CONNECTED SUMS OF SELF-DUAL MANIFOLDS AND
EQUIVARIANT RELATIVE SMOOTHINGS

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In this paper we address a problem in differential geometry using tools from algebraic geometry and the theory of singular complex spaces. We obtain examples of compact four dimensional self-dual conformal manifolds with torus symmetry and positive scalar curvature from twistor spaces with divisors and we study the local moduli of such geometries.

1. Introduction.

Recall that a conformal 4-manifold is called *self-dual* if its Weyl curvature, considered as a bundle valued 2-form, is in the +1 eigenspace of the Hodge star-operator [**1**]. Due to Schoen's proof [**19**] of the Yamabe conjecture it is known that within any conformal class on a compact manifold is a metric whose scalar curvature is constant and the sign of this constant is a conformal invariant. The main objective of this paper is to ensure that the scalar curvature is positive for the self-dual structures on the connected sums $n\mathbb{CP}^2$ of complex projective planes found in [**13**]. The metrics admit a torus T^2 of orientation preserving conformal isometries.

For self-dual metrics the total space Z of the bundle of anti-self-dual 2-forms of unit length is a complex 3-manifold. This complex manifold is the *twistor space* [**1**] and it gives an alternative description of self-duality. Indeed, Donaldson and Friedman [**2**] used a desingularisation of a singular model of the desired twistor space to prove existence of self-dual structures on $n\mathbb{CP}^2$. The self-dual metric on $\mathbb{CP}^2$ is the Fubini-Study metric and the full moduli on $2\mathbb{CP}^2$ had previously been obtained [**15**] via a different twistor construction. In [**13**] we adapted the theory of Donaldson and Friedman to obtain equivariant connected sums of compact self-dual manifolds.

If the symmetry group is at least three-dimensional it is known [**16**] that the conformal metric is of non-negative type. In contrast Kim [**7**] obtained S^1-symmetric examples of negative scalar curvature while LeBrun [**11**] gave examples on $n\mathbb{CP}^2$ of positive scalar curvature and with symmetry group S^1. These metrics were obtained via an ansatz involving monopoles on hyperbolic 3-space. Similarly, Joyce [**6**] obtained T^2-symmetric metrics on $n\mathbb{CP}^2$ of positive type using hyperbolic monopoles in two dimensions. These constructions give relatively easy access to knowledge about scalar curvature

while such insight is absent from our equivariant smoothings [**13**]. However, it is known [**3**] that if the complex manifold Z carries effective divisors, then the corresponding metric is of non-negative type and if futhermore the intersection form of the 4-manifold is positive definite the scalar curvature is in fact positive [**10**].

Kim and Pontecorvo [**8**] extended the work of Donaldson and Friedman obtaining a way of constructing scalar-flat Kähler surfaces based on relative complex deformations of singular twistor spaces with divisors of degree 1. We combine the equivariant and the relative smoothing programme to obtain a local moduli space of dimension $n-1$ of T^2-symmetric self-dual structures on $n\mathbb{CP}^2$ such that the associated twistor spaces all have smooth degree 1 and 2 divisors. In particular, all these self-dual metrics have positive scalar curvature.

We also use our equivariant relative smoothing approach to obtain anti-self-dual Hermitian metrics with non-semi-free circle action on the blow up $(S^1 \times S^3) \# n\overline{\mathbb{CP}}^2$, for $n \geq 3$, of the Hopf suface in more than two points. This should be compared with LeBrun's examples on the same spaces [**12**] but with semi-free circle symmetry.

2. The Geometrical Construction.

Our study of self-duality on a 4-manifold M takes place on the associated twistor space Z. As a smooth manifold Z is the total space of the sphere bundle $S(\Lambda^2_-) \to M$ of the bundle of anti-self-dual 2-forms. Equivalently, Z is the total space of the bundle $P(V^-) \xrightarrow{\pi} M$ of projectivised half spinors. The twistor space is a complex 3-manifold and for any point $x \in M$ the *twistor line* in Z, $\pi^{-1}(x) = L_x$, is a rational curve with normal bundle N the sum $\mathcal{O}(1) \oplus \mathcal{O}(1)$ of degree 1 line bundles. Invariantly $N = H^- \otimes V^+_x$ where $H^- \to P(V^-_x)$ is the hyperplane bundle. The antipodal map on the fibers of $\pi : Z \to M$ induces a *real structure* on Z [**1**]. The anti-canonical bundle $K^{-1} = \Lambda^{3,0}TZ$ of Z has a square root $K^{-\frac{1}{2}}$ which, when restricted onto a twistor line, coincides with the degree 2 holomorphic line bundle $\mathcal{O}(2)$. The zero set in the twistor space of a real holomorphic section of $K^{-\frac{1}{2}}$ is called a *degree* 2 *divisor*. Such a divisor S may either be irreducible, in which case it is smooth [**14**], or it may decompose into a conjugate pair $D, \bar{D}$ of smooth degree 1 divisors meeting in exactly one twistor line [**17**].

Now, assume we have two self-dual manifolds M_1 and M_2 with associated twistor spaces Z_1 and Z_2. Furthermore, assume Z_1 contains an irreducible degree 2 divisor S and let $L_1 = L_{x_1}$ be a twistor line above $x_1 \in M_1$ meeting S transversely in a conjugate pair of points q and $\bar{q}$. Also, let $D_2, \overline{D_2}$ be a conjugate pair of divisors in Z_2 meeting in the twistor line $L_2 = L_{x_2}$ above $x_2 \in M_2$. Then, following Donaldson and Friedman [**2**], we blow up the twistor spaces Z_i along the twistor lines L_i, $i = 1, 2$. The proper transform

$\tilde{S}$ of S in the blow-up $\tilde{Z}_1$ of Z_1 meets the exceptional divisor Q_1 in two rational curves C_1 and $\bar{C}_1$. The normal bundles $N_{C_1}^{\tilde{S}}, N_{\bar{C}_1}^{\tilde{S}}$ relative to $\tilde{S}$ are both equal to $\mathcal{O}(-1)$. In the blow-up $\tilde{Z}_2$ of Z_2 the proper transforms $\tilde{D}_2, \tilde{\bar{D}}_2$ of $D_2, \bar{D}_2$, respectively, meet the exceptional divisor Q_2 in a conjugate pair of rational curves $C_2, \bar{C}_2$ with normal bundles $N_{C_2}^{\tilde{D}_2} = \mathcal{O}(1)$ and $N_{\bar{C}_2}^{\tilde{\bar{D}}_2} = \mathcal{O}(1)$.

From an orientation reversing isometry $\phi : T_{x_1}M_1 \to T_{x_2}M_2$ we obtain a holomorphic isomorphism

$$\phi = \phi_+ \times \phi_- : Q_1 = P(V_{x_1}^+) \times P(V_{x_1}^-) \to Q_2 = P(V_{x_2}^-) \times P(V_{x_2}^+)$$

and we construct the singular complex space $\mathcal{Z} = \tilde{Z}_1 \cup_\phi \tilde{Z}_2$ with normal crossing singularity along $Q = Q_1 = \phi^{-1}(Q_2)$. In $\mathcal{Z}$ we have the singular divisor $\mathcal{S}$ with normal crossing singularities along $C = C_1 = \phi^{-1}(C_2)$ and $\bar{C} = \bar{C}_1 = \phi^{-1}(\bar{C}_2)$.

Inspired by Kim and Pontecorvo [8] we now proceed to study smoothings of the pair $(\mathcal{Z}, \mathcal{S})$. The aim is to show that in some cases the connected sum has a twistor space with a degree 2 divisor. Let us introduce the necessary notation: If $Y_1, \ldots, Y_n$ are smooth submanifolds of a compact complex manifold X we consider the sheaf $\Theta_{XY_1,\ldots,Y_n}$ of holomorphic vector fields on X which are tangent to Y_i along Y_i, $i = 1, \ldots, n$. It is well known [9] that infinitesimal relative deformations are given by the first cohomology of this sheaf and that obstructions lie in the second cohomology. For a compact singular complex space $\mathcal{X}$ the deformation theory is described in terms of global extension groups $T_{\mathcal{X}}^i = \mathrm{Ext}^i(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ where $\Omega_{\mathcal{X}}$ is the sheaf of Kähler differentials and $\mathcal{O}_{\mathcal{X}}$ is the structure sheaf. These groups are computed from the sheaves $\tau_{\mathcal{X}}^i = \mathcal{E}xt^i(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ using the local to global spectral sequence $E_2^{p,q} = H^p(\tau_{\mathcal{X}}^q) \Rightarrow T_{\mathcal{X}}^{p+q}$ [2], [4]. Here $T_{\mathcal{X}}^0$ is the Lie algebra of the group of automorphisms of $\mathcal{X}$, the first order deformations lie in $T_{\mathcal{X}}^1$ and obstructions are in $T_{\mathcal{X}}^2$. For normal crossings $\tau_{\mathcal{X}}^2 = 0$ and the local to global spectral sequence is given as

$$(2.1) \qquad 0 \to H^1(\tau_{\mathcal{X}}^0) \to T_{\mathcal{X}}^1 \to H^0(\tau_{\mathcal{X}}^1) \to H^2(\tau_{\mathcal{X}}^0) \to T_{\mathcal{X}}^2 \to H^1(\tau_{\mathcal{X}}^1).$$

In the situations we are going to investigate, we have $H^1(\tau_{\mathcal{X}}^1) = 0$. Furthermore, we impose conditions implying the vanishing of $H^2(\tau_{\mathcal{X}}^0)$ so that $T_{\mathcal{X}}^2 = 0$ and the deformations are unobstructed and parametrised by $T_{\mathcal{X}}^1$. The subspace $H^1(\tau_{\mathcal{X}}^0)$ corresponds to deformations for which the singularities remain locally a product. If the image of an element in the projection $T_{\mathcal{X}}^1 \to H^0(\tau_{\mathcal{X}}^1)$ does not vanish the corresponding deformed space is smooth. Furthermore, for $\mathcal{X}$ equal to the singular twistor space $\mathcal{Z}$ the smoothing results in a twistor space with real structure [2].

The singular theory above will be applied to $\mathcal{X} = \mathcal{Z}$ and $\mathcal{X} = \mathcal{S}$. To study deformations of the singular pair $f : \mathcal{S} \hookrightarrow \mathcal{Z}$ we employ the theory of Ran [18]: Let $T_{\mathcal{Z}|\mathcal{S}}^i$ denote the extension groups $\mathrm{Ext}^i(\Omega_{\mathcal{Z}|\mathcal{S}}, \mathcal{O}_{\mathcal{S}})$ [8] with the local

to global spectral sequence $E_2^{p,q} = H^p\big(\mathcal{E}xt^q(\Omega_{\mathcal{Z}|\mathcal{S}}, \mathcal{O}_{\mathcal{S}})\big)$ where $E_2^{p,q} \Rightarrow T_{\mathcal{Z}|\mathcal{S}}^{p+q}$. This gives the sequence

$$(2.2) \qquad 0 \to H^1(\tau_{\mathcal{Z}|\mathcal{S}}^0) \to T_{\mathcal{Z}|\mathcal{S}}^1 \to H^0(\tau_{\mathcal{Z}|\mathcal{S}}^1) \to H^2(\tau_{\mathcal{Z}|\mathcal{S}}^0).$$

Furthermore, the groups $T_{\mathcal{Z}|\mathcal{S}}^i$ fit into a sequence

$$(2.3) \qquad \to T_f^1 \to T_{\mathcal{S}}^1 \oplus T_{\mathcal{Z}}^1 \to T_{\mathcal{Z}|\mathcal{S}}^1 \to T_f^2 \to T_{\mathcal{S}}^2 \oplus T_{\mathcal{Z}}^2 \to$$

where T_f^i is a natural derived functor and T_f^1 gives infinitesimal deformations of the singular pair. Also, T_f^0 is the Lie algebra of the symmetries of the pair $(\mathcal{Z}, \mathcal{S})$ and T_f^2 is the obstruction space. For pairs $(\mathcal{X}, \mathcal{Y})$ of complex spaces we also have the sheaf $\tau_{\mathcal{X}\mathcal{Y}}^0$ of derivations of $\mathcal{X}$ preserving the ideal sheaf of $\mathcal{Y}$. This sheaf of relative derivations coincide with Θ_{XY} for pairs of smooth manifolds. From inclusions and restrictions we obtain the exact sequence

$$(2.4) \qquad 0 \to \tau_{\mathcal{X}\mathcal{Y}}^0 \to \tau_{\mathcal{Y}}^0 \oplus \tau_{\mathcal{X}}^0 \to \tau_{\mathcal{X}|\mathcal{Y}}^0 \to 0.$$

In the associated sequence on cohomology

$$(2.5) \qquad H^1(\tau_{\mathcal{X}\mathcal{Y}}^0) \to H^1(\tau_{\mathcal{Y}}^0) \oplus H^1(\tau_{\mathcal{X}}^0) \to H^1(\tau_{\mathcal{X}|\mathcal{Y}}^0) \to H^2(\tau_{\mathcal{X}\mathcal{Y}}^0)$$

the first term corresponds to deformations for which the singularities remain locally a product so $H^1(\tau_{\mathcal{X}\mathcal{Y}}^0)$ sits naturally inside T_f^1. Call the quotient space Δ. Then we have the short exact sequence

$$(2.6) \qquad 0 \to H^1(\tau_{\mathcal{X}\mathcal{Y}}^0) \to T_f^1 \to \Delta \to 0.$$

Now put together (2.1) with $\mathcal{X} = \mathcal{Z}$ and $\mathcal{X} = \mathcal{S}$, (2.2), (2.3), (2.5) with $\mathcal{X} = \mathcal{Z}$ and $\mathcal{Y} = \mathcal{S}$ and (2.6) to get the following commutative diagram.

