

*Pacific
Journal of
Mathematics*

COHOMOLOGY OF COMPLETE INTERSECTIONS IN
TORIC VARIETIES

ANVAR R. MAVLYUTOV

COHOMOLOGY OF COMPLETE INTERSECTIONS IN TORIC VARIETIES

ANVAR R. MAVLYUTOV

We explicitly describe cohomology of complete intersections in compact simplicial toric varieties.

In this paper we will study intersections of hypersurfaces in compact simplicial toric varieties \mathbf{P}_Σ . The main purpose is to relate naturally the Hodge structure of a complete intersection $X_{f_1} \cap \dots \cap X_{f_s}$ in \mathbf{P}_Σ to a graded ring. Originally this idea appears in [Gr], [St], [Dol], [PS]. The case of a hypersurface in a toric variety has been treated in [BC]. Also the Hodge structure of complete intersections in a projective space was described in [Te], [Ko], [L], [Di], [Na]. The common approach was to reduce studying of the Hodge structure on a complete intersection to studying of the Hodge structure on a hypersurface in a higher dimensional projective variety. This is the idea of a “Cayley trick”. About a Cayley trick in the toric context see [GKZ], [DK], [BB]. A special case of a complete intersection (when it is empty) in a complete simplicial toric variety was elaborated in [CCD]. The basic references on toric varieties are [F1], [O], [Da], [C].

The paper is organized as follows:

Section 1 establishes notation and studies cohomology of subvarieties in a complete simplicial toric variety. In Section 2 we describe a Cayley trick for toric varieties. In Section 3 we prove the main result where we relate the Hodge components $H^{d-s-p,p}(X_{f_1} \cap \dots \cap X_{f_s})$ in the middle cohomology group to homogeneous components of a graded ring. Section 4 treats a special case of complete intersections: a nondegenerate intersection.

Acknowledgment. I would like to thank D. Cox for his advice and useful comments.

1. Quasi-smooth intersections.

We first fix some notation. Let M be a lattice of rank d , $N = \text{Hom}(M, \mathbf{Z})$ the dual lattice; $M_{\mathbf{R}}$ (resp. $N_{\mathbf{R}}$) denotes the \mathbf{R} -scalar extension of M (resp. of N). Let Σ be a rational simplicial complete d -dimensional fan in $N_{\mathbf{R}}$ [BC], \mathbf{P}_Σ a complete simplicial toric variety associated with this fan.

Such a toric variety can be described as a geometric quotient [C]. Let $S(\Sigma) = \mathbf{C}[x_1, \dots, x_n]$ be the polynomial ring over \mathbf{C} with variables x_1, \dots, x_n

corresponding to the integral generators e_1, \dots, e_n of the 1-dimensional cones of Σ . For $\sigma \in \Sigma$ let $\hat{x}_\sigma = \prod_{e_i \notin \sigma} x_i$, and let $B(\Sigma) = \langle \hat{x}_\sigma : \sigma \in \Sigma \rangle \subset S$ be the ideal generated by the \hat{x}_σ 's. This ideal gives the variety $Z(\Sigma) = \mathbf{V}(B(\Sigma)) \subset \mathbf{A}^n$. The toric variety $\mathbf{P} = \mathbf{P}_\Sigma$ will be a geometric quotient of $U(\Sigma) := \mathbf{A}^n \setminus Z(\Sigma)$ by the group $\mathbf{D} := \text{Hom}_{\mathbf{Z}}(A_{d-1}(\mathbf{P}), \mathbf{C}^*)$, where $A_{d-1}(\mathbf{P})$ is the Chow group of Weil divisors modulo rational equivalence.

Each variable x_i in the coordinate ring $S(\Sigma)$ corresponds to a torus-invariant irreducible divisor D_i of \mathbf{P} . As in [C], we grade $S = S(\Sigma)$ by assigning to a monomial $\prod_{i=1}^n x_i^{a_i}$ its degree $[\sum_{i=1}^n a_i D_i] \in A_{d-1}(\mathbf{P})$. A polynomial f in the graded piece S_α corresponding to $\alpha \in A_{d-1}(\mathbf{P})$ is said to be \mathbf{D} -homogeneous of degree α .

Let f_1, \dots, f_s be \mathbf{D} -homogeneous polynomials. They define a zero set $\mathbf{V}(f_1, \dots, f_s) \subset \mathbf{A}^n$, moreover $\mathbf{V}(f_1, \dots, f_s) \cap U(\Sigma)$ is stable under the action of \mathbf{D} and hence descends to a closed subset $X \subset \mathbf{P}$, because \mathbf{P} is a geometric quotient.

Definition 1.1. We say that X is a *quasi-smooth intersection* if $\mathbf{V}(f_1, \dots, f_s) \cap U(\Sigma)$ is either empty or a smooth subvariety of codimension s in $U(\Sigma)$.

Remark 1.2. This notion generalizes a nonsingular complete intersection in a projective space. Notice that since the $(n - d)$ -dimensional group \mathbf{D} has only zero dimensional stabilizers [BC], X is of pure dimension $d - s$ or empty.

We can now relate this notion to a V -submanifold (see Definition 3.2 in [BC]).

Proposition 1.3. *If $X \subset \mathbf{P}$ is a closed subset of codimension s defined by \mathbf{D} -homogeneous polynomials f_1, \dots, f_s , then X is a quasi-smooth intersection if and only if X is a V -submanifold of \mathbf{P} .*

The proof of this is very similar to the proof of the Proposition 3.5 in [BC].

The next result is a Lefschetz-type theorem.

