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Suppose X is a simply connected mod p H -space such that the mod p cohomology $H^*(\Omega X)$ is a finitely generated algebra. We show that the loop space ΩX is homotopy equivalent to a finite product of Eilenberg-MacLane spaces $K(\mathbb{Z}, 1)$, $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}/p^i, 1)$ for $i \geq 1$. This is a generalization of the result due to Lin, in which the same result was proved under the assumption that X is an A_p -space.

1. Introduction.

Let p be an odd prime. We assume that all spaces are completed at p in the sense of Bousfield-Kan [2], and the cohomologies are taken with \mathbb{Z}/p -coefficients unless otherwise specified. In this paper, we investigate the homotopy type for the loop space of an H -space whose cohomology is finitely generated as an algebra. In the case of the cohomology is finite dimensional, there is the following theorem due to Aguadé-Smith:

Theorem 1.1 ([1]). *If X is a simply connected mod p H -space such that $H^*(\Omega X)$ is finite dimensional, then ΩX has the homotopy type of a torus.*

The above theorem is known as the mod p torus theorem, and some generalizations of Theorem 1.1 are investigated by Hemmi [8] and McGibbon [15]. Hemmi showed that a connected finite quasi C_p -space has the homotopy type of a torus, where a quasi C_p -space is defined as an H -space which has certain higher homotopy associativity and commutativity (see [8, Def. 2.1]).

Our main result is stated as follows:

Theorem A. *If X is a simply connected mod p H -space such that $H^*(\Omega X)$ is finitely generated as an algebra, then ΩX is homotopy equivalent to a finite product of Eilenberg-MacLane spaces $K(\mathbb{Z}, 1)$, $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}/p^i, 1)$ for $i \geq 1$.*

Theorem A generalizes Theorem 1.1 since $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}/p^i, 1)$ for $i \geq 1$ do not have the finite cohomology. Our theorem also generalizes a result of Lin [12] who has shown Theorem A under the assumption that X is an A_p -space in the sense of Stasheff [19]. We owe much to the results

in [12] and [13] (see §2). From the result of Hemmi, it may be possible to generalize our result to the case of quasi C_p -spaces instead of loop spaces on H -spaces.

For $p = 2$, there is the following more general result due to Slack and Broto-Crespo:

Theorem 1.2 ([18, Cor. 0.2], [3, Cor. 1.5]). *If X is a connected homotopy commutative mod 2 H -space such that the mod 2 cohomology $H^*(X)$ is finitely generated as an algebra, then X is homotopy equivalent to a finite product of Eilenberg-MacLane spaces $K(\mathbb{Z}, 1)$, $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}/2^i, 1)$ for $i \geq 1$.*

We remark that for the odd prime case, the corresponding result of Theorem 1.2 does not hold. In fact, Iriye-Kono [9] have shown that for an odd prime p , any mod p H -space possesses a multiplication which is homotopy commutative. Moreover, one may guess that a homotopy commutative mod p loop space which has the finitely generated cohomology is homotopy equivalent to a product of Eilenberg-MacLane spaces. However, we note that $Sp(2)$ for $p = 3$ and S^3 for $p \geq 5$ are counterexamples (see [14, Thm. 2]).

In the proof of Theorem A, we use a technique for H -fibrations introduced by Broto-Crespo [3]. Their observation was concentrated on the mod 2 case, and some parts of their proof have generalizations to the odd prime cases with simple modifications (see Proposition 3.3 and Proposition 3.6). We combine these results with the computations in §2 for the cohomology of ΩX to establish a proof of Theorem A (see §4).

Now we provide an outline of the proof of Theorem A so that the reader has an overview of the ideas and strategy.

For a mod p H -space X satisfying the assumption, we consider the three-connected cover \tilde{X} . Then we have a fibration

$$\Omega\tilde{X} \longrightarrow \Omega X \longrightarrow K,$$

where K is a finite product of Eilenberg-MacLane spaces of degrees 1 and 2. We see that $H^*(\Omega\tilde{X})$ is free commutative, finitely generated as an algebra which has generators in degrees $2p, 2p + 1, 2p^2$ and $2p^2 + 1$ with certain Steenrod relations induced from $H^*(\Omega X)$ (see Proposition 2.3). For a generator x of degree $2p$, using the Lannes theory, we construct an H -map $\phi : B\mathbb{Z}/p \rightarrow \Omega\tilde{X}$ such that $\phi^*(x) = \omega^p$, where $\omega \in H^2(B\mathbb{Z}/p)$ denotes the generator. We construct an H -fibration

$$B\mathbb{Z}/p \xrightarrow{\phi} \Omega\tilde{X} \longrightarrow E_1,$$

where E_1 is an H -space given by the Borel construction for ϕ . By repeating this construction, we have a sequence of H -spaces and H -maps

$$\Omega\tilde{X} \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots,$$

and if we set $Y = \varinjlim_s E_s$, then the three-connected cover $Y\langle 3 \rangle \simeq \Omega\tilde{X}$, and the cohomology $H^*(Y)$ is related to $H^*(\Omega\tilde{X})$ in that $H^*(Y)$ has an additional three dimensional generator and one less $2p$ -dimensional generators (see Proposition 4.1). Applying this procedure a finite number of times, we obtain a mod p H -space Z such that $Z\langle 3 \rangle \simeq \Omega\tilde{X}$ and the cohomology $H^*(Z)$ has no $2p$ -dimensional generator.

By using the same methods, we can knock off the $2p^2$ -dimensional generators, and thus we obtain a mod p H -space W such that $W\langle 3 \rangle \simeq \Omega\tilde{X}$ and the cohomology $H^*(W)$ is an exterior algebra with generators in degrees 3 and $2p + 1$ (see Proposition 4.8).

By the localization theory due to Dror Farjoun and Neisendorfer, we can show that W is also the loop space on an H -space, and so W is contractible by Theorem 1.1. This implies that $\Omega\tilde{X}$ is also contractible and therefore $\Omega X \simeq K$. The ideas and strategy come from [3].

This paper is organized as follows: In §2, we prove Theorem A using Theorem 1.1, Proposition 2.6 and results for the localization theory due to Dror Farjoun [6] and Neisendorfer [17]. Here Proposition 2.6 is the key to the proof of Theorem A, and we postpone the proof until §4. In §3, we recall the Lannes theory and show some properties for H -fibrations. In §4, we prove Proposition 2.6 using the results of §3.

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2. Proof of Theorem A.

In this section we prove Theorem A. Thus, throughout this section, the space X is always assumed to satisfy the hypothesis of Theorem A. First, we recall the following result due to Lin:

Theorem 2.1 ([12, Thm. A]). *$H^*(\Omega X)$ is free commutative, primitively generated on generators in degrees 1, 2, $2p$, $2p + 1$, $2p^2$ and $2p^2 + 1$.*

Remark 2.2. Lin has proved that if X is a simply connected mod p H -space such that $H^*(\Omega X)$ is finitely generated as an algebra, then $H^*(\Omega X)$ is primitively generated on generators in degrees 1, 2, $2p$, $2p + 1$, $2p^2$ and $2p^2 + 1$, and under the assumption that $H_*(X)$ is associative, $H^*(\Omega X)$ is free commutative. We note that his proof does not use this assumption to show that $H^*(\Omega X)$ is primitively generated (see [12, Cor. 2.2, Thm. 2.3]). But we see that the last statement also holds without this assumption. In fact, since ΩX is homotopy associative, homotopy commutative H -space and $H^*(\Omega X)$ is primitively generated, by a theorem of Browder [4, Thm. 8.15], $H^*(\Omega X)$ is free commutative.

