c_{1,M+j} \geq c_{1,M}$ for $j \geq 0$, and

$$\begin{aligned} \sum_{n=M+h+1}^{\infty} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{c_{1,n}} &\leq \sum_{n=M+h+1}^{\infty} \frac{2^{K^{n+\gamma-(\sqrt{2}+\epsilon)\sqrt{n}+3}}}{2^{K^{n-2}}} \\ &\leq 2 \frac{2^{K^{M+\gamma-(\sqrt{2}+\epsilon)\sqrt{M}+3+h}}}{2^{K^{M+h-1}}}. \end{aligned}$$

So

$$T_M \leq (h+1) \frac{2^{K^{M+\gamma-(\sqrt{2}+\epsilon)\sqrt{M}+3+h}}}{c_{1,M}} + 2 \frac{2^{K^{M+\gamma-(\sqrt{2}+\epsilon)\sqrt{M}+3+h}}}{2^{K^{M+h-1}}}.$$

Now the inequality is proven if

$$\begin{aligned} &\left(2^{K^{M+\gamma-(\sqrt{2}+\epsilon)\sqrt{M}+h+4}} - (h+1) 2^{K^{M+\gamma-(\sqrt{2}+\epsilon)\sqrt{M}+3+h}}\right) 2^{K^{M+h-1}} \\ &\geq c_{1,M} 2^{K^{M+\gamma-(\sqrt{2}+\epsilon)\sqrt{M}+3+h}+1} \end{aligned}$$

which is true for M large by the choice of h, and the fact $c_{1,j} \leq 2^{(V+\delta)K^j}$ for all large j. The proof of inequality (12) is finished. It follows from (11) and (12) that

$$\sum_{n=M}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}} \leq \frac{2^{K^{M+\beta-(\sqrt{2}+\epsilon)\sqrt{M}+4}}}{c_{1,M}} \tag{13}$$

for every sufficiently large natural number M. Hence, we have

$$\begin{aligned} B = \frac{p}{q} &= \sum_{n=N}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}} \\ &= \sum_{n=N}^{M-1} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}} + \sum_{n=M}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}}. \end{aligned}$$

Thus

$$\begin{aligned} &p.lcm(c_{1,N}, \dots, c_{R,N}, c_{1,N+1}, \dots, c_{R,N+1}, \dots, c_{1,M-1}, \dots, c_{R,M-1}) \\ &= q.lcm(c_{1,N}, \dots, c_{R,M-1}) \sum_{n=N}^{M-1} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}} \\ &\quad + q.lcm(c_{1,N}, \dots, c_{R,M-1}) \sum_{n=M}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}}, \end{aligned}$$

where $lcm(x_1, \dots, x_n)$ denotes the least common multiple of numbers $x_1, ...,$ x_n. Thus, the number

$$C = q.lcm(c_{1,N}, \dots, c_{R,M-1}) \sum_{n=M}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}}$$

is a positive integer. From this and (13) we obtain

$$\begin{aligned} C &= q.lcm(c_{1,N}, \dots, c_{R,M-1}) \sum_{n=M}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}} \\ &\leq \frac{lcm(c_{1,N}, \dots, c_{R,M-1})}{c_{1,M}} 2^{K^{M+\beta+4-(\sqrt{2}+\epsilon)\sqrt{M}}} = \frac{D}{c_{1,M}} \end{aligned} \tag{14}$$

for every sufficiently large natural number M. From (1) and the definition of the sequence $\{c_{1,n}\}_{n=1}^{\infty}$ we have

$$\begin{aligned} D &= lcm(c_{1,N}, \dots, c_{R,M-1}) 2^{K^{M+\beta+4-(\sqrt{2}+\epsilon)\sqrt{M}}} \\ &\leq \left(\prod_{n=N}^{M-2} 2^{K^{n-2}} \right)^{-1} \left(\prod_{n=N}^{M-1} \prod_{j=1}^{R} c_{j,n} \right) 2^{K^{M+\beta+4-(\sqrt{2}+\epsilon)\sqrt{M}}}. \end{aligned}$$