Proposition 1.4. *Let $X \subset \mathbf{P}$ be a closed subset, defined by \mathbf{D} -homogeneous polynomials f_1, \dots, f_s , in a complete simplicial toric variety \mathbf{P} . If $f_1, \dots, f_s \in B(\Sigma)$, then the natural map $i^* : H^i(\mathbf{P}) \rightarrow H^i(X)$ is an isomorphism for $i < d - s$ and an injection for $i = d - s$. In particular, this is valid if X is an intersection of ample hypersurfaces.*

Proof. We can present $X = X_{f_1} \cap \dots \cap X_{f_s}$, where $X_{f_i} \subset \mathbf{P}$ is a hypersurface defined by f_i . As it was shown in the proof of the Proposition 10.8 [BC], if $f \in B(\Sigma)$ then $\mathbf{P} \setminus X_f = (\mathbf{A}^n \setminus \mathbf{V}(f))/\mathbf{D}(\Sigma)$ is affine, hence $H^i(\mathbf{P} \setminus X_f) = 0$ for $i > d$. We will prove by induction on s that $H^i(\mathbf{P} \setminus (X_{f_1} \cap \dots \cap X_{f_s})) = 0$ for $i > d + s - 1$. Consider the Mayer-Vietoris sequence

$$\dots \rightarrow H^i(U \cap V) \rightarrow H^{i+1}(U \cup V) \rightarrow H^{i+1}(U) \oplus H^{i+1}(V) \rightarrow H^{i+1}(U \cap V) \rightarrow \dots$$

with $U = \mathbf{P} \setminus (X_{f_1} \cap \dots \cap X_{f_{s-1}})$, $V = \mathbf{P} \setminus X_{f_s}$. Notice that $U \cup V = \mathbf{P} \setminus (X_{f_1} \cap \dots \cap X_{f_s})$ and $U \cap V = \bigcup_{i=1}^{s-1} \mathbf{P} \setminus (X_{f_i} \cup X_{f_s}) = \mathbf{P} \setminus (X_{f_1 \cdot f_s} \cap \dots \cap X_{f_{s-1} \cdot f_s})$. So, using the induction and the above sequence, we obtain that $H^i(\mathbf{P} \setminus X) = 0$ for $i > d + s - 1$. As a consequence of this, X is nonempty unless $s > d$ because the dimension $h^{2d}(\mathbf{P}) = 1$. Since $\mathbf{P} \setminus X$ is a V-manifold, Poincaré duality implies that $H_c^i(\mathbf{P} \setminus X) = 0$ for $i \leq d - s$. Now the desired result follows from the long exact sequence of the cohomology with compact supports (X and \mathbf{P} are compact):

$$\dots \rightarrow H_c^i(\mathbf{P} \setminus X) \rightarrow H_c^i(\mathbf{P}) \rightarrow H_c^i(X) \rightarrow H_c^{i+1}(\mathbf{P} \setminus X) \rightarrow H_c^{i+1}(\mathbf{P}) \rightarrow \dots$$

If X is an intersection of ample hypersurfaces defined by f_1, \dots, f_s , then Lemma 9.15 [BC] gives us that f_1, \dots, f_s belong to $B(\Sigma)$. □

Corollary 1.5. *A quasi-smooth intersection $X = X_{f_1} \cap \dots \cap X_{f_s}$, defined by $f_1, \dots, f_s \in B(\Sigma)$, has pure dimension $d - s$.*

Since the dimension of $H^0(X, \mathbf{C})$ is the number of connected components of X , we obtain another important result.

Corollary 1.6. *An intersection $X_{f_1} \cap \dots \cap X_{f_s}$, defined by $f_1, \dots, f_s \in B(\Sigma)$, in a complete simplicial toric variety \mathbf{P}_Σ is connected provided $s < \dim \mathbf{P}_\Sigma$.*

Remark 1.7. If the polynomials f_1, \dots, f_s have ample degrees, then this corollary follows from a more general statement in [FL1] (see also [FL2] and [FH] for connectedness theorems).

2. “Cayley trick”.

We will explore a Cayley trick to reduce studying of the cohomology of quasi-smooth intersections to results already known for hypersurfaces.

Let L_1, \dots, L_s be line bundles on a complete d -dimensional toric variety $\mathbf{P} = \mathbf{P}_\Sigma$, and let $\pi : \mathbf{P}(E) \rightarrow \mathbf{P}$ be the projective space bundle associated to the vector bundle $E = L_1 \oplus \dots \oplus L_s$. Then the \mathbb{P}^{s-1} -bundle $\mathbf{P}(E)$ is a toric variety. The fan corresponding to it can be described as follows [O, p. 58]. Suppose that support functions h_1, \dots, h_s give rise to the isomorphism classes of line bundles $[L_1], \dots, [L_s] \in \text{Pic}(\mathbf{P})$, respectively. Introduce a \mathbf{Z} -module N' with a \mathbf{Z} -basis $\{n_2, \dots, n_s\}$ and let $\tilde{N} := N \oplus N'$ and $n_1 := -n_2 - \dots - n_s$. Denote by $\tilde{\sigma}$ the image of each $\sigma \in \Sigma$ under the \mathbf{R} -linear map $N_{\mathbf{R}} \rightarrow \tilde{N}_{\mathbf{R}}$ which sends $y \in N_{\mathbf{R}}$ to $y - \sum_{j=1}^s h_j(y)n_j$. On the other hand, let σ'_i be the cone in $N'_{\mathbf{R}}$ generated by $n_1, \dots, n_i, n_{i+1}, \dots, n_s$ and let Σ' be the fan in $N'_{\mathbf{R}}$ consisting of the faces of $\sigma'_1, \dots, \sigma'_s$. Then $\mathbf{P}(E)$ corresponds to the fan $\tilde{\Sigma} := \{\tilde{\sigma} + \sigma' : \sigma \in \Sigma, \sigma' \in \Sigma'\}$. From this description it is easy to see that if Σ is a complete simplicial fan then $\mathbf{P}(L_1 \oplus \dots \oplus L_s)$ is a complete simplicial toric variety. We see that the integral generators of

the 1-dimensional cones in $\tilde{\Sigma}$ are given by

$$\begin{aligned} \tilde{e}_i &= e_i - \sum_{1 \leq j \leq s} h_j(e_i)n_j, \quad i = 1, \dots, n, \\ \tilde{n}_1 &= -n_2 - \dots - n_s, \\ \tilde{n}_j &= n_j, \quad j = 2, \dots, s, \end{aligned}$$

where e_1, \dots, e_n are the integral generators of the 1-dimensional cones in Σ .