By [12, Cor. 1.2, Thm. 2.1], $H^*(X)$ is generated by odd degree generators in degrees $2p^j + 1$ for some $j \geq 0$ and even degree generators in degrees $2, 2p^j + 2$ for some $j \geq 1$. We choose the basis of $H^2(X)$ and $H^3(X)$ as $B_2 = \{x_{j_0}\} \cup \{x_{j_1}\} \cup \dots \cup \{x_{j_r}\}$ and $B_3 = \{\beta_1(x_{j_1})\} \cup \dots \cup \{\beta_r(x_{j_r})\} \cup \{y_k\}$, respectively, where x_{j_0} are the mod p reductions of the integral classes and β_s denotes the s -th Bockstein operation. We define a generalized Eilenberg-MacLane space K as

$$K = \prod_{\{x_{j_0}\}} K(\mathbb{Z}, 2) \times \prod_{\{x_{j_1}\}} K(\mathbb{Z}/p, 2) \times \dots \times \prod_{\{x_{j_r}\}} K(\mathbb{Z}/p^r, 2) \times \prod_{\{y_k\}} K(\mathbb{Z}, 3).$$

Let $f : X \rightarrow K$ be an H -map which represents the generators of the integral cohomology of dimension 2 and 3, and \tilde{X} denote the homotopy fiber of f . Then, \tilde{X} is an H -space and $2p$ -connected. By the spectral sequence argument, we see that $H^*(\Omega\tilde{X})$ is finitely generated as an algebra, and so \tilde{X} satisfies the same conditions as X .

Now we define an algebra A as

$$A = \mathbb{Z}/p[x_1, \dots, x_m, y_1, \dots, y_n] \otimes \Lambda(z_1, \dots, z_{m+n}, w_1, \dots, w_n),$$

where $|x_i| = 2p$ for $1 \leq i \leq m$, $|y_j| = 2p^2$ for $1 \leq j \leq n$, $|z_k| = 2p + 1$ for $1 \leq k \leq m + n$, and $|w_l| = 2p^2 + 1$ for $1 \leq l \leq n$.

Then we can prove the following proposition:

Proposition 2.3. $H^*(\Omega\tilde{X}) \cong A$ as algebras, and the following operations act on $H^*(\Omega\tilde{X})$:

$$(2.4) \quad \begin{cases} \beta(x_i) = z_i & \text{for } 1 \leq i \leq m, \\ \beta(y_j) = \mathcal{P}^p(z_{m+j}) = w_j & \text{for } 1 \leq j \leq n. \end{cases}$$

For a mod p H -space Y , we denote the primitive and indecomposable modules of the Hopf algebra $H^*(Y)$ by $PH^*(Y)$ and $QH^*(Y)$, respectively. We need the following fact for the proof of Proposition 2.3.

Lemma 2.5 ([16, Thm. 4.21]). *If Y is a connected mod p H -space, then there is the following exact sequence:*

$$0 \rightarrow P(\xi H^*(Y)) \rightarrow PH^*(Y) \rightarrow QH^*(Y),$$

where $\xi : H^*(Y) \rightarrow H^*(Y)$ is a map defined as $\xi(x) = x^p$.

Proof of Proposition 2.3. By Theorem 2.1, $H^*(\Omega\tilde{X})$ is free commutative, and has generators x_i, y_j, z_k and w_l with $|x_i| = 2p, |y_j| = 2p^2, |z_k| = 2p + 1$ and $|w_l| = 2p^2 + 1$ for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq q$ and $1 \leq l \leq r$, where generators are primitive.

Since $\beta(x_i) \in PH^{2p+1}(\Omega\tilde{X})$ for $1 \leq i \leq m$, we see that $\beta(x_i) \in QH^{2p+1}(\Omega\tilde{X})$ by Lemma 2.5. By [13, Cor. E], we have that $\beta(x_i) \neq 0$, and if $i_1 \neq i_2$, then $\beta(x_{i_1}) \neq \beta(x_{i_2})$. Thus, we can set $\beta(x_i) = z_i$ for $1 \leq i \leq m$. Similarly, we can set $\beta(y_j) = w_j$ for $1 \leq j \leq n$.

Since the suspension map $\sigma^* : QH^{2p^2+2}(\tilde{X}) \rightarrow PH^{2p^2+1}(\Omega\tilde{X})$ is an epimorphism, and $\beta : QH^{2p^2+1}(\tilde{X}) \rightarrow QH^{2p^2+2}(\tilde{X})$ is also an epimorphism by [12, Thm. 1.10], we have that $w_l \in \beta PH^{2p^2}(\Omega\tilde{X})$ for $1 \leq l \leq r$. Thus, we have that $w_l \in \beta QH^{2p^2}(\Omega\tilde{X})$ by Lemma 2.5, which implies that $r = n$.

Using [12, Thm. 1.9], the similar arguments show that $w_l \in \mathcal{P}^p QH^{2p+1}(\Omega\tilde{X})$ for $1 \leq l \leq n$. We can assume that $\mathcal{P}^p(z_{m+l}) = w_l$ for $1 \leq l \leq n$ since $\mathcal{P}^p(z_k) = \mathcal{P}^1 \beta \mathcal{P}^{p-1}(x_k) + \beta \mathcal{P}^p(x_k) = 0$ for $1 \leq k \leq m$.

If we set

$$\mathcal{P}^p(z_{m+n+1}) = \sum_{l=1}^n \sigma_l w_l$$

for $\sigma_l \in \mathbb{Z}/p$, then for

$$\zeta = z_{m+n+1} - \sum_{l=1}^n \sigma_l z_{m+l},$$

we have that $\mathcal{P}^p(\zeta) = 0$. Since $\sigma^* : QH^{2p+2}(\tilde{X}) \rightarrow PH^{2p+1}(\Omega\tilde{X})$ is an epimorphism, $\zeta = \sigma^*(\mu)$ for some $\mu \in QH^{2p+2}(\tilde{X})$. Since $\sigma^*(\mathcal{P}^p(\mu)) = \mathcal{P}^p(\zeta) = 0$, by [10, Thm. B], there exists $\nu \in QH^{2p+1}(\tilde{X})$ such that $\mathcal{P}^p(\mu) = \beta \mathcal{P}^p(\nu)$ in $QH^{2p^2+2}(\tilde{X})$. Applying the Adem relation $\mathcal{P}^p \beta = \mathcal{P}^1 \beta \mathcal{P}^{p-1} + \beta \mathcal{P}^p$ to ν , we have that $\mathcal{P}^p(\mu) = \mathcal{P}^p(\beta(\nu))$, which implies that $\mu = \beta(\nu)$ by [12, Thm. 1.9]. Then, $\zeta = \sigma^*(\mu) = \beta(\sigma^*(\nu)) \in \beta QH^{2p}(\Omega\tilde{X})$ by Lemma 2.5, which implies that

$$\zeta = \sum_{k=1}^m \tau_k z_k$$

for $\tau_k \in \mathbb{Z}/p$. Therefore, we have that

$$z_{m+n+1} = \sum_{k=1}^m \tau_k z_k + \sum_{l=1}^n \sigma_l z_{m+l},$$

which implies that $q = m + n$. This completes the proof. □

The following proposition is crucial for our study, which will be proved in §4 using the Lannes theory.

