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We obtain a 3-fold Selberg integral formula. As a consequence we are able to compute the explicit value of the sharp constant in a trilinear fractional integral inequality due to Beckner.

1. Introduction.

Multilinear fractional integral inequalities have been used in connection with restriction theorems of the Fourier transform and also in obtaining estimates for the k-plane and the x-ray transform. See for instance [**C1**], [**C2**], and [**D**].

In this article we are interested in a sharp form of a multilinear fractional integral inequality obtained by [**B**] (Theorem 6).

Theorem ([**B**]). *Let $1 < p_1, \dots, p_k < \infty$, $\sum_{j=1}^k p_j^{-1} > 1$, and $0 \le \gamma_{ij} = \gamma_{ji} < n$ be real numbers satisfying*

$$(0) \qquad \sum_{\substack{1\le j\le k,\\ j\ne s}} \gamma_{js} = \frac{2n}{p_s'} \qquad \text{and} \qquad \frac{1}{n}\sum_{1\le i<j\le k} \gamma_{ij} + \sum_{j=1}^k \frac{1}{p_j} = k,$$

where p_j and p_j' are dual exponents. Then

$$(1) \quad \left| \int_{\mathbf{R}^{nk}} \prod_{j=1}^k f_j(x_j) \prod_{1\le i<j\le k} |x_i - x_j|^{-\gamma_{ij}} dx_1 \dots dx_k \right| \le A(\gamma_{ij}, n) \prod_{j=1}^k \|f\|_{p_j}.$$

Moreover, the best constant $A(\gamma_{ij}, n)$ in (1) is attained for the extremal functions $f_j(x) = C(1+|x|^2)^{-n/p_j}$ up to a conformal automorphism.

The second condition in (0) is necessary to ensure conformal invariance of the variational inequality (1). It is worth mentioning that the one dimensional form of inequality (1) above when all the exponents are equal was obtained by [**C1**] without sharp constants (and without the first restriction in (0)).

The value of the best constant in (1) was computed in [**B**]:

$$(2) \qquad A(\gamma_{ij}, n) = |S^n|^{-k+\frac{1}{n}\sum\limits_{1\le i<j\le k}\gamma_{ij}} \int_{(S^n)^k} \prod_{1\le i<j\le k} |\xi_i - \xi_j|^{-\gamma_{ij}} d\xi_1 \dots d\xi_k,$$

where $|S^n| = (4\pi)^{n/2}\Gamma(n/2)\Gamma(n)^{-1}$ is the Lebesgue measure of the unit sphere in $\mathbf{R}^{n+1}$. This formula brings a connection between multilinear fractional integral inequalities and Selberg integrals.

Multiple integrals such as the one in (2) are known as *Selberg's integrals* and their exact values are useful in representation theory and in mathematical physics. These integrals have only been computed in special cases, for instance by Selberg himself when $n = 1$ and $\gamma_{ij} = \gamma$ (see [**Se**]), or when $n = 2$ and $\gamma_{ij} = 1$ (see [**Ca**]), but not in general. For a treatment of Selberg integrals, the reader could consult [**Me**], Section 17.11.

The question we would like to address is the following:

Question. *Can the constant $A(\gamma_{ij}, n)$ be computed explicitly?*

In this paper we give an answer to this question when $k = 3$. We are able to compute the three-fold Selberg integral (2) when $\gamma_{12} + \gamma_{23} + \gamma_{31} = n$ for $n \geq 1$.