From this, (7), (10), and the fact $\beta = \gamma + h$ we obtain

$$D \leq 2^{\frac{-1}{K-1}(K^{M-3}-K^N)} \left(\prod_{n=N}^{M-1} \prod_{j=1}^{R} 2^{(V+\delta)K^n} 2^{K^{n+\beta+2-(\sqrt{2}+\epsilon)\sqrt{n}}} \right) \cdot 2^{K^{M+\beta+4-(\sqrt{2}+\epsilon)\sqrt{M}}} S(N_1, N_2, \delta),$$

where $S(N_1, N_2, \delta)$ does not depend on M. It follows that

$$\begin{aligned} D &\leq 2^{-\frac{(K^{M-3}-K^N)}{K-1}} S(N_1, N_2, \delta) \left(\prod_{n=N}^{M-1} 2^{R(V+\delta)K^n} 2^{RK^{n+\beta+2-(\sqrt{2}+\epsilon)\sqrt{n}}} \right) \\ &\quad \cdot 2^{K^{M+\beta+4-(\sqrt{2}+\epsilon)\sqrt{M}}} \\ &\leq 2^{-\frac{K^{M-3}-K^N}{K-1}} S(N_1, N_2, \delta) 2^{R(V+\delta)\frac{K^M-K^N}{K-1}} 2^{K^{M+\beta+5-(\sqrt{2}+\epsilon)\sqrt{M}+\log M}} \\ &\leq 2^{-\frac{K^{M-3}-K^N}{K-1}} s(N_1, N_2, \delta) 2^{(V+\delta)K^M} 2^{K^{M+\beta+5-(\sqrt{2}+\epsilon)\sqrt{M}+\log M}}. \end{aligned}$$

Hence,

$$D \leq 2^{(V+\delta-K^{-4})K^M}$$

for every sufficiently large M. This, (8), and (14) imply that

$$C = \frac{D}{c_{1,M}} \leq 2^{(V+\delta-K^{-4})K^M} . 2^{-(V-\delta)K^M} = 2^{(2\delta-K^{-4})K^M}$$

for infinitely many natural numbers M. But this is impossible for a sufficiently small δ and a sufficiently large M.

2. Secondly, let us assume that

$$\limsup_{n\to\infty} c_{1,n}^{1/K^n} = \infty. \tag{15}$$

Let Q be a sufficiently large positive integer. Let the number of $c_{1,n}$ such that $c_{1,n} < 2^{K^Q}$ be Z. (The definition of the sequence $\{c_{1,n}\}_{n=N}^{\infty}$ and (1) imply that $Z - 1 < Q$.) Let $g(X, Y)$ be the number of $c_{1,n}$ satisfying $c_{1,n} \in [2^{K^Y}, 2^{K^X})$ and put $f(X, Y) = X - g(X, Y)$. Then (15) yields

$$\limsup_{X\to\infty} f(X, Y) = \infty \tag{16}$$

and

$$f(X + 1, Y) - f(X, Y) \leq 1. \tag{17}$$

Because of (16) and (17) there is a least positive integer P such that

$$g(P, Q) = P - Q - Z - 2. \tag{18}$$

It follows that for every S ($Q \leq S < P$)

$$g(P, S) \leq P - S - 1. \tag{19}$$

(Otherwise $g(S,Q) = g(P,Q) - g(P,S) \leq P - Q - Z - 2 - (P - S) = S - Q - Z - 2$ and the number P would not be the least.) Now (18) and (19) imply that for every $j = 0, 1, \dots, P - Q - Z - 3$,

$$c_{1,P-Q-3-j+N} \leq 2^{K^{P-j-1}}.$$

Thus,

$$\prod_{c_{1,j}<2^{K^P}} c_{1,j} = \prod_{j=N}^{P-Q-3+N} c_{1,j} = \prod_{j=N}^{N+Z-1} c_{1,j} \prod_{j=N+Z}^{P-Q-3+N} c_{1,j} \tag{20}$$

$$< 2^{ZK^Q} \prod_{j=N+Z}^{P-Q-3+N} 2^{K^{Q+j-N+2}}$$

$$= 2^{ZK^Q} 2^{\frac{1}{K-1}(K^P - K^{Q+Z+2})} \leq 2^{\frac{1}{K-1}K^P}.$$

Now we define a sequence $\{S_n\}_{n=0}^{\infty}$ by induction in the following way. Let us put $S_0 = P$. Suppose that we have $S_0, S_1, \dots, S_{k-1}$. Because of (16) and (17) there is a least positive integer S_k such that

$$g(S_k, S_{k-1}) = S_k - S_{k-1} - 1. \tag{21}$$

Similarly (21) implies that for every S $(S_{k-1} \leq S \leq S_k)$

$$g(S_k, S) \leq S_k - S - 1. \tag{22}$$

The last inequality implies that for every $j = 1, \dots, S_k - S_{k-1} - 1$

$$c_{1,N+S_{k-1}-Q-2-k+j} \leq 2^{K^{S_{k-1}+j}}.$$

Hence, it follows that

$$\prod_{c_{1,j}\in(2^{K^{S_{k-1}}}, 2^{K^{S_k}})} c_{1,j} = \prod_{j=1}^{S_k-S_{k-1}-1} c_{1,N+S_{k-1}-Q-2-k+j} \tag{23}$$

$$\leq \prod_{j=1}^{S_k-S_{k-1}-1} 2^{K^{S_{k-1}+j}} = 2^{\frac{1}{K-1}(K^{S_k} - K^{S_{k-1}+1})}.$$