(2.7)

$$
\begin{array}{ccccc}
0 & & H^2(\tau_{\mathcal{S}}^0) \oplus H^2(\tau_{\mathcal{Z}}^0) & & H^2(\tau_{\mathcal{Z}|\mathcal{S}}^0) \\
\uparrow & & \uparrow & & \uparrow \\
\Delta & \xrightarrow{\alpha} & H^0(\tau_{\mathcal{S}}^1) \oplus H^0(\tau_{\mathcal{Z}}^1) & \xrightarrow{\beta} & H^0(\tau_{\mathcal{Z}|\mathcal{S}}^1) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
T_f^1 & \longrightarrow & T_{\mathcal{S}}^1 \oplus T_{\mathcal{Z}}^1 & \xrightarrow{\gamma} & T_{\mathcal{Z}|\mathcal{S}}^1 & \longrightarrow & T_f^2 & \longrightarrow T_{\mathcal{S}}^2 \oplus T_{\mathcal{Z}}^2 \\
\uparrow & & \uparrow & & \uparrow \\
H^1(\tau_{\mathcal{Z}\mathcal{S}}^0) & \longrightarrow & H^1(\tau_{\mathcal{S}}^0) \oplus H^1(\tau_{\mathcal{Z}}^0) & \longrightarrow & H^1(\tau_{\mathcal{Z}|\mathcal{S}}^0) & \longrightarrow H^2(\tau_{\mathcal{Z}\mathcal{S}}^0) \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}
$$

The morphism α is the quotient of the morphism $T_f^1 \to T_{\mathcal{S}}^1 \oplus T_{\mathcal{Z}}^1$ and the squares in the diagram are commutative by naturality. The row with the morphisms α and β needs more explanation: The sheaf $\tau_{\mathcal{Z}}^1$ is supported on

$Q \subseteq \mathcal{Z}$ and is the sheaf of sections of $N_{Q_1}^{\tilde{Z}_1} \otimes \phi^* N_{Q_2}^{\tilde{Z}_2} = \mathcal{O}_Q$ [**2**]. Likewise, a choice of trivialization $\tau_{\mathcal{Z}}^1 \cong \mathcal{O}_Q$ induces trivializations $\tau_{\mathcal{S}}^1 \cong \mathcal{O}_{C\bar{C}}$ and $\tau_{\mathcal{Z}|\mathcal{S}}^1 \cong \mathcal{O}_{Q|\mathcal{S}} \cong \mathcal{O}_{C\bar{C}}$. Then $H^0(\tau_{\mathcal{S}}^1) = H^0(\mathcal{O}_C) \oplus H^0(\mathcal{O}_{\bar{C}})$, $H^0(\tau_{\mathcal{Z}}^1) = H^0(\mathcal{O}_Q)$, $H^0(\tau_{\mathcal{Z}|\mathcal{S}}^1) = H^0(\mathcal{O}_C) \oplus H^0(\mathcal{O}_{\bar{C}})$ and $\beta\big((a,b),c\big) = (a-c,b-c)$ which certainly is surjective. The kernel of β is $\langle c,c,c \rangle \cong \mathbb{C}$.

Now we shall make the following assumptions which are satisfied in examples to be considered later:

Assumption 2.1. *In the case where the degree 2 divisor is irreducible we assume the vanishing of the following cohomology groups:* $H^2(Z_i, \Theta_{Z_i})$, $i = 1, 2$; $H^2(S, \Theta_S)$; $H^2(D_2, \Theta_{D_2})$; $H^2(\bar{D}_2, \Theta_{\bar{D}_2})$; $H^2(Z_1, \Theta_{Z_1 S})$ *and* $H^2(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})$.

We shall show how these assumptions lead to the vanishing of $H^2(\tau_{\mathcal{S}}^0)$, $H^2(\tau_{\mathcal{Z}}^0)$ and $H^2(\tau_{\mathcal{Z}\mathcal{S}}^0)$ and therefore from (2.1) we have the vanishing of the obstruction spaces $T_{\mathcal{Z}}^2$ and $T_{\mathcal{S}}^2$. We also see that Δ is equal to the kernel of β. Now a diagram chasing gives the surjectivity of the morphism γ so $T_f^2 = 0$. Therefore, any element ω in the complement of the hyperplane $H^1(\tau_{\mathcal{Z}\mathcal{S}}^0)$ of T_f^1 gives smoothings of $\mathcal{Z}$ and $\mathcal{S}$ as ω maps to non-zero elements in $H^0(\tau_{\mathcal{S}}^1)$ and $H^0(\tau_{\mathcal{Z}}^1)$. Thus the twistor space of the connected sum $M_1 \# M_2$ has a degree 2 divisor. Later we return to a calculation of the dimension of the space T_f^1 of infinitesimal deformations.

We shall also consider the situation where the degree 2 divisor S decomposes into a conjugate pair $D_1, \bar{D}_1$ of degree 1 divisors meeting in a twistor line L. We still blow up Z_1 along a twistor line L_1 intersecting $D_1 + \bar{D}_1$ transversely in a conjugate pair of points. Also the proper transforms $\tilde{D}_1, \tilde{\bar{D}}_1$ meet Q_1 in C_1 and $\bar{C}_1$. As before we construct the singular twistor space $\mathcal{Z}$ with the singular divisors $\mathcal{D}, \bar{\mathcal{D}}$ having normal crossing singularities along C and $\bar{C}$ respectively. The divisors $\mathcal{D}$ and $\bar{\mathcal{D}}$ meet in L. In this situation $\mathcal{S}$ is the singular space $\mathcal{D} \cup_L \bar{\mathcal{D}}$ with normal crossing singularities along L, C and $\bar{C}$. We make the

Assumption 2.2. *In the case where the degree 2 divisor is reducible we assume the vanishing of the following cohomology groups:* $H^2(Z_i, \Theta_{Z_i})$, $H^2(D_i, \Theta_{D_i})$, $H^2(\bar{D}_i, \Theta_{\bar{D}_i})$, $i = 1, 2$; $H^2(Z_1, \Theta_{Z_1 S})$ *and* $H^2(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})$.

Furthermore, we get a diagram similar to (2.7) and as in the case where S is irreducible the smoothing is unobstructed. However, we do not control the singularities of $\mathcal{S}$ along L. Therefore, a smoothing of $\mathcal{Z}$ may produce an irreducible smooth degree 2 divisor. We shall discuss this problem at the end of Section 5.

3. Symmetries.

In this section we assume M_1 and M_2 have a torus group T^2 of orientation preserving conformal transformations with fixed points x_1 and x_2 respectively.

Furthermore, the orientation reversing isometry $\phi : T_{x_1}M_1 \rightarrow T_{x_2}M_2$, used to make the connected sum, is assumed to be an equivariant map. In this situation we proved in [**13**] that if the smoothing is unobstructed then also the equivariant smoothing is unobstructed. We refer to that paper for details. Now we want to impose the symmetries onto the relative smoothing described in Section 2.

Recall [**13**] that we may choose metrics in M_1 and M_2 for which the torus acts as isometries. Therefore, the isotropy representation ι at the fixed points map into $SO(4)$. Also, the symmetries lift to the twistor space to give real holomorphic automorphisms. If x_1 is a fixed point in M_1, and L_1 is the twistor line with normal bundle N, then the well known isomorphisms from twistor theory

$$(3.1) \qquad (T_{x_1}M_1)^{\mathbb{C}} = H^0(L_1, N) = V_{x_1}^+ \otimes V_{x_1}^-$$

are equivariant. This means the isotropy representation on the complexified tangent space coincide with the induced holomorphic action on the sections of the normal bundle and with the tensor product $\iota_+ \otimes \iota_-$ on $V_{x_1}^+ \otimes V_{x_1}^-$. Here $\iota_\pm$ are defined modulo $\mathbb{Z}_2$ from the isotropy representation ι and the projections onto the factors of $SU(2) \times SU(2)$.

Since the twistor line L_1 intersects the divisor S transversely at two distinct points, we assume that S is reducible to $D_1 + \bar{D}_1$ when we consider the twistor geometry in a neighbourhood of L_1. Now suppose the divisors $D_i, \bar{D}_i$ in Z_i, $i = 1, 2$ are invariant. Then the points $q = P(V_{x_1}^-) \cap D_1$ and $\bar{q} = P(V_{x_1}^-) \cap \bar{D}_1$ are fixed by the action on Z_1. Therefore if $v \in V_{x_1}^-$ represents q we have $V_{x_1}^- = \text{span}(v) \oplus \text{span}(\bar{v})$ as a decomposition into T^2-invariant subspaces. Furthermore, from the definition of the complex structure on Z_1 the horizontal space $T_q D_1$ is equal to $V_{x_1}^+ \otimes \left(\text{span}(v)\right)^*$ with the action $\iota_+ \otimes \iota_-^*$. We have $C_1 = P(T_q D_1)$ and the normal bundle $N_{C_1}^{D_1}$ is the universal subbundle of $T_q D_1$ over C_1.

On the other side we have $L_2 = P(V_{x_2}^-)$ and the normal bundles $N_{C_2}^{\tilde{D}_2} = N_{L_2}^{D_2}$, $N_{\bar{C}_2}^{\tilde{\bar{D}}_2} = N_{L_2}^{\bar{D}_2}$. As L_2 is equal to the transversal intersection $D_2 \cap \bar{D}_2$ we have $N_{L_2}^{Z_2} = N_{L_2}^{D_2} \oplus N_{L_2}^{\bar{D}_2}$ and this is a decomposition into invariant subbundles. From twistor theory we have $N_{L_2}^{Z_2} = V_{x_2}^+ \otimes H^-$. Also, the torus action on $V_{x_2}^+$ decomposes, $V_{x_2}^+ = E \oplus \bar{E}$, and the only decomposition of $V_{x_2}^+ \otimes H^-$ into invariant subspaces is $(E \otimes H^-) \oplus (\bar{E} \otimes H^-)$. We may therefore assume that $N_{L_2}^{D_2} = E \otimes H^-$ and $N_{L_2}^{\bar{D}_2} = \bar{E} \otimes H^-$.

The T^2-equivariant isomorphism $\phi_- : V_{x_1}^- \to V_{x_2}^+$ induced by $\phi : T_{x_1}M_1 \to T_{x_2}M_2$ can be assumed to satisfy $\phi_-\big(\mathrm{span}(v)\big) = E$ and $\phi_-\big(\mathrm{span}(\bar{v})\big) = \bar{E}$. Now we have $\mathcal{D} = \tilde{D}_1 \cup_{\phi_+} \tilde{D}_2$ where

$$\phi_+ : C_1 = P(T_q D_1) = P(V_{x_1}^+) \to P(V_{x_2}^-) = C_2$$

and similarly with $\bar{\mathcal{D}}$. Recall the fact [13] that $H^0(\tau_{\mathcal{Z}}^1)_{T^2} = H^0(\tau_{\mathcal{Z}}^1) \cong \mathbb{C}$ which is true because $\tau_{\mathcal{Z}}^1 = N_{Q_1}^{\tilde{Z}_1} \otimes \phi^*(N_{Q_2}^{\tilde{Z}_2}) = \mathcal{O}_Q$ and the action is trivial. Also, $H^0(\tau_{\mathcal{Z}|\mathcal{S}}^1)_{T^2} = H^0(\tau_{\mathcal{Z}|\mathcal{S}}^1) \cong \mathbb{C}^2$ because $\tau_{\mathcal{Z}}^1$ is supported by Q and Q intersects $\mathcal{S}$ transversely along C and $\bar{C}$.