Before we state our first result we would like to discuss the case $k = 2$. The bilinear version of (1) is the well known Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbf{R}^{2n}} |x-y|^{-\gamma} f_1(x) f_2(y)\,dx\,dy \leq E(\gamma, p_1, p_2, n)\|f_1\|_{p_1}\|f_2\|_{p_2} \tag{3}$$

which holds when $1/p_1 + 1/p_2 > 1$, $1/p_1 + 1/p_2 + d/n = 2$, and $0 < \gamma < n$. The sharp constant in inequality (3) was derived by [**L**] when $p_1 = p_2 = 2n/(2n-\gamma)$ and also when $p_1 = 2$ or $p_2 = 2$. When $p_1 = p_2 = 2n/(2n-\gamma)$, the sharp constant in (3) is

$$E(\gamma, p_1, p_2, n) = |S^n|^{(\gamma-2n)/n} \int_{(S^n)^2} |\xi - \eta|^{-\gamma} d\xi d\eta,$$

which can be easily computed since

$$\int_{S^n} |\xi - \eta|^{-\gamma} d\xi = 2^{n-\gamma}\pi^{n/2} \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{\Gamma\left(n - \frac{\gamma}{2}\right)}, \tag{4}$$

for all given $\eta \in S^n$.

We now turn our attention to the case $k = 3$. It turns out that in this case we can find a closed form for the constant in (2) when $1/p_1 + 1/p_2 + 1/p_3 = 2$ or, equivalently, when $\gamma_{12} + \gamma_{23} + \gamma_{31} = n$. It will be convenient to slightly change our notation in this case. We set $-\gamma_{12} = d_1 - n$, $-\gamma_{23} = d_2 - n$, and $-\gamma_{31} = d_3 - n$. Now for $0 < d_1, d_2, d_3 < n$ we denote by

$$\begin{aligned} &Q_{d_1,d_2,d_3}[f_1, f_2, f_3] \\ &= \int_{\mathbf{R}^{3n}} |x-y|^{d_1-n}|y-z|^{d_2-n}|z-x|^{d_3-n} f_1(x) f_2(y) f_3(z)\,dx\,dy\,dz \end{aligned} \tag{5}$$

the trilinear fractional integral that appears in (1). With this notation, inequality (1) is just

$$Q_{d_1,d_2,d_3}[f_1,f_2,f_3] \le C(d_1,d_2,d_3,n)\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3}, \tag{6}$$

where $0 < d_1, d_2, d_3 < n$ are real numbers satisfying $d_1 + d_2 + d_3 > n$, and $p_1 = 2n/(d_1 + d_3)$, $p_2 = 2n/(d_1 + d_2)$, $p_3 = 2n/(d_2 + d_3)$. The best constant in the inequality above can be written as

$$C(d_1,d_2,d_3,n) = |S^n|^{-\frac{d_1+d_2+d_3}{n}} \int_{(S^n)^3} |\xi-\eta|^{d_1-n}|\eta-\zeta|^{d_2-n}|\zeta-\xi|^{d_3-n} d\xi d\eta d\zeta. \tag{7}$$

We now state our first result:

Theorem 1. *Let $0 < d_1, d_2, d_3 < n$, and $d_1 + d_2 + d_3 = 2n$. Then, for any distinct $x, y, z \in \mathbf{R}^n$, the following formula holds*

$$\begin{aligned}\int_{\mathbf{R}^n} &|x-t|^{-d_2}|y-t|^{-d_3}|z-t|^{-d_1} dt \\ &= B(d_1,d_2,d_3,n)|x-y|^{d_1-n}|y-z|^{d_2-n}|z-x|^{d_3-n},\end{aligned} \tag{8}$$

where

$$B(d_1,d_2,d_3,n) = \pi^{n/2} \prod_{j=1}^{3} \frac{\Gamma\big(\frac{n-d_j}{2}\big)}{\Gamma\big(\frac{d_j}{2}\big)}.$$

Similarly, for any distinct $\xi, \eta, \zeta \in S^n$ we have

$$\begin{aligned}\int_{S^n} &|\xi-\tau|^{-d_2}|\eta-\tau|^{-d_3}|\zeta-\tau|^{-d_1} d\tau \\ &= B(d_1,d_2,d_3,n)|\xi-\eta|^{d_1-n}|\eta-\zeta|^{d_2-n}|\zeta-\xi|^{d_3-n}.\end{aligned} \tag{9}$$