Now we will prove that there are infinitely many positive integers $T \geq P$ such that

$$lcm(c_{1,j}, c_{1,j} < 2^{K^T}) \leq 2^{\frac{1}{K-1}(K^T - K^{T-(\sqrt{2}+\frac{\epsilon}{4})\sqrt{T}})} \tag{24}$$

and

$$\prod_{c_{1,j}<2^{K^T}} c_{1,j} \leq 2^{\frac{1}{K-1}K^T}. \tag{25}$$

To prove this, we will consider three cases.

2.1. First, let us assume that

$$S_k - S_{k-1} < \sqrt{2S_k} \tag{26}$$

for infinitely many numbers k. Then (20), (23), and (26) yield

$$\begin{aligned}
\prod_{c_{1,j}<2^{K^{S_k}}} c_{1,j} &= \left(\prod_{c_{1,j}<2^{K^P}} c_{1,j}\right)\left(\prod_{i=1}^{k}\prod_{c_{1,j}\in[2^{K^{S_{i-1}}},2^{K^{S_i}})} c_{1,j}\right) \\
&\le 2^{\frac{1}{K-1}K^P}.\prod_{i=1}^{k} 2^{\frac{1}{K-1}(K^{S_i}-K^{S_{i-1}+1})} \\
&= 2^{\frac{1}{K-1}(K^{S_0}+K^{S_1}-K^{S_0+1}+\dots+K^{S_k}-K^{S_{k-1}+1})} \\
&\le 2^{\frac{1}{K-1}(K^{S_k}-K^{S_{k-1}})} < 2^{\frac{1}{K-1}(K^{S_k}-K^{S_k-\sqrt{2S_k}})}.
\end{aligned}$$

Thus (24) and (25) hold under condition (26).

2.2. Secondly, let us assume that for every positive integer k

$$S_k - S_{k-1} \ge \sqrt{2S_k}.$$

It follows that

$$S_k - \sqrt{2S_k} - S_{k-1} \ge 0.$$

Thus,

$$S_k \ge \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2} + S_{k-1}}\right)^2 = 1 + S_{k-1} + \sqrt{1 + 2S_{k-1}}. \tag{27}$$

Now, by mathematical induction we prove that

$$S_k \ge \frac{1}{2}k^2. \tag{28}$$

For $k = 0$ (28) holds. Suppose that (28) holds for $k-1$. Then (27) and (28) imply

$$\begin{aligned}
S_k &\ge 1 + S_{k-1} + \sqrt{1 + 2S_{k-1}} \\
&\ge 1 + \frac{1}{2}(k-1)^2 + \sqrt{1 + 2\frac{1}{2}(k-1)^2} \\
&> 1 + \frac{1}{2}k^2 - k + \frac{1}{2} + (k-1) > \frac{1}{2}k^2.
\end{aligned}$$

From (18) and (21) the number of $c_{1,j}$ such that $c_{1,j} < 2^{K^{S_k}}$ is equal to

$$(29)\qquad g(S_k, 0) = Z + g(S_0, Q) + \sum_{j=1}^{k} g(S_j, S_{j-1})$$
$$= Z + P - Q - Z - 2 + \sum_{j=1}^{k} (S_j - S_{j-1} - 1)$$
$$= P - Q - 2 + S_k - S_0 - k = S_k - k - Q - 2.$$

Now, (28) and (29) imply that

$$(30)\qquad g(S_k, 0) = S_k - k - Q - 2$$
$$\geq S_k - \sqrt{2S_k} - Q - 2 \geq S_k - \left(\sqrt{2} + \frac{\epsilon}{2}\right)\sqrt{S_k} + 2$$

for every sufficiently large k. Also (20), (23), and (30) yield

$$\prod_{c_{1,j} < 2^{K^{S_k}}} c_{1,j} = \prod_{c_{1,j} < 2^{K^P}} c_{1,j} \prod_{i=1}^{k} \prod_{c_{1,j} \in [2^{K^{S_{i-1}}}, 2^{K^{S_i}})} c_{1,j}$$
$$\leq 2^{\frac{1}{K-1} K^P} \prod_{i=1}^{k} 2^{\frac{1}{K-1}(K^{S_i} - K^{S_{i-1}+1})}$$
$$= 2^{\frac{1}{K-1}(K^P + \sum_{i=1}^{k}(K^{S_i} - K^{S_{i-1}+1}))} \leq 2^{\frac{1}{K-1} K^{S_k}}$$

