Pacific Journal of Mathematics

EXPLICIT CAYLEY TRIPLES IN REAL FORMS OF E_7

Dragomir Ž. Đoković

Volume 191 No. 1 November 1999

PACIFIC JOURNAL OF MATHEMATICS
Vol. 191, No. 1, 1999

EXPLICIT CAYLEY TRIPLES IN REAL FORMS OF E_7

Dragomir Ž. Đoković

Let $\mathfrak{g}$ be a noncompact real form of the simple complex Lie algebra $\mathfrak{g}^c$ of type E_7. Up to isomorphism, there are exactly three such algebras: EV, EVI, and EVII in Cartan notations. For each of these algebras we obtain a list of representatives of the adjoint orbits of standard triples (E, H, F), i.e., triples $\{E, H, F\} \subset \mathfrak{g}$ spanning a subalgebra isomorphic to $\mathfrak{sl}_2(\mathrm{R})$, and such that $[H, E] = 2E$, $[H, F] = -2F$, and $[F, E] = H$. These representative standard triples are chosen to be Cayley triples with respect to a fixed Cartan decomposition of $\mathfrak{g}$.

1. Introduction.

The nilpotent adjoint orbits in noncompact real forms of exceptional simple Lie algebras have been classified in our papers [4, 5]. This classification is based on the so-called Kostant-Sekiguchi bijection for which we refer to [2] and [7]. See Section 5 for a more detailed discussion of this bijection. In our first two papers mentioned above, we did not compute the representatives of the nilpotent orbits.

This paper is a sequel to [7] and we shall freely use the notations introduced there. In that paper we have compiled a list of representatives of G-orbits of standard triples (E, H, F) in $\mathfrak{g}$ where $\mathfrak{g}$ is a noncompact real form of $\mathfrak{g}^c$, and $\mathfrak{g}^c$ is a simple complex Lie algebra of type G_2, F_4, or E_6. In fact these representative triples were chosen to be real Cayley triples with respect to a fixed Cartan decomposition of $\mathfrak{g}$. In the present paper we accomplish the same objective for noncompact real forms $\mathfrak{g}$ of $\mathfrak{g}^c$ when the latter is of the type E_7. Up to G^c-conjugacy, there are exactly 3 such real forms. They are denoted by EV, EVI, EVII or $E_{7(7)}, E_{7(-5)}, E_{7(-25)}$, respectively. The nilpotent elements E of these representative Cayley triples are representatives of the nonzero nilpotent adjoint orbits.

By using a result from our recent note [6] it is easy to determine which complex nilpotent adjoint orbits possess real points. It is more delicate to determine the number of real nilpotent orbits that are contained in a given complex orbit.

We record here that D.R. King [8, p. 254] has detected an error in [5, Table VIII]. Namely the last entry for orbit 5 of that table should be $\mathfrak{sl}(3, \mathbf{C})$ instead of $2\mathfrak{su}(2,1)$.

Several misprints in our paper [4] have been mentioned in [7]. There is one more: Namely on p. 515, in Table XII, the labels "020220 0" of the orbit No. 31 (given in the second column) should be replaced by "020220 2". Consequently these labels should also be corrected in [2, p. 158].

2. The root system of E_7.

We denote by $\mathfrak{h}$ a maximally split Cartan subalgebra of $\mathfrak{g}$ which is stable under the Cartan involution θ, and by $\mathfrak{h}^c$ its complexification. The number of positive roots of $\mathfrak{g}^c = E_7$ is $N = 63$. The positive roots are enumerated as $\alpha_1, \alpha_2, \ldots, \alpha_N$ with $\Pi = \{\alpha_1, \ldots, \alpha_7\}$ as a base. The enumeration is chosen so that the heights increase, i.e., $\mathrm{ht}(\alpha_i) \leq \mathrm{ht}(\alpha_j)$ for $i < j$. The negative root $-\alpha_i$ is written also as α_{-i}. The extended Dynkin diagram of E_7 is given in Fig. 1.

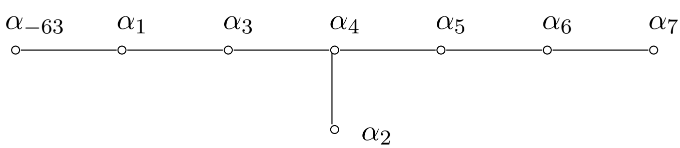

Figure 1.

Since E_7 is simply laced, if $\alpha_i = k_1\alpha_1 + \cdots + k_7\alpha_7$ is a positive root, then the corresponding coroot H_i is given by $H_i = k_1 H_1 + \cdots + k_7 H_7$. Our enumeration of positive roots α_i (and coroots H_i) is given in Table 1.

Table 1.
Positive roots of E_7.

i	α_i, H_i	i	α_i, H_i	i	α_i, H_i
1	1000000	22	0111100	43	1112210
2	0100000	23	0101110	44	1112111
3	0010000	24	0011110	45	0112211
4	0001000	25	0001111	46	1122210
5	0000100	26	1111100	47	1122111
6	0000010	27	1011110	48	1112211
7	0000001	28	0112100	49	0112221
8	1010000	29	0111110	50	1123210
9	0101000	30	0101111	51	1122211
10	0011000	31	0011111	52	1112221
11	0001100	32	1112100	53	1223210
12	0000110	33	1111110	54	1123211
13	0000011	34	1011111	55	1122221
14	1011000	35	0112110	56	1223211
15	0111000	36	0111111	57	1123221
16	0101100	37	1122100	58	1223221
17	0011100	38	1112110	59	1123321
18	0001110	39	1111111	60	1223321
19	0000111	40	0112210	61	1224321
20	1111000	41	0112111	62	1234321
21	1011100	42	1122110	63	2234321

3. The structure constants of E_7.

As in [7], we use an algorithm of J. Kurtzke [9] to fix the choice of a Chevalley basis of $\mathfrak{g}^c$:

$$H_i, \quad 1 \le i \le 7; \quad X_i, X_{-i}, \quad 1 \le i \le 63.$$

If $\alpha_i + \alpha_j = \alpha_k$, then

$$[X_i, X_j] = N(i,j)X_k.$$

As all roots of E_7 have the same length, the nonzero structure constants are ± 1, i.e., $N(i,j) = \varepsilon(i,j)$.

We specify that $N(1,3) = +1$. Then, by Kurtzke's algorithm,

$$N(3,4) = N(5,6) = -1, \; N(4,2) = N(4,5) = N(6,7) = +1,$$

and $N(i,j) = +1$ whenever $\alpha_i + \alpha_j$ is a root and $1 \le i \le 7 < j \le 63$. Furthermore all other $N(i,j)$'s are uniquely determined.

For the convenience of the reader, we list in the Appendix, the nonzero structure constants $N(i,j)$ for all $i > 0$. For $i < 0$ one can use the formula

$$N(-i,-j) = N(i,j).$$

4. The conjugation σ.

We recall that σ denotes the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$, and that $\mathfrak{h}^c$ is σ-invariant. The action of σ on $\mathfrak{h}^c$ induces naturally an action on the dual space of $\mathfrak{h}^c$ which preserves Φ. As σ acts on Φ as an automorphism, it suffices to know the action of σ on Π. If $\sigma(\alpha_i) = \alpha_j$ we also write $\sigma(i) = j$. Note that $\sigma(i) = j$ implies that $\sigma(-i) = -j$.

One can further assume that the Chevalley basis has been chosen so that, in addition to the properties mentioned earlier, the action of σ on the X_i's is given by

$$\sigma(X_i) = \xi_i X_{\sigma(i)}$$

where $\xi_i = \pm 1$. We recall that $\xi_i = 1$ whenever $\alpha_i \in \Phi_0$, and $\xi_{-i} = \xi_i$ for all i. For all three noncompact real forms $\mathfrak{g}$ of $\mathfrak{g}^c$ we may choose $\xi_i = 1$ for $1 \le i \le 7$. With this information, one can compute the coefficients ξ_i for arbitrary i.

For reader's convenience we list the vectors $\sigma(X_i)$, $i > 0$, in Table 2 for EVI and Table 3 for EVII. When $\mathfrak{g}$ is of type EV, i.e., $\mathfrak{g}$ is the split real form of $\mathfrak{g}^c$, then the action of σ on Φ is trivial. Hence in that case we have $\sigma(X_i) = X_i$ for all i.

Table 2.