Likewise we shall need the following:

Lemma 3.1. *The sheaf $\tau_{\mathcal{D}}^1$ is supported on the curve C and is trivial. All the sections are T^2-invariant, i.e. $H^0(\tau_{\mathcal{D}}^1)_{T^2} = H^0(\tau_{\mathcal{D}}^1) = \mathbb{C}$. Similarly $H^0(\tau_{\mathcal{S}}^1)_{T^2} = H^0(\tau_{\mathcal{S}}^1) = \mathbb{C}^2$ when the degree 2 divisor S is irreducible.*

Proof. Since the singularity is a normal crossing we have $\tau_{\mathcal{D}}^1 = N_{C_1}^{\tilde{D}_1} \otimes \phi^* N_{C_2}^{\tilde{D}_2}$ [2]. The curve C_1 is the exceptional divisor of the blowing up of q in D_1 so $N_{C_1}^{\tilde{D}_1} = \mathcal{O}(-1)$. Also, $N_{C_2}^{\tilde{D}_2}$ is isomorphic to $N_{L_2}^{D_2}$ so $N_{C_2}^{\tilde{D}_2} = \mathcal{O}(1)$. Thus $\tau_{\mathcal{D}}^1 = \mathcal{O}_C$. Furthermore, the torus action is trivial on $\tau_{\mathcal{D}}^1$: the action on $N_{C_1}^{\tilde{D}_1}$ is $\iota_+ \otimes \iota_-^*$ as this normal bundle is the universal subbundle of $T_q D_1 = V_{x_1}^+ \otimes \big(\mathrm{span}(v)\big)^*$. On the other hand $\phi^* N_{C_2}^{\tilde{D}_2} = \phi^* N_{L_2}^{D_2} = \phi^*(E \otimes H^-) = (\phi_-^* E) \otimes (\phi_+^* H^-) = \mathrm{span}(v) \otimes H^+ = \big((\mathrm{span}(v))^* \otimes U^+\big)^* = (N_{C_1}^{\tilde{D}_1})^*$. The torus action on $N_{L_2}^{D_2} = E \otimes H^-$ is $\iota_+ \otimes \iota_-^*$. Therefore the action on $\phi^* N_{L_2}^{D_2}$ is $\iota_- \otimes \iota_+^*$ and we get the trivial action on $\tau_{\mathcal{D}}^1 = N_{C_1}^{\tilde{D}_1} \otimes \phi^* N_{C_2}^{\tilde{D}_2}$ from the tensor product $\iota_+ \otimes \iota_-^* \otimes \iota_- \otimes \iota_+^*$. $\square$

4. Vanishing Theorems and the Unobstructed Case.

We consider an irreducible degree 2 divisor S and address the problem of proving the vanishing of $H^2(\tau_{\mathcal{Z}}^0)$, $H^2(\tau_{\mathcal{S}}^0)$ and $H^2(\tau_{\mathcal{Z}\mathcal{S}}^0)$. This will lead to a theorem on the unobstructed T^2-equivariant relative smoothing.

First some technical lemmae:

Lemma 4.1. *Let $Y \subseteq X$ be a smooth hypersurface in a complex manifold X. Then for any sufficiently small open set $U \subseteq X$ we have $H^i(U, \Theta_X) = 0$ and $H^i(U, \Theta_{XY}) = 0$, $i \geq 1$.*

Proof. We have $H^i(U, \Theta_X) = 0$, $i \geq 1$, as Θ_X is locally free and $H^i(U, \mathcal{O}) = 0$ by the $\bar{\partial}$-Poincaré Lemma. Similarly $H^i\big(U \cap Y, \mathcal{O}_Y(N_Y^X)\big) = 0$, $i \geq 1$.

From the sequence

$$0 \to \Theta_{XY} \to \Theta_X \to \mathcal{O}_Y(N_X^Y) \to 0$$

we get $H^i(U, \Theta_{XY}) = 0$, $i \geq 2$. We may assume U is a coordinate patch where the vanishing $x_1 = 0$ of the first coordinate defines Y. Therefore

$$0 \to H^0(U, \Theta_{XY}) \to H^0(U, \Theta_X) \to H^0\big(U \cap Y, \mathcal{O}_Y(N_Y^X)\big) \to 0$$

is exact and it follows that $H^1(U, \Theta_{XY}) = 0$. $\qquad\qquad\square$

Lemma 4.2. *Let $b_i : \tilde{Z}_i \to Z_i$ be the blow-up of Z_i along the real twistor line L_i, $i = 1, 2$. Then for all $j \geq 0$: $H^j(\tilde{Z}_i, \Theta_{\tilde{Z}_i}) = H^j(Z_i, \Theta_{Z_i L_i})$.*

Proof. We drop the subscript "i". The direct image sheaf $b_* \Theta_{\tilde{Z}}$ is isomorphic to Θ_{ZL} via the differential of b

$$db : H^0(b^{-1}(U), \Theta_{\tilde{Z}}) \to H^0(U, \Theta_{ZL})$$

for $U \subseteq Z$ open. Certainly db maps into $H^0(U, \Theta_{ZL})$ if we can prove that any vector field on $b^{-1}(U)$ is tangential to Q along Q. But this follows from the vanishing of $H^0(Q, [Q]) = H^0\big(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1, -1)\big)$ and from the sequence

$$0 \to TQ \to T\tilde{Z}|_Q \to [Q]|_Q \to 0.$$

Conversely, let L be given in local coordinates (z_1, z_2, z_3) by $z_2 = z_3 = 0$. Then, a vector field $X = a_1 \frac{\partial}{\partial z_1} + (a_2 z_2 + a_3 z_3) \frac{\partial}{\partial z_2} + (a_4 z_2 + a_5 z_3) \frac{\partial}{\partial z_3}$ in $H^0(U, \Theta_{ZL})$ can be lifted to $\tilde{X} = a_1 \frac{\partial}{\partial z_1} + (a_2 z_2 + a_3 z_2 \ell) \frac{\partial}{\partial z_2} + (a_4 + a_5 \ell) \frac{\partial}{\partial \ell}$ in $H^0(b^{-1}(U), \Theta_{\tilde{Z}})$ where $\ell = \frac{\ell_3}{\ell_2}$ is a coordinate on $b^{-1}(U) = \big\{ \big((z_1, z_2, z_3), [\ell_2, \ell_3]\big) \in U \times \mathbb{CP}^1 \mid \ell_2 z_3 = \ell_3 z_2 \big\}$.

The lemma now follows from the Leray spectral sequence [**4**] if we can prove that the sheaf $R^i b_*(\Theta_{\tilde{Z}})$ vanishes for all $i \geq 1$ where $R^i b_*(\Theta_{\tilde{Z}})(U) = H^i(b^{-1}(U), \Theta_{\tilde{Z}})$. Thus, consider the Leray cover $V_1 \cup V_2$ of $b^{-1}(U)$ given by coordinates (z_1, z_2, ℓ) and $(z_1, z_3, \frac{1}{\ell})$ respectively. Then $H^i(b^{-1}(U), \Theta_{\tilde{Z}}) = 0$, $i \geq 2$, and $H^1(b^{-1}(U), \Theta_{\tilde{Z}}) = H^1(V_1 \cap V_2, \Theta_{\tilde{Z}}) = H^1(\mathbb{C}^3 \backslash \mathbb{C}^2, \Theta_{\tilde{Z}}) = 0$ as $H^1(\mathbb{C}^3 \backslash \mathbb{C}^2, \mathcal{O}) = 0$ by the $\bar{\partial}$-Poincaré lemma. $\qquad\square$

Now, let us study the cohomology groups $H^j(\tau^0_{\mathcal{Z}\mathcal{S}})$. Let $\mathcal{Z}', \mathcal{S}'$ be the normalizations of $\mathcal{Z}, \mathcal{S}$ respectively and let also Q' denote the disjoint union of Q_1 and Q_2. Then we have the following sequence on $\mathcal{Z}$

$$(4.1) \qquad\qquad 0 \to \tau^0_{\mathcal{Z}\mathcal{S}} \to q_* \Theta_{\mathcal{Z}'\mathcal{S}'Q'} \to i_* \Theta_{QC\bar{C}} \to 0$$

where $q : \mathcal{Z}' \to \mathcal{Z}$ is the identification map and $i : Q \hookrightarrow \mathcal{Z}$ is the inclusion.

Lemma 4.3. *For all $j \geq 0$ we have:*

$$H^j(\mathcal{Z}, q_* \Theta_{\mathcal{Z}'\mathcal{S}'Q'}) = H^j(\mathcal{Z}', \Theta_{\mathcal{Z}'\mathcal{S}'Q'})$$
$$= H^j(\tilde{Z}_1, \Theta_{\tilde{Z}_1 \tilde{S} Q_1}) \oplus H^j(\tilde{Z}_2, \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2 Q_2}).$$

Proof. This follows from the Leray Spectral sequence if we can prove the vanishing of $R^i q_* \Theta_{\mathcal{Z}'\mathcal{S}'Q'}$, $i \geq 1$, or equivalently we need the vanishing of $H^i(U_1, \Theta_{\tilde{Z}_1 \tilde{S} Q_1})$ and $H^i(U_2, \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{\bar{D}}_2 Q_2})$, $i \geq 1$ for small open sets U_k, $k = 1, 2$. On U_1, where $U_1 \cap \tilde{S} \cap Q_1$ is non-empty, we have

$$(4.2) \qquad 0 \to \Theta_{\tilde{Z}_1 \tilde{S} Q_1} \xrightarrow{i} \Theta_{\tilde{Z}_1 \tilde{S}} \xrightarrow{\pi} \mathcal{O}_{Q_1}(N_{Q_1}^{\tilde{Z}_1}) \longrightarrow 0$$

which is exact by the transversality of Q_1 and $\tilde{S}$: Choose coordinates (x_1, x_2, x_3) such that $x_1 = 0$ defines $\tilde{S}$ and $x_2 = 0$ defines Q_1. Then on U_1 $\pi = r \circ dx_2 : \Theta_{\tilde{Z}_1 \tilde{S}} \to \mathcal{O}_{Q_1}(N_{Q_1}^{\tilde{Z}_1})$ where r is the restriction onto Q_1 and the exactness is now easily seen. Then, from Lemma 4.1 and the sequence (4.2) we get $H^i(U_1, \Theta_{\tilde{Z}_1 \tilde{S} Q_1}) = 0$, $i \geq 2$ and as (4.2) is true on the level of presheafs we have also the vanishing for $i = 1$. The vanishing of $H^i(U_2, \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{\bar{D}}_2 Q_2})$, $i \geq 1$, may be obtained as above by choosing U_2 so small that it meets only $\tilde{D}_2$ (or $\tilde{\bar{D}}_2$). $\square$

Lemma 4.4. *For all $j \geq 0$, $H^j(\tilde{Z}_1, \Theta_{\tilde{Z}_1 \tilde{S} Q_1}) = H^j(Z_1, \Theta_{Z_1 SL_1})$.*

Proof. As in the proof of Lemma 4.2 we have $b_{1*}\Theta_{\tilde{Z}_1 \tilde{S} Q_1} = \Theta_{Z_1 SL_1}$. The result follows from the Leray spectral sequence once we have established the vanishing of $R^i b_{1*}(\Theta_{\tilde{Z}_1 \tilde{S} Q_1})$ for $i \geq 1$. We consider the sequences (4.2) and

$$(4.3) \qquad 0 \to \Theta_{\tilde{Z}_1 \tilde{S}} \to \Theta_{\tilde{Z}_1} \to \mathcal{O}_{\tilde{S}}(N_{\tilde{S}}^{\tilde{Z}_1}) \to 0.$$

Assume $\bar{q} \notin U \subseteq Z_1$ and that the bundles are trivial on U. Then on $b_1^{-1}(U)$ we have $\mathcal{O}_{\tilde{S}}(N_{\tilde{S}}^{\tilde{Z}_1}) = [\tilde{S}]|_{\tilde{S}} = b_1^*[S]|_{\tilde{S}} \otimes Q_1^{-1}|_{\tilde{S}} = b_1^*[S]|_{\tilde{S}} \otimes C_1^{-1}$. Thus on $b_1^{-1}(U) \cap \tilde{S}$

$$0 \to \mathcal{O}_{\tilde{S}}(N_{\tilde{S}}^{\tilde{Z}_1}) \to b_1^*[S] \to \mathcal{O}_{C_1}(b_1^*[S]) \to 0$$

is exact. Also, $H^j(b_1^{-1}(U) \cap C_1, b_1^*[S]) = H^j(q, [S]) = 0$ for $j \geq 1$, $H^0(q, [S]) = [S]_q = \mathbb{C}$ and $H^j(b_1^{-1}(U) \cap \tilde{S}, b_1^*[S]) = H^j(U \cap S, [S]) = 0$, for $j \geq 1$, by the $\bar{\partial}$-Poincaré Lemma. Thus, $H^j(b_1^{-1}(U) \cap \tilde{S}, \mathcal{O}_{\tilde{S}}(N_{\tilde{S}}^{\tilde{Z}_1})) = 0$, $j \geq 1$, if we can prove surjectivity of

$$H^0(b_1^{-1}(U) \cap \tilde{S}, b_1^*[S]) \to H^0(b_1^{-1}(U) \cap C_1, b_1^*[S]).$$

This corresponds via b_1 to the map $H^0(U \cap S, [S]) \to H^0(q, [S])$ which is onto by evaluation. From the proof of Lemma 4.2 we also have $H^j(b_1^{-1}(U), \Theta_{\tilde{Z}_1}) = 0$, $j \geq 1$. Thus, from (4.3), $H^j(b_1^{-1}(U), \Theta_{\tilde{Z}_1 \tilde{S}}) = 0$, $j \geq 1$, once we have proved that

$$H^0(b_1^{-1}(U), \Theta_{\tilde{Z}_1}) \to H^0(b_1^{-1}(U) \cap \tilde{S}, \mathcal{O}_{\tilde{S}}(N_{\tilde{S}}^{\tilde{Z}_1}))$$

is onto. However, it is not hard to describe the vector fields on $b_1^{-1}(U)$ in local coordinates on patches V_1, V_2 as in the proof of Lemma 4.2 and to see that the map is surjective.