The homogeneous coordinate ring of $\mathbf{P}(E)$ is the polynomial ring

$$R = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_s],$$

where x_i corresponds to \tilde{e}_i and y_j corresponds to \tilde{n}_j . This ring has a grading by the Chow group $A_{d+s-2}(\mathbf{P}(E))$. Since \mathbf{P} is a normal variety, there is an embedding of the Picard group $\text{Pic}(\mathbf{P}) \hookrightarrow A_{d-1}(\mathbf{P})$. We want to show that if some polynomials $f_j \in S(\Sigma) = \mathbf{C}[x_1, \dots, x_n]$ have the property $\text{deg}(f_j) = [L_j] \in \text{Pic}(\mathbf{P})$, then the polynomials $y_j f_j$ all have the same degree in R . This will allow us to consider a hypersurface defined by the homogeneous polynomial $F = \sum_{j=1}^s y_j f_j$.

Lemma 2.1. *Let $f_1, \dots, f_s \in S(\Sigma)$ be \mathbf{D} -homogeneous polynomials, such that $\text{deg}(f_j) = [L_j]$ for some line bundles L_1, \dots, L_s . Then $F = \sum_{j=1}^s y_j f_j$ is homogeneous in R and its degree is the isomorphism class $[O_{\mathbf{P}(E)}(1)]$ of the canonical line bundle on $\mathbf{P}(E) = \mathbf{P}(L_1 \oplus \dots \oplus L_s)$.*

Proof. To prove that F is a homogeneous polynomial we will repeat the arguments in the proof of Lemma 3.5 in [CCD]. Let D_1, \dots, D_n be the torus-invariant divisors on $\mathbf{P} = \mathbf{P}_\Sigma$ corresponding to the 1-dimensional cones of the fan Σ . Then the pullback π^*D_i is the torus-invariant divisor of $\mathbf{P}(E)$ corresponding to the cone generated by \tilde{e}_i . Also denote by D'_j the torus-invariant divisor corresponding to \tilde{n}_j . Let $\tilde{M} = M \oplus M'$ be the lattice dual to $\tilde{N} = N \oplus N'$ with $M' = \text{Hom}(N', \mathbf{Z})$ having $\{n_2^*, \dots, n_s^*\}$ as a basis dual to $\{n_2, \dots, n_s\}$. The divisor corresponding to the character $\chi^{n_j^*}$ is

$$\begin{aligned} \text{div}(\chi^{n_j^*}) &= \sum_{i=1}^n \langle n_j^*, \tilde{e}_i \rangle \pi^*D_i + \sum_{k=1}^s \langle n_j^*, \tilde{n}_k \rangle D'_k \\ &= \sum_{i=1}^n (h_1(e_i) - h_j(e_i))\pi^*D_i - D'_1 + D'_j. \end{aligned}$$

Therefore, $[D'_j] + [\pi^*L_j]$ all have the same degree in the Chow group $A_{d+s-2}(\mathbf{P}(E))$, and, consequently, F is a homogeneous polynomial.

Now consider the following exact sequence [M]:

$$0 \rightarrow O_{\mathbf{P}(E)} \rightarrow \pi^*E^* \otimes O_{\mathbf{P}(E)}(1) \rightarrow T_{\mathbf{P}(E)} \rightarrow \pi^*T_{\mathbf{P}} \rightarrow 0,$$

where T_X denotes the tangent bundle, E^* is the dual bundle. From here we can compute the Chern class

$$\begin{aligned} c_1(T_{\mathbf{P}(E)}) &= c_1(\pi^*T_{\mathbf{P}}) + c_1(\pi^*E^* \otimes O_{\mathbf{P}(E)}(1)) \\ &= \pi^*c_1(T_{\mathbf{P}}) - \pi^*c_1(E) + s \cdot c_1(O_{\mathbf{P}(E)}(1)). \end{aligned}$$

Hence, $s \cdot c_1(O_{\mathbf{P}(E)}(1)) = \pi^*c_1(L_1) + \dots + \pi^*c_1(L_s) + c_1(T_{\mathbf{P}(E)}) - \pi^*c_1(T_{\mathbf{P}})$. On the other hand, from the generalized Euler exact sequence [BC, §12] we get

$$0 \rightarrow O_{\mathbf{P}}^{n-d} \rightarrow \bigoplus_{i=1}^n O_{\mathbf{P}}(D_i) \rightarrow T_{\mathbf{P}} \rightarrow 0.$$

This implies that $c_1(T_{\mathbf{P}}) = [D_1] + \dots + [D_n]$. Similarly we have $c_1(T_{\mathbf{P}(E)}) = [\pi^*D_1] + \dots + [\pi^*D_n] + [D'_1] + \dots + [D'_s]$. Under the identification $\text{Pic}(\mathbf{P}(E)) \hookrightarrow A_{d+s-2}(\mathbf{P}(E))$ the first Chern class of a line bundle on $\mathbf{P}(E)$ is exactly its isomorphism class in the Picard group $\text{Pic}(\mathbf{P}(E))$. Therefore

$$s \cdot [O_{\mathbf{P}(E)}(1)] = [\pi^*L_1] + \dots + [\pi^*L_s] + [D'_1] + \dots + [D'_s] = s \cdot ([\pi^*L_2] + [D'_2]).$$