Proposition 2.6. *If Y is a mod p H -space with $H^*(Y) \cong A$ as algebras, and the operations (2.4) act on $H^*(Y)$, then there is a simply connected mod p finite H -space W such that $Y \simeq W\langle 3 \rangle$, where $W\langle 3 \rangle$ is the three-connected cover of W .*

Using Proposition 2.6, we can prove Theorem A as follows:

Proof of Theorem A. By Proposition 2.3 and Proposition 2.6, there exists a simply connected mod p finite H -space W such that $\Omega\tilde{X} \simeq W\langle 3 \rangle$. Let L_g denote the localization functor with respect to a map g constructed by

Dror Farjoun [6]. For the constant map $c : B\mathbb{Z}/p \rightarrow *$, $L_c(\Omega\tilde{X}) \simeq W$ by the results due to Neisendorfer [17, Thm. 0.1]. Since $L_c(\Omega\tilde{X}) \simeq \Omega L_{\Sigma c}(\tilde{X})$ by [6, Thm. 3.A.1], and $L_{\Sigma c}$ preserves the H -structure, we see that the space W is the loop space of an H -space. By Theorem 1.1, W is contractible, and so $\Omega\tilde{X} \simeq W\langle 3 \rangle$ is also contractible. Therefore, $\Omega X \simeq \Omega K$, and we have the required conclusion. This completes the proof of Theorem A. \square

3. Lannes T -functor and H -fibrations.

In this section we recall some results concerning the Lannes theory and the H -fibrations, which will be used in the next section.

Let \mathcal{K} denote the category of unstable \mathcal{A}_p -algebras. The objects of \mathcal{K} are called \mathcal{K} -algebras. It is known that $H^*(X)$ is a \mathcal{K} -algebra for any space X .

The Lannes T -functor $T : \mathcal{K} \rightarrow \mathcal{K}$ is a left adjoint of the functor $H^*(B\mathbb{Z}/p) \otimes -$, that is, there is the adjoint isomorphism $\text{Hom}_{\mathcal{K}}(T(A), B) \cong \text{Hom}_{\mathcal{K}}(A, H^*(B\mathbb{Z}/p) \otimes B)$ for \mathcal{K} -algebras A and B .

For a \mathcal{K} -map $f : A \rightarrow H^*(B\mathbb{Z}/p)$, its adjoint restricts to a \mathcal{K} -map $T(A)^0 \rightarrow \mathbb{Z}/p$, where $T(A)^0$ is the subalgebra of $T(A)$ of elements of degree 0. The connected component of $T(A)$ corresponding to f is defined by $T_f(A) = T(A) \otimes_{T(A)^0} \mathbb{Z}/p$, and there is the natural \mathcal{K} -map $\epsilon_f : A \rightarrow T_f(A)$.

The evaluation map $e : B\mathbb{Z}/p \times \text{Map}(B\mathbb{Z}/p, X) \rightarrow X$ induces a \mathcal{K} -map e^* , and taking the adjoint of this yields a \mathcal{K} -map $\lambda : T(H^*(X)) \rightarrow H^*(\text{Map}(B\mathbb{Z}/p, X))$. On the component level, for a map $\phi : B\mathbb{Z}/p \rightarrow X$, there is a \mathcal{K} -map $\lambda_{\phi^*} : T_{\phi^*}(H^*(X)) \rightarrow H^*(\text{Map}(B\mathbb{Z}/p, X)_{\phi})$. The composite $\lambda_{\phi^*}\epsilon_{\phi^*}$ is induced by the evaluation at the base point $e_{\phi} : \text{Map}(B\mathbb{Z}/p, X)_{\phi} \rightarrow X$. The following theorem is due to Lannes:

Theorem 3.1 ([11, Thm. 3.2.1]). *Let X be a space and $\phi : B\mathbb{Z}/p \rightarrow X$ be a map. If $T_{\phi^*}(H^*(X))^1 = 0$, then $\lambda_{\phi^*} : T_{\phi^*}(H^*(X)) \rightarrow H^*(\text{Map}(B\mathbb{Z}/p, X)_{\phi})$ is an isomorphism.*

For the cohomology of an H -space, Dwyer-Wilkerson have proved the following:

Proposition 3.2 ([7, Thm. 3.2, Lemma 4.5]). *If X is a mod p H -space with finitely generated cohomology and $f : H^*(X) \rightarrow H^*(B\mathbb{Z}/p)$ is a \mathcal{K} -map, then $\epsilon_f : H^*(X) \rightarrow T_f(H^*(X))$ is an isomorphism.*

Recently, an important theory of H -fibrations using the Lannes theory was introduced by Broto-Crespo [3]. Their observation was concentrated on the mod 2 case. However, we also have the corresponding results for the odd prime case.

Proposition 3.3. *Let X be a mod p H -space with finitely generated cohomology, and $\phi : B\mathbb{Z}/p \rightarrow X$ be an H -map with $H^*(B\mathbb{Z}/p)$ is finitely*

generated $H^*(X)$ -module induced by ϕ^* . If

$$B\mathbb{Z}/p \xrightarrow{\phi} X \xrightarrow{\psi} Y$$

is a principal fibration, then Y is an H -space and ψ is an H -map.

Lemma 3.4. *Let $c : B\mathbb{Z}/p \times B\mathbb{Z}/p \rightarrow Y$ denote the constant map, where Y comes from Proposition 3.3. Then the base point evaluation map $e_c : \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, Y)_c \rightarrow Y$ is a homotopy equivalence.*

Proof. We have the following commutative diagram of fibrations:

$$\begin{array}{ccc} \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, B\mathbb{Z}/p)_S & \xrightarrow{e_c} & B\mathbb{Z}/p \\ \downarrow & & \downarrow \phi \\ \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, X)_c & \xrightarrow{e_c} & X \\ \downarrow & & \downarrow \psi \\ \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, Y)_c & \xrightarrow{e_c} & Y, \end{array}$$

where $S = \{g : B\mathbb{Z}/p \times B\mathbb{Z}/p \rightarrow B\mathbb{Z}/p \mid \phi g \simeq c\}$ and e_c denote the base point evaluation maps.

Since X has the finitely generated cohomology, $e_c : \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, X)_c \rightarrow X$ is a homotopy equivalence by [7, Thm. 3.2]. It is known that $H^*(B\mathbb{Z}/p) \cong \Lambda(\theta) \otimes \mathbb{Z}/p[\omega]$ with $\beta(\theta) = \omega$. For a map $g : B\mathbb{Z}/p \times B\mathbb{Z}/p \rightarrow B\mathbb{Z}/p$ with $\phi g \simeq c$, there exists some $n \geq 1$ so that $g^*(\omega)^n = g^*(\omega^n) = 0$ since $H^*(B\mathbb{Z}/p)$ is finitely generated $H^*(X)$ -module induced by ϕ^* and $g^*\phi^* = 0$, which implies that $g^*(\omega) = 0$. If we put $g^*(\theta) = a_1\theta_1 + a_2\theta_2$ for $a_1, a_2 \in \mathbb{Z}/p$, then $g^*(\omega) = \beta(g^*(\theta)) = a_1\omega_1 + a_2\omega_2 = 0$, and we must have $a_1 = a_2 = 0$, which implies that $g^*(\theta) = 0$. By a result of Lannes [11, Thm. 3.1.1], we obtain that $g \simeq c$.