for every sufficiently large k. From this, (1), (30), and the definition of the sequence $\{c_{1,n}\}_{n=N}^{\infty}$ it follows that

$$lcm(c_{1,j}, c_{1,j} < 2^{K^{S_k}}) \leq 2^{\frac{-1}{K-1}(K^{g(S_k,0)-1} - K^N)} . \prod_{c_{1,j} < 2^{K^{S_k}}} c_{1,j}$$
$$\leq 2^{\frac{1}{K-1}(K^{S_k} - K^{S_k - (\sqrt{2} + \frac{\epsilon}{3})\sqrt{S_k}})}$$

for every sufficiently large k.

2.3. Third, let us assume that $S_k - S_{k-1} \leq \sqrt{2S_k}$, and $S_j - S_{j-1} \geq \sqrt{2S_j}$ for every $j > k$. Let us put $P' = S_k = S'_0$, and $S'_j = S_{k+j}$. We now proceed as in the second case with $\{S'_j\}_{j=0}^{\infty}$ in place of $\{S_j\}_{j=0}^{\infty}$. Thus (24) and (25) hold. Now let T be a positive integer such that (24) and (25) hold. Then we obtain from (5) that

$$B.q.lcm(c_{1,N}, \dots, c_{1,N+g(T,0)-1}, c_{2N}, \dots, c_{R,N+g(T,0)-1})$$
$$= q.lcm(c_{1,N}, \dots, c_{R,N+g(T,0)-1}) \sum_{n=N}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}}.$$

Thus, there is a positive integer E such that

$$E = q.lcm(c_{1,N}, \dots, c_{R,N+g(T,0)-1}) \sum_{n=N+g(T,0)}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}}. \tag{31}$$

From (1), (4), the definition of the sequence $\{c_{1,n}\}_{n=N}^{\infty}$, (18), (21), (24), and (25) it follows that for infinitely many sufficiently large T

$$\begin{aligned} &lcm(c_{1,N}, \dots, c_{R,N+g(T,0)-1}) \\ &\le lcm(c_{1,N}, \dots, c_{1,N+g(T,0)-1}) \left(\prod_{j=N}^{N+g(T,0)-1} c_{1,j} 2^{K^{T+2-(\sqrt{2}+\epsilon)\sqrt{T}}} \right)^{K-2} \\ &= lcm(c_{1,j}, c_{1,j} < 2^{K^T}) \left(\prod_{c_{1,j}<2^{K^T}} c_{1,j} 2^{K^{T+2-(\sqrt{2}+\epsilon)\sqrt{T}}} \right)^{K-2} \\ &\le 2^{\frac{1}{K-1}\left(K^T - K^{T-(\sqrt{2}+\frac{\epsilon}{4})\sqrt{T}}\right)} \left(2^{\frac{1}{K-1}K^T} 2^{TK^{T+2-(\sqrt{2}+\epsilon)\sqrt{T}}} \right)^{K-2} \\ &= 2^{K^T - \frac{1}{K-1}K^{T-(\sqrt{2}+\frac{\epsilon}{4})\sqrt{T}} + T(K-2)K^{T+2-(\sqrt{2}+\epsilon)\sqrt{T}}} \le 2^{K^T - K^{T-(\sqrt{2}+\frac{\epsilon}{3})\sqrt{T}}}. \end{aligned} \tag{32}$$

On the other hand (1), (2), (4), the definition of the sequence $\{c_{1,n}\}_{n=N}^{\infty}$, (18), and (21) imply that

$$\begin{aligned} \sum_{n=N+g(T,0)}^{\infty} \sum_{j=1}^{R} \alpha_j \frac{d_{j,n}}{c_{j,n}} &\le \frac{T.K. \max_{j=1,\dots,R} |\alpha_j| . 2^{K^{T+2-(\sqrt{2}+\epsilon)\sqrt{T}}}}{2^{K^T}} \\ &\le 2^{K^{T-(\sqrt{2}+\frac{\epsilon}{2})\sqrt{T}} - K^T} \end{aligned} \tag{33}$$

for all sufficienly large T. Finally (31)-(33) imply that

$$\begin{aligned} E &\le q.2^{K^T - K^{T-(\sqrt{2}+\frac{\epsilon}{3})\sqrt{T}}} 2^{K^{T-(\sqrt{2}+\frac{\epsilon}{2})\sqrt{T}} - K^T} \\ &= q.2^{K^{T-(\sqrt{2}+\frac{\epsilon}{2})\sqrt{T}} - K^{T-(\sqrt{2}+\frac{\epsilon}{3})\sqrt{T}}} \end{aligned}$$

for infinitely many natural numbers T. But this is impossible for a positive integer E and a sufficiently large T. □

Example 1. Let $a_{j,n} = 2^{K^n}$, $b_{j,n} = (j+n)!$ $(j = 1, 2, \dots, K-1)$. Then the sequences $\{a_{j,n}/b_{j,n}\}_{n=1}^{\infty}$ are linearly unrelated.