Now, as $H^j\big(b_1^{-1}(U), \mathcal{O}_{Q_1}(N_{Q_1}^{\tilde{Z}_1})\big) = 0$, $j \geq 1$, we get from (4.2) that $H^j(b_1^{-1}(U), \Theta_{\tilde{Z}_1 \tilde{S} Q_1}) = 0$, $j \geq 2$. The map

$$H^0(b_1^{-1}(U), \Theta_{\tilde{Z}_1 \tilde{S}}) \to H^0\big(b_1^{-1}(U), \mathcal{O}_{Q_1}(N_{Q_1}^{\tilde{Z}_1})\big)$$

is onto which again is seen using local coordinates on $b_1^{-1}(U)$. Thus we have $R^i b_*(\Theta_{\tilde{Z}_1 \tilde{S} Q_1}) = 0$, $i \geq 1$ and the lemma is proved. $\qquad\square$

Similarly we have:

Lemma 4.5. *For all $j \geq 0$, $H^j(\tilde{Z}_2, \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2 Q_2}) = H^j(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})$.*

Proof. Again $b_{2*}\Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2 Q_2} = \Theta_{Z_2 D_2 \bar{D}_2}$ via the differential db_2. Furthermore the sheaf $R^i b_{2*}(\Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2 Q_2})$ vanishes for $i \geq 1$: First we prove the vanishing of $H^j(b_2^{-1}(U_2), \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2})$, $j \geq 1$, $U_2 \subseteq Z_2$. Use coordinates (z_1, z_2, ℓ), $(z_1, z_3, \frac{1}{\ell})$ on a cover $V_1 \cup V_2$ of $b_2^{-1}(U_2)$ such that D is given by $z_2 = 0$, $\bar{D}$ by $z_3 = 0$ and L_2 by $z_2 = z_3 = 0$. Lemma 4.1 gives that this is a Leray cover of $b_2^{-1}(U_2)$ because $\Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2}(V_2) = \Theta_{\tilde{Z}_2 \tilde{D}_2}(V_2)$ and $\tilde{D}_2$ in V_2 is a smooth hypersurface. Also, $\Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2}(V_1 \cap V_2) = \Theta_{\tilde{Z}_2}(\mathbb{C}^3 \backslash \mathbb{C}^2)$. Therefore $H^j(b_2^{-1}(U_2), \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2})$ vanishes, $j \geq 1$. Then consider the sequence

$$0 \to \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2 Q_2} \to \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2} \to \mathcal{O}_{Q_2}(N_{Q_2}^{\tilde{Z}_2}) \to 0.$$

The vanishing of $H^j\big(b_2^{-1}(U_2) \cap Q_2, \mathcal{O}_{Q_2}(N_{Q_2}^{\tilde{Z}_2})\big)$, $j \geq 1$ is seen using the Leray cover V_1, V_2. Also, using coordinates on V_1, V_2 we can show that vector fields may be lifted so we have surjectivity of

$$H^0\big(b_2^{-1}(U_2), \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2}\big) \to H^0\big(b_2^{-1}(U_2), \mathcal{O}_{Q_2}(N_{Q_2}^{\tilde{Z}_2})\big).$$

Then the short exact sequence gives $H^j(b_2^{-1}(U_2), \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2 Q_2}) = 0$, $j \geq 1$, and we have proved the lemma. $\qquad\square$

To make use of the sequence (4.1) we prove the following lemma.

Lemma 4.6. *The cohomology groups $H^j(Q, \Theta_{QC\bar{C}})$ vanishes for $j \geq 1$ and $H^0(Q, \Theta_{QC\bar{C}}) = su(V^+) \oplus u(1)$.*

Proof. $H^j(Q, \Theta_Q) = 0$, $j \geq 1$ and $H^0(Q, \Theta_Q) = H^0\big(P(V_{x_1}^+), \mathcal{O}(2)\big) \oplus H^0\big(P(V_{x_1}^-), \mathcal{O}(2)\big) = S^2(V_{x_1}^+)^* \oplus S^2(V_{x_1}^-)^* = su(V_{x_1}^-) \oplus su(V_{x_1}^-)$. From the sequence

$$(4.4) \qquad\qquad 0 \to \Theta_{QC\bar{C}} \to \Theta_Q \to \mathcal{O}_{C\bar{C}}(N_{C\bar{C}}^Q) \to 0$$

and the vanishing of $H^j\big(C, \mathcal{O}_C(N_C^Q)\big)$, $j \geq 1$ we get $H^2(Q, \Theta_{QC\bar{C}}) = 0$. Now, $H^0(Q, \Theta_Q)$ is generated by global $SU(2) \times SU(2)$ holomorphic transformations. Elements in $H^0(Q, \Theta_{QC\bar{C}})$ are the vector fields generated by actions leaving C and $\bar{C}$ invariant, i.e. by rotations on L_1 leaving q and $\bar{q}$ fixed. Thus $H^0(Q, Q_{C\bar{C}})$ is generated by $SU(V_{x_1}^+) \times U(1)$ where $U(1) \leq SU(V_{x_1}^-)$. Also, the map

$$H^0(Q, \Theta_Q) \to H^0\big(C, \mathcal{O}_C(N_C^Q)\big) \oplus H^0\big(\bar{C}, \mathcal{O}_{\bar{C}}(N_{\bar{C}}^Q)\big)$$

is surjective so $H^1(Q, \Theta_{QC\bar{C}}) = 0$. □

We are now able to prove the vanishing of the relative obstruction group.

Proposition 4.7. *Assume $H^2(Z_1, \Theta_{Z_1 S}) = 0$ and $H^2(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = 0$. Then the obstruction group $H^2(\tau_{\mathcal{Z}S}^0)$ vanishes.*

Proof. The long exact sequence associated to (4.1) together with Lemmae 4.3, 4.4, 4.5 and 4.6 gives

$$H^2(\mathcal{Z}, \tau_{\mathcal{Z}S}^0) = H^2(Z_1, \Theta_{Z_1 S L_1}) \oplus H^2(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}).$$

The second summand vanishes due to the assumption. Consider the exact sequence

(4.5) $$0 \to \Theta_{Z_1 S L_1} \to \Theta_{Z_1 S} \to \mathcal{O}_{L_1}(N_{L_1}^{Z_1}) \to 0.$$

We have $H^1\big(L_1, \mathcal{O}_{L_1}(N_{L_1}^{Z_1})\big) = 0$ and also $H^2(Z_1, \Theta_{Z_1 S}) = 0$ so the first summand also vanishes. □

As explained in Section 2 we also need the following vanishing results.

Proposition 4.8. *Assume $H^2(Z_i, \Theta_{Z_i}) = 0$, $i = 1, 2$, $H^2(S, \Theta_S) = 0$ and $H^2(D_2, \Theta_{D_2}) = 0$. Then the obstruction groups $H^2(\tau_{\mathcal{Z}}^0)$ and $H^2(\tau_{\mathcal{S}}^0)$ vanishes.*

Proof. Consider the normalization in the following sequence

(4.6) $$0 \to \tau_{\mathcal{Z}}^0 \to q_* \Theta_{\mathcal{Z}'Q'} \to i_* \Theta_Q \to 0.$$

Then, from Lemma 4.1 and the Leray spectral sequence, we get

$$H^2(\mathcal{Z}, q_* \Theta_{\mathcal{Z}'Q'}) = H^2(\tilde{Z}_1, \Theta_{\tilde{Z}_1 Q_1}) \oplus H^2(\tilde{Z}_2, \Theta_{\tilde{Z}_2 Q_2}).$$

From the sequence

$$0 \to \Theta_{\tilde{Z}_i Q_i} \to \Theta_{\tilde{Z}_i} \to \mathcal{O}_{Q_i}(N_{Q_i}^{\tilde{Z}_i}) \to 0$$

and the vanishing of $H^j\big(Q_i, \mathcal{O}_{Q_i}(N_{Q_i}^{\tilde{Z}_i})\big)$, $j \geq 1$, we have $H^2(\tilde{Z}_i, \Theta_{\tilde{Z}_i Q_i}) = H^2(\tilde{Z}_i, \Theta_{\tilde{Z}_i})$ which is equal to $H^2(Z_i, \Theta_{Z_i L_i})$ by Lemma 4.2. Since $H^j\big(L_i, \mathcal{O}_{L_i}(N_{L_i}^{Z_i})\big) = 0$, $j \geq 1$, we have $H^2(Z_i, \Theta_{Z_i L_i}) = H^2(Z_i, \Theta_{Z_i})$, $i = 1, 2$, and since $H^j(Q_i, \Theta_{Q_i}) = 0$, $i = 1, 2$, $j \geq 1$, the long exact sequence

associated to (4.6) gives $H^2(\tau_{\mathcal{Z}}^0) = H^2(Z_i, \Theta_{Z_1}) \oplus H^2(Z_2, \Theta_{Z_2})$ which vanishes by assumptions. This was just a repetition of the result of Donaldson and Friedman [**2**].

Next we concentrate on the divisor. Again we have a normalization sequence

(4.7) $$0 \to \tau_{\mathcal{S}}^0 \to q_*\Theta_{\mathcal{S}'C'\bar{C}'} \to i_*\Theta_{C\bar{C}} \to 0.$$

Lemma 4.1 together with the Leray spectral sequence give

$$H^2(\mathcal{S}, q_*\Theta_{\mathcal{S}'C'\bar{C}'}) = H^2(\tilde{S}, \Theta_{\tilde{S}C_1\bar{C}_1}) \oplus H^2(\tilde{D}_2, \Theta_{\tilde{D}_2C_2}) \oplus H^2(\tilde{\bar{D}}_2, \Theta_{\tilde{D}_2\bar{C}_2}).$$

Then, as $H^j\big(C_1, \mathcal{O}_{C_1}(N_{C_1}^{\tilde{S}})\big) = H^j\big(\mathbb{CP}^1, \mathcal{O}(-1)\big) = 0$, $j \geq 0$, the sequence

$$0 \to \Theta_{\tilde{S}C_1\bar{C}_1} \to \Theta_{\tilde{S}} \to \mathcal{O}_{C_1}(N_{C_1}^{\tilde{S}}) \oplus \mathcal{O}_{\bar{C}_1}(N_{\bar{C}_1}^{\tilde{S}}) \to 0$$

gives $H^j(\tilde{S}, \Theta_{\tilde{S}C_1\bar{C}_1}) = H^j(\tilde{S}, \Theta_{\tilde{S}})$, $j \geq 0$. Essentially by repeating the arguments in Lemma 4.2 we get $H^j(\tilde{S}, \Theta_{\tilde{S}}) = H^j(S, \Theta_{Sq\bar{q}})$, $j \geq 0$. For dimensional reasons we have $H^j\big(q\bar{q}, \mathcal{O}_{q\bar{q}}(N_{q\bar{q}}^S)\big) = 0$, $j \geq 1$, so $H^2(S, \Theta_{Sq\bar{q}}) = H^2(S, \Theta_S)$. For $\tilde{D}_2$ we have $N_{C_2}^{\tilde{D}} \cong \mathcal{O}(1)$ so the sequence

$$0 \to \Theta_{\tilde{D}_2C_2} \to \Theta_{\tilde{D}_2} \to \mathcal{O}_{C_2}(N_{C_2}^{\tilde{D}_2}) \to 0$$

gives $H^2(\tilde{D}_2, \Theta_{\tilde{D}C_2}) = H^2(\tilde{D}_2, \Theta_{\tilde{D}_2})$. Then as $\Theta_{C_2} = \mathcal{O}(2)$ the long exact sequence associated to (4.7) gives

$$H^2(\tau_{\mathcal{S}}^0) = H^2(S, \Theta_S) \oplus H^2(D_2, \Theta_{D_2}) \oplus H^2(\bar{D}_2, \Theta_{\bar{D}_2})$$

which vanishes by assumptions. $\qquad\square$

This ends our proof of the fact that under the general Assumption 2.1, the twistor space of the connected sum $M_1 \# M_2$ has an irreducible degree 2 divisor.