Then we have that $S = \{c\}$, and thus $e_c : \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, B\mathbb{Z}/p)_S \rightarrow B\mathbb{Z}/p$ is a homotopy equivalence. Using the five lemma, $e_c : \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, Y)_c \rightarrow Y$ is a homotopy equivalence, and thus we have the required conclusion. \square

For the proof of Proposition 3.3, we need the following fact which is known as the Zabrodsky lemma:

Lemma 3.5 ([21, Lemma 3.1]). *Let*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a principal fibration, and Y be a space which satisfies that $e_c : \text{Map}(F, Y)_c \rightarrow Y$ is a homotopy equivalence. Then the induced map $\text{Map}(B, Y) \rightarrow \text{Map}(E, Y)_S$ is a homotopy equivalence, where $S = \{g : E \rightarrow Y \mid gi \simeq c\}$.

Now we can prove Proposition 3.3 as follows:

Proof of Proposition 3.3. By Lemma 3.4, the evaluation map $e_c : \text{Map}(B\mathbb{Z}/p \times B\mathbb{Z}/p, Y)_c \rightarrow Y$ is a homotopy equivalence. Then, applying Lemma 3.5 to a principal fibration

$$B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{\phi \times \phi} X \times X \xrightarrow{\psi \times \psi} Y \times Y,$$

we have that $\text{Map}(Y \times Y, Y) \simeq \text{Map}(X \times X, Y)_S$, where $S = \{g : X \times X \rightarrow Y \mid g(\phi \times \phi) \simeq c\}$. If we denote the multiplication of the H -space X as μ_X , then there is a map $\mu_Y : Y \times Y \rightarrow Y$ so that $\psi\mu_X \simeq \mu_Y(\psi \times \psi)$. Using Lemma 3.5 again, we see that the map μ_Y gives an H -structure on Y . This completes the proof. \square

Proposition 3.6. *Suppose that there is an H -fibration*

$$(3.7) \quad B\mathbb{Z}/p \xrightarrow{\phi_i} X_i \xrightarrow{\psi_i} X_{i+1}$$

for $i \geq 0$, and we put $Y = \varinjlim_i X_i$. If $H^*(Y)$ is finitely generated as an algebra, then the space Y has an H -structure.

Proof. We set $\mu = \varinjlim_i \mu_i : Y \times Y \rightarrow Y$ for the multiplication $\mu_i : X_i \times X_i \rightarrow X_i$ of the H -space X_i . Let $\iota_j : Y \rightarrow Y \times Y$ denote the inclusion map on the j -th factor for $j = 1, 2$. If we show that $\mu\iota_j \simeq 1_Y$ for $j = 1, 2$, then we have the required conclusion.

We denote the inclusion map as $\kappa_i : X_i \rightarrow Y$ for $i \geq 0$. Since μ_i is a multiplication for $i \geq 0$, we have that $\mu\iota_j\kappa_i \simeq \kappa_i\mu_i\iota_j^i \simeq \kappa_i$, where $\iota_j^i : X_i \rightarrow X_i \times X_i$ denotes the inclusion map on the j -th factor for $j = 1, 2$. By [20, Prop. 4], the obstruction to construct a homotopy between $\mu\iota_j$ and 1_Y lies in

$$(3.8) \quad \varprojlim_i^k \pi_k(\text{Map}(X_i, Y)_{\kappa_i})$$

for $k \geq 1$. Since $H^*(Y)$ is a finitely generated algebra, $\text{Map}(B\mathbb{Z}/p, Y)_c \simeq Y$ by [7, Thm. 3.2]. Then, applying Lemma 3.5 to the fibration (3.7), we have that $\text{Map}(X_i, Y)_{\kappa_i} \simeq \text{Map}(X_{i+1}, Y)_{\kappa_{i+1}}$ for $i \geq 0$, and so the obstruction group (3.8) vanishes. This completes the proof. \square

Now we introduce a result which is useful to compute the Serre spectral sequence for an H -fibration, which will be used in §4. Let X and Y be H -spaces and

$$X \longrightarrow Y \longrightarrow B^2\mathbb{Z}/p$$

be an H -fibration. We consider the Serre spectral sequence for the fibration whose E_2 -term is given as

$$(3.9) \quad E_2^{*,*} = H^*(B^2\mathbb{Z}/p) \otimes H^*(X).$$

Then we see that the spectral sequence has a differential Hopf algebra structure, and for $r \geq 2$, if we put $A_r = E_r^{*,0}$ and $B_r = E_r^{0,*}$, then they have Hopf algebra structures induced from the E_2 -term.

Proposition 3.10. (1) If $d_r(B_r) \neq 0$, then the transgression $\tau : B_r^{r-1} \rightarrow P^r(A_r)$ is non-trivial.

(2) For $x \in B_r^q$, $d_r(x) \in P^r(A_r) \otimes B_r^{q-r+1}$, where $P^n(A_r)$ denotes the primitive module of A_r^n .

We need the following lemma to show Proposition 3.10:

Lemma 3.11. (1) For $r \geq 2$, the E_r -term is given as

$$E_r^{*,*} \cong A_r \otimes B_r \otimes \Lambda(\alpha_1, \dots, \alpha_k),$$

where $\alpha_i \in E_r^{s_i, t_i}$ with $s_i < r$ and $|\alpha_i| = 2m_i + 1$ with $p|m_i$ for $1 \leq i \leq k$.

(2) If $x \in P^{2s}(A_r)$ with $2s \geq r$, then x has the infinite height.

Proof. We show (1) and (2) by induction. For $r = 2$, by (3.9) and since $H^*(B^2\mathbb{Z}/p)$ is free commutative, the results (1) and (2) hold. We assume that the results (1) and (2) have already shown for the E_r -term.

By a result of Browder [5, Thm. 5.8], the E_{r+1} -term is described as

$$E_{r+1}^{*,*} \cong A_{r+1} \otimes B_{r+1} \otimes \Lambda(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l),$$

where α_i are elements stated in the lemma, $\beta_j \in E_{r+1}^{s_j, t_j}$ with $t_j < r - 1$ and $|\beta_j| = 2n_j - 1$ with $p|n_j$ for $1 \leq j \leq l$. From the proof of [5, Thm. 5.8], we see that $\beta_j = \{x_j \cdot d_r(x_j)^{p-1}\}$ for some $x_j \in E_r^{*,*}$ with $d_r(x_j) \in P(A_r)$. But by assumption, $d_r(x_j)$ has the infinite height, and so the element of the form β_j cannot occur, which shows (1).

For a non-trivial element $x \in P^{2s}(A_{r+1})$ with $2s \geq r + 1$, we assume that $x^{p^k} = 0$ for some $k \geq 1$, and obtain a contradiction from this assumption. By inductive hypothesis, $x^{p^k} \neq 0 \in A_r$, and then there exists an element $y \in E_r^{*,*}$ so that $d_r(y) = x^{p^k}$. By the form of the E_r -term, we have either a generator $z \in B_r^{r-1}$ with $d_r(z) = x^{p^{k_1}}$ for some $k_1 \leq k$ or a generator $\alpha \in E_r^{*,*}$ with $d_r(\alpha) = x^{p^{k_2}}$ for some $k_2 \leq k$. On the one hand, if $d_r(z) = x^{p^{k_1}}$, then $|x^{p^{k_1}}| = r < |x|$, which causes a contradiction. On the other hand, if $d_r(\alpha) = x^{p^{k_2}}$, then $|x^{p^{k_2}}| = 2m + 2$ for some $m \geq 1$ with $p|m$. This shows that $k_2 = 0$, and so $\{x\} = 0$ in the E_{r+1} -term, which also causes a contradiction. This completes the proof. \square

Now we can prove Proposition 3.10 as follows:

Proof of Proposition 3.10. First we show (1). By assumption, there is an element $x \in B_r^q$ so that $d_r(x) \neq 0$. We can assume that if $y \in B_r^{\bar{q}}$ with $\bar{q} < q$, then $d_r(y) = 0$. If we set that

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i \bar{x}_i \otimes \bar{\bar{x}}_i,$$

then $\Delta(d_r(x)) = d_r(\Delta(x)) = d_r(x) \otimes 1 + 1 \otimes d_r(x)$, and so $d_r(x) \in P(E_r^{r, q-r+1})$. By Lemma 3.11, the primitive elements of $E_r^{*,*}$ consist of $P(A_r)$, $P(B_r)$ and

$\alpha_i \in E_r^{s,t}$ with $s < r$, and then we have that $q = r - 1$ and $d_r(x) \in P^r(A_r)$. This implies the required conclusion.