Corollary 1. *Let $0 < d_1, d_2, d_3 < n$ and $d_1 + d_2 + d_3 = 2n$. Then the following Selberg integral formula holds:*

$$\begin{aligned}\int_{(S^n)^3} &|\xi-\eta|^{d_1-n}|\eta-\zeta|^{d_2-n}|\zeta-\xi|^{d_3-n} d\xi d\eta d\zeta \\ &= |S^n|(2\pi)^n \prod_{j=1}^{3} \frac{\Gamma\big(\frac{d_j}{2}\big)}{\Gamma\big(n-\frac{d_j}{2}\big)},\end{aligned} \tag{10}$$

and thus the exact value of the best constant in (6) when $d_1 + d_2 + d_3 = 2n$ is

$$C(d_1,d_2,d_3,n) = (2\pi)^n|S^n|^{-1} \prod_{j=1}^{3} \frac{\Gamma\big(\frac{d_j}{2}\big)}{\Gamma\big(n-\frac{d_j}{2}\big)}. \tag{11}$$

We point out that the kernel formula (8) is a trilinear version of the standard beta integral on $\mathbf{R}^n$:

$$\begin{aligned}(12)\qquad &\int_{\mathbf{R}^n} |x-t|^{-\alpha_1}|y-t|^{-\alpha_2}dt\\ &= \pi^{n/2}\,\frac{\Gamma\big(\frac{n-\alpha_1}{2}\big)\Gamma\big(\frac{n-\alpha_2}{2}\big)\Gamma\big(\frac{\alpha_1+\alpha_2-n}{2}\big)}{\Gamma\big(\frac{\alpha_1}{2}\big)\Gamma\big(\frac{\alpha_2}{2}\big)\Gamma\big(n-\frac{\alpha_1+\alpha_2}{2}\big)}\;|x-y|^{n-\alpha_1-\alpha_2},\end{aligned}$$

which is valid when $0<\alpha_1,\alpha_2<n$, $\alpha_1+\alpha_2>n$. It is still unclear to us whether or not there is a corresponding k-fold analogue of (8) and (9).

2. The proof of Theorem 1.

Clearly both sides of (8) are invariant under translations, dilations, and rotations of x,y,z. Therefore, by a translation we can assume that $z=0$, by a dilation that $|y|=1$, and by a rotation that $y=e_1=(1,0,\dots,0)$. Let us denote by $\widehat{f}(\xi)=\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot x}f(x)dx$ the Fourier transform of f. Recall that

$$(|x|^{d-n})^\wedge(\xi)=\pi^{n/2-d}\frac{\Gamma\big(\frac{d}{2}\big)}{\Gamma\big(\frac{n-d}{2}\big)}\;|\xi|^{-d}:=c(d)|\xi|^{-d}$$

in the sense of distributions (see [**GS**]).

After these reductions, we prove (8) by showing that the Fourier transform of both sides coincide. The function $x\to|x-e_1|^{d_1-n}|x|^{d_3-n}$ has Fourier transform

$$\begin{aligned}(13)\qquad &\big(|x|^{d_3-n}|x-e_1|^{d_1-n}\big)^\wedge(\xi)\\ &=c(d_3)c(d_1)|\xi|^{-d_3}*\big(|\xi|^{-d_1}e^{-2\pi i\xi\cdot e_1}\big)\\ &=c(d_3)c(d_1)\int_{\mathbf{R}^n}|\xi-\eta|^{-d_3}|\eta|^{-d_1}e^{-2\pi i\eta\cdot e_1}d\eta\\ &=c(d_3)c(d_1)|\xi|^{n-d_1-d_3}\int_{\mathbf{R}^n}|\xi'-t|^{-d_3}|t|^{-d_1}e^{-2\pi i|\xi|t\cdot e_1}dt,\end{aligned}$$

where $\xi'=\xi/|\xi|$. Now, for given ξ find a rotation A_ξ so that $A_\xi e_1=\xi'$. Clearly $|\xi'-t|=|e_1-A_\xi^{-1}t|$, $|t|=|A_\xi^{-1}t|$, and $t\cdot e_1=t\cdot A_\xi^{-1}\xi'=A_\xi^{-1}t\cdot A_\xi^{-2}\xi'$. Hence, with $s=A_\xi^{-1}t$ the expression in (13) is equal to