3. Irrational Sequences.

Definition 3.1. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the series

$$\sum_{n=1}^{\infty} \frac{1}{A_n c_n}$$

is irrational, then the sequence $\{A_n\}_{n=1}^{\infty}$ is irrational. If $\{A_n\}_{n=1}^{\infty}$ is not an irrational sequence, then it is a rational sequence.

Theorem 3.1. *Let $\epsilon > 0$, and let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that*

$$a_n \geq 2^{2^n}$$

and

$$b_n \leq 2^{2^{n-(\sqrt{2}+\epsilon)\sqrt{n}}}.$$

Then the sequence $\left\{\frac{\prod_{i=1}^{n} a_i}{b_n}\right\}_{n=1}^{\infty}$ is irrational and the series $\sum_{n=1}^{\infty} \frac{b_n}{\prod_{i=1}^{n} a_i}$ is irrational too.

This theorem is an immediate consequence of Theorem 2.1. It is enough to put $K = 2$.

Example 2. The sequences $\{2^{2^n - n^2}\}_{n=1}^{\infty}$, $\{2^{2^n}/n\}_{n=1}^{\infty}$, and $\{2^{2^n - n}\}_{n=1}^{\infty}$ are irrational sequences.

Open Problem. Is the sequence $\left\{2^{[2^n(1-\frac{1}{n})]}\right\}_{n=1}^{\infty}$ irrational or not? ($[x]$ denotes the greatest integer less than or equal x.)

Remark. Let us put in Theorem 3.1 $a_n = 2^{2^n}$ and $b_n = 1$ for every natural number n. Then we obtain the very famous result of Erdös (see [5]) which states that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is irrational.

From the last theorem we also obtain the following criterion for the so-called Cantor sequences.

Theorem 3.2. *Let $\epsilon > 0$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that*

$$b_n \leq 2^{2^{n-(\sqrt{2}+\epsilon)\sqrt{n}}}.$$

Let us put

$$a_n = \left[2^{n\left(1-\frac{1}{n}\log_2\left(\frac{n}{\frac{\log_2 n}{n}+1}\right)\right)} \right].$$

Then the sequence $\{\frac{a_n!}{b_n}\}_{n=1}^{\infty}$ is irrational.

This theorem is an immediate consequence of Theorem 3.1.

Example 3. The sequences $\left\{2^{[n(1-\frac{1}{\sqrt{n}})]}!\right\}_{n=1}^{\infty}$ and $\left\{2^{[n(1-\frac{1}{\sqrt{n}})]}!/n!\right\}_{n=1}^{\infty}$ are irrational.

Thank you very much to reviewer and to Professor Carter for their correction of this paper.

References

[1] R. Apery, *Irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Soc. Math. France, Asterisque, **61** (1979), 11-13.

[2] J.P. Bezivin, *Linear independence of the values of transcendental solutions of some functional equations*, Manuscripta Math., **61**(**1**) (1988), 103-129.

[3] P.B. Borwein, *On the irrationality of* $\sum(1/q^n+r)$, J. Num. Theory, **37** (1991), 253-259.

[4] P. Bundschuh, *A criterion for algebraic independence with some applications*, Osaka J. Math., **25** (1988), 849-858.

[5] P. Erdös, *Some problems and results on the irrationality of the sum of infinite series*, J. Math. Sci., **10** (1975), 1-7.

[6] J. Hančl, *Criterion for irrational sequences*, J. Num. Theory **43**(**1**) (1993), 88-92.

[7] ———, *Continued fractional algebraic independence of sequences*, Publ. Math., **46**(**1**-**2**) (1995), 27-31.

[8] V.N. Sorokin, *Linear independence of logarithm of some rational numbers*, Mat. Zametki, **46**(**3**) (1989), 74-79, 127, translated in Math. Notes, **46**(**3**-**4**) (1989), 727-730.

Received May 20, 1997 and revised February 26, 1998. This work was supported by Grant no. 201/97/0433 of the Czech Grant Agency.

University of Ostrava
701 03 Ostrava 1
Czech Republic
E-mail address: hancl@osu.cz