Next to show (2), let $x \in B_r^q$. By Lemma 3.11, we can set that

$$d_r(x) = \sum_{i \geq 1} a_i \cdot b_i \in E_r^{r,q-r+1},$$

where $a_i \in A_r \otimes \Lambda(\alpha_1, \dots, \alpha_k)$, $b_i \in B_r$ with $|a_i| + |b_i| = q + 1$ for $i \geq 1$. We can assume that the elements b_i are linearly independent for $i \geq 1$. Then we have that

$$\begin{aligned} \Delta(d_r(x)) &= \sum_i \Delta(a_i)\Delta(b_i) \\ &= \sum_i \left(a_i \otimes 1 + 1 \otimes a_i + \sum_j \bar{a}_{i,j} \otimes \bar{\bar{a}}_{i,j} \right) \\ &\quad \cdot \left(b_i \otimes 1 + 1 \otimes b_i + \sum_k \bar{b}_{i,k} \otimes \bar{\bar{b}}_{i,k} \right), \end{aligned}$$

where $0 < |\bar{a}_{i,j}|, |\bar{\bar{a}}_{i,j}| < |a_i|$ and $0 < |\bar{b}_{i,k}|, |\bar{\bar{b}}_{i,k}| < |b_i|$. On the other hand, we obtain that

$$\Delta(d_r(x)) = d_r(\Delta(x)) \in \bigoplus_{s+t=q} (E_r^{r,s-r+1} \otimes B_r^t) \oplus (B_r^s \otimes E_r^{r,t-r+1}).$$

For the dimensional reason, we see that $\sum_{i,j} \bar{a}_{i,j} b_i \otimes \bar{\bar{a}}_{i,j} = 0$, which implies that $\sum_j \bar{a}_{i,j} \otimes \bar{\bar{a}}_{i,j} = 0$ for $i \geq 1$ since b_i are linearly independent. This implies that

$$a_i \in P(A_r \otimes \Lambda(\alpha_1, \dots, \alpha_k)) = P(A_r) \oplus \{\alpha_1, \dots, \alpha_k\},$$

and then $a_i \in P^r(A_r)$ for $i \geq 1$. Thus we can conclude that $d_r(x) \in P^r(A_r) \otimes B_r^{q-r+1}$. This completes the proof. \square

Remark 3.12. We note that by Proposition 3.10, for $r \geq 2$, if either $P^r(A_r) = 0$ or $Q^{r-1}(B_r) = 0$, then $d_r(B_r) = 0$.

4. Proof of Proposition 2.6.

In this section we prove Proposition 2.6, and thus we assume that Y is a mod p H -space such that $H^*(Y) \cong A$, and the operations (2.4) act on $H^*(Y)$.

For $1 \leq t \leq m + 1$, we set an algebra K_t as

$$\begin{aligned} K_t &= \mathbb{Z}/p[x_t, \dots, x_m, y_1, \dots, y_n] \\ &\quad \otimes \Lambda(v_1, \dots, v_{t-1}, z_t, \dots, z_{m+n}, w_1, \dots, w_n) \end{aligned}$$

with x_i, y_j, z_k and w_l are as in A , $|v_q| = 3$ for $1 \leq q \leq t - 1$. First, we prove the following proposition:

Proposition 4.1. *For $1 \leq t \leq m + 1$, there is a mod p H -space Y_t such that $Y_t\langle 3 \rangle \simeq Y$ and $H^*(Y_t) \cong K_t$ with the following operations:*

$$\begin{cases} \beta(x_i) = z_i & \text{for } t \leq i \leq m, \\ \beta(y_j) = w_j & \text{for } 1 \leq j \leq n, \\ \mathcal{P}^p(z_{m+l}) = w_l + \delta_l & \text{for } 1 \leq l \leq n, \end{cases}$$

where δ_l is some decomposable element of K_t for $1 \leq l \leq n$.

For $1 \leq t \leq m$, we set an algebra C_t as

$$C_t = \mathbb{Z}/p[u, x_{t+1}, \dots, x_m, y_1, \dots, y_n] \quad \otimes \Lambda(v_1, \dots, v_t, z_{t+1}, \dots, z_{m+n}, w_1, \dots, w_n)$$

with x_i, y_j, z_k, w_l and v_q for $1 \leq q \leq t - 1$ are as in K_t , $|u| = 2$ and $|v_t| = 3$.

An algebra A is said to be a \mathcal{K} -Hopf algebra if A is a \mathcal{K} -algebra and has a Hopf algebra structure compatible with the \mathcal{K} -structure, namely the diagonal map of A becomes a \mathcal{K} -map. It is known that for an H -space X , $H^*(X)$ is a \mathcal{K} -Hopf algebra. We see that if K_t and C_t have \mathcal{K} -Hopf algebra structures, then for the dimensional reason, v_q is primitive for $1 \leq q \leq t$.

Lemma 4.2. *Suppose that the algebras K_t and C_t are \mathcal{K} -Hopf algebras with the following operations:*

$$(4.3) \quad \begin{cases} \beta(u) = \lambda v_t & \text{for } \lambda = 0 \text{ or } 1, \\ \beta(x_i) = z_i & \text{for } t \leq i \leq m, \\ \beta(y_j) = w_j & \text{for } 1 \leq j \leq n, \\ \mathcal{P}^p(z_{m+l}) = w_l + \delta_l & \text{for } 1 \leq l \leq n, \end{cases}$$

where δ_l is some decomposable element of K_t for $1 \leq l \leq n$. Then the following hold:

(1) *There is a map of \mathcal{K} -Hopf algebra $f : K_t \rightarrow H^*(B\mathbb{Z}/p)$ such that $f(x_t) = \omega^p$ and $f = 0$ on the other generators of K_t , where $H^*(B\mathbb{Z}/p) \cong \Lambda(\theta) \otimes \mathbb{Z}/p[\omega]$ with $\beta(\theta) = \omega$.*

(2) *There is a map of \mathcal{K} -Hopf algebra $g : C_t \rightarrow H^*(B\mathbb{Z}/p)$ such that $g(u) = \omega$ and $g = 0$ on the other generators of C_t .*

Proof. We show only (2), since (1) is proved by similar arguments.

Let I denote the ideal of C_t generated by odd degree generators. For the dimensional reason, we see that I is a Hopf ideal of C_t . We show that I is closed under the action of \mathcal{A}_p .

For the dimensional reason, $\mathcal{P}^a(I) \subset I$ for $a \geq 1$, and using the relation $\beta\beta = 0$, we have that $\beta(z_k) = \beta(w_l) = 0$ for $t + 1 \leq k \leq m$ and $1 \leq l \leq n$. Thus, it suffices to show that $\beta(v_q), \beta(z_k) \in I$ for $1 \leq q \leq t$ and $m + 1 \leq k \leq m + n$. We see that $\beta(v_q)$ is primitive since v_q is primitive, and so $\beta(v_q) = 0$ since $\mathcal{P}^4(C_t) = 0$.