$$\begin{aligned}&c(d_3)c(d_1)|\xi|^{n-d_1-d_3}\int_{\mathbf{R}^n}|e_1-s|^{-d_3}|s|^{-d_1}e^{-2\pi i s\cdot A_\xi^{-2}\xi}ds\\ &\;=c(d_3)c(d_1)|\xi|^{n-d_1-d_3}\widehat{h}(A_\xi^{-2}\xi),\end{aligned}$$

where $h(t):=|t-e_1|^{-d_3}|t|^{-d_1}$.

On the other hand, let us denote by $g(x)$ the left hand side of (8) when $z=0$ and $y=(1,0,\dots,0)$. We have that

$$\widehat{g}(\xi)=\big(h*|t|^{-d_2}\big)^\wedge(\xi)=c(n-d_2)\widehat{h}(\xi)|\xi|^{d_2-n}.$$

Using that $d_1+d_2+d_3=2n$ and that $c(n-d)^{-1}=c(d)$ we deduce that the Fourier transforms of the two sides of (8) are equal if and only if

$$\widehat{h}(\xi)=\widehat{h}(A_\xi^{-2}\xi) \qquad \text{for almost all } \ \xi \in \mathbf{R}^n. \tag{14}$$

We now use the fact that if a function is reflection invariant with respect to a hyperplane then so is its Fourier transform. Modulo rotations and translations it is enough to check this for hyperplanes of the form $x_j=0$. But the function h is constant along circles orthogonal to e_1; in particular h is reflection invariant with respect to the hyperplanes $x_j=0$, for $j=2,3,\dots,n$, and hence so is $\widehat{h}$. But $A_\xi^{-2}\xi$ can be obtained from ξ by finitely many reflections with respect to the above hyperplanes, and this concludes the proof of (8).

To prove (9) we use the stereographic projection $\bar{\pi}:\mathbf{R}^n\to S^n$. Recall that the Jacobian of $\bar{\pi}$ is

$$|J_{\bar{\pi}}(t)|=2^n(1+|t|^2)^{-n}, \tag{15}$$

and that for any a, b in $\mathbf{R}^n$ we have

$$|\bar{\pi}(a)-\bar{\pi}(b)|=2|a-b|(1+|a|^2)^{-\frac{1}{2}}(1+|b|^2)^{-\frac{1}{2}}. \tag{16}$$

Now let $\xi=\bar{\pi}(x)$, $\eta=\bar{\pi}(y)$, $\zeta=\bar{\pi}(z)$, and $\tau=\bar{\pi}(t)$ in the integral on the left hand side of (7). Using formulas (12) and (13) one can obtain (7) by simply rewriting (6) in terms of the coordinates ξ,η,ζ,τ. □

Proof of Corollary 1. Formula (10) follows by integrating (9) with respect to ξ, ζ, η and using (3). Formula (11) is immediate from (7). □

From analyticity considerations it follows that the upper bound for the d_j in Corollary 1 can be extended to $2n$ instead of n. Of course this is not the case in Theorem 1 since the integral in (8) may diverge if $d_j\geq n$.

3. Application to a sharp Sobolev imbedding.

The purpose if this section is to bring out some connections between multilinear integrals of type (1) with equal exponents and Sobolev imbeddings. These connections enter also in the context of conformal deformations of the metric structure of S^n and spectral theory. Indeed, as shown in [**Mo**], multilinear fractional integrals arise as explicit computations of zeta functions of natural pseudodifferential operators, in the conformal class of the standard metric on S^n.