EVI $= E_{7(-5)}$: Action of σ on the X_i's.

i	$\sigma(X_i)$	i	$\sigma(X_i)$	i	$\sigma(X_i)$
1	X_1	22	$-X_{10}$	43	X_{44}
2	X_{-2}	23	$-X_{25}$	44	X_{43}
3	X_3	24	X_{36}	45	$-X_{35}$
4	X_{16}	25	$-X_{23}$	46	X_{47}
5	X_{-5}	26	$-X_{14}$	47	X_{46}
6	X_{19}	27	X_{39}	48	$-X_{38}$
7	X_{-7}	28	X_{28}	49	X_{49}
8	X_8	29	$-X_{31}$	50	X_{56}
9	X_{11}	30	X_{18}	51	$-X_{42}$
10	$-X_{22}$	31	$-X_{29}$	52	X_{52}
11	X_9	32	X_{32}	53	$-X_{54}$
12	X_{13}	33	$-X_{34}$	54	$-X_{53}$
13	X_{12}	34	$-X_{33}$	55	X_{55}
14	$-X_{26}$	35	$-X_{45}$	56	X_{50}
15	X_{17}	36	X_{24}	57	$-X_{60}$
16	X_4	37	X_{37}	58	X_{59}
17	X_{15}	38	$-X_{48}$	59	X_{58}
18	X_{30}	39	X_{27}	60	$-X_{57}$
19	X_6	40	X_{41}	61	X_{61}
20	X_{21}	41	X_{40}	62	X_{62}
21	X_{20}	42	$-X_{51}$	63	X_{63}

Table 3.
EVII $= E_{7(-25)}$: Action of σ on the X_i's.

i	$\sigma(X_i)$	i	$\sigma(X_i)$	i	$\sigma(X_i)$
1	X_{37}	22	X_{-22}	43	$-X_{42}$
2	X_{-2}	23	X_{24}	44	X_{51}
3	X_{-3}	24	X_{23}	45	$-X_{13}$
4	X_{-4}	25	$-X_{36}$	46	X_{38}
5	X_{-5}	26	$-X_{14}$	47	$-X_{48}$
6	X_{40}	27	X_{53}	48	$-X_{47}$
7	X_7	28	X_{-28}	49	X_{49}
8	X_{32}	29	$-X_{18}$	50	$-X_{33}$
9	X_{-9}	30	X_{31}	51	X_{44}
10	X_{-10}	31	X_{30}	52	$-X_{62}$
11	X_{-11}	32	X_8	53	X_{27}
12	X_{35}	33	$-X_{50}$	54	$-X_{39}$
13	$-X_{45}$	34	X_{56}	55	X_{61}
14	$-X_{26}$	35	X_{12}	56	X_{34}
15	X_{-15}	36	$-X_{25}$	57	$-X_{60}$
16	X_{-16}	37	X_1	58	X_{59}
17	X_{-17}	38	X_{46}	59	X_{58}
18	$-X_{29}$	39	$-X_{54}$	60	$-X_{57}$
19	X_{41}	40	X_6	61	X_{55}
20	X_{21}	41	X_{19}	62	$-X_{52}$
21	X_{20}	42	$-X_{43}$	63	X_{63}

5. The Kostant-Sekiguchi bijection.

We refer the reader to our previous paper [7] for definition of G, K, G^c, K^c, standard triples, normal triples, real and complex Cayley triples, etc.

In the following diagram we exhibit several important sets on which some of the above groups act and some natural maps between these sets

Set	Group
Nonzero nilpotent elements in $\mathfrak{g}$	G
$\uparrow \alpha$	
Standard triples in $\mathfrak{g}$	G
$\uparrow \beta$	
Real Cayley triples in $\mathfrak{g}$	K
$\downarrow \gamma$	
Complex Cayley triples in $\mathfrak{g}^c$	K
$\downarrow \delta$	
Normal triples in $\mathfrak{g}^c$	K^c
$\downarrow \varepsilon$	
Nonzero nilpotent elements in $\mathfrak{p}^c$	K^c

The map α (resp. ε) sends the standard (resp. normal) triple (E,H,F) to its nilpositive part E. The maps β and δ are the inclusion maps. The map γ is the Cayley transformation. Clearly γ is bijective and K-equivariant. The maps β and δ are also K-equivariant. The map α is G-equivariant while ε is K^c-equivariant. We shall prove below that, on the level of orbits, each of these maps induces a bijection.

Since α is G-equivariant, it induces a map α^* from the set of G-orbits of standard triples in $\mathfrak{g}$ to the set of G-orbits of nonzero nilpotent elements of $\mathfrak{g}$. One defines similarly the maps β^*, γ^*, δ^*, and ε^*.

Proposition 1. *Each of the maps $\alpha^*,\ldots,\varepsilon^*$ is a bijection.*

Proof. All the references in this proof are to the book [2]. Since γ is bijective, so is γ^*. The map α^* is surjective by Theorem 9.2.1, and injective by Theorem 9.2.3. The map ε^* is surjective by Theorem 9.4.2, and injective by Theorem 9.4.3. The map β^* is surjective by Theorem 9.4.1. The fact that δ^* is surjective is shown in the proof of Theorem 9.5.1.

We shall prove now that β^* is injective. Thus, if (E,H,F) and (E',H',F') are real Cayley triples which are G-conjugate, we have to show that they are also K-conjugate. As $H,H'\in\mathfrak{p}$, Lemma 9.4.5 shows that H and H' are K-conjugate. Hence without any loss of generality we may assume that $H'=H$. By our hypothesis, there exists $g\in G$ such that $g\cdot(E,H,F)=(E',H,F')$. In particular, $g\in Z_G(H)$ and $g\cdot(E+F)=E'+F'$. Since H is semisimple, the centralizer $Z_G(H)$ is reductive. By applying the proof of Lemma 9.4.5 and observing that $E+F,E'+F'\in\mathfrak{k}$, we conclude that there exists $k\in Z_K(H)$ such that $k\cdot(E+F)=E'+F'$. The formula $[H,E+F]=2(E-F)$ now implies that $k\cdot(E-F)=E'-F'$. It follows that $k\cdot E=E'$ and $k\cdot F=F'$, i.e., the real Cayley triples (E,H,F) and (E',H,F') are K-conjugate.

By Theorem 9.5.1 the composite map $\varepsilon^*\circ\delta^*\circ\gamma^*\circ(\beta^*)^{-1}\circ(\alpha^*)^{-1}$ is a bijection. It follows that δ^* must be also injective. □

The composite map $\alpha^*\circ\beta^*\circ(\gamma^*)^{-1}\circ(\delta^*)^{-1}\circ(\varepsilon^*)^{-1}$ is the Kostant–Sekiguchi bijection from the set of nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ to the set of nonzero nilpotent G-orbits in $\mathfrak{g}$. Explicitly, if (E,H,F) is a real Cayley triple and (E',H',F') its Cayley transform, then the orbit $G\cdot E$ corresponds to the orbit $K^c\cdot E'$.

Define a partial order in the set of nilpotent K^c-orbits in $\mathfrak{p}^c$ by setting $\mathcal{O}_1\geq\mathcal{O}_2$ if $\mathcal{O}_2$ is contained in the closure of the orbit $\mathcal{O}_1$. Define similarly the partial order in the set of nilpotent G-orbits in $\mathfrak{g}$. It was shown very recently [1] that the Kostant–Sekiguchi bijection preserves these partial orders.

6. Two invariants.

Let (E, H, F) be a real Cayley triple and (E', H', F') its Cayley transform. In order to distinguish between various G-orbits in $\mathfrak{g}$ which are contained in the same nonzero nilpotent G^c-orbit in $\mathfrak{g}^c$, we use two invariants:

$$\mathrm{tr} := \text{ trace } \left(\mathrm{ad}(H')^2|_{\mathfrak{k}^c}\right)$$

and

$$\mathrm{inv} := \dim Z_{\mathfrak{k}^c}(H').$$

The second one was used in our previous paper [7], while the first one is easier to compute. It is evident from our tables that in some instances tr fails to distinguish two orbits and we have to use inv. In a few instances inv fails, while tr succeeds to distinguish two orbits. Our method for computing the representative real Cayley triples (E, H, F) in $\mathfrak{g}$ is described in detail in [7]. In a relatively few cases the method fails and we had to do extensive computations to find the desired representatives. We shall describe the difficulties that arise on one such example in Section 9 (the most difficult case).

7. Pairs of orbits $G \cdot E$ and $-G \cdot E$.

If $E \neq 0$ is a nilpotent element in $\mathfrak{g}$, then there exists an automorphism of $\mathfrak{g}$ which maps E to $-E$ (see [3]). In general, this automorphism is not inner and so E and $-E$ may belong to different G-orbits. One can decide whether or not $G \cdot E = -G \cdot E$ by means of the following proposition.