For the dimensional reason, we can put

$$\beta(z_k) = \kappa u^{p+1} + \sum_{i=t+1}^m \rho_i u x_i \pmod I$$

for $\kappa, \rho_i \in \mathbb{Z}/p$.

If $\lambda = 1$, then using the relation $\beta\beta(z_k) = 0$, $\kappa = \rho_i = 0$ for $t + 1 \leq i \leq m$, which implies that $\beta(z_k) \in I$.

When $\lambda = 0$, using the relation $\beta\beta(z_k) = 0$, we have that $\beta(z_k) = \kappa u^{p+1} \pmod I$. For the dimensional reason, we have that $\beta(z_k)$ is primitive since $\beta(u) = \beta(v_q) = 0$, which implies that $\kappa = 0$, and so $\beta(z_k) \in I$.

From the above considerations, C_t/I is a \mathcal{K} -Hopf algebra, and the quotient map $\pi : C_t \rightarrow C_t/I$ becomes a map of \mathcal{K} -Hopf algebra. Since C_t/I is a polynomial algebra, there is a monomorphism of \mathcal{K} -Hopf algebra $\sigma : C_t/I \rightarrow H^*(BV)$ by [1], where V is a $(m + n - t + 1)$ -dimensional vector space over \mathbb{Z}/p . It is known that $H^*(BV) \cong \Lambda(\theta_1, \dots, \theta_{m+n-t+1}) \otimes \mathbb{Z}/p[\omega_1, \dots, \omega_{m+n-t+1}]$ with $\beta(\theta_k) = \omega_k$ for $1 \leq k \leq m + n - t + 1$.

Taking a suitable basis of V , and by the argument of [1], we can assume that $\sigma(u) = \omega_1$, $\sigma(x_i) = \omega_{i-t+1}^p$ for $t + 1 \leq i \leq m$, and $\sigma(y_j) = \omega_{m+j-t+1}^{p^2}$ for $1 \leq j \leq n$. If we define a map $g = (Bi)^*\sigma\pi$, where $i : \mathbb{Z}/p \rightarrow V$ is the inclusion on the first factor, then g is a map of \mathcal{K} -Hopf algebra which satisfies the required properties. This completes the proof. \square

Proof of Proposition 4.1. We proceed by an induction on t . For $t = 1$, if we put $Y_1 = Y$, then $Y\langle 3 \rangle \simeq Y$ since Y is 3-connected, and by assumption, $H^*(Y) \cong A \cong K_1$ with the operations (2.4). Now we assume that there exists an H -space Y_t with the required properties.

From now on, we construct an H -space Y_{t+1} satisfying the required properties. For the map f of Lemma 4.2, a result of Lannes [11, Thm. 3.1.1] implies that there is a map $\phi : B\mathbb{Z}/p \rightarrow Y_t$ such that $\phi^* = f$. We see that the evaluation map $e_\phi : \text{Map}(B\mathbb{Z}/p, Y_t)_\phi \rightarrow Y_t$ becomes a homotopy equivalence by Theorem 3.1 and Proposition 3.2. Let $\iota : B\mathbb{Z}/p \rightarrow \text{Map}(B\mathbb{Z}/p, Y_t)_\phi$ be the adjoint of $\phi\mu$, where μ is the multiplication of an H -structure of $B\mathbb{Z}/p$. Then we have the following commutative diagram of fibrations:

$$(4.4) \quad \begin{array}{ccccccc} B\mathbb{Z}/p & \xlongequal{\quad} & B\mathbb{Z}/p & \longrightarrow & EB\mathbb{Z}/p & \longrightarrow & B^2\mathbb{Z}/p \\ \phi \downarrow & & \iota \downarrow & & \downarrow & & \parallel \\ Y_t & \xleftarrow[\simeq]{e_\phi} & \text{Map}(B\mathbb{Z}/p, Y_t)_\phi & \xrightarrow{\kappa} & E_1 & \xrightarrow{\zeta} & B^2\mathbb{Z}/p, \end{array}$$

where $E_1 = EB\mathbb{Z}/p \times_{B\mathbb{Z}/p} \text{Map}(B\mathbb{Z}/p, Y_t)_\phi$ denotes the Borel construction.

Since f is a map of \mathcal{K} -Hopf algebra, ϕ is an H -map, and so the bottom fibration becomes an H -fibration by Proposition 3.3. The E_2 -term of the

Serre spectral sequence for this fibration is given as

$$E_2^{*,*} = H^*(B^2\mathbb{Z}/p) \otimes H^*(Y_t)$$

for $H^*(B^2\mathbb{Z}/p) \cong \mathbb{Z}/p[\eta, \beta\mathcal{P}^{\Delta^i}\beta(\eta) \mid i \geq 0] \otimes \Lambda(\beta(\eta), \mathcal{P}^{\Delta^i}\beta(\eta) \mid i \geq 0)$, where $\mathcal{P}^{\Delta^i} = \mathcal{P}^{p^i} \cdots \mathcal{P}^1$ and η denotes the fundamental class. Now we use the notations from Proposition 3.10.

For the dimensional reason and by Remark 3.12, we have that $E_{2p+1}^{*,*} \cong E_2^{*,*}$. The generator x_i is transgressive for $t \leq i \leq m$, and then by using the naturality of the diagram (4.4), and by Proposition 3.10, we obtain that

$$(4.5) \quad d_{2p+1}(x_i) = \begin{cases} \mathcal{P}^1\beta(\eta) & \text{for } i = t, \\ 0 & \text{for } t + 1 \leq i \leq m. \end{cases}$$

By the Kudo transgression theorem, there are the following differentials:

$$(4.6) \quad \begin{cases} d_{2p^{k+1}+1}(x_t^{p^k}) = \mathcal{P}^{\Delta^k}\beta(\eta) & \text{for } k \geq 1, \\ d_{2p^k(p-1)+1}(\mathcal{P}^{\Delta^{k-1}}\beta(\eta) \otimes x_t^{p^{k-1}(p-1)}) = \beta\mathcal{P}^{\Delta^k}\beta(\eta) & \text{for } k \geq 1. \end{cases}$$

In particular, we see that

$$(4.7) \quad d_{2p(p-1)+1}(\mathcal{P}^1\beta(\eta) \otimes x_t^{p-1}) = \beta\mathcal{P}^{\Delta^1}\beta(\eta)$$

in the $E_{2p(p-1)+1}$ -term. Since $H^1(Y_t) = 0$, $d_{2p+1}(z_k) = 0$ for $1 \leq k \leq m + n$. If $d_{2p+1}(y_j) \neq 0$, then we can replace the generator y_j so that $d_{2p+1}(y_j) = 0$ for $1 \leq j \leq n$. In fact, by Proposition 3.10, we can write

$$d_{2p+1}(y_j) = \mathcal{P}^1\beta(\eta) \otimes \sum_{s=0}^{p-1} b_s x_t^s,$$

where b_s are polynomials of generators of $H^*(Y_t)$ other than x_t for $0 \leq s \leq p - 1$. If we put \bar{y}_j as

$$\bar{y}_j = y_j - \sum_{s=0}^{p-2} \frac{1}{s+1} b_s x_t^{s+1},$$

then by (4.5), $d_{2p+1}(\bar{y}_j) = \mathcal{P}^1\beta(\eta) \otimes b_{p-1}x_t^{p-1}$, and applying the differential $d_{2p(p-1)+1}$ to $\{\mathcal{P}^1\beta(\eta) \otimes b_{p-1}x_t^{p-1}\} = 0$ in the $E_{2p(p-1)+1}$ -term, we have that $\{\beta\mathcal{P}^{\Delta^1}\beta(\eta) \otimes b_{p-1}\} = 0$. This implies that $b_{p-1} = 0$, and so $d_{2p+1}(\bar{y}_j) = 0$. Similarly, we can replace the generators w_l so that $d_{2p+1}(w_l) = 0$ for $1 \leq l \leq n$. Then the E_{2p+2} -term of the spectral sequence is given as

$$E_{2p+2}^{*,*} \cong A_{2p+2} \otimes B_{2p+2} \otimes \Lambda(\mathcal{P}^1\beta(\eta) \otimes x_t^{p-1}),$$

where $A_{2p+2} \cong A_{2p+1}/(\mathcal{P}^1\beta(\eta))$ and B_{2p+2} is generated by the generators of B_{2p+1} other than x_t .