It can be easily checked that D'_2 is a Cartier divisor on $\mathbf{P}(E)$. Hence all classes $[O_{\mathbf{P}(E)}(1)]$, $[\pi^*L_2]$ and $[D'_2]$ lie in the Picard group $\text{Pic}(\mathbf{P}(E))$. But this group is free abelian, because $\mathbf{P}(E)$ is complete. So the above equality is divisible by s : $[O_{\mathbf{P}(E)}(1)] = [\pi^*L_2] + [D'_2] = \text{deg}(F)$. \square

From now on we assume that $\mathbf{P} = \mathbf{P}_{\Sigma}$ is a complete simplicial toric variety and that $\text{deg}(f_j) \in \text{Pic}(\mathbf{P})$, $j = 1, \dots, s$. Denote by Y the hypersurface in $\mathbf{P}(E)$ defined by $F = \sum_{j=1}^s y_j f_j$.

Lemma 2.2. *$X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection iff the hypersurface Y is quasi-smooth.*

Proof. $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection means that whenever $x \in \mathbf{V}(f_1, \dots, f_s) \setminus Z(\Sigma)$, the $\text{rank}(\frac{\partial f_j}{\partial x_i}(x))_{i,j} = s$. And Y is quasi-smooth iff $z = (x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$ implies that one of the partial derivatives $\frac{\partial F}{\partial y_j}(z) = f_j(x)$, $j = 1, \dots, s$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \dots, n$, is nonzero.

So let $(x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$, then there is a cone $\tilde{\sigma} + \sigma' \in \tilde{\Sigma}$ with $\sigma \in \Sigma$, $\sigma' \in \Sigma'$, such that $\prod_{\tilde{e}_i \notin \tilde{\sigma}} x_i \prod_{\tilde{n}_j \notin \sigma'} y_j \neq 0$ where x_i, y_j are the coordinates of (x, y) . If $f_1(x) = \dots = f_s(x) = 0$, then $x \in \mathbf{V}(f_1, \dots, f_s) \setminus Z(\Sigma)$ because $\prod_{e_i \in \sigma} x_i \neq 0$. And if $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \dots, n$, is nonzero.

Conversely, suppose Y is quasi-smooth. Pick any $x \in \mathbf{V}(f_1, \dots, f_s) \setminus Z(\Sigma)$, then $(x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$ for each $y = (y_1, \dots, y_s) \neq 0$. Therefore $\sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some i , which means the $\text{rank}(\frac{\partial f_j}{\partial x_i}(x))_{i,j}$ is maximal. \square

3. Cohomology of quasi-smooth intersections.

Since a quasi-smooth intersection is a compact V-manifold (Proposition 1.3), the cohomology on it has a pure Hodge structure. Using Proposition 1.4 and the Poincaré duality, we can compute the cohomology of a quasi-smooth intersection except for the cohomology in the middle dimension $d - s$. So we introduce the following definition.

Definition 3.1. The *variable cohomology group* $H_{\text{var}}^{d-s}(X)$ is $\text{coker}(H^{d-s}(\mathbf{P}) \xrightarrow{i^*} H^{d-s}(X))$.

The variable cohomology group also has a pure Hodge structure.

Proposition 3.2. *Let $X = X_{f_1} \cap \dots \cap X_{f_s}$ be a quasi-smooth intersection of ample hypersurfaces. Then there is an exact sequence of mixed Hodge structures*

$$0 \rightarrow H^{d-s-1}(\mathbf{P}) \xrightarrow{\cup[X]} H^{d+s-1}(\mathbf{P}) \rightarrow H^{d+s-1}(\mathbf{P} \setminus X) \rightarrow H_{\text{var}}^{d-s}(X) \rightarrow 0,$$

where $[X] \in H^{2s}(\mathbf{P})$ is the cohomology class of X .

Proof. Consider the Gysin exact sequence:

$$(1) \quad \dots \rightarrow H^{i-2s}(X) \xrightarrow{i_!} H^i(\mathbf{P}) \rightarrow H^i(\mathbf{P} \setminus X) \rightarrow H^{i-2s+1}(X) \xrightarrow{i_!} H^{i+1}(\mathbf{P}) \rightarrow \dots$$

Since i^* is Poincaré dual to the Gysin map $i_!$, it follows that $H_{\text{var}}^{d-s}(X)$ is isomorphic to the kernel of $i_! : H^{d-s}(X) \rightarrow H^{d+s}(\mathbf{P})$. So we get an exact sequence

$$H^{d-s-1}(X) \xrightarrow{i_!} H^{d+s-1}(\mathbf{P}) \rightarrow H^{d+s-1}(\mathbf{P} \setminus X) \rightarrow H_{\text{var}}^{d-s}(X) \rightarrow 0.$$

Now we use a commutative diagram

$$\begin{array}{ccc} H^{d-s-1}(X) & \xrightarrow{i_!} & H^{d+s-1}(\mathbf{P}) \\ i^* \uparrow & \nearrow \cup[X] & \\ H^{d-s-1}(\mathbf{P}) & & \end{array}$$

By Proposition 1.4 i^* is an isomorphism in this diagram, so it suffices to prove that the Gysin map $i_!$ is injective in the above diagram.