Proposition 2. *Let (E, H, F) be a real Cayley triple and (E', H', F') its Cayley transform. Then $G \cdot E = -G \cdot E$ if and only if $K^c \cdot H' = -K^c \cdot H'$.*

Proof. The triple $(-E, H, -F)$ is another real Cayley triple, and its Cayley transform is $(-F', -H', -E')$. We have $G \cdot E = -G \cdot E$ if and only if the real Cayley triples (E, H, F) and $(-E, H, -F)$ are G-conjugate. By the properties of the Sekiguchi bijection, this is the case if and only if the corresponding complex Cayley triples, namely (E', H', F') and $(-F', -H', -E')$ are K^c-conjugate. The latter condition is equivalent to the K^c-conjugacy of H' and $-H'$. □

The characteristics H' of the nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ are known [4]. Since $H' \in \mathfrak{h}^c$, H' and $-H'$ are K^c-conjugate if and only if they belong to the same orbit of the Weyl group $W(\mathfrak{k}^c, \mathfrak{g}^c)$. Hence it is easy to check whether or not $G \cdot E = -G \cdot E$.

When $\mathfrak{g}$ is of Cartan type EVI, then Aut $(\mathfrak{g}) = G$, and so $G \cdot E = -G \cdot E$ for all nonzero nilpotent elements $E \in \mathfrak{g}$. In the other two cases, when $\mathfrak{g}$ is of Cartan type EV or EVII, there exist nonzero nilpotent elements $E \in \mathfrak{g}$ such that $G \cdot E \neq -G \cdot E$. Such pairs of orbits are easily recognizable from Tables 4 and 6 because we give their representatives jointly as $\pm E$.

8. Tables of real Cayley triples.

We give here lists of representatives (E, H, F) for K-orbits of real Cayley triples in $\mathfrak{g}$, a noncompact real form of the simple complex Lie algebra $\mathfrak{g}^c$ of type E_7.

We record only the elements E and H because F can be easily computed by using

$$F = \theta(E) = \theta\sigma(E) = \sigma_u(E)$$

and $\sigma_u(X_i) = X_{-i}$ (for all i).

For the neutral element H, we list both the labels $\alpha_i(H)$ for $1 \leq i \leq 7$ and the coefficients k_i in the linear combination $H = k_1 H_1 + \cdots + k_7 H_7$.

The nilpositive element E is written explicitly as a linear combination of the root vectors X_i ,$i > 0$.

Note that the element H is always the characteristic of the nilpotent orbit $G^c \cdot E$. On the other hand there is no natural choice for the nilpositive element E. Our preference was to choose E so that its support (i.e. the number of nonzero coefficients) is minimal even though this may have a drawback of introducing irrational coefficients.

Table 4.
Cayley triples in EV $= E_{7(7)}$.

	$\alpha_i(H)$	k_i	E	tr	inv
1	1000000	$2,2,3,4,3,2,1$	X_{63}	32	31
2	0000010	$2,3,4,6,5,4,2$	$X_{49}+X_{63}$	64	23
3, 4	0000002	$2,3,4,6,5,4,3$	$\pm(X_7+X_{49}+X_{63})$	96	39
5	0010000	$3,4,6,8,6,4,2$	$X_{37}+X_{55}+X_{61}$	96	19
6	2000000	$4,4,6,8,6,4,2$	$X_1+X_{37}+X_{55}+X_{61}$	128	37
7	2000000	$4,4,6,8,6,4,2$	$\sqrt{2}(X_1+X_{62})$	128	31
8, 9	0100001	$3,5,6,9,7,5,3$	$\pm(X_{30}+X_{47}+X_{53}+X_{59})$	128	19
10, 11	1000010	$4,5,7,10,8,6,3$	$\pm(X_{27}+X_{39}+X_{49}$ $+X_{53}-X_{54})$	160	21
12	1000010	$4,5,7,10,8,6,3$	$X_{27}+X_{39}+X_{49}$ $+X_{53}+X_{54}$	160	15
13, 14	0001000	$4,6,8,12,9,6,3$	$\pm(X_{28}+X_{38}+X_{47}$ $-X_{48}+X_{49})$	192	25
15	0001000	$4,6,8,12,9,6,3$	$X_{28}+X_{38}+X_{46}$ $+X_{47}+X_{48}+X_{49}$	192	13
16, 17	0200000	$4,7,8,12,9,6,3$	$\pm(X_2+X_{28}-X_{38}+X_{46}$ $+X_{47}+X_{48}+X_{49})$	224	49
18, 19	0200000	$4,7,8,12,9,6,3$	$\pm(X_2+X_{28}+X_{38}+X_{46}$ $+X_{47}+X_{48}+X_{49})$	224	25
20	2000010	$6,7,10,14,11,8,4$	$\sqrt{3}(X_1+X_{37})+2X_{49}$	320	15
21	0000020	$4,6,8,12,10,8,4$	$\sqrt{2}(X_6+X_{19}+X_{50}+X_{56})$	256	23
22, 23	2000002	$6,7,10,14,11,8,5$	$\pm[X_7+2X_{49}$ $+\sqrt{3}(X_1+X_{37})]$	352	23
24	0010010	$5,7,10,14,11,8,4$	$X_{37}+\sqrt{2}(X_{24}+X_{33}$ $+X_{41}+X_{48})$	288	11
25	1001000	$6,8,11,16,12,8,4$	$X_{28}+2X_{49}+\sqrt{3}(X_{14}+X_{26})$	352	11
26	0020000	$6,8,12,16,12,8,4$	$X_3-X_{28}+2X_{49}$ $+\sqrt{3}(X_{14}+X_{26})$	384	19
27	0020000	$6,8,12,16,12,8,4$	$X_3+X_{28}+2X_{49}$ $+\sqrt{3}(X_{14}+X_{26})$	384	21
28, 29	1000101	$6,8,11,16,13,9,5$	$\pm[X_{19}+X_{40}+2X_{41}$ $+\sqrt{3}(X_{21}+X_{33})]$	384	11
30	2020000	$10,12,18,24,18,$ $12,6$	$\sqrt{10}X_1+\sqrt{6}(X_3$ $+X_{28}+X_{49})$	896	19
31, 32	0110001	$6,9,12,17,13,9,5$	$\pm[X_{15}+X_{30}+2X_{31}-X_{40}$ $+\sqrt{3}(X_{26}+X_{38})]$	416	11
33, 34	0110001	$6,9,12,17,13,9,5$	$\pm[X_{15}+X_{30}+2X_{31}+X_{40}$ $+\sqrt{3}(X_{26}+X_{38})]$	416	13
35, 36	0001010	$6,9,12,18,14,$ $10,5$	$\pm[-X_{18}+X_{30}+2X_{32}+X_{33}$ $+X_{34}+\sqrt{3}(X_{29}-X_{31})]$	448	17

Table 4.
(continued)