Let us discuss again the case $k=2$. Inequality (3) can also be written as

$$\|I_\alpha(f)\|_q\leq N_{p,\alpha,n}\|f\|_p, \tag{17}$$

where $\alpha - n = -\gamma$, $1/q = 1/p - \alpha/n$, $1 < p, q < \infty$, and I_α denotes fractional integration given by

$$I_\alpha(f)(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)dy. \tag{18}$$

When $q' = p$, the sharp constant in (17) is

$$N_{p,\alpha,n} = 2^{-\alpha}\pi^{-\alpha/2} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right]^{\alpha/n} \tag{19}$$

as computed in [**L**]. Inequality (17) expresses the sharp imbedding from $L^p(\mathbf{R}^n) \hookrightarrow \dot{L}^{p'}_{-\alpha}(\mathbf{R}^n)$, where $\alpha = n(2/p-1) > 0$, $1 < p < 2$, and

$$\dot{L}^q_{-\alpha}(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) : \ \|f\|_{\dot{L}^q_{-\alpha}} = \|I_\alpha(|f|)\|_q < \infty\}$$

is a homogeneous Sobolev space.

We now consider the case $k = 3$. When $d_1 = d_2 = d_3 = d$ and $f_1 = f_2 = f_3 = f$, inequality (6) is a special case of a more general sharp inequality derived in [**Mo**]. In the special case $0 < d \le 2$ it is possible to write $Q_d[f]$ as a certain path integral, which is an L^3 norm with an appropriate Wiener measure. This allows us to conclude that the expression

$$f \to Q_d[f] = \left(Q_{d,d,d}[|f|,|f|,|f|]\right)^{1/3}$$

is a norm when $0 < d \le 2$. It is quite natural to expect that Q_d is a norm also when $2 < d < n$, although we are not quite certain how to prove this in general. When $d = 2n/3$, however, it is an easy consequence of Theorem 1 that Q_d is a norm since it is the L^3 norm of a fractional integral.

Corollary 2. *We have that for all $f \ge 0$,*

$$Q_{2n/3}\big[f\big] = (2\pi)^{n/3}\|I_{n/3}(f)\|_3 = (2\pi)^{n/3}\|f\|_{\dot{L}^3_{-n/3}}. \tag{20}$$

To prove Corollary 2, take $d_1 = d_2 = d_3 = 2n/3$ in (8). Then multiply (8) by $f(x)f(y)f(z)$ and integrate with respect to dx, dy, and dz. Apply Fubini's theorem and use (18) to obtain (20). $\square$

Thus, in the special case $d_1 = d_2 = d_3 = 2n/3$ inequality (6) is the same as inequality (17), with $q = 3 = p'$ and $\alpha = n/3$. Observe that the constants also coincide since

$$\begin{aligned}(|S^n|^{-2}B(2n/3,2n/3,2n/3,n))^{1/3} &= C(2n/3,2n/3,2n/3,n)\\ &= (2\pi)^{n/3}N_{3/2,n/3,n},\end{aligned}$$

as it should be. This gives a relationship between the sharp imbedding given by (6) and the sharp Sobolev imbedding given in (17) when $d = 2n/3$.

It is fairly routine to check that the expression $Q_d f$ remains unchanged if f is replaced by $(f \circ U)|J_U|^{d/n}$, where U is a conformal transformation of $\mathbf{R}^n$

and J_U is its Jacobian. This transformation is scaled suitably to preserve $L^{n/d}$. We will denote by B_d the space of all measurable functions f on $\mathbf{R}^n$ such that $Q_d f < \infty$. The main feature of the space B_d is the conformal invariance of its 'norm' Q_d. It is reasonable to ask whether B_d is related to any L^3-based homogeneous Sobolev space. By homogeneity it can only be compared to $\dot{L}^3_{n/3-d}$. We have the following:

Theorem 2. *The space B_d is contained in $\dot{L}^3_{n/3-d}$ when $n > d > 2n/3$ but B_d contains $\dot{L}^3_{n/3-d}$ when $2n/3 > d > n/3$; furthermore both inclusions are strict. Quantitatively speaking, for any $n > d > 2n/3$ there exists a constant $C = C_{d,n}$ such that for all measurable functions f we have*

$$\|I_{d-n/3}(|f|)\|_3 \le C Q_d[f]. \tag{21}$$

For any $2n/3 > d > n/3$ there exists a constant $C = C_{d,n}$ such that for all measurable functions f we have

$$Q_d[f] \le C \|I_{d-n/3}(|f|)\|_3. \tag{22}$$

4. The proof of Theorem 2.

Observe that the cube of the left hand side of (21) is equal to

$$C_{d,n} \int_{\mathbf{R}^{3n}} K_d(x,y,z) f(x) f(y) f(z) dx dy dz,$$

where

$$K_d(x,y,z) = \int_{\mathbf{R}^n} |x-t|^{d-\frac{4n}{3}} |y-t|^{d-\frac{4n}{3}} |z-t|^{d-\frac{4n}{3}} dt.$$