Lemma 3.3. *If $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection of ample hypersurfaces, then the Gysin map $H^{d-s-1}(X) \xrightarrow{i_!} H^{d+s-1}(\mathbf{P})$ is injective.*

Proof. Since the odd dimensional cohomology of a complete simplicial toric variety vanishes [F1, pp. 92-94] and $i^* : H^{d-s-1}(\mathbf{P}) \rightarrow H^{d-s-1}(X)$ is an isomorphism by Proposition 1.4, it follows that $H^{d-s-1}(X) = H^{d-s-1}(\mathbf{P}) = H^{d+s-1}(\mathbf{P}) = 0$ when $d + s - 1$ is odd. So by the Gysin exact sequence (1) it is enough to show that $H^{d+s-2}(\mathbf{P} \setminus X) = 0$ when $d + s - 2$ is odd. To prove this we use the Cayley trick again. Let Y be the hypersurface defined by

$F = \sum_{j=1}^s y_j f_j$. Then the natural map $\mathbf{P}(E) \setminus Y \rightarrow \mathbf{P} \setminus X$, induced by the projection $\pi : \mathbf{P}(E) \rightarrow \mathbf{P}$, is a \mathbf{C}^{s-1} bundle in the Zariski topology. Notice that $\mathbf{P} \setminus X$ is simply connected, because \mathbf{P} is simply connected [F1, p. 56] and X has codimension at least 2 in \mathbf{P} . Hence, the Leray-Serre spectral sequence implies that $H^i(\mathbf{P}(E) \setminus Y) = H^i(\mathbf{P} \setminus X)$ for $i \geq 0$. We have that $H^{d+s-2}(\mathbf{P}(E)) = 0$ for $d + s - 2$ odd and Y is quasi-smooth by Lemma 2.2. So from the Gysin exact sequence

$$H^{d+s-2}(\mathbf{P}(E)) \rightarrow H^{d+s-2}(\mathbf{P}(E) \setminus Y) \rightarrow H^{d+s-3}(Y) \xrightarrow{j_!} H^{d+s-1}(\mathbf{P}(E))$$

(here the Gysin map $j_!$ is induced by the inclusion $j : Y \hookrightarrow \mathbf{P}(E)$) it follows that we need to show injectivity of $j_! : H^{d+s-3}(Y) \rightarrow H^{d+s-1}(\mathbf{P}(E))$. Consider the commutative diagram

$$\begin{array}{ccc} H^{d+s-3}(Y) & \xrightarrow{j_!} & H^{d+s-1}(\mathbf{P}(E)) \\ j^* \uparrow & \nearrow_{\cup[Y]} & \\ H^{d+s-3}(\mathbf{P}(E)) & & \end{array}$$

where $[Y] \in H^2(\mathbf{P}(E))$ is the cohomology class of Y . The canonical line bundle $O_{\mathbf{P}(E)}(1)$ is ample [H, III, §1], whence by Lemma 2.1, Y is ample. So by Proposition 10.8 [BC] $j^* : H^{d+s-3}(\mathbf{P}(E)) \rightarrow H^{d+s-3}(Y)$ is an isomorphism and by Hard Lefschetz $\cup[Y] : H^{d+s-3}(\mathbf{P}(E)) \rightarrow H^{d+s-1}(\mathbf{P}(E))$ is injective. Thus, from the above diagram the lemma follows. \square

Definition 3.4. For a nonzero polynomial $F \in R = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_s]$ the *Jacobian ring* $R(F)$ denotes the quotient of R by the ideal generated by the partial derivatives $\frac{\partial F}{\partial y_j}, j = 1, \dots, s, \frac{\partial F}{\partial x_i}, i = 1, \dots, n$.

Remark 3.5. If $F = y_1 f_1 + \dots + y_s f_s$ is as in Lemma 2.1 with $f_j \in S_{\alpha_j}$, then $R(F)$ carries a natural grading by the Chow group $A_{d+s-2}(\mathbf{P}(E))$. Moreover, there are canonical isomorphisms $A_{d+s-2}(\mathbf{P}(E)) \cong A_{d-1}(\mathbf{P}) \oplus A_d(\mathbf{P}) \cong A_{d-1}(\mathbf{P}) \oplus \mathbf{Z}$ ([F2]). With respect to this bigrading of the Chow group $A_{d+s-2}(\mathbf{P}(E))$ we have that $\deg(F) = (0, 1), \deg(f_j) = (\alpha_j, 0), \deg(y_j) = (-\alpha_j, 1)$, which is very similar to the case when \mathbf{P} is a projective space.

We now can state the main result.

Theorem 3.6. *Let \mathbf{P} be a d -dimensional complete simplicial toric variety, and let $X \subset \mathbf{P}$ be a quasi-smooth intersection of ample hypersurfaces defined by $f_j \in S_{\alpha_j}, j = 1, \dots, s$. If $F = y_1 f_1 + \dots + y_s f_s$, then for $p \neq \frac{d+s-1}{2}$, we have a canonical isomorphism*

$$R(F)_{(d+s-p)\beta-\beta_0} \cong H_{\text{var}}^{p-s, d-p}(X)$$

where $\beta_0 = \deg(x_1 \cdots x_n \cdot y_1 \cdots y_s), \beta = \deg(F) = \deg(f_j) + \deg(y_j)$. In the case $p = \frac{d+s-1}{2}$ there is an exact sequence

$$0 \rightarrow H^{d-s-1}(\mathbf{P}) \xrightarrow{\cup[X]} H^{d+s-1}(\mathbf{P}) \rightarrow R(F)_{\frac{d+s+1}{2}\beta-\beta_0} \rightarrow H_{\text{var}}^{\frac{d-s-1}{2}, \frac{d-s+1}{2}}(X) \rightarrow 0.$$

Proof. Since $H^i(\mathbf{P})$ vanishes for i odd and has a pure Hodge structure of type (p, p) for i even, from Proposition 3.2 we get $\mathrm{Gr}_F^p H^{d+s-1}(\mathbf{P} \setminus X) \cong H_{var}^{p-s, d-p}(X)$ if $p \neq \frac{d+s-1}{2}$, and in case $p = \frac{d+s-1}{2}$ the following sequence

$$0 \rightarrow H^{d-s-1}(\mathbf{P}) \xrightarrow{U[X]} H^{d+s-1}(\mathbf{P}) \rightarrow \mathrm{Gr}_F^{\frac{d+s-1}{2}} H^{d+s-1}(\mathbf{P} \setminus X) \rightarrow H_{var}^{\frac{d-s-1}{2}, \frac{d-s+1}{2}}(X) \rightarrow 0$$

is exact.