	$\alpha_i(H)$	k_i	E	tr	inv
37	0001010	$6, 9, 12, 18, 14, 10, 5$	$2X_{37} + \sqrt{2}(X_{29} + X_{34}) + \sqrt{3}(X_{18} + X_{30})$	448	9
38	2000020	$8, 10, 14, 20, 16, 12, 6$	$2(X_1 + X_{37}) + \sqrt{3}(X_6 + X_{19} + X_{40} + X_{41})$	640	19
39, 40	0000200	$6, 9, 12, 18, 15, 10, 5$	$\pm[X_5 + 2X_{18} + X_{28} + X_{29} - X_{30} - X_{31} + \sqrt{3}(X_{26} + X_{47})]$	480	33
41, 42	0000200	$6, 9, 12, 18, 15, 10, 5$	$\pm[X_{30} + 2X_{47} + \sqrt{2}(X_{28} + X_{33}) + \sqrt{3}(X_5 + X_{18})]$	480	17
43	2000020	$8, 10, 14, 20, 16, 12, 6$	$2(X_1 + X_{37}) + \sqrt{3}(-X_6 + X_{19} + X_{40} + X_{41})$	640	15
44, 45	2000022	$10, 13, 18, 26, 21, 16, 9$	$\pm[3X_7 + 2\sqrt{2}(X_6 + X_{40}) + \sqrt{5}(X_1 + X_{37})]$	1120	15
46, 47	2110001	$10, 13, 18, 25, 19, 13, 7$	$\pm[X_{30} + \sqrt{10}X_1 + \sqrt{6}(X_{15} + X_{31} + X_{40})]$	928	11
48, 49	1001010	$8, 11, 15, 22, 17, 12, 6$	$\pm[X_{28} + 2(X_{14} + X_{26}) + \sqrt{3}(X_{18} + X_{29} + X_{30} - X_{31})]$	672	11
50	1001010	$8, 11, 15, 22, 17, 12, 6$	$X_{28} + 2(X_{14} + X_{26}) + \sqrt{6}(X_{18} + X_{36})$	672	7
51, 52	2001010	$10, 13, 18, 26, 20, 14, 7$	$\pm[X_{18} - X_{30} + \sqrt{10}X_1 + \sqrt{6}(X_{28} + X_{29} + X_{31})]$	960	13
53	2001010	$10, 13, 18, 26, 20, 14, 7$	$X_{18} - X_{30} + \sqrt{10}X_1 + \sqrt{6}(X_{28} + X_{29} + X_{31})$	960	7
54	0002000	$8, 12, 16, 24, 18, 12, 6$	$2(X_4 + X_{16}) + \sqrt{2}(X_{15} + X_{21}) + \sqrt{6}(X_{24} + X_{39})$	768	13
55, 56	2000200	$10, 13, 18, 26, 21, 14, 7$	$\pm[X_5 - X_{18} + X_{30} + \sqrt{10}X_1 \sqrt{6}(X_{28} + X_{29} + X_{31})]$	992	25
57, 58	2000200	$10, 13, 18, 26, 21, 14, 7$	$\pm[X_5 + X_{18} + X_{30} + \sqrt{10}X_1 \sqrt{6}(X_{28} + X_{29} + X_{31})]$	992	13
59	1001020	$10, 14, 19, 28, 22, 16, 8$	$3X_{28} + 2\sqrt{2}(X_6 + X_{19}) + \sqrt{5}(X_{14} + X_{26})$	1120	7
60, 61	1001012	$10, 14, 19, 28, 22, 16, 9$	$\pm[3X_7 + X_{28} + 2\sqrt{2}(X_{18} + X_{29}) + \sqrt{2}(X_{18} + X_{29}) + \sqrt{5}(X_{14} + X_{26})]$	1152	7
62	0020020	$10, 14, 20, 28, 22, 16, 8$	$X_3 - 3X_{28} + 2\sqrt{2}(X_6 + X_{19}) + \sqrt{5}(X_{14} + X_{26})$	1152	11

Table 4.
(continued)

	$\alpha_i(H)$	k_i	E	tr	inv
63	0020020	$10, 14, 20, 28, 22, 16, 8$	$X_3 + 3X_{28} + 2\sqrt{2}(X_6 + X_{19}) + \sqrt{5}(X_{14} + X_{26})$	1152	13
64, 65	0110102	$10, 15, 20, 29, 23, 16, 9$	$\pm[X_{13} - X_{15} - 3X_{23} + X_{24} + 2\sqrt{2}(X_{17} - X_7) + \sqrt{5}(X_{20} + X_{27})]$	1216	11
66	2020020	$14, 18, 26, 36, 28, 20, 10$	$\sqrt{2}(2X_3 + 3X_{28}) + \sqrt{14}X_1 + \sqrt{10}(X_6 + X_{19})$	1920	11
67, 68	0002002	$10, 15, 20, 30, 23, 16, 9$	$\pm[3X_7 + \sqrt{5}(X_4 + X_{16}) + \sqrt{2}(X_{15} + X_{21} + 2X_{24} + 2X_{33})]$	1248	11
69, 70	0002002	$10, 15, 20, 30, 23, 16, 9$	$\pm[2X_4 + X_{27} - \sqrt{10}X_{17} + \sqrt{6}(X_{13} + X_{20} - X_{23}) + \sqrt{3}(X_{26} - X_{19})]$	1248	21
71	0002020	$12, 18, 24, 36, 28, 20, 10$	$2\sqrt{3}(X_{15} + X_{21}) + \sqrt{6}(X_4 + X_{16}) + \sqrt{10}(X_6 + X_{19})$	1792	9
72, 73	2110102	$14, 19, 26, 37, 29, 20, 11$	$\pm[X_{13} + X_{23} + \sqrt{2}(2X_{15} + 3X_{24}) + \sqrt{10}(X_{16} - X_7) + \sqrt{14}X_1]$	1984	9
74, 75	2110110	$14, 19, 26, 37, 29, 20, 10$	$\pm[X_{16} + \sqrt{2}(3X_{15} + 2X_{17}) + \sqrt{10}(X_{12} + X_{25}) + \sqrt{14}X_1]$	1952	7
76, 77	2002002	$14, 19, 26, 38, 29, 20, 11$	$\pm[X_{10} + X_{13} + X_{29} + \sqrt{14}X_8 + \sqrt{2}(3X_9 + 2X_{18}) + \sqrt{10}(X_{17} - X_{19})]$	2016	9
78, 79	2002002	$14, 19, 26, 38, 29, 20, 11$	$\pm[X_4 + X_{13} + \sqrt{14}X_1 + X_{23} + \sqrt{2}(2X_{15} + 3X_{24}) + \sqrt{10}(X_7 - X_{16})]$	2016	17
80	2002020	$16, 22, 30, 44, 34, 24, 12$	$4X_1 + 2\sqrt{3}(X_6 + X_{19}) + \sqrt{7}(X_4 + X_{16}) + \sqrt{15}(X_{15} + X_{17})$	2688	9
81	2002020	$16, 22, 30, 44, 34, 24, 12$	$4X_1 + 2\sqrt{3}(X_{12} + X_{13}) + \sqrt{7}(X_4 + X_{16}) + \sqrt{15}(X_{10} + X_{22})$	2688	7
82, 83	2110122	$18, 25, 34, 49, 39, 28, 15$	$\pm[3\sqrt{2}X_1 + 2\sqrt{6}X_{17} + 2\sqrt{7}X_6 + \sqrt{10}X_{15} + \sqrt{15}(X_7 + X_{16})]$	3520	7

Table 4.
(continued)

	$\alpha_i(H)$	k_i	E	tr	inv
84	2022020	$22,30,42,60,$ $46,32,16$	$4(X_6+X_{19})+\sqrt{22}X_1$ $+\sqrt{30}(X_4+X_{16})+\sqrt{42}X_3$	4992	7
$85,86$	2002022	$18,25,34,50,$ $39,28,15$	$\pm[X_{11}+3\sqrt{2}X_1+2\sqrt{6}X_{10}$ $+2\sqrt{7}X_{12}+\sqrt{10}X_{22}$ $+\sqrt{15}(X_7+X_9)]$	3552	7
$87,88$	2002022	$18,25,34,50,$ $39,28,15$	$\pm[-X_{11}+3\sqrt{2}X_1+2\sqrt{6}X_{10}$ $+2\sqrt{7}X_{12}+\sqrt{10}X_{22}$ $+\sqrt{15}(X_7+X_9)]$	3552	13
$89,90$	2220202	$22,31,42,60,$ $47,32,17$	$\pm[-X_2-4X_5+4X_7$ $+X_{12}+X_{13}+\sqrt{22}X_1$ $+\sqrt{30}(X_9+X_{18})+\sqrt{42}X_3]$	5088	11
$91,92$	2220222	$26,37,50,72,$ $57,40,21$	$\pm[\sqrt{26}X_1+5\sqrt{2}X_3$ $+2\sqrt{10}X_6+\sqrt{21}X_7$ $+\frac{1}{\sqrt{5}}(\sqrt{33}X_5-\sqrt{77}X_2$ $-6\sqrt{3}X_9-6\sqrt{7}X_{11})]$	7392	9
$93,94$	2222222	$34,49,66,96,$ $75,52,27$	$\pm[\sqrt{34}X_1+7X_2+\sqrt{66}X_3$ $+4\sqrt{6}X_4+\sqrt{3}(5X_5+3X_7)$ $+2\sqrt{13}X_6]$	12768	7

Table 5.
Cayley triples in EVI $= E_{7(-5)}$.