If we establish that for $2n/3 < d < 2n$ we have

$$K_d(x,y,z) \le C_{d,n} |x-y|^{d-n} |y-z|^{d-n} |z-x|^{d-n}, \tag{23}$$

then (21) will follow immediately. Similarly, if we prove that for $n/3 < d < 2n/3$ we have

$$|x-y|^{d-n} |y-z|^{d-n} |z-x|^{d-n} \le C_{d,n} K_d(x,y,z), \tag{24}$$

then (22) will follow as well. Now a simple dilation implies that (23) and (24) are valid for $|x|, |y|, |z| \le 1$, then they are valid for $|x|, |y|, |z| \le R$ with the same constant for all $R > 0$. Letting $R \to \infty$ we conclude that (23) and (24) are valid for all $R > 0$. Therefore, it suffices to prove (23) and (24) for $|x|, |y|, |z| \le 1$.

Given any three points x, y, z in $\mathbf{R}^n$, let $M(x,y,z) = \max(|x-y|, |y-z|, |z-x|)$ be their maximum and $m(x,y,z) = \min(|x-y|, |y-z|, |z-x|)$ be their minimum. Let us also call $\mu(x,y,z)$ the number in the middle. Then we have that $\mu(x,y,z) \ge \frac{1}{2} M(x,y,z)$. The following lemma gives us asymptotic estimates for $K_d(x,y,z)$.

Lemma. *Let $|x|, |y|, |z| \leq 1$. Then for $5n/6 < d < n$ we have*

$$K_d(x, y, z) \sim M(x, y, z)^{3(d-n)}. \tag{25}$$

For $d = 5n/6$ we have

$$K_d(x, y, z) \sim M(x, y, z)^{-\frac{n}{2}} \log \frac{M(x, y, z)}{m(x, y, z)}, \tag{26}$$

and for $n/3 < d < 5n/6$ we have

$$K_d(x, y, z) \sim m(x, y, z)^{-n+2(d-\frac{n}{3})} M(x, y, z)^{d-\frac{4n}{3}}. \tag{27}$$

Now (23) and (24) are easy consequences of this lemma and of the observation that $\mu(x, y, z)$ is always comparable to $M(x, y, z)$.

Let us now give sketch the proof of the lemma above. Since the problem is translation invariant, it suffices to study the asymptotic behavior of the integral below as $|\alpha|$, $|\beta| \to 0$

$$\int_{\mathbf{R}^n} |\alpha - t|^{-n+\lambda} |\beta - t|^{-n+\lambda} |t|^{-n+\lambda} dt, \tag{28}$$

where we set $\lambda = d - n/3$ and $\alpha = z - x$ and $\beta = z - y$. Since both $|\alpha|$, $|\beta| \leq 2$ the problem is local and we consider the following five cases:

Case 1. $|\alpha| \to 0$, $|\beta| \sim |\beta - \alpha| \sim 1$.

Case 2. $|\alpha - \beta| \to 0$, $|\beta| \sim |\alpha| \sim 1$.

Case 3. $|\beta - \alpha| \ll |\alpha| \sim |\beta| \to 0$.

Case 4. $|\beta - \alpha| \sim |\alpha| \sim |\beta| \to 0$.

Case 5. $|\alpha| \ll |\beta| \sim |\beta - \alpha| \to 0$.