Now use the isomorphism of mixed Hodge structures $H^i(\mathbf{P} \setminus X) \cong H^i(\mathbf{P}(E) \setminus Y)$ and by the Theorem 10.6 [BC] the desired result follows. \square

4. Cohomology of nondegenerate intersections.

In this section we consider a special case of quasi-smooth intersections.

Definition 4.1. A closed subset $X = X_{f_1} \cap \dots \cap X_{f_s}$, defined by \mathbf{D} -homogeneous polynomials f_1, \dots, f_s , is called a *nondegenerate intersection* if $X_{f_{j_1}} \cap \dots \cap X_{f_{j_k}} \cap \mathbf{T}_\tau$ is a smooth subvariety of codimension k in \mathbf{T}_τ for any $\{j_1, \dots, j_k\} \subset \{1, \dots, s\}$ and $\tau \in \Sigma$. (Here \mathbf{T}_τ denotes the torus in \mathbf{P}_Σ associated with a cone $\tau \in \Sigma$.)

We will show how to define a nondegenerate intersection in terms of the polynomials f_1, \dots, f_s . For $\sigma \in \Sigma$, let $U_\sigma = \{x \in \mathbf{A}^n : \hat{x}_\sigma \neq 0\}$. We know that \mathbf{P}_Σ has an affine toric open cover by $\mathbf{A}_\sigma = U_\sigma / \mathbf{D}(\Sigma)$, $\sigma \in \Sigma$ [BC]. Also $\mathbf{T}_\tau = (U_\tau \setminus \cup_{\gamma \prec \tau} U_\gamma) / \mathbf{D}(\Sigma)$. Notice that $U_\tau \setminus \cup_{\gamma \prec \tau} U_\gamma = \{x \in \mathbf{A}^n : \hat{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i \subset \tau\}$ is a torus. So each \mathbf{T}_τ is a quotient of a torus by a D -subgroup, because \mathbf{D} is diagonalizable [BC].

Lemma 4.2. *Let $T = (\mathbf{C}^*)^n / G$ be the quotient of a torus by a D -subgroup G . Suppose that $X \subset (\mathbf{C}^*)^n$ is an invariant subvariety with respect to the action of G . Then the geometric quotient X/G is smooth iff X is smooth.*

Proof. By the structure theorem of a D -group [Hu, §16.2] we can assume that $(\mathbf{C}^*)^n = G^\circ \times (\mathbf{C}^*)^k$, where $G^\circ \cong (\mathbf{C}^*)^{n-k}$ is the identity component of G , and $G = G^\circ \times H$ for some finite subgroup H in $(\mathbf{C}^*)^k$. Now it suffices to show the Lemma if G is a torus or a finite group. If $G = G^\circ$ then $X = (\mathbf{C}^*)^{n-k} \times p(X)$, where by $p(X)$ we mean the projection of X onto $(\mathbf{C}^*)^k$. Notice that $p(X) \cong X/G$, hence X is smooth iff X/G is smooth. In the case $G = H$ is a finite group it can be easily checked that $X \rightarrow X/G$ is an unramified cover [Sh, p. 346]. So X and X/G are smooth simultaneously. \square

From this Lemma it follows that $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection iff $\mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$ is a smooth subvariety of codimension k in the torus $V_\tau = \{x \in \mathbf{A}^n : \hat{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i \subset \tau\}$.

As in Section 2 we can consider the hypersurface $Y \subset \mathbf{P}(E)$ defined by $F = \sum_{j=1}^s y_j f_j$. □

Lemma 4.3. $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection iff Y is a nondegenerate hypersurface.

Proof. As shown above, $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection if the rank $(\frac{\partial f_j}{\partial x_i}(x))_{\substack{j \in \{j_1, \dots, j_k\} \\ i \in \{i: e_i \notin \tau\}}} = k$ for all $x \in \mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$, $\tau \in \Sigma$ and $\{j_1, \dots, j_k\} \subset \{1, \dots, s\}$. Similarly Y is nondegenerate iff $z = (x, y) \in \mathbf{V}(F) \cap V_{\tilde{\tau} + \tau'}$, $\tilde{\tau} + \tau' \in \tilde{\Sigma}$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$ (recall the definition of $\mathbf{P}(E)$ associated with $\tilde{\Sigma}$ in the Section 2) implies that one of the partial derivatives $\frac{\partial F}{\partial y_j}(z) = f_j(x)$, $j \in \{j : \tilde{n}_j \notin \tau'\}$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde{e}_i \notin \tilde{\tau}\}$, is nonzero.