Finally we want to summarize and at the same time bring the symmetries back into considerations.

Theorem 4.9. *Let M_1, M_2 be compact self-dual conformal 4-manifolds with torus symmetry and fixed points. Assume the isotropy representations at the fixed points are intertwined via an orientation reversing isometry. If the Assumption 2.1 is satisfied for T^2-invariant divisors S, D_2 and $\bar{D}_2$, then there is a complex equivariant smoothing of the singular twistor space $\mathcal{Z}$ and the singular divisor $\mathcal{S}$ into a twistor space with torus action and an invariant irreducible degree 2 divisor.*

Proof. Due to the equivariance of the various sheaf morphisms we get an analogue of diagram (2.7) where the morphisms are between the G-invariant part of the groups. Consider the vanishing results in Propositions 4.7 and 4.8. The spectral sequence (2.1) now gives the vanishing of the obstruction spaces $T_{\mathcal{Z}}^2$ and $T_{\mathcal{S}}^2$. Then, chasing the diagram (2.7) gives surjectivity of the

morphism γ so $T_f^2 = 0$. Therefore, any element ω in the complement of the hyperplane $H^1(\tau_{\mathcal{ZS}}^0)$ of T_f^1 gives smoothings of $\mathcal{Z}$ and $\mathcal{S}$ as ω maps to non-zero elements in $H^0(\tau_{\mathcal{S}}^1)$ and $H^0(\tau_{\mathcal{Z}}^1)$. Thus we obtain a smooth twistor space with an irreducible degree 2 divisor. Together with Lemma 3.1 this shows that there are T^2-equivariant deformations of $\mathcal{Z}$ and $\mathcal{S}$. $\qquad\square$

Remark. We can also prove $H^2(\tau_{\mathcal{Z}|\mathcal{S}}^0) = 0$ if we assume that each component M_i, $i = 1, 2$, has positive scalar curvature. As this vanishing result is not used in this paper, we only give a short outline of the proof: Start with the normalization sequence

$$(4.8) \qquad 0 \to \tau_{\mathcal{Z}|\mathcal{S}}^0 \to q_*\Theta_{\mathcal{Z}'Q'|\mathcal{S}} \to i_*\Theta_{Q|C\bar{C}} \to 0.$$

Then, as $(q_*\Theta_{\mathcal{Z}'Q'})|_{\mathcal{S}} = q_*(\Theta_{\mathcal{Z}'Q'|\mathcal{S}'})$, we get

$$H^j\left(\mathcal{S}, q_*\Theta_{\mathcal{Z}'Q'|\mathcal{S}}\right)$$
$$= H^j\left(\tilde{S}, \Theta_{\tilde{Z}_1 Q_1|\tilde{S}}\right) \oplus H^j\left(\tilde{D}_2, \Theta_{\tilde{Z}_2 Q_2|\tilde{D}_2}\right) \oplus H^j\left(\tilde{\bar{D}}_2, \Theta_{\tilde{Z}_2 Q_2|\tilde{\bar{D}}_2}\right), \quad j \geq 0.$$

We study each component in this sum. From the sequence

$$(4.9) \qquad 0 \to \Theta_{\tilde{Z}_1 Q_1|\tilde{S}} \to \Theta_{\tilde{Z}_1|\tilde{S}} \to \mathcal{O}_{C_1\bar{C}_1}\left(N_{Q_1}^{\tilde{Z}_1}\right) \to 0,$$

and the vanishing $H^j\left(\tilde{S}, \mathcal{O}_{C_1\bar{C}_1}(N_{Q_1}^{\tilde{Z}_1})\right) = 0$, $j \geq 0$, we get $H^j(\tilde{S}, \Theta_{\tilde{Z}_1 Q_1|\tilde{S}}) = H^j(\tilde{S}, \Theta_{\tilde{Z}_1|\tilde{S}})$, $j \geq 0$. Consider therefore the sequence

$$(4.10) \qquad 0 \to \Theta_{\tilde{S}} \to \Theta_{\tilde{Z}_1|\tilde{S}} \to \mathcal{O}_{\tilde{S}}\left(N_{\tilde{S}}^{\tilde{Z}_1}\right) \to 0.$$

As $N_{\tilde{S}}^{\tilde{Z}_1} = b_1^*[S]|_{\tilde{S}} \otimes [C_1]^{-1} \otimes [\bar{C}_1]^{-1}$ and $\mathcal{O}_{C_1\bar{C}_1}(b_1^*[S]) = \mathcal{O}_{C_1\bar{C}_1}$ we have

$$(4.11) \qquad 0 \to N_{\tilde{S}}^{\tilde{Z}_1} \to b_1^*[S]|_{\tilde{S}} \to \mathcal{O}_{C_1\bar{C}_1} \to 0.$$

Also, $H^j(\tilde{S}, b_1^*[S]) = H^j(S, [S]) = H^j(S, K^{-\frac{1}{2}})$, $j \geq 0$. Then, due to the positive scalar curvature and the vanishing results of Hitchin [**5**], the sequence

$$0 \to \mathcal{O}_{Z_1} \to K^{-\frac{1}{2}} \to \mathcal{O}_S(K^{-\frac{1}{2}}) \to 0$$

gives $H^2(\tilde{S}, b_1^*[S]) = 0$. Then (4.11) gives $H^2\left(\tilde{S}, \mathcal{O}_{\tilde{S}}(N_{\tilde{S}}^{\tilde{Z}_1})\right) = 0$. From the sequence (4.10) and from $H^2(\tilde{S}, \Theta_{\tilde{S}}) = H^2(S, \Theta_{Sq\bar{q}}) = H^2(S, \Theta_S) = 0$ we get $H^2(\tilde{S}, \Theta_{\tilde{Z}_1|\tilde{S}}) = 0$. Thus $H^2(\tilde{S}, \Theta_{\tilde{Z}_1 Q_1|\tilde{S}}) = 0$. To prove the vanishing of $H^2(\tilde{D}_2, \Theta_{\tilde{Z}_2 Q_2|\tilde{D}_2})$ we proceed as above with sequences similar to (4.9), (4.10), (4.11) and the use of vanishing results due to the positive scalar curvature of M_2. Thus we have $H^2(\mathcal{S}, q_*\Theta_{\mathcal{Z}'Q'|\mathcal{S}}) = 0$ and (4.8) then gives $H^2(\mathcal{S}, \tau_{\mathcal{Z}|\mathcal{S}}^0) = 0$.

5. A Local Moduli Space of Self-Dual Metrics on $n\mathbb{CP}^2$.

We shall continue our study [**13**] of the local moduli of T^2-symmetric self-dual structures on the connected sums of n copies of the complex projective plane. As in [**13**] it is assumed that the only orbits are tori on which T^2 acts freely, circles stabilized by some S^1-subgroup inside T^2 and isolated fixed points. We construct $n\mathbb{CP}^2$ from $\mathbb{CP}^2$ by attaching planes step by step and at the same time keeping divisors and symmetries. However, let us take the general approach a little further in the case of an irreducible degree 2 divisor.

From the equivariant version of (2.7) we get the sequence

$$(5.1) \qquad 0 \to H^1(\tau^0_{\mathcal{ZS}})_{T^2} \to (T^1_f)_{T^2} \to \Delta_{T^2} \to 0.$$

Similarly (4.1) and Lemmae 4.3, 4.4, 4.5 and 4.6 give the sequence

$$(5.2) \qquad 0 \to H^0(\tau^0_{\mathcal{ZS}})_{T^2} \to H^0(Z_1, \Theta_{Z_1 S L_1})_{T^2} \oplus H^0(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})_{T^2}$$
$$\to \big(su(V^+) \oplus u(1)\big)_{T^2}$$
$$\to H^1(\tau^0_{\mathcal{ZS}})_{T^2} \to H^1(Z_1, \Theta_{Z_1 S L_1})_{T^2} \oplus H^1(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})_{T^2} \to 0.$$

Furthermore, from (4.5) we have the sequence

$$(5.3) \qquad 0 \to H^0(Z_1, \Theta_{Z_1 S L_1})_{T^2} \to H^0(Z_1, \Theta_{Z_1 S})_{T^2} \to H^0\big(L_1, \mathcal{O}(N^{Z_1}_{L_1})\big)_{T^2}$$
$$\to H^1(Z_1, \Theta_{Z_1 S L_1})_{T^2} \to H^1(Z_1, \Theta_{Z_1 S})_{T^2} \to 0.$$

Thus, putting (5.1), (5.2) and (5.3) together, we get a formula for the dimensions: Let $\chi_1(\mathcal{F}_X)_{T^2} = h^0(X, \mathcal{F}_X)_{T^2} - h^1(X, \mathcal{F}_X)_{T^2}$ for a sheaf $\mathcal{F}_X$ on a complex manifold X.

Proposition 5.1. *Let $(T_{x_1} M_1)_{T^2}$ denote the real vector space of T^2-invariant tangent vectors at x_1 and let $C(T^2)$ denote the Lie algebra of the centralizer of T^2 in $SO(3) \times SO(3)$. Then*

$$\begin{aligned}
\dim(T^1_f)_{T^2} &= 1 + h^1(\tau^0_{\mathcal{ZS}})_{T^2} \\
&= 1 + h^0(\tau^0_{\mathcal{ZS}})_{T^2} + \dim_{\mathbb{R}}(T_{x_1} M_1)_{T^2} + \dim_{\mathbb{R}} C(T^2) \\
&\quad - \chi_1(\Theta_{Z_1 S})_{T^2} - \chi_1(\Theta_{Z_2 D_2 \bar{D}_2})_{T^2}.
\end{aligned}$$

Proof. We have used the Kodaira equivalence (3.1) between sections of the normal bundle of the line L_1 in the twistor space and tangent vectors at the corresponding point x_1 in M_1. The symmetry group T^2 sits in $SO(3) \times SO(3)$ via the composition of the isotropy representation $i : T^2 \to SO(4)$ and the representation of $SO(4)$ on $\Lambda^2_+ \oplus \Lambda^2_-$. $\qquad \square$

Now we need a series of lemmae which will ensure that we have the general Assumption 2.1 satisfied and will make it possible to find the number $\dim(T^1_f)_{T^2}$ in the case of irreducible degree 2 divisors in the connected sums of complex projective planes.

Lemma 5.2. *Let Z_2 be the twistor space of $\mathbb{CP}^2$. Then $H^2(Z_2, \Theta_{Z_2}) = 0$, $H^2(D_2, \Theta_{D_2}) = 0$, $H^2(\bar{D}_2, \Theta_{\bar{D}_2}) = 0$, $H^0(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) \cong \big(u(1) \oplus su(2)\big)_{\mathbb{C}} \subseteq su(3)_{\mathbb{C}}$ and $H^1(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = H^2(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = 0$.*

Proof. The space $H^0(Z_2, \Theta_{Z_2})$ is generated by the complexifications $SU(3)_{\mathbb{C}}$ of the lifts of the isometries on $\mathbb{CP}^2$ to the flag

$$(5.4) \qquad Z_2 = \Big\{ ([v_0, v_1, v_2], [\ell_0, \ell_1, \ell_2]) \mid \sum v_i \ell_i = 0 \Big\}.$$

Then $H^0(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})$ consists of the vector fields generated by the group that leaves $D_2 \cup \bar{D}_2$ invariant. For D_2 given by $v_0 = 0$ and $\bar{D}_2$ by $\ell_0 = 0$ it is clear that $H^0(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})$ must contain and therefore be equal to the maximal subalgebra $\big(u(1) \oplus su(2)\big)_{\mathbb{C}}$ of $su(3)_{\mathbb{C}}$. Since Z_2 is a flag manifold and D_2 is the blow-up of $\mathbb{CP}^2$ at one point, $H^2(Z_2, \Theta_{Z_2})$ and $H^2(D_2, \Theta_{D_2})$ vanish.