By (4.5), we have that

$$d_{2p+2}(z_k) = \begin{cases} \beta\mathcal{P}^1\beta(\eta) & \text{for } k = t, \\ 0 & \text{for } t + 1 \leq k \leq m, \end{cases}$$

and for $m + 1 \leq k \leq m + n$, if $d_{2p+2}(z_k) \neq 0$, then by Proposition 3.10, $d_{2p+2}(z_k) = a_k\beta\mathcal{P}^1\beta(\eta)$ for some $a_k \in \mathbb{Z}/p$. If we set $\bar{z}_k = z_k - a_k z_t$, then $d_{2p+2}(\bar{z}_k) = 0$. If $d_{2p+2}(y_j) \neq 0$, then by Proposition 3.10, we can write that

$$d_{2p+2}(y_j) = \beta\mathcal{P}^1\beta(\eta) \otimes (b_0 + b_1 z_t),$$

where b_s are polynomials of generators of B_{2p+2} other than z_t for $s = 0, 1$. If we set $\bar{y}_j = y_j - b_0 z_t$, then $d_{2p+2}(\bar{y}_j) = \beta\mathcal{P}^1\beta(\eta) \otimes b_1 z_t$, and applying d_{2p+2} to $d_{2p+2}(\bar{y}_j)$, we have that $(\beta\mathcal{P}^1\beta(\eta))^2 \otimes b_1 = 0$, which implies that $b_1 = 0$, and so $d_{2p+2}(\bar{y}_j) = 0$. By the same arguments, we can replace the generators w_l so that $d_{2p+2}(w_l) = 0$ for $1 \leq l \leq n$. Then we obtain that

$$E_{2p+3}^{*,*} \cong A_{2p+3} \otimes B_{2p+3} \otimes \Lambda(\mathcal{P}^1\beta(\eta) \otimes x_t^{p-1}),$$

where $A_{2p+3} \cong A_{2p+2}/(\beta\mathcal{P}^1\beta(\eta))$ and B_{2p+3} is generated by the generators of B_{2p+2} other than z_t . For the dimensional reason and by Remark 3.12, $E_{2p(p-1)+1}^{*,*} \cong E_{2p+3}^{*,*}$, and by (4.7), $E_{2p(p-1)+2}^{*,*} \cong A_{2p(p-1)+2} \otimes B_{2p(p-1)+2}$, where $A_{2p(p-1)+2} \cong A_{2p(p-1)+1}/(\beta\mathcal{P}^{\Delta_1}\beta(\eta))$ and $B_{2p(p-1)+2} \cong B_{2p(p-1)+1}$. Furthermore, for the dimensional reason and by Remark 3.12, $E_{2p^2+1}^{*,*} \cong E_{2p(p-1)+2}^{*,*}$, and so we conclude that

$$E_{2p^2+1}^{*,*} \cong A_{2p^2+1} \otimes B_{2p^2+1}$$

for

$$A_{2p^2+1} \cong H^*(B^2\mathbb{Z}/p)/(\beta\mathcal{P}^1\beta(\eta), \mathcal{P}^1\beta(\eta))$$

and

$$B_{2p^2+1} \cong \mathbb{Z}/p[x_t^p, x_{t+1}, \dots, x_m, y_1, \dots, y_n] \otimes \Lambda(v_1, \dots, v_{t-1}, z_{t+1}, \dots, z_{m+n}, w_1, \dots, w_n).$$

By iterating this process, we can compute the spectral sequence. In particular, the differentials are completely determined by (4.6), and so we have that for $k \geq 1$,

$$E_{2p^k+1}^{*,*} \cong A_{2p^k+1} \otimes B_{2p^k+1},$$

where

$$A_{2p^k+1} \cong H^*(B^2\mathbb{Z}/p)/(\beta\mathcal{P}^{\Delta_j}\beta(\eta), \mathcal{P}^{\Delta_j}\beta(\eta) \mid 0 \leq j \leq k - 2)$$

and

$$B_{2p^{k+1}} \cong \mathbb{Z}/p \left[x_t^{p^{k-1}}, x_{t+1}, \dots, x_m, y_1, \dots, y_n \right] \\ \otimes \Lambda(v_1, \dots, v_{t-1}, z_{t+1}, \dots, z_{m+n}, w_1, \dots, w_n).$$

This implies that $H^*(E_1) \cong C_t$ as algebras, where u and v_t represent the generators η and $\beta(\eta)$ in $H^*(B^2\mathbb{Z}/p)$. Since

$$\left\{ \begin{array}{ll} \kappa^*(x_i) = x_i & \text{for } t+1 \leq i \leq m, \\ \kappa^*(y_j) = y_j & \text{for } 1 \leq j \leq n, \\ \kappa^*(z_k) = z_k & \text{for } t+1 \leq k \leq m, \\ \kappa^*(z_k) = z_k - a_k z_t & \text{for } m+1 \leq k \leq m+n, a_k \in \mathbb{Z}/p, \\ \kappa^*(w_l) = w_l & \text{for } 1 \leq l \leq n \end{array} \right.$$

up to decomposable elements and $\mathcal{P}^p(z_t) = 0$, we can take the generators of $H^*(E_1)$ satisfying the condition (4.3) with $\lambda = 1$.

Next we apply same arguments to the H -space E_1 . For the map g of Lemma 4.2, a result of Lannes [11, Thm. 3.1.1] implies that there is a map $\psi_1 : B\mathbb{Z}/p \rightarrow E_1$ such that $\psi_1^* = g$. The evaluation map $e_{\psi_1} : \text{Map}(B\mathbb{Z}/p, E_1)_{\psi_1} \rightarrow E_1$ is a homotopy equivalence by Theorem 3.1 and Proposition 3.2. Let $\iota_1 : B\mathbb{Z}/p \rightarrow \text{Map}(B\mathbb{Z}/p, E_1)_{\psi_1}$ be the adjoint of $\psi_1 \mu$. Then, we have the following H -fibration by the same construction as above:

$$E_1 \xleftarrow{\simeq} \text{Map}(B\mathbb{Z}/p, E_1)_{\psi_1} \xrightarrow{\kappa_1} E_2 \longrightarrow B^2\mathbb{Z}/p,$$

where $E_2 = (\text{Map}(B\mathbb{Z}/p, E_1)_{\psi_1})_{hB\mathbb{Z}/p}$ denotes the Borel construction. Computing the spectral sequence for this fibration as above, we conclude that $H^*(E_2) \cong C_t$ with the operations (4.3) with $\lambda = 0$.