	$\alpha_i(H)$	k_i	E	tr	inv
1	1000000	$2,2,3,4,3,2,1$	X_{63}	32	37
2	0000010	$2,3,4,6,5,4,2$	$X_{49}-X_{63}$	48	33
3	0000010	$2,3,4,6,5,4,2$	$X_{49}+X_{63}$	80	25
4	0010000	$3,4,6,8,6,4,2$	$X_{37}-X_{55}+X_{61}$	48	37
5	0010000	$3,4,6,8,6,4,2$	$X_{37}+X_{55}+X_{61}$	112	21
6	2000000	$4,4,6,8,6,4,2$	$X_1-X_{37}+X_{55}-X_{61}$	32	67
7	2000000	$4,4,6,8,6,4,2$	$X_1+X_{37}+X_{55}-X_{61}$	128	37
8	2000000	$4,4,6,8,6,4,2$	$X_1+X_{37}+X_{55}+X_{61}$	160	35
9	1000010	$4,5,7,10,8,6,3$	$X_{27}+X_{39}+X_{49}+X_{53}-X_{54}$	192	19
10	0001000	$4,6,8,12,9,6,3$	$X_{28}+X_{38}+X_{46}$ $+X_{47}-X_{48}+X_{49}$	240	19
11	0001000	$4,6,8,12,9,6,3$	$-X_{28}+X_{38}+X_{46}$ $+X_{47}-X_{48}+X_{49}$	208	15
12	2000010	$6,7,10,14,11,8,4$	$\sqrt{3}(X_1-X_{37})+2X_{49}$	272	21
13	2000010	$6,7,10,14,11,8,4$	$\sqrt{3}(X_1+X_{37})+2X_{49}$	368	21
14	0000020	$4,6,8,12,10,8,4$	$\frac{1}{\sqrt{2}}[X_6+X_{12}+X_{13}+X_{19}$ $+X_{35}+X_{40}+X_{41}-X_{45}$ $+i(X_{38}+X_{42}+X_{43}-X_{44}$ $-X_{46}+X_{47}+X_{48}+X_{51})]$	320	49
15	0000020	$4,6,8,12,10,8,4$	$\sqrt{2}(X_6+X_{19}+X_{50}+X_{56})$	320	25
16	0010010	$5,7,10,14,11,8,4$	$\sqrt{2}(X_{24}+X_{36}+X_{38}-X_{48})$ $+X_{37}$	352	13
17	1001000	$6,8,11,16,12,8,4$	$X_{28}+2X_{49}+\sqrt{3}(X_{14}-X_{26})$	432	21
18	1001000	$6,8,11,16,12,8,4$	$X_{28}-2X_{49}+\sqrt{3}(X_{14}-X_{26})$	368	13
19	0020000	$6,8,12,16,12,8,4$	$-X_3+X_{28}+2X_{49}$ $+\sqrt{3}(X_{14}-X_{26})$	480	39
20	0020000	$6,8,12,16,12,8,4$	$X_3-X_{28}+2X_{49}$ $+\sqrt{3}(X_{14}-X_{26})$	352	23
21	0020000	$6,8,12,16,12,8,4$	$X_3+X_{28}+2X_{49}$ $+\sqrt{3}(X_{14}-X_{26})$	448	21
22	2020000	$10,12,18,24,$ $18,12,6$	$\sqrt{10}X_1+\sqrt{6}(X_3-X_{28}+X_{49})$	608	37
23	2020000	$10,12,18,24,$ $18,12,6$	$\sqrt{10}X_1+\sqrt{6}(X_3+X_{28}+X_{49})$	992	21
24	0001010	$6,9,12,18,14,10,5$	$X_{29}-X_{31}+2X_{37}+iX_{27}$ $-X_{39}+\sqrt{3}(X_{18}+X_{30})$	528	11

Table 5.
(continued)

	$\alpha_i(H)$	k_i	E	tr	inv
25	2000020	$8,10,14,20,$ $16,12,6$	$2(X_1-X_{37})$ $+\sqrt{3}(X_6+X_{19}+X_{40}+X_{41})$	672	33
26	2000020	$8,10,14,20,$ $16,12,6$	$2(X_1+X_{37})$ $+\sqrt{3}(X_6+X_{19}+X_{40}+X_{41})$	800	17
27	1001010	$8,11,15,22,$ $17,12,6$	$X_{28}+2(X_{14}-X_{26})$ $+\sqrt{3}(X_{18}+X_{29}+X_{30}-X_{31})$	832	9
28	2001010	$10,13,18,26,$ $20,14,7$	$\sqrt{10}X_1+X_{18}+X_{30}$ $+\sqrt{6}(X_{28}+X_{29}-X_{31})$	1072	11
29	0002000	$8,12,16,24,$ $18,12,6$	$X_4+X_{15}+X_{16}+X_{17}$ $+X_{18}+X_{29}+X_{30}-X_{31}$ $+\sqrt{2}(X_9+X_{10}+X_{11}-X_{22})$ $+2i(X_{27}-X_{39})$	960	27
30	1001020	$10,14,19,28,$ $22,16,8$	$3X_{28}+2\sqrt{2}(X_6+X_{19})$ $+\sqrt{5}(X_{14}-X_{26})$	1328	13
31	0020020	$10,14,20,28,$ $22,16,8$	$X_3+2\sqrt{2}(X_6+X_{19})$ $+3X_{28}+\sqrt{5}(X_{14}-X_{26})$	1408	13
32	0020020	$10,14,20,28,$ $22,16,8$	$-X_3+2\sqrt{2}(X_6+X_{19})$ $+3X_{28}+\sqrt{5}(X_{14}-X_{26})$	1312	23
33	2020020	$14,18,26,36,$ $28,20,10$	$\sqrt{14}X_1+\sqrt{2}(2X_3+3X_{28})$ $+\sqrt{10}(X_6+X_{19})$	2272	13
34	2020020	$14,18,26,36,$ $28,20,10$	$\sqrt{14}X_1+\sqrt{2}(3X_{28}-2X_3)$ $+\sqrt{10}(X_6+X_{19})$	1888	21
35	0002020	$12,18,24,36,$ $28,20,10$	$\sqrt{6}(X_4+X_{15}+X_{16}$ $+X_{17}+iX_{20}-iX_{21})$ $+\sqrt{10}(X_6+X_{19})$	2240	19
36	2002020	$16,22,30,44,$ $34,24,12$	$4X_1+2\sqrt{3}(X_6+X_{19})$ $+\sqrt{7}(X_4+X_{16})$ $+\sqrt{15}(X_{15}+X_{17})$	3232	15
37	2022020	$22,30,42,60,$ $46,32,16$	$4(X_6+X_{19})+\sqrt{22}X_1$ $+\sqrt{42}X_3+\sqrt{30}(X_4+X_{16})$	5728	13

Table 6.
Cayley triples in EVII $= E_{7(-25)}$.

	$\alpha_i(H)$	k_i	E	tr	inv
$1,2$	1000000	$2,2,3,4,3,2,1$	$\pm X_{63}$	32	47
$3,4$	0000010	$2,3,4,6,5,4,2$	$\pm(X_{49}-X_{63})$	32	47
5	0000010	$2,3,4,6,5,4,2$	$X_{49}+X_{63}$	96	31
$6,7$	0000002	$2,3,4,6,5,4,3$	$\pm(X_7-X_{49}-X_{63})$	0	79
$8,9$	0000002	$2,3,4,6,5,4,3$	$\pm(X_7+X_{49}+X_{63})$	128	47
10	2000000	$4,4,6,8,6,4,2$	$X_1+X_{37}+X_{55}+X_{61}$	192	37
$11,12$	1000010	$4,5,7,10,8,$ $6,3$	$\pm(X_{27}+X_{39}+X_{49}$ $+X_{53}-X_{54})$	224	21
$13,14$	2000010	$6,7,10,14,$ $11,8,4$	$\pm[2X_{49}+\sqrt{3}(X_1+X_{37})]$	416	31
15	0000020	$4,6,8,12,$ $10,8,4$	$\sqrt{2}(X_6+X_{34}+X_{40}+X_{56})$	384	31
$16,19$	2000002	$6,7,10,14,$ $11,8,5$	$\pm[-X_7+2X_{49}+\sqrt{3}(X_1+X_{37})]$	384	31
$17,18$	2000020	$6,7,10,14,$ $11,8,5$	$\pm[X_7+2X_{49}+\sqrt{3}(X_1+X_{37})]$	512	47
20	2000020	$8,10,14,20,$ $16,12,6$	$2(X_1+X_{37})$ $+\sqrt{3}(X_6+X_{19}+X_{40}+X_{41})$	960	19
$21,22$	2000022	$10,13,18,26,$ $21,16,9$	$\pm[\sqrt{5}(X_1+X_{37})$ $+2\sqrt{2}(X_6+X_{40})+3X_7]$	1536	31

9. An elaborate example.

Let $\mathfrak{g}$ be of type $E_{7(-5)}$ and $\mathcal{O}^c$ the complex nilpotent orbit of $\mathfrak{g}^c$ with characteristic 0000020, i.e., $H = 2H_6$. Then $\mathcal{O}^c \cap \mathfrak{g}$ is the union of two G-orbits, namely orbits No. 14 and 15 in Table 5.