Lemma 4.5 and the vanishing of $H^j(Q_2, \mathcal{O}_{Q_2}(N_{Q_2}^{\tilde{Z}_2}))$, $j \geq 0$, give $H^j(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = H^j(\tilde{Z}_2, \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2 Q_2}) = H^j(\tilde{Z}_2, \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2})$. Consider the sequence

$$(5.5) \qquad 0 \to \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2} \to \Theta_{\tilde{Z}_2} \to \mathcal{O}_{\tilde{D}_2 \tilde{D}_2} \left(N_{\tilde{D}_2 \tilde{D}_2}^{\tilde{Z}_2} \right) \to 0.$$

Here $H^j\big(\tilde{Z}_2, \mathcal{O}_{\tilde{D}_2 \tilde{D}_2}(N_{\tilde{D}_2 \tilde{D}_2}^{\tilde{Z}_2})\big)$ is equal to the sum of $H^j(D_2, [D_2] \otimes [L_2]^{-1})$ and the conjugate part. This follows because $N_{\tilde{D}_2}^{\tilde{Z}_2} = b_2^*[D_2]|_{\tilde{D}_2} \otimes [C_2]^{-1}$ and b_2 gives isomorphisms $\tilde{D}_2 \cong D_2$, $C_2 \cong L_2$. Identify D_2 to the blow-up of $\mathbb{CP}^2$ at one point with hyperplane class H and exceptional divisor E, then $D_2|_{D_2} = H - E$ and $\bar{D}_2|_{D_2} = L_2 = H$ [17]. It follows that $H^j(D_2, [D_2] \otimes [L_2]^{-1})$ vanishes for $j \geq 0$ as $H^0(D_2, -E) = 0$, $H^0(D_2, -3H + 2E) = 0$ and $\chi(-E) = 0$ by the Riemann-Roch formula. The sequence (5.5) now shows that $H^j(\tilde{Z}_2, \Theta_{\tilde{Z}_2 \tilde{D}_2 \tilde{D}_2}) = H^j(\tilde{Z}_2, \Theta_{\tilde{Z}_2})$ which coincide with $H^j(Z_2, \Theta_{Z_2 L_2})$ as shown in Lemma 4.2. Tracing these identities we now have $H^j(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = H^j(Z_2, \Theta_{Z_2 L_2})$, $j \geq 0$. As $H^j(L_2, \mathcal{O}_{L_2}(N_{L_2}^{Z_2})) = 0$, $j \geq 1$, the sequence

$$(5.6) \qquad 0 \to \Theta_{Z_2 L_2} \to \Theta_{Z_2} \to N_{L_2}^{Z_2} \to 0$$

gives $H^2(Z_2, \Theta_{Z_2 L_2}) = H^2(Z_2, \Theta_{Z_2})$ which vanishes for the flag. Thus, $H^2(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = 0$ as claimed. Also, as $D_2 \cap \bar{D}_2 = L_2$, $H^0(Z_2, \Theta_{Z_2 L_2})$ contains $H^0(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = \big(u(1) \oplus su(2)\big)_{\mathbb{C}}$, which is maximal, so $H^0(Z_2, \Theta_{Z_2 L_2})$ is 4-dimensional. Since $H^0\big(L_2, \mathcal{O}_{L_2}(N_{L_2}^{Z_2})\big)$ is 4-dimensional so (5.6) gives $H^1(Z_2, \Theta_{Z_2 L_2}) = H^1(Z_2, \Theta_{Z_2})$ which vanishes for the flag. Thus we get $H^1(Z_2, \Theta_{Z_2 D_2 \bar{D}_2}) = 0$ and the lemma has been proved. $\square$

Consider the twistor space (5.4) of $\mathbb{CP}^2$ with the action of T^2 given by

$$\big([v_0 e^{i\Phi_1}, v_1 e^{i\Phi_2}, v_2 e^{-i(\Phi_1 + \Phi_2)}], [\ell_0 e^{-i\Phi_1}, \ell_1 e^{-i\Phi_2}, \ell_2 e^{i(\Phi_1 + \Phi_2)}]\big)$$

where $\Phi_i = p_i\theta + q_i\psi$, $i = 1, 2$, $(\theta, \psi) \in T^2$ and the integers p_i, q_i satisfy $p_1q_2 - p_2q_1 = \pm 1$ [**13**]. Then the irreducible degree 2 divisor S given by $\sum \lambda_i v_i \ell_i = 0$, where $\lambda_i \neq \lambda_j$, $i \neq j$, is T^2-invariant. Indeed T^2 is exactly the symmetry group of S. The twistor fibration onto $\mathbb{CP}^2$ is $[z_0, z_1, z_2] = [v_1\bar\ell_2 - v_2\bar\ell_1, v_2\bar\ell_0 - v_0\bar\ell_2, v_0\bar\ell_1 - v_1\bar\ell_0]$ and it induces the action $[z_0e^{-i\Phi_1}, z_1e^{-i\Phi_2}, z_2e^{i(\Phi_1+\Phi_2)}]$ on $\mathbb{CP}^2$. Take $x = [0, 0, 1] \in \mathbb{CP}^2$. Then the twistor line L_x is given by $v_2 = \ell_2 = 0$ with two fixed points $q = ([0, 1, 0], [1, 0, 0])$ and $\bar q = ([1, 0, 0], [0, 1, 0])$. A generic T^2-invariant degree 2 divisor is transversal to L_x and contains q and $\bar q$.

We get a reducible invariant divisor with D_1 given by $v_0 = 0$ and $\bar D_1$ given by $\ell_0 = 0$. We have $L_x \cap D_1 = q$ and $L_x \cap \bar D_1 = \bar q$. Finally take D_2 given by $v_2 = 0$ and $\bar D_2$ corresponding to $\ell_2 = 0$. Then $L_x = D_2 \cap \bar D_2$. Thus we have the following building blocks.

Lemma 5.3. *The twistor space Z of $\mathbb{CP}^2$ contains T^2-invariant divisors S, D_i, $\bar D_i$, $i = 1, 2$ as in Section 2 with arbitrary isotropy representation at fixed points and satisfying $H^0(Z, \Theta_{ZS})_{T^2} = \mathbb{C}^2$, $H^2(S, \Theta_S) = 0$, $H^2(D_i, \Theta_{D_i}) = 0$, $H^2(\bar D_i, \Theta_{\bar D_i}) = 0$ and $H^2(Z, \Theta_{ZD_i\bar D_i}) = 0$.*

Proof. The vanishing of $H^2(S, \Theta_S)$ follows because $H^2(Z, \Theta_Z) = 0$ and because S is contained in Z with positive normal bundle. $H^0(Z, \Theta_{ZS})_{T^2}$ contains at least the algebra generated by T^2 as S is invariant and for the generic S described above where $\lambda_i \neq \lambda_j$, $i \neq j$, the symmetry group can at most be two dimensional. With Lemma 5.2 and the discussion above in mind, the lemma is proved. $\square$

Note that in order to fulfill all the vanishing conditions stated in Assumptions 2.1 and 2.2 we still need to prove $H^2(Z, \Theta_{ZS}) = 0$ for the flag manifold Z. This could be done, mutatis mutandis, as for $\Theta_{ZD\bar D}$ in Lemma 5.2. However, as we focus on the T^2-equivariant situation we only prove the vanishing of the T^2-invariant part of this cohomology group. Indeed it follows from the next more general lemma.

Lemma 5.4. *Let Z_1 be the twistor space of $n\mathbb{CP}^2$ and let $S \subseteq Z_1$ be an irreducible degree 2 divisor with canonical bundle K_S. Then $H^0(S, \mathcal{O}(K_S^{-1}))_{T^2} \cong \mathbb{C}$, $H^1(S, \mathcal{O}(K_S^{-1}))_{T^2} = 0$ and $H^2(Z_1, \Theta_{Z_1S})_{T^2} = 0$.*

Proof. It is known [**14**] that S is the blow-up S_n of a real quadric S_0 n times in a pair of conjugate points. Let $b_k : S_k \to S_{k-1}$, $1 \leq k \leq n$, be the blow-down map from the blow-up of S_0 k times to the blow-up $k - 1$ times.

The torus action has only isolated fixed points on S_n so the action on S_k, $1 \leq k \leq n$, induced by blowing down, also only has isolated fixed points. Therefore, the points $q_{k-1}, \bar q_{k-1}$ of blowing-up from S_{k-1} to S_k is at the intersection of invariant divisors. For topological reasons the action on S_0 has four isolated fixed points.

Let $K_k = K_{S_k}$ and let $E_k, \bar{E}_k$ be the exceptional divisor of the blowing-up b_k. Then $K_k^{-1} = b_k^* K_{(k-1)}^{-1} \otimes E_k^{-1} \otimes \bar{E}_k^{-1}$ and

$$0 \to \mathcal{O}(K_k^{-1}) \to \mathcal{O}(b_k^* K_{(k-1)}^{-1}) \to \mathcal{O}_{E_k \cup \bar{E}_k}(b_k^* K_{(k-1)}^{-1}) \to 0$$

is exact and induces

$$(5.7) \quad 0 \;\to\; H^0(S_k, \mathcal{O}(K_k^{-1}))_{T^2} \to H^0\left(S_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1})\right)_{T^2}$$
$$\to\; H^0\left(E_k \cup \bar{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1})\right)_{T^2}$$
$$\to\; H^1(S_k, \mathcal{O}(K_k^{-1}))_{T^2} \to H^1\left(S_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1})\right)_{T^2} \to 0.$$

Now, $H^0\left(E_k \cup \bar{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1})\right)$ is isomorphic to $K_{(k-1),q_{k-1}}^{-1} \oplus K_{(k-1),\bar{q}_{k-1}}^{-1}$. Furthermore, the T^2-action on $H^0\left(E_k \cup \bar{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1})\right)$ is given as non-trivial rotations on each factor: Let $q_{k-1} \in S_{k-1}$ and let A, B be two invariant divisors such that $q_{k-1} \in A \cap B$ and such that in local coordinates (z_1, z_2), A is given by $z_2 = 0$ and B is given as $z_1 = 0$. As the T^2-action does not have 2-dimensional fixed point set but only isolated fixed points, we may assume the action on $T_{q_{k-1}} S_{q_{k-1}} = T_{q_{k-1}} A \oplus T_{q_{k-1}} B$ is given as $\mathrm{diag}\,(e^{i\Phi_1}, e^{i\Phi_2})$. On $K_{k-1,q_{k-1}}^{-1} = \Lambda^2 T_{q_{k-1}} S_{q_{k-1}}$ the action is $e^{i(\Phi_1 + \Phi_2)}$. This action will not be trivial due to the assumption $p_1 q_2 - p_2 q_1 = \pm 1$. On the blow-up the action is $(e^{i\Phi_1}, e^{i(\Phi_2 - \Phi_1)})$ near the intersection of the exceptional divisor and the proper transform $\tilde{A}$ and near $\tilde{B}$ it is $(e^{i\Phi_2}, e^{i(\Phi_1 - \Phi_2)})$. Therefore the weights must be linear independent at all the points q_{k-1}.

Thus $H^0\left(E_k \cup \bar{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1})\right)_{T^2}$ vanishes. By an inductive argument we get from (5.7) that $H^j\left(S, \mathcal{O}(K_S^{-1})\right)_{T^2} = H^j\left(S_0, \mathcal{O}(K_0^{-1})\right)_{T^2}$, $j = 0, 1$. As S_0 is a quadric surface we have $H^1\left(S_0, \mathcal{O}(K_0^{-1})\right) = 0$. Choose coordinates $([s_0, s_1], [t_0, t_1])$ on $S_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ such that the action on S_0 is $\left([s_0, s_1 e^{i\Phi_1}], [t_0, t_1 e^{i\Phi_2}]\right)$. The nine monomials $(s_0^2 t_0^2, \ldots, s_1^2 t_1^2)$ of order four give a basis of $H^0\left(S_0, \mathcal{O}(K_0^{-1})\right) = H^0\left(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(2,2)\right)$. The weights of the induced T^2-action vanishes on $s_0^2 t_0^2$ and is of the form $n\Phi_1 + m\Phi_2$ on the eight other monomials. We claim that $s_0^2 t_0^2$ is the only invariant section: Note that if the weights (Φ_1, Φ_2) of the isotropy representation at $\left([1, 0], [1, 0]\right) \in S_0$ are linearly dependent then the weights of the isotropy representations at the other three fixed points on S_0 are also linearly dependent. Since the weights at the fixed points of the T^2-action on the blow-up are linear combinations of Φ_1 and Φ_2 the weights on S will be linearly dependent too. It follows, by restricting the twistor projection onto S, that on $n\mathbb{CP}^2$ there is a fixed point where $p_1 q_2 - p_2 q_2 = 0$ which is a contradiction. Thus, $H^0\left(S, \mathcal{O}(K_S^{-1})\right)_{T^2} \cong \mathbb{C}$. Finally, as $[S] = K_S^{-1}$, the sequence

$$(5.8) \qquad\qquad 0 \to \Theta_{Z_1 S} \to \Theta_{Z_1} \to \mathcal{O}_S(K_S^{-1}) \to 0,$$

gives $H^2(Z_1, \Theta_{Z_1 S})_{T^2} = 0$. $\square$

We are now able to find the number $\dim(T_f^1)_{T^2}$ in a concrete example.