Iterating this process, we have the following sequence of H -spaces and H -maps:

$$Y_t \xrightarrow{\kappa} E_1 \xrightarrow{\kappa_1} E_2 \xrightarrow{\kappa_2} \dots$$

satisfying $H^*(Y_t) \cong K_t$, $H^*(E_s) \cong C_t$ with the operations (4.3) with $\lambda = 1$ for $s = 1$ and $\lambda = 0$ for $s > 1$, $\kappa_s^*(u) = 0$ and

$$\kappa_s^* : H^*(E_{s+1})/(u) \longrightarrow H^*(E_s)/(u)$$

is an isomorphism for $s \geq 1$.

If we set $Y_{t+1} = \varinjlim_s E_s$, then there is the Milnor exact sequence

$$0 \rightarrow \varinjlim_s H^{s+1}(E_s) \rightarrow H^*(Y_{t+1}) \rightarrow \varinjlim_s H^*(E_s) \rightarrow 0.$$

Since $\varinjlim_s H^{s+1}(E_s) = 0$ by the Mittag-Leffler condition, we have that $H^*(Y_{t+1}) \cong \varinjlim_s H^*(E_s) \cong K_{t+1}$, and by Proposition 3.6, we see that Y_{t+1} has an \bar{H} -structure. Let F be the homotopy fiber of the composite $E_1 \rightarrow Y_{t+1}$, then $H^*(F) \cong H^*(K(\mathbb{Z}, 2))$ by the spectral sequence argument, and this implies that $F \simeq K(\mathbb{Z}, 2)$. By the cohomology, E_1 is homotopy

equivalent to the homotopy fiber of $[p]v_t : Y_{t+1} \rightarrow K(\mathbb{Z}, 3)$. Therefore, we have the following commutative diagram of fibrations:

$$\begin{array}{ccccc}
 Y_t & \xrightarrow{\kappa} & E_1 & \xrightarrow{\zeta} & B^2\mathbb{Z}/p \\
 \parallel & & \downarrow & & \downarrow \\
 Y_{t+1}\langle v_t \rangle & \longrightarrow & Y_{t+1} & \xrightarrow{v_t} & K(\mathbb{Z}, 3) \\
 \downarrow & & \downarrow [p]v_t & & \downarrow [p] \\
 * & \longrightarrow & K(\mathbb{Z}, 3) & \xlongequal{\quad} & K(\mathbb{Z}, 3),
 \end{array}$$

which implies that $Y_t \simeq Y_{t+1}\langle v_t \rangle$, where $Y_{t+1}\langle v_t \rangle$ denotes the homotopy fiber of the map $v_t : Y_{t+1} \rightarrow K(\mathbb{Z}, 3)$. By the induction hypothesis, $Y_t\langle 3 \rangle \simeq Y$, and so we have that $Y_{t+1}\langle 3 \rangle \simeq (Y_{t+1}\langle v_t \rangle)\langle 3 \rangle \simeq Y$. This completes the proof. \square

Next, for $1 \leq t \leq n + 1$, we set an algebra L_t as

$$\begin{aligned}
 L_t &= \mathbb{Z}/p[y_t, \dots, y_n] \\
 &\quad \otimes \Lambda(v_1, \dots, v_{m+t-1}, z_{m+t}, \dots, z_{m+n}, c_1, \dots, c_{t-1}, w_t, \dots, w_n)
 \end{aligned}$$

with y_j, z_k and w_l are as in A , $|v_q| = 3$ for $1 \leq q \leq m + t - 1$, and $|c_r| = 2p + 1$ for $1 \leq r \leq t - 1$. Then we have the following proposition:

Proposition 4.8. *For $1 \leq t \leq n + 1$, there is a mod p H -space Z_t such that $Z_t\langle 3 \rangle \simeq Y$ and $H^*(Z_t) \cong L_t$ with the following operations:*

$$(4.9) \quad \begin{cases} \beta(y_j) = w_j & \text{for } t \leq j \leq n, \\ \mathcal{P}^1(v_{m+r}) = c_r & \text{for } 1 \leq r \leq t - 1, \\ \mathcal{P}^p(z_{m+l}) = w_l + \delta_l & \text{for } t \leq l \leq n, \end{cases}$$

where δ_l is some decomposable element of L_t for $t \leq l \leq n$.

Proposition 4.8 is proved by same arguments as in Proposition 4.1, and so we give an outline of the proof.

We proceed by an induction on $1 \leq t \leq n + 1$. For $t = 1$, if we set $Z_1 = Y_{m+1}$, then by Proposition 4.1, Z_1 satisfies the required properties. We assume that there exists an H -space Z_t with the conditions of Proposition 4.8, and construct an H -space Z_{t+1} satisfying the required properties.

We can construct a \mathcal{K} -Hopf algebra map $h : H^*(Z_t) \rightarrow H^*(B\mathbb{Z}/p)$ such that $h(y_t) = \omega^{p^2}$ and $h = 0$ on the other generators. By a result of Lannes, there is an H -map $\xi : B\mathbb{Z}/p \rightarrow Z_t$ such that $\xi^* = h$, and we see that the evaluation map $e_\xi : \text{Map}(B\mathbb{Z}/p, Z_t)_\xi \rightarrow Z_t$ becomes a homotopy equivalence. For an H -structure μ of $B\mathbb{Z}/p$, if $\iota : B\mathbb{Z}/p \rightarrow \text{Map}(B\mathbb{Z}/p, Z_t)_\xi$ denotes the adjoint of $\xi\mu$, then we have the following fibration:

$$Z_t \xleftarrow{\simeq} \text{Map}(B\mathbb{Z}/p, Z_t)_\xi \longrightarrow F_1 \longrightarrow B^2\mathbb{Z}/p,$$

where F_1 is an H -space given by the Borel construction for ι .

For $1 \leq t \leq n + 1$, we set an algebra D_t as

$$D_t = \mathbb{Z}/p[u, y_{t+1}, \dots, y_n] \otimes \Lambda(v_1, \dots, v_{m+t}, z_{m+t+1}, \dots, z_{m+n}, c_1, \dots, c_t, w_{t+1}, \dots, w_n)$$

with y_j, z_k, w_l and v_q for $1 \leq q \leq m+t-1$ are as in L_t , $|u| = 2$ and $|v_{m+t}| = 3$. Then, using the Serre spectral sequence, we have that $H^*(F_1) \cong D_t$ with the operations (4.9) and $\beta(u) = v_{m+t}$. Iterating this process, we have a sequence of H -spaces and H -maps

$$Z_t \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \dots$$

such that $H^*(Z_t) \cong L_t$, $H^*(F_s) \cong D_t$ with the operations (4.9). If we set $Z_{t+1} = \varinjlim_s F_s$, then Z_{t+1} has an H -structure, and using the Milnor exact sequence, we obtain that $H^*(Z_{t+1}) \cong L_{t+1}$ with the operations (4.9). We can show that the homotopy fiber $Z_{t+1}\langle v_{m+t} \rangle \simeq Z_t$, and so by the induction hypothesis, the three-connected cover $Z_{t+1}\langle 3 \rangle \simeq Y$. This establishes the proof of Proposition 4.8.

Now we set $W = Z_{n+1}$. Then W is a simply connected mod p finite H -space such that

$$H^*(W) \cong \Lambda(v_1, \dots, v_{m+n}, c_1, \dots, c_n)$$

with $\mathcal{P}^1(z_{m+r}) = c_r$ for $1 \leq r \leq n$, and $Y \simeq W\langle 3 \rangle$ by Proposition 4.8. This completes the proof of Proposition 2.6. □

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