The characteristic H defines a gradation of $\mathfrak{g}^c$ (and $\mathfrak{g}$) such that

$$\mathfrak{g}^c = \bigoplus_{k=-2}^{2} \mathfrak{g}(2k)^c.$$

The subspace $\mathfrak{g}(0)^c$ is a reductive subalgebra with 1-dimensional center and the derived subalgebra of type $A_1 + D_5$. The derived subgroup of $G(0)^c$ is isomorphic to

$$(\mathrm{Spin}_{10} \times \mathrm{SL}_2)/Z_2 \ .$$

The dimensions of the spaces $\mathfrak{g}(2k)^c$ are as follows:

$$\begin{aligned}
&\dim \mathfrak{g}(0)^c = 49 \ ,\\
&\dim \mathfrak{g}(-2)^c = \dim \mathfrak{g}(2)^c = 32 \ ,\\
&\dim \mathfrak{g}(-4)^c = \dim \mathfrak{g}(4)^c = 10 \ .
\end{aligned}$$

The subspace $\mathfrak{g}(2)^c$ has a basis consisting of the root vectors X_k, $k \in I$, where $I = I_1 \cup I_2$ and

$$I_1 = \{6, 12, 18, 23, 24, 27, 29, 33, 35, 38, 40, 42, 43, 46, 50, 53\},$$

$$I_2 = \{13, 19, 25, 30, 31, 34, 36, 39, 41, 44, 45, 47, 48, 51, 54, 56\}.$$

As a $G(0)^c$-module, $\mathfrak{g}(2)^c$ is the tensor product of a half-spin module of Spin_{10} and the 2-dimensional simple module of SL_2. The subspace of $\mathfrak{g}(2)^c$ spanned by the vectors X_k with $k \in I_1$ (or $k \in I_2$) is a half-spin module of Spin_{10}.

We find several subsets $J \subset I$ such that the subspace of $\mathfrak{g}(2)^c$ spanned by X_k with $k \in J$ is σ-stable, for $k, j \in J$ the difference $\alpha_k - \alpha_j$ is not a root, and H belongs to the subspace spanned by all coroots H_k with $k \in J$. All of them have size 4. For instance, the set $J = \{6, 19, 50, 56\}$ satisfies all the conditions mentioned above. As $\sigma(X_6) = X_{19}$ and $\sigma(X_{50}) = X_{56}$ (see Table 2), the vector

$$E = (aX_6 + \bar{a}X_{19}) + (bX_{50} + \bar{b}X_{56})$$

is real in the sense that $\sigma(E) = E$, i.e., $E \in \mathfrak{g}(2)$.

The equation $[F, E] = H$, where $F = \theta(E) = \sigma_u(E)$, implies that $|a|^2 = |b|^2 = 2$. All possible choices for a and b produce an element E belonging to the orbit 15. This is established by computing the invariant inv which turns out to be 25 in all cases. The other sets J also produce only representatives for the orbit 15.

In order to find a representative for the orbit 14 we had to undertake the following tedious calculation. Since $E \in \mathfrak{g}(2)$, i.e., $E \in \mathfrak{g}(2)^c$ and $\sigma(E) = E$, the representative E must have the form

$$\begin{aligned} E = {} & (aX_6 + \bar{a}X_{19}) + (bX_{12} + \bar{b}X_{13}) + (cX_{18} + \bar{c}X_{30}) \\ & + (dX_{23} - \bar{d}X_{25}) + (eX_{24} + \bar{e}X_{36}) + (fX_{27} + \bar{f}X_{39}) \\ & + (gX_{29} - \bar{g}X_{31}) + (hX_{33} - \bar{h}X_{34}) + (\alpha X_{35} - \bar{\alpha}X_{45}) \\ & + (\beta X_{38} - \bar{\beta}X_{48}) + (\gamma X_{40} + \bar{\gamma}X_{41}) + (\delta X_{42} - \bar{\delta}X_{51}) \\ & + (\varepsilon X_{43} + \bar{\varepsilon}X_{44}) + (\zeta X_{46} + \bar{\zeta}X_{47}) + (\eta X_{50} + \bar{\eta}X_{56}) + (\theta X_{53} - \bar{\theta}X_{54}) \end{aligned}$$

where $a, b, \ldots, \theta$ are some complex numbers.

Since $F = \theta(E) = \sigma_u(E)$, we must have

$$\begin{aligned} F = {} & (\bar{a}X_{-6} + aX_{-19}) + (\bar{b}X_{-12} + bX_{-13}) + (\bar{c}X_{-18} + cX_{-30}) \\ & + (\bar{d}X_{-23} - dX_{-25}) + (\bar{e}X_{-24} + eX_{-36}) + (\bar{f}X_{-27} + fX_{-39}) \\ & + (\bar{g}X_{-29} - gX_{-31}) + (\bar{h}X_{-33} - hX_{-34}) + (\bar{\alpha}X_{-35} - \alpha X_{-45}) \\ & + (\bar{\beta}X_{-38} - \beta X_{-48}) + (\bar{\gamma}X_{-40} + \gamma X_{-41}) + (\bar{\delta}X_{-42} - \delta X_{-51}) \\ & + (\bar{\varepsilon}X_{-43} + \varepsilon X_{-44}) + (\bar{\zeta}X_{-46} + \zeta X_{-47}) + (\bar{\eta}X_{-50} + \eta X_{-56}) \\ & + (\bar{\theta}X_{-53} - \theta X_{-54}). \end{aligned}$$

Next we have computed the bracket $[F, E]$ by using the above expressions for E and F and the list of the structure constants given in the Appendix. As $[F, E] = H$ where

$$H = 4H_1 + 6H_2 + 8H_3 + 12H_4 + 10H_5 + 8H_6 + 4H_7 \ ,$$

we obtain the following system of equations.

$$\begin{aligned}
&\mathrm{Re}\,(e\bar{f} + g\bar{h} + \alpha\bar{\beta} + \gamma\bar{\varepsilon}) = 0\\
&\mathrm{Re}\,(c\bar{e} + d\bar{g} + \beta\bar{\delta} + \varepsilon\bar{\zeta}) = 0\\
&\mathrm{Re}\,(c\bar{f} + d\bar{h} - \alpha\bar{\delta} - \gamma\bar{\zeta}) = 0\\
&\mathrm{Re}\,(-a\bar{\alpha} + b\bar{\gamma} + f\bar{\eta} + h\bar{\theta}) = 0\\
&\mathrm{Re}\,(a\bar{\beta} - b\bar{\varepsilon} + e\bar{\eta} + g\bar{\theta}) = 0\\
&\mathrm{Re}\,(-a\bar{\delta} + b\bar{\zeta} + c\bar{\eta} + d\bar{\theta}) = 0
\end{aligned}$$

$$\begin{aligned}
&\bar{b}c - a\bar{d} + \bar{g}\alpha + \bar{h}\beta + e\bar{\gamma} + f\bar{\varepsilon} + \bar{\zeta}\eta + \delta\bar{\theta} = 0\\
&a\bar{c} + \bar{b}d - \bar{e}\alpha - \bar{f}\beta + g\bar{\gamma} + h\bar{\varepsilon} + \bar{\zeta}\theta - \delta\bar{\eta} = 0\\
&a\bar{g} - \bar{b}e + c\bar{\gamma} + \bar{d}\alpha - f\bar{\zeta} - \bar{h}\delta + \beta\bar{\theta} + \bar{\varepsilon}\eta = 0\\
&a\bar{h} - \bar{b}f + c\bar{\varepsilon} + \bar{d}\beta + e\bar{\zeta} + \bar{g}\delta - \alpha\bar{\theta} - \bar{\gamma}\eta = 0\\
&a\bar{e} + \bar{b}g + \bar{c}\alpha - d\bar{\gamma} - \bar{f}\delta + h\bar{\zeta} + \beta\bar{\eta} - \bar{\varepsilon}\theta = 0\\
&a\bar{f} + \bar{b}h + \bar{c}\beta - d\bar{\varepsilon} + \bar{e}\delta - g\bar{\zeta} - \alpha\bar{\eta} + \bar{\gamma}\theta = 0
\end{aligned}$$

$$\begin{aligned}
&|f|^2 + |h|^2 + |\beta|^2 + |\delta|^2 + |\varepsilon|^2 + |\zeta|^2 + |\theta|^2 + |\eta|^2 = 2\\
&|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 + |g|^2 + |\alpha|^2 + |\gamma|^2 = 2\\
&|e|^2 + |g|^2 + |\alpha|^2 + |\gamma|^2 + |\delta|^2 + |\zeta|^2 + |\theta|^2 + |\eta|^2 = 2\\
&|c|^2 + |d|^2 + |e|^2 + |g|^2 + |\theta|^2 + |\eta|^2 + 2|\alpha|^2 + 2|\gamma|^2 = 2 + |f|^2 + |h|^2.
\end{aligned}$$