Proposition 5.5. *It is possible to construct a T^2-symmetric twistor space of $n\mathbb{CP}^2$ with invariant irreducible degree 2 divisors by equivariant relative smoothings using $\mathbb{CP}^2$ as building blocks and proceeding step by step. Furthermore, in the final step we have $\dim(T_f^1)_{T^2} = h^1(Z, \Theta_{ZS})_{T^2} = n$, where Z is the smooth twistor space of $n\mathbb{CP}^2$.*

Proof. The assumptions which give unobstructed equivariant relative smoothings of two planes are satisfied. This follows from Lemmae 5.2, 5.3 and 5.4. Using upper semi-continuity of the dimension of cohomology we may indeed proceed step by step to get T^2-symmetric twistor spaces of $n\mathbb{CP}^2$ with degree 2 irreducible divisors.

With the assumption, $p_1 q_2 - p_2 q_1 = \pm 1$, on the T^2-action we have $\dim_{\mathbb{R}}(T_{x_1} M_1)_{T_2} = 0$ and $\dim_{\mathbb{R}} C(T^2) = 2$. Also, from Lemma 5.2, we get $\chi_1(\Theta_{Z_2 D_2 \bar{D}_2})_{T^2} = \dim\big(u(1) \oplus su(2)\big)_{T^2} = 2$. Lemma 5.3 and upper semi-continuity gives $h^0(Z_1, \Theta_{Z_1 S})_{T^2} = h^0(\tau^0_{ZS})_{T^2} = 2$ in each step. From the sequence (5.8) and Lemma 5.4 we get

$$(5.9) \quad 0 \to H^0\big(S, \mathcal{O}(K_S^{-1})\big)_{T^2} \to H^1(Z_1, \Theta_{Z_1 S})_{T^2} \to H^1(Z_1, \Theta_{Z_1})_{T^2} \to 0$$

and $h^0\big(S, \mathcal{O}(K_S^{-1})\big)_{T^2} = 1$. In [13] we got $h^1(Z_1, \Theta_{Z_1})_{T^2} = n - 2$ if Z_1 corresponds to $(n-1)\mathbb{CP}^2$, so $h^1(Z_1, \Theta_{Z_1 S})_{T^2} = n - 1$. Now, putting all these data into the formula in Proposition 5.1, we get $\dim(T_f^1)_{T^2} = n$ which by upper semi-continuity coincide with $h^1(Z, \Theta_{ZS})_{T^2}$ for the smooth twistor space Z of $n\mathbb{CP}$ with smooth irreducible degree 2 divisor S. $\square$

Thus, we may summarize and formulate the following result concerning the local moduli space of self-dual structures on $n\mathbb{CP}^2$ constructed in [13].

Theorem 5.6. *Consider the local moduli space of T^2-symmetric self-dual structures on $n\mathbb{CP}^2$ obtained by equivariant smoothings. The self-dual structures are all associated to T^2-symmetric twistor spaces Z with invariant irreducible degree 2 divisors. In particular, the scalar curvature of each Yamabe metric is positive. For $n \geq 3$, the T^2-action can be chosen such that all S^1 subgroups are non-semi-free. The dimension of the local moduli is equal to $h^1(Z, \Theta_Z)_{T^2} = n - 1$.*

Proof. Note that we have ensured that all twistor spaces have symmetry group of dimension 2 so a local moduli space is well-defined.

To prove this theorem, we need to prove that the deformation is target stable [18]. By taking the second components of the maps from the first

column to the second column in diagram (2.7), we have

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\tau^0_{\mathcal{ZS}})_{T^2} & \longrightarrow & (T^1_f)_{T^2} & \longrightarrow & \Delta_{T^2} & \longrightarrow & 0 \\
& & \varepsilon \downarrow & & \delta \downarrow & & \hat{\alpha} \downarrow & & \\
0 & \longrightarrow & H^1(\tau^0_{\mathcal{Z}})_{T^2} & \longrightarrow & (T^1_{\mathcal{Z}})_{T^2} & \longrightarrow & H^0(\tau^1_{\mathcal{Z}})_{T^2} & \longrightarrow & 0
\end{array}
$$

The target stability is equivalent to δ being surjective. Since $\hat{\alpha}$ is the identity map, δ is surjective if ε is surjective.

By Lemma 5.3 and upper semi-continuity, the exact sequence (5.2) gives the isomorphism

$$
H^1(\tau^0_{\mathcal{ZS}})_{T^2} \xrightarrow{\cong} H^1(Z_1, \Theta_{Z_1 S L_1})_{T^2}.
$$

Also (5.3) gives the isomorphism

$$
H^1(Z_1, \Theta_{Z_1 S L_1})_{T^2} \xrightarrow{\cong} H^1(Z_1, \Theta_{Z_1 S})_{T^2}.
$$

Similarly, one has the isomorphisms [**13**]

$$
H^1(\tau^0_{\mathcal{Z}})_{T^2} \xrightarrow{\cong} H^1(Z_1, \Theta_{Z_1 L_1})_{T^2} \ \text{ and } \ H^1(Z_1, \Theta_{Z_1 L_1})_{T^2} \xrightarrow{\cong} H^1(Z_1, \Theta_{Z_1})_{T^2}.
$$

Since the two compositions

$$
H^1(\tau^0_{\mathcal{ZS}})_{T^2} \xrightarrow{\cong} H^1(Z_1, \Theta_{Z_1 S})_{T^2} \quad \text{and} \quad H^1(\tau^0_{\mathcal{Z}})_{T^2} \xrightarrow{\cong} H^1(Z_1, \Theta_{Z_1})_{T^2}
$$

are induced by normalizations and restrictions, the inclusions induce a commutative diagram

$$
\begin{array}{ccc}
H^1(\tau^0_{\mathcal{ZS}})_{T^2} & \xrightarrow{\varepsilon} & H^1(\tau^0_{\mathcal{Z}})_{T^2} \\
\| & & \| \\
H^1(Z_1, \Theta_{Z_1 S})_{T^2} & \xrightarrow{\lambda} & H^1(Z_1, \Theta_{Z_1})_{T^2}.
\end{array}
$$

By (5.9), λ is surjective. Therefore ε is surjective.

Thus, the local moduli space obtained in [**13**] does indeed correspond to twistor spaces carrying divisors as claimed. Therefore the corresponding Yamabe metrics are of non-negative type [**3**]. In fact the scalar curvature must be positive because the intersection form for $n\mathbb{CP}^2$ is positive definite [**10**]. The fact that we may assume the S^1 subgroups are all non-semi-free was proved in [**13**]. $\qquad\square$

Turning to the problem of finding degree 1 divisors we first prove:

Lemma 5.7. *If L is the twistor line over a fixed point of a T^2-symmetric $n\mathbb{CP}^2$ and if S is an invariant degree 2 divisor containing L, then S is reducible.*

Proof. Assume on the contrary that S is irreducible. Then S is the blow-up of a quadric surface with L a smooth fiber [**14**]. In particular it does not pass through any points of blowing-up and this implies that L is not invariant: On the quadric the only T^2-invariant curves are the two conjugate pair of generator lines passing through the four fixed points which are the only points of blowing-up as S has T^2-symmetry. □

Proposition 5.8. *Given a T^2-symmetric twistor space over $n\mathbb{CP}^2$ with an invariant irreducible divisor S. Let L be a twistor line above an isolated fixed point. Then there exists a conjugate pair $D, \bar{D}$ of invariant degree 1 divisors intersecting along L.*

Proof. Let V_1, V_2 be a pair of independent holomorphic vector fields generated by the torus action on the twistor space Z. Since S is invariant the restrictions of V_1 and V_2 to S are tangential to S. Therefore we have an invariant section $\hat{s} = V_1 \wedge V_2|_S \in H^0(S, K_S^{-1})$. We claim that this section is nontrivial. From Lemma 5.7 L intersects S transversely at $q, \bar{q}$. Since the T^2-action only has isolated fixed points, there exist complex coordinates (z_1, z_2) centered at q such that the T^2-action is given as $(e^{in\theta}z_1, e^{im\psi}z_2)$ with n and m both non-zero. Therefore $\hat{s} = nmz_1z_2\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$ near q.

Since $H^0(S, K_S^{-1})$ is non-trivial and $h^1(Z, \mathcal{O}) = h^1(n\mathbb{CP}^2, \mathbb{R}) = 0$ [**5**], the exact sequence

$$0 \to \mathcal{O} \to K^{-\frac{1}{2}} \to \mathcal{O}_S(K_S^{-1}) \to 0$$

implies that the section $\hat{s}$ lifts to an invariant section of $K^{-\frac{1}{2}}$.

Let $s \in H^0(Z, K^{-\frac{1}{2}})$ be the section such that $s^{-1}(0) = S$. Then the subspace $\mathcal{V} = \text{span}\{s, \hat{s}\}$ in $H^0(Z, K^{-\frac{1}{2}})$ is T^2-invariant. Every element of this system contains q and $\bar{q}$ and we may choose an element $S' \in |\mathcal{V}|$ containing $p \in L$ where $p \neq q$ and $p \neq \bar{q}$. As S' is of degree 2 it also contains L. We may assume S' is real. The divisor S' is invariant, otherwise as L is invariant, L would be contained in the base locus of the system $|\mathcal{V}|$ but S does not contain L. Then by Lemma 5.7, S' must be reducible. □

Corollary 5.9. *The twistor spaces described in Theorem 5.6 contain T^2-invariant reducible degree 2 divisors.*

Remark. Given a T^2-symmetric twistor space over $n\mathbb{CP}^2$ with an invariant reducible degree 2 divisor as above, we construct T^2-symmetric singular twistor spaces with degree 2 divisors over $(n+1)\mathbb{CP}^2$ as outlined at the end of Section 2. As in Section 4, mutatis mutandis, we can prove that the obstructions to the equivariant relative smoothing vanish provided Assumption 2.2 is satisfied. The cohomology group $H^2(Z_i, \Theta_{Z_i})$ vanishes by upper semi-continuity. The groups $H^2(D_i, \Theta_{D_i})$, $H^2(\bar{D}_i, \Theta_{\bar{D}_i})$, $i = 1, 2$; $H^2(Z_1, \Theta_{Z_1 S})_{T^2}$ and $H^2(Z_2, \Theta_{Z_2 D_2 \bar{D}_2})_{T^2}$ is proved to vanish as outlined at the end of the

paper. However, as we do not control the singularity of the degree 2 divisors in the smoothing process, we cannot conclude a priori that there are reducible degree 2 divisors after the smoothing. Furthermore, the notion of local moduli becomes dubious when the degree 2 divisor is reducible. We do not pursue this issue. However, the existence of reducible degre 2 divisors is secured by Corollary 5.9. This has applications as shown in the next section.
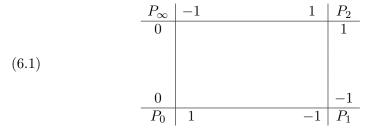
6. Anti-Self-Dual Hermitian Surfaces.

In [**13**] we constructed self-dual metrics on $(S^1 \times S^3) \# n\mathbb{CP}^2$, $n \geq 3$, such that the symmetry group is S^1 and the action is non-semi-free. We begin with a $\mathbb{CP}^2$ with S^1-action $[z_0 e^{i\theta}, z_1, z_2]$. We label the fixed points as $P_0 = [0, 1, 0]$, $P_1 = [1, 0, 0]$ and $A_1 = [0, 0, 1]$. Then we attach a $\mathbb{CP}^2$ with S^1-action $[z_0 e^{-i\theta}, z_1, z_2]$ and fixed points $A_2 = [0, 0, 1]$, $P_2 = [1, 0, 0]$, $P_\infty = [0, 1, 0]$. We attach A_1 to A_2 via an orientation reversing isometry and of course using the smoothing of the twistor space. Note that the isotropy representation of P_0 is $(e^{i\theta} x, y)$ while near P_∞ it is $(e^{-i\theta} x, y)$. Therefore, we may consider the possibility of making a self sum by identifying P_0 and P_∞. Note that at this stage the S^1-action is semi-free but if we equivariantly attach a $\mathbb{CP}^2$ to P_2 we get a S^1-symmetric $3\mathbb{CP}^2$ with non-semi-free action and we can still make a self-sum at P_0 and P_∞. Indeed we can go on attaching more $\mathbb{CP}^2$-blocks away from P_0 and P_∞ and then make a self-sum at P_0 and P_∞ to obtain self-dual structures on $(S^1 \times S^3) \# n\mathbb{CP}^2$, $n \geq 3$, with non-semi-free S^1-symmetry.