Let $(x, y) \in \mathbf{V}(F) \cap V_{\tilde{\tau} + \tau'}$, where $\tilde{\tau} + \tau' \in \tilde{\Sigma}$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$. Then $\prod_{\tilde{e}_i \notin \tilde{\tau}} x_i \prod_{\tilde{n}_j \notin \tau'} y_j \neq 0$ and $x_i = 0$ if $\tilde{e}_i \in \tilde{\tau}$, $y_j = 0$ if $\tilde{n}_j \in \tau'$. If $f_j(x) = 0$ for all $j \in \{j : \tilde{n}_j \notin \tau'\}$, then $x \in \mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$ where $\{j_1, \dots, j_k\} = \{j : \tilde{n}_j \notin \tau'\}$. So if $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde{e}_i \notin \tilde{\tau}\}$, is nonzero.

Conversely, suppose Y is nondegenerate. Take any $x \in \mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$ with $\tau \in \Sigma$, $\{j_1, \dots, j_k\} \subset \{1, \dots, s\}$. Then $(x, y) \in \mathbf{V}(F) \cap V_{\tilde{\tau} + \tau'}$ for each $y \in V_{\tau'} = \{y \in \mathbf{A}^s : y_j \neq 0 \text{ if } \tilde{n}_j \notin \tau', y_j = 0 \text{ if } \tilde{n}_j \in \tau'\}$ where τ' is the cone generated by the complement of $\{\tilde{n}_{j_1}, \dots, \tilde{n}_{j_k}\}$ in the set $\{\tilde{n}_1, \dots, \tilde{n}_s\}$. Therefore $\sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some i , which means the rank $(\frac{\partial f_j}{\partial x_i}(x))_{\substack{j \in \{j_1, \dots, j_k\} \\ i \in \{i: e_i \notin \tau\}}} = k$. □

Since a nondegenerate hypersurface is quasi-smooth [BC], Lemma 2.2 shows that a nondegenerate intersection is quasi-smooth.

Definition 4.4 ([BC]). Given a polynomial $f \in S = \mathbf{C}[x_1, \dots, x_n]$, we get the ideal quotient $J_1(f) = \langle x_1 \partial f / \partial x_1, \dots, x_n \partial f / \partial x_n \rangle : x_1 \cdots x_n$ (see [CLO, p. 193]) and the ring $R_1(f) = S / J_1(f)$.

Remark 4.5. If $F = \sum_{j=1}^s y_j f_j \in R$ is as in Lemma 2.1, then $R_1(F) = R / J_1(F)$ has a natural grading by the Chow group $A_{d+s-2}(\mathbf{P}(E)) \cong A_{d-1}(\mathbf{P}) \oplus \mathbf{Z}$.

Theorem 4.6. Let $X = X_{f_1} \cap \dots \cap X_{f_s}$ be a nondegenerate intersection of ample hypersurfaces given by $f_j \in S_{\alpha_j}$, $j = 1, \dots, s$. If $F = \sum_{j=1}^s y_j f_j \in R$, then there is a canonical isomorphism

$$H_{\text{var}}^{p-s, d-p}(X) = R_1(F)_{(d+s-p)\beta - \beta_0},$$

where $\beta_0 = \deg(x_1 \cdots x_n \cdot y_1 \cdots y_s)$, $\beta = \deg(F)$.

Proof. First we will show that there is an isomorphism of Hodge structures $H_{\text{var}}^{d-s}(X)(1-s) \cong H_{\text{var}}^{d+s-2}(Y)$. Let $\varphi : Y \rightarrow \mathbf{P}$ be the composition of the inclusion $j : Y \hookrightarrow \mathbf{P}(E)$ and the projection $\pi : \mathbf{P}(E) \rightarrow \mathbf{P}$. As in [Te], consider the following morphism of the Leray spectral sequences

$$\begin{array}{ccc} E_2^{p,q} = H^p(\mathbf{P}, R^q\pi_*\mathbf{C}) & \Rightarrow & H^{p+q}(\mathbf{P}(E)) \\ & \downarrow & \downarrow \\ {}'E_2^{p,q} = H^p(\mathbf{P}, R^q\varphi_*\mathbf{C}) & \Rightarrow & H^{p+q}(Y). \end{array}$$

Since

$$\varphi^{-1}(X) = \begin{cases} \mathbb{P}^{s-1} & \text{if } x \in X, \\ \mathbb{P}^{s-2} & \text{if } x \notin X, \end{cases}$$

we have that (see [Go, p. 202], [De])

$$R^q\varphi_*\mathbf{C} = \begin{cases} \mathbf{C}_{\mathbf{P}}(-\frac{q}{2}) & \text{if } q \text{ is even and } 0 \leq q < 2s - 2, \\ \mathbf{C}_X(1-s) & \text{if } q = 2s - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Also we have

$$R^q\pi_*\mathbf{C} = \begin{cases} \mathbf{C}_{\mathbf{P}}(-\frac{q}{2}) & \text{if } q \text{ is even and } 0 \leq q \leq 2s - 2, \\ 0 & \text{otherwise.} \end{cases}$$

The first spectral sequence degenerates at E_2 , because for p or q odd $E_r^{p,q}$ vanishes. The second spectral sequence also degenerates at E_2 :

$$h^{l-2s-2}(X) + \sum_{q=0}^{2s-4} h^{l-q}(\mathbf{P}) = \sum_{p+q=l} \dim {}'E_2^{p,q} \geq \sum_{p+q=l} \dim E_{\infty}^{p,q} = h^l(Y).$$

To show the degeneracy of $'E_2^{p,q}$ it suffices to show that the above inequality is an equality. From Proposition 10.8 [BC] and Proposition 3.2 we get

$$\begin{aligned} h^{d+s-2}(Y) &= h^{d+s-2}(\mathbf{P}(E)) + h^{d+s-1}(\mathbf{P}(E) \setminus Y) \\ &\quad - h^{d+s-1}(\mathbf{P}(E)) + h^{d+s-3}(\mathbf{P}(E)), \\ h^{d-s}(X) &= h^{d-s}(\mathbf{P}) + h^{d+s-1}(\mathbf{P} \setminus X) - h^{d+s-1}(\mathbf{P}) + h^{d-s-1}(\mathbf{P}). \end{aligned}$$