After some judicious specializations and experimentations using the Maple's *solve* routine, we have found the following solution of the above system:

$$\begin{aligned}
&a = b = 1/\sqrt{2}\ ,\ c = d = e = f = g = h = 0\ ,\\
&\alpha = \gamma = 1/\sqrt{2}\ ,\ \beta = \delta = \varepsilon = i/\sqrt{2}\ ,\\
&\zeta = -i/\sqrt{2}\ ,\ \eta = \theta = 0\ .
\end{aligned}$$

This means that we have a real Cayley triple (E, H, F) with

$$\begin{aligned}
E = \frac{1}{\sqrt{2}}[&X_6 + X_{12} + X_{13} + X_{19} + X_{35} + X_{40} + X_{41} - X_{45}\\
&+ i(X_{38} + X_{42} + X_{43} - X_{44} - X_{46} + X_{47} + X_{48} + X_{51})].
\end{aligned}$$

For this triple we find that inv $= 49$, and so E is a representative of the orbit 14.

10. Appendix.

We list here the nonzero structure constants $N(i, j)$ of E_7 for $i > 0$ and j arbitrary. The i-th entry in this list contains two sequences separated by a semicolon. The first (resp., second) sequence consists of those j for which $N(i, j)$ is $+1$ (resp., -1). Note that each of these sequences has length 16.

The nonzero structure constants of E_7

1 3,10,15,17,22,24,28,29,31,35,36,40,41,45,49,62;
-8,-14,-20,-21,-26,-27,-32,-33,-34,-38,-39,-43,-44,-48,-52,-63

2 10,11,14,17,18,21,24,25,27,31,34,50,54,57,59,-9;
4,-15,-16,-20,-22,-23,-26,-29,-30,-33,-36,-39,-53,-56,-58,-60

3 9,11,16,18,23,25,30,32,38,43,44,48,52,61,-8,-10;
1,4,-15,-17,-22,-24,-29,-31,-36,-37,-42,-46,-47,-51,-55,-62

4 2,3,5,8,12,19,22,26,29,33,36,39,46,51,55,60;
-9,-10,-11,-14,-18,-25,-28,-32,-35,-38,-41,-44,-50,-54,-57,-61

5 9,10,13,14,15,20,35,38,41,42,44,47,57,58,-11,-12;
4,6,-16,-17,-19,-21,-22,-26,-40,-43,-45,-46,-48,-51,-59,-60

6 5,7,11,16,17,21,22,26,28,32,37,45,48,51,54,56;
-12,-13,-18,-23,-24,-27,-29,-33,-35,-38,-42,-49,-52,-55,-57,-58

7 12,18,23,24,27,29,33,35,38,40,42,43,46,50,53,-13;
6,-19,-25,-30,-31,-34,-36,-39,-41,-44,-45,-47,-48,-51,-54,-56

8 9,11,16,18,23,25,30,61,-1,-14,-37,-42,-46,-47,-51,-55;
4,28,35,40,41,45,49,-3,-20,-21,-26,-27,-33,-34,-39,-63

9 17,21,24,27,31,34,59,-4,-15,-16,-20,-23,-30,-53,-56,-58;
3,5,8,12,19,46,51,55,-2,-28,-32,-35,-38,-41,-44,-61

10 16,23,30,43,48,52,-4,-14,-15,-17,-24,-31,-37,-42,-47,-62;
1,2,5,12,19,26,33,39,60,-3,-28,-35,-41,-50,-54,-57

11 13,15,20,42,47,58,-4,-16,-17,-18,-21,-40,-43,-45,-48,-59;
2,3,6,8,29,33,36,39,55,-5,-25,-28,-32,-50,-54,-61

12 9,10,14,15,20,41,44,47,-6,-18,-19,-40,-43,-46,-59,-60;
4,7,28,32,37,54,56,-5,-23,-24,-27,-29,-33,-49,-52,-55

13 40,43,46,50,53,-6,-19,-25,-30,-31,-34,-36,-39,-41,-44,-47;
5,11,16,17,21,22,26,28,32,37,-7,-49,-52,-55,-57,-58

14 16,22,23,29,30,36,-1,-4,-20,-21,-27,-34,-50,-54,-57,-63;
2,5,12,19,40,45,49,60,-8,-10,-32,-37,-38,-42,-44,-47

15 21,27,34,43,48,52,59,-2,-3,-20,-22,-28,-29,-35,-36,-41;
1,5,11,12,18,19,25,-9,-10,-37,-42,-47,-53,-56,-58,-62

16 13,24,27,31,34,42,47,-2,-5,-22,-23,-26,-28,-32,-60,-61;
3,6,8,10,14,55,57,-9,-11,-30,-40,-43,-45,-48,-53,-56

17 13,20,23,30,52,58,-3,-5,-21,-22,-24,-28,-46,-50,-51,-54;
1,2,6,9,33,38,39,44,-10,-11,-31,-37,-40,-45,-59,-62

18 15,20,22,26,47,51,-4,-6,-23,-24,-25,-27,-49,-50,-52,-61;
2,3,7,8,36,37,39,56,-11,-12,-35,-38,-40,-43,-57,-59

19 9,10,14,15,20,50,53,-5,-7,-25,-45,-48,-49,-51,-52,-55;
4,28,32,35,37,38,42,-12,-13,-30,-31,-34,-36,-39,-59,-60

20 59,-1,-2,-8,-26,-32,-33,-37,-38,-39,-42,-44,-47,-53,-56,-58;
5,11,12,17,18,19,24,25,31,40,45,49,-9,-14,-15,-63

21 13,23,29,30,35,36,41,58,-1,-5,-8,-26,-27,-32,-37,-59;
2,6,9,15,49,-11,-14,-17,-34,-43,-46,-48,-50,-51,-54,-63

22 13,27,34,52,-2,-3,-5,-26,-28,-29,-37,-40,-45,-53,-56,-62;
1,4,6,14,18,25,38,44,57,-15,-16,-17,-36,-46,-51,-60

23 31,34,47,51,54,-2,-6,-12,-29,-30,-33,-35,-38,-40,-43,-53;
3,7,8,10,14,17,21,37,-9,-16,-18,-49,-52,-58,-60,-61

24 20,26,30,32,-3,-6,-12,-27,-29,-31,-35,-40,-55,-57,-59,-62;
1,2,7,9,16,39,44,48,56,-10,-17,-18,-42,-46,-49,-50

25 15,20,22,26,29,33,53,-4,-7,-11,-30,-31,-34,-54,-57,-59;
2,3,8,37,42,46,-13,-18,-19,-41,-44,-45,-48,-49,-52,-61

26 10,13,35,41,-1,-2,-5,-8,-32,-33,-43,-46,-48,-51,-60,-63;
4,6,18,24,25,31,49,57,-16,-20,-21,-22,-37,-39,-53,-56

27 30,36,41,45,-1,-6,-8,-12,-33,-34,-38,-42,-43,-46,-50,-63;
2,7,9,15,16,22,28,56,-14,-18,-21,-24,-52,-55,-57,-59

28 8,12,13,19,27,33,34,39,52,55,-4,-9,-10,-11,-32,-35;
1,6,-15,-16,-17,-22,-37,-40,-41,-45,-50,-53,-54,-56,-61,-62

29 11,32,34,54,-2,-3,-6,-12,-33,-35,-36,-42,-46,-49,-58,-60;
1,4,7,14,21,25,44,48,-15,-22,-23,-24,-40,-53,-55,-62

30 -2,-7,-16,-19,-36,-39,-41,-44,-45,-48,-49,-52,-56,-58,-60,-61;
3,8,10,14,17,21,24,27,37,42,46,50,-9,-13,-23,-25

31 20,26,32,33,38,43,53,-3,-7,-17,-19,-34,-36,-41,-45,-49;
1,2,9,16,23,-10,-13,-24,-25,-47,-51,-54,-55,-57,-59,-62

32 12,13,19,55,-1,-4,-9,-11,-14,-37,-38,-50,-53,-54,-56,-61;
3,6,24,29,31,36,49,-16,-20,-21,-26,-28,-43,-44,-48,-63

33 10,11,17,41,45,54,-1,-2,-6,-8,-12,-38,-39,-52,-53,-55;
4,7,25,28,31,-20,-23,-26,-27,-29,-42,-43,-46,-58,-60,-63