To get a complex structure on $(S^1 \times S^3) \# n\mathbb{CP}^2$ compatible with the conformal metric but with opposite orientation we bring in the relative smoothing: Due to Corollary 5.9, we may assume that on $n\mathbb{CP}^2$ we have a non-semi-free S^1-symmetric twistor space with a reducible divisor $D + \bar{D}$ such that $D \cap \bar{D} = L_{P_0}$ and such that the twistor line above P_∞ intersect D and $\bar{D}$ transversely. Then make an equivariant self-sum at P_0, P_∞ relative to the divisor $D + \bar{D}$. Following the notation from Section 2 we have curves $C_1, \bar{C}_1$ in $\tilde{D}, \tilde{\bar{D}}$ above P_∞. Also, there are curves $C_2, \tilde{\bar{C}}_2$ above P_0 which are both mapped onto L_{P_0} by the blowing-down.
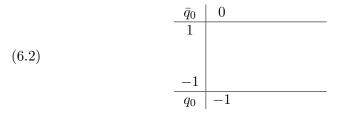
Theorem 6.1. *There exist anti-self-dual conformal Hermitian metrics on $(S^1 \times S^3) \# n\overline{\mathbb{CP}}^2$, for $n \geq 3$, such that the symmetry group is S^1 and the action is non-semi-free.*

Proof. We claim the isotropy data ensures that C_1 in $\tilde{D}$ is identified to C_2 in $\tilde{D}$ and not to $\bar{C}_2$ in $\tilde{\bar{D}}$: On $2\mathbb{CP}^2$ we have the four fixed points P_0, P_1, P_2, P_∞.
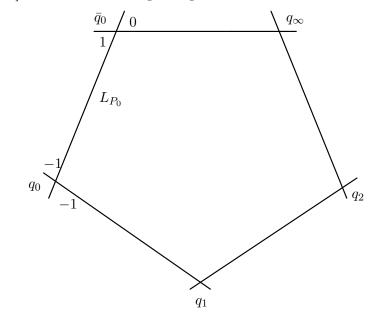
We draw the spheres between points as lines and get a diagram:

(6.1)

P_∞	-1		1	P_2
0				1
0				-1
P_0	1		-1	P_1

Here the numbers represent the weights of the S^1-action along the spheres near the fixed points. These numbers are easily obtained from the information above about the S^1 action. The twistor space of $\mathbb{CP}^2$ was described in (5.4). We see that the twistor line L_{P_0} is given by $v_1 = 0 = \ell_1$ and the divisor corresponds to $v_1 = 0$ and its conjugate corresponds to $\ell_1 = 0$. On L_{P_0} the S^1-action has the two fixed points $q_0 = ([0,0,1],[1,0,0])$ and $\bar{q}_0 = ([1,0,0],[0,0,1])$. On the divisor the isotropy data is given by

(6.2)

$\bar{q}_0$	0
1	
-1	
q_0	-1

Now, the divisor D in the twistor space of $2\mathbb{CP}^2$ is the blow up of $\mathbb{CP}^2$ twice [17]. It has the following configuration
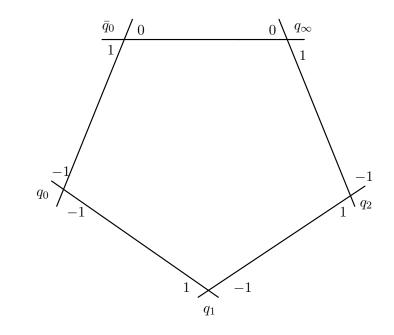

(6.3)

with $\pi(q_i) = p_i$ where π is the restriction of the twistor projection
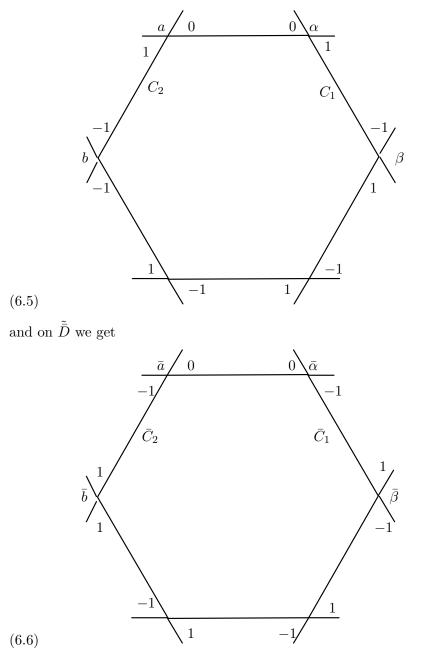
$$\pi : D \to 2\mathbb{CP}^2.$$

Note that π maps the whole line L_{P_0} to P_0 and otherwise is an orientation reversing equivariant diffeomorphism. Therefore, by comparing with the data (6.1) on $2\mathbb{CP}^2$, we obtain the full isotropy data on D:

(6.4)

From the real structure $\sigma : D \to \bar{D}$ we can obtain the isotropy data on $\bar{D}$ near $q_0, \bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_\infty$. Then, as in the proof of Lemma 5.4 we get the data

on the blow-up of q_∞: on $\tilde{D}$ we get

(6.5)

and on $\tilde{\tilde{D}}$ we get

(6.6)

Here the points a, $\bar{b}$ are mapped to $\bar{q}_0$ by the blow-down and $\bar{a}, b$ are mapped to q_0. The points $\alpha, \bar{\alpha}$ corresponds to $q_\infty, \bar{q}_\infty$ while $\beta, \bar{\beta}$ correspond to $q_2, \bar{q}_2$. For the sake of completeness we may compute the isotropy data at $a, b, \alpha, \beta, \bar{a}, \bar{b}, \bar{\alpha}, \bar{\beta}$ also in directions transversal to the surfaces $\tilde{D}, \tilde{\tilde{D}}$: At

each point let (z, v, w) denote coordinates with (v, w) coordinates along the exceptional divisor, the quadric surface, and with (z, v) coordinates along the blown up divisor. Then with respect to such coordinates the weights of the isotropy is $a(0, 1, 1)$, $b(-1, -1, 1)$, $\alpha(0, 1, 1)$, $\beta(1, -1, 1)$ and $\bar{a}(0, -1, -1)$, $\bar{b}(1, 1, -1)$, $\bar{\alpha}(0, -1, -1)$, $\bar{\beta}(-1, 1, -1)$. The identification map $(z, v, w) \mapsto (\bar{z}, v, w)$ gives the orientation change on the divisor and it is now clear that a is identified to α, and b to β.

This shows that we need to attach C_2 to C_1 and $\bar{C}_2$ to $\bar{C}_1$. For $n\mathbb{CP}^2$, in a neighbourhood of the invariant sphere joining the fixed points P_0 and P_∞, the isotropy data is identical to the $2\mathbb{CP}^2$ case. Therefore, the arguments above can be applied to prove the claim also for $n\mathbb{CP}^2$.

It follows that $\tilde{D}$ and $\tilde{\bar{D}}$ are glued to themselves respectively. Therefore, the singular divisor $\mathcal{S}$ in the singular twistor space $\mathcal{Z}$ is a disjoint union of two degree 1 divisors $\mathcal{D}$ and $\bar{\mathcal{D}}$. In the next paragraph, we prove that the obstructions to smoothing the pair $(\mathcal{Z}, \mathcal{S})$ vanish. The resulting smooth twistor space carries a conjugate pair of disjoint degree 1 divisors intersecting all twistor lines transversely. This pair corresponds to complex structures $\pm I$ on $(S^1 \times S^3)\#n\mathbb{CP}^2$ compatible with the metric but inducing the opposite orientation. $\qquad\square$

To prove that the obstructions to the S^1-equivariant smoothing of the pair $(\mathcal{Z}, \mathcal{S})$ vanish, we proceed as in Section 4. It suffices to prove the vanishing of $H^2(Z, \Theta_{ZD\bar{D}})_{S^1}$ and $H^2(D, \Theta_D)_{S^1}$.

By Serre duality, $h^2(D, \Theta_D) = h^0(D, K_D \otimes \Omega^1)$. Since D is T^2-invariant, $(V_1 \wedge V_2)_{|D}$ is a non-trivial section of K_D^{-1}, where V_1 and V_2 is a pair of independent vector fields generated by the torus action on the twistor space. Therefore, if $h^0(D, K_D \otimes \Omega^1)$ was not equal to zero, there would have been non-trivial holomorphic 1-forms on D. Since D is a rational surface, it follows that $H^2(D, \Theta_D)$ vanishes.

To deal with $H^2(Z, \Theta_{ZD\bar{D}})_{S^1}$, we consider the following exact sequences

$$0 \to \Theta_{ZD\bar{D}} \to \Theta_{Z\bar{D}} \to \mathcal{O}_D(N_D^Z) \to 0,$$

and

$$0 \to \Theta_{Z\bar{D}} \to \Theta_Z \to \mathcal{O}_{\bar{D}}(N_{\bar{D}}^Z) \to 0.$$

By upper semi-continuity, $H^2(Z, \Theta_Z)$ vanishes. With the next lemma, we conclude that $H^2(Z, \Theta_{ZD\bar{D}})_{S^1}$ vanishes.

Lemma 6.2. *Let Z be the twistor space of $n\mathbb{CP}^2$ with T^2 symmetry and let $D \subseteq Z$ be an invariant degree 1 divisor. Then $H^j(D, \mathcal{O}(N_D^Z))_{S^1} = 0$ and $H^j(\bar{D}, \mathcal{O}_{\bar{D}}(N_{\bar{D}}^Z)) = 0$, $j \geq 1$.*

Proof. Recall [17] that D is the blow-up of $\mathbb{CP}^2$ n times. For $1 \leq k \leq n$ consider the sequence $b_k : D_k \to D_{k-1}$ of blowing down to $D_0 = \mathbb{CP}^2$. Let $p_{k-1} \in D_{k-1}$ be the point which is blown up and let E_k be the exceptional

divisor. On $D = D_n$, $[D_n]$ denotes the line bundle of $H - \sum_{i=1}^{n} E_i$ and on D_k, $[D_k]$ is the line bundle of $H - \sum_{i=1}^{k} E_i$ [**17**]. Then $[D_k] = b_k^*[D_{k-1}] \otimes E_k^{-1}$ and from

$$0 \to [D_k] \to b_k^*[D_{k-1}] \to \mathcal{O}_{E_k}(b_k^*[D_{k-1}]) \to 0$$

we get

$$(6.7) \quad 0 \quad \to \quad H^0(D_k, [D_k])_{S^1} \to H^0(D_{k-1}, [D_{k-1}])_{S^1} \to ([D_{k-1}]_{p_{k-1}})_{S^1}$$
$$\to \quad H^1(D_k, [D_k])_{S^1} \to H^1(D_{k-1}, [D_{k-1}])_{S^1} \to 0$$

and $H^2(D_k, [D_k]) = H^2(D_{k-1}, [D_{k-1}])$. Thus by induction we have

$$H^2(D, \mathcal{O}(N_D^{Z_1})) = H^2(D_n, [D_n]) = H^2(D_0, [D_0]) = H^2(\mathbb{CP}^2, [H]) = 0.$$

Now we prove $([D_k]_{p_k})_{S^1} = 0$, $0 \le k \le n - 1$. Let g_k be the blow-down from D_k to $\mathbb{CP}^2$. Then on D_k we have [**17**]

$$K_{D_k}^{-1} = [\bar{D}_k]|_{D_k} \otimes [D_k]|_{D_k} = (g_k^* H)^2 \otimes [D_k]|_{D_k}$$

so $(K_{D_k}^{-1})_{p_k} = (g_k^* H)_{p_k}^2 \otimes [D_k]_{p_k}$. In general, we may assume that the action on $\mathbb{CP}^2$ is $[z_0, z_1 e^{i\Phi_1}, z_2 e^{i\Phi_2}]$ in homogeneous coordinates. Then for $p