It is easy to see that in Case 1, the integral (28) behaves like a constant when $\lambda > n/2$, blows up like $|\alpha|^{-n+2\lambda}$ when $\lambda < n/2$ and also blows up like $\log |\alpha|^{-1}$ when $\lambda = n/2$.

Case 2 is similar to Case 1 where the roles of $|\alpha|$ and $|\beta - \alpha|$ are interchanged.

In Case 3 the situation is slightly different. The integral (28) behaves asymptotically like $|\alpha|^{-n+\lambda} |\alpha - \beta|^{-n+2\lambda}$ when $\lambda < n/2$, as $|\alpha|^{-2n+3\lambda}$ when $\lambda > n/2$, and as $|\alpha|^{-2n+3\lambda} \log(|\alpha||\alpha - \beta|^{-1})$ when $\lambda = n/2$.

In Case 4, the integral (28) behaves asymptotically like $|\alpha - \beta|^{-2n+3\lambda}$.

Finally, Case 5 follows from Case 1. In this case one has asymptotic behavior $|\alpha|^{-n+2\lambda} |\beta|^{-n+\lambda}$ when $\lambda < n/2$, $|\beta|^{-2n+3\lambda}$ when $\lambda > n/2$, and $|\beta|^{-2n+3\lambda} \log(|\beta|/|\alpha|)$ when $\lambda = n/2$.

The derivation of the asymptotics of (28) in each case involves different splitting of the integral (28) and use of formula (12). The details are rather tedious and are omitted.

The exceptional case $\lambda = n/2$ corresponds to $d = 5n/6$ and only in this case a logarithmic term appears.

We now indicate how the behavior of $K(x, y, z)$ follows from the asymptotic behavior of the integral (28). First take $5n/6 < d < n$, equivalently $n/2 < \lambda < 2n/3$. Recalling that $|\alpha| = |z - x|$ and $|\beta| = |z - y|$, we observe that the asymptotics in the five cases above, (i.e. C in Cases 1 and 2, $|x-z|^{-2n+3\lambda}$ in Case 3, $|x-y|^{-2n+3\lambda}$ in Case 4, and $|y-z|^{-2n+3\lambda}$ in Case 5) is a restatement of (25). Likewise, the statements in the five cases above can be summarized in (26) when $d = 5n/6$, and in (27) when $n/3 < d < 5n/6$.

Using again the asymptotics for K_d, one can construct examples to show that the converse inequalities to (21) and (22) are false. The details are omitted. This concludes the proof of Theorem 2. $\square$

5. An application to fractional integrals.

Formula (8) can be used to give an alternative proof of inequality (6) in the particular case $d_1 + d_2 + d_3 = 2n$. Observe that in this case $1/p_1 + 1/p_2 + 1/p_3 = 2$ while $1/p_1' + 1/p_2' + 1/p_3' = 1$. Use (8) to rewrite $Q_{d_1,d_2,d_3}(f_1, f_2, f_3)$ as

$$\frac{1}{B(d_1, d_2, d_3, n)} \int_{\mathbf{R}^n} (f_1 * |\cdot|^{-d_2})(t)\ (f_2 * |\cdot|^{-d_3})(t)\ (f_3 * |\cdot|^{-d_1})(t)\ dt. \tag{29}$$

Apply Hölder's inequality to estimate (29) by

$$\frac{1}{B(d_1, d_2, d_3, n)} \|f_1 * |\cdot|^{-d_2}\|_{p_1'} \|f_2 * |\cdot|^{-d_3}\|_{p_2'} \|f_3 * |\cdot|^{-d_1}\|_{p_3'}. \tag{30}$$

(17), (18), and (19) now imply that (30) is bounded by

$$\frac{1}{B(d_1, d_2, d_3, n)} \prod_{j=1}^{3} \pi^{\frac{d_j}{2}} \frac{\Gamma(\frac{n-d_j}{2})}{\Gamma(n - \frac{d_j}{2})} \left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{-1+\frac{d_j}{n}} \|f_j\|_{p_j}$$

which is nothing else than $C(d_1, d_2, d_3, n) \prod_{j=1}^{3} \|f_j\|_{p_j}$.

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