34 53,-1,-7,-8,-19,-21,-39,-44,-47,-48,-51,-52,-54,-55,-57,-59;
2,9,15,16,22,23,28,29,35,40,-13,-14,-25,-27,-31,-63

35 8,19,34,39,-4,-6,-9,-10,-18,-38,-40,-41,-50,-53,-61,-62;
1,5,7,21,26,48,51,-15,-23,-24,-28,-29,-42,-49,-57,-58

36 11,18,32,38,43,-2,-3,-7,-19,-22,-39,-41,-47,-51,-55,-62;
1,4,14,21,27,50,-13,-15,-29,-30,-31,-45,-49,-56,-58,-60

37 12,13,18,19,23,25,30,-3,-14,-15,-17,-26,-28,-42,-62,-63;
6,49,52,-8,-10,-20,-21,-22,-32,-46,-47,-50,-51,-53,-54,-56

38 17,19,22,45,-1,-4,-6,-9,-14,-18,-42,-43,-44,-57,-58,-63;
3,5,7,31,36,51,-20,-23,-27,-32,-33,-35,-50,-52,-53,-61

39 10,11,17,18,24,-1,-2,-7,-8,-19,-26,-44,-56,-58,-60,-63;
4,28,35,40,50,-13,-20,-30,-33,-34,-36,-47,-48,-51,-52,-55

40 8,14,20,34,39,44,47,-5,-16,-17,-18,-28,-29,-43,-45,-49;
1,7,13,-11,-12,-22,-23,-24,-35,-46,-50,-53,-59,-60,-61,-62

41 8,43,46,-4,-7,-9,-10,-25,-28,-44,-45,-49,-54,-56,-57,-58;
1,5,12,21,26,27,33,-13,-15,-30,-31,-35,-36,-47,-61,-62

42 19,25,30,45,48,-3,-6,-14,-15,-24,-33,-35,-46,-47,-50,-53;
5,7,11,16,-8,-10,-20,-27,-29,-37,-38,-55,-57,-58,-62,-63

43 47,-1,-5,-16,-18,-21,-32,-33,-46,-48,-50,-52,-53,-59,-60,-61;
3,7,10,13,15,31,36,41,-11,-12,-23,-26,-27,-38,-40,-63

44 17,22,24,29,46,-1,-4,-7,-9,-14,-25,-32,-47,-48,-52,-61;
3,5,12,40,-13,-20,-30,-34,-38,-39,-41,-54,-56,-57,-58,-63

45 8,14,20,-5,-7,-16,-17,-25,-28,-36,-48,-49,-59,-60,-61,-62;
1,6,27,33,38,42,-11,-19,-22,-30,-31,-40,-41,-51,-54,-56

46 9,25,30,-3,-5,-21,-22,-24,-33,-37,-40,-50,-51,-55,-62,-63;
4,7,13,41,44,-8,-12,-17,-26,-27,-29,-42,-43,-53,-59,-60

47 -3,-7,-14,-15,-31,-37,-39,-41,-51,-54,-55,-56,-57,-58,-62,-63;
5,11,12,16,18,23,40,43,-8,-10,-13,-20,-34,-36,-42,-44

48 24,29,35,-1,-5,-7,-16,-21,-25,-32,-39,-51,-52,-54,-56,-63;
3,6,10,15,42,-11,-19,-26,-30,-34,-43,-44,-45,-59,-60,-61

49 8,14,20,21,26,32,37,-6,-12,-13,-23,-24,-25,-35,-36,-52;
1,-18,-19,-29,-30,-31,-40,-41,-45,-55,-57,-58,-59,-60,-61,-62

50 30,36,39,-4,-10,-11,-21,-24,-28,-37,-38,-40,-53,-54,-57,-59;
2,7,13,19,-14,-17,-18,-27,-32,-35,-42,-43,-46,-61,-62,-63

51 9,35,38,-3,-5,-7,-21,-22,-31,-37,-39,-45,-54,-55,-59,-60;
4,6,18,23,-8,-17,-19,-26,-34,-36,-46,-47,-48,-56,-62,-63

52 37,-1,-6,-12,-13,-23,-25,-27,-38,-39,-55,-57,-58,-59,-60,-61;
3,10,15,17,22,28,-18,-19,-30,-33,-34,-43,-44,-48,-49,-63

53 -2,-15,-16,-26,-28,-29,-37,-38,-40,-46,-56,-58,-60,-61,-62,-63;
7,13,19,25,31,34,-9,-20,-22,-23,-32,-33,-35,-42,-43,-50

54 12,-4,-7,-10,-11,-21,-28,-31,-37,-44,-45,-56,-57,-61,-62,-63;
2,6,23,29,33,-14,-17,-25,-32,-34,-41,-47,-48,-50,-51,-59

55 9,11,16,-3,-6,-12,-13,-27,-29,-31,-39,-42,-49,-57,-62,-63;
4,28,32,-8,-19,-24,-33,-34,-36,-46,-47,-51,-52,-58,-59,-60

56 12,18,24,27,-2,-7,-15,-16,-26,-28,-36,-37,-44,-45,-51,-58;
6,-9,-20,-22,-30,-32,-39,-41,-47,-48,-53,-54,-60,-61,-62,-63

57 16,22,26,-4,-6,-10,-13,-18,-27,-31,-35,-42,-44,-49,-58,-59;
2,5,-14,-24,-25,-34,-38,-41,-47,-50,-52,-54,-55,-61,-62,-63

58 -2,-6,-13,-15,-23,-33,-35,-36,-42,-44,-49,-55,-60,-61,-62,-63;
5,11,17,21,-9,-20,-29,-30,-38,-39,-41,-47,-52,-53,-56,-57

59 -5,-17,-18,-19,-27,-31,-40,-46,-48,-49,-54,-55,-60,-61,-62,-63;
2,9,15,20,-11,-12,-21,-24,-25,-34,-43,-45,-50,-51,-52,-57

60 10,14,-2,-5,-19,-22,-23,-33,-36,-40,-46,-48,-49,-55,-56,-61;
4,-12,-16,-26,-29,-30,-39,-43,-45,-51,-52,-53,-58,-59,-62,-63

61 -4,-9,-11,-23,-25,-28,-38,-40,-41,-48,-49,-50,-56,-57,-62,-63;
3,8,-16,-18,-30,-32,-35,-43,-44,-45,-52,-53,-54,-58,-59,-60

62 -3,-15,-17,-28,-29,-31,-40,-41,-42,-49,-50,-51,-56,-57,-60,-63;
1,-10,-22,-24,-35,-36,-37,-45,-46,-47,-53,-54,-55,-58,-59,-61

63 -1,-8,-20,-21,-32,-33,-34,-42,-43,-44,-50,-51,-52,-56,-57,-60;
-14,-26,-27,-37,-38,-39,-46,-47,-48,-53,-54,-55,-58,-59,-61,-62

References

[1] D. Barbasch and M.R. Sepanski, *Closure ordering and the Kostant-Sekiguchi correspondence*, Proc. Amer. Math. Soc., **126** (1998), 311-317.

[2] D.H. Collingwood and W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold, New York, 1992.

[3] D. Ž. Đoković , *On automorphisms and adjoint orbits of real semisimple Lie algebras*, J. Algebra, **94** (1985), 19-29.

[4] ———, *Classification of nilpotent elements in simple exceptional real Lie algebras of inner type and description of their centralizers*, J. Algebra, **112** (1988), 503-524.

[5] ———, *Classification of nilpotent elements in simple real Lie algebras $E_{6(6)}$ and $E_{6(-26)}$ and description of their centralizers*, J. Algebra, **116** (1988), 196-207.

[6] ———, *Note on rational points in nilpotent orbits of semisimple groups*, Indag. Math., **9** (1998), 31-34.

[7] ———, *Explicit Cayley triples in real forms of G_2, F_4, and E_6*, Pacific J. Math., **184** (1998), 231-255.

[8] D.R. King, *The component groups of nilpotents in exceptional simple real Lie algebras*, Comm. Algebra, **20** (1992), 219-284.

[9] J.F. Kurtzke, *Centers of centralizers in reductive algebraic groups*, Comm. Algebra, **19** (1991), 3393-3410.

[10] R. Steinberg, *Lectures on Chevalley groups*, Yale University, 1967.

Received January 14, 1998 and revised April 24, 1998. The author was supported in part by the NSERC Grant A-5286.

University of Waterloo
Waterloo, Ontario N2L 3G1
Canada
E-mail address: dragomir@herod.uwaterloo.ca