Hence, using the spectral sequence $E_2^{p,q}$, we can easily compute the Hodge numbers of $\mathbf{P}(E)$ and check that $h^{l-2s-2}(X) + \sum_{q=0}^{2s-4} h^{l-q}(\mathbf{P}) = h^l(Y)$ for $l = d + s - 2$. Using Proposition 1.4, we can similarly show the above equality for $l \neq d + s - 2$ as well. So the spectral sequence $'E_2^{p,q}$ degenerates at E_2 . Since $E_2^{d+s-2-q,q} = {}'E_2^{d+s-2-q,q}$ for $q \neq 2s - 2$ and, by Proposition 1.4, $E_2^{d-s,2s-2} \hookrightarrow {}'E_2^{d-s,2s-2}$, we get an isomorphism of Hodge structures (for details see [Te]):

$$H_{\text{var}}^{d+s-2}(Y) \cong {}'E_2^{d-s,2s-s} / E_2^{d-s,2s-2} \cong H_{\text{var}}^{d-s}(X)(1-s).$$

Now we only need to apply Theorem 11.8 [BC] to finish the proof. \square

References

- [BB] V.V. Batyrev and L. Borisov, *Dual cones and mirror symmetry for generalized Calabi-Yau manifolds*, in *Mirror Symmetry II*, Ed. B. Greene and S.-T. Yau, AMS-IP, Providence, 1996.
- [BC] V.V. Batyrev and D.A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, *Duke Math. J.*, **75** (1994), 293-338.
- [CCD] E. Cattani, D. Cox and A. Dickenstein, *Residues in toric varieties*, *Compositio Mathematica*, **108** (1997), 35-76.
- [C] D. Cox, *The homogeneous coordinate ring of a toric variety*, *J. Algebraic Geom.*, **4** (1995), 17-50.
- [CLO] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms*, Springer-Verlag, New-York, 1992.
- [Da] V. Danilov, *The geometry of toric varieties*, *Russian Math. Surveys*, **33** (1978), 97-154.
- [DK] V. Danilov and A. Khovanskii, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, *Math. USSR-Izv.*, **29** (1987), 279-298.
- [De] P. Deligne, *Théorie de Hodge*, II, III, *Inst. Hautes Études Sci. Publ. Math.*, **40** (1971), 5-58; **44** (1975), 5-77.
- [Di] A. Dimca, *Residues and cohomology of complete intersections*, *Duke Math. J.*, **78** (1995), 89-100.
- [Dol] I. Dolgachev, *Weighted projective varieties*, in 'Lecture Notes in Math.', **956**, Springer-Verlag, Berlin, (1982), 34-71.
- [F1] W. Fulton, *Introduction to Toric Varieties*, Princeton Univ. Press, Princeton, NJ, 1993.
- [F2] ———, *Intersection Theory*, Springer-Verlag, Berlin, 1984.
- [FH] W. Fulton and J. Hansen, *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, *Annals of Math.*, **110** (1979), 159-166.
- [FL1] W. Fulton and R. Lazarsfeld, *On the connectedness of degeneracy loci and special divisors*, *Acta Math.*, **146** (1981), 271-283.
- [FL2] ———, *Connectivity and its Applications in Algebraic Geometry*, *Lecture Notes in Math.*, **862**, Springer-Verlag, Berlin-Heidelberg, (1981), 26-92.
- [GKZ] I. Gelfand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser Verlag, Basel-Boston, 1994.
- [Go] R. Godement, *Topologie Algébrique et Théorie des Faceaux*, Hemann, Paris, 1958.
- [Gr] P. Griffiths, *On the periods of certain rational integrals*, I, II, *Ann. of Math.*, **90**(2) (1969), 460-495, 498-541.
- [H] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*, *Lecture Notes in Math.*, **156**, Springer-Verlag, Berlin-Heidelberg, 1970.
- [Hu] J. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, Berlin, 1977.
- [Ko] K. Konno, *On the variational Torelli problem for complete intersections*, *Comp. Math.*, **78** (1991), 271-296.
- [L] A. Libgober, *Differential forms on complete intersections and related quotient module*, *Israel Mathematical Conference Proceedings*, **9** (1996), 295-305.

- [M] Yu. Manin, *Lectures on the K-functor in algebraic geometry*, Russ. Math. Surveys, **24(5)** (1969), 1-89.
- [Na] J. Nagel, *The Abel-Jacobi map for complete intersections*, Indag. Math., **8** (1997), 95-113.
- [O] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, Berlin, 1988.
- [PS] C. Peters and J. Steenbrink, *Infinitesimal Variations of Hodge Structure and the Generic Torelli Problem for Projective Hypersurfaces (after Carlson, Donagi, Green, Griffiths, Harris)*, Classification of Algebraic and Analytic Manifolds, Ed. K. Ueno, Progr. Math., **39**, Birkhäuser, Boston, (1983), 399-463.
- [Sh] I.R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, New York-Berlin, 1974.
- [St] J. Steenbrink, *Intersection form for quasi-homogeneous singularities*, Compositio Math., **34** (1977), 211-223.
- [Te] T. Terasoma, *Infinitesimal variation of Hodge structures and the weak global Torelli theorem for complete intersections*, Ann. of Math., **132** (1990), 213-235.

Received February 18, 1998.

UNIVERSITY OF MASSACHUSETTS

AMHERST, MA 01003

E-mail address: anvar@math.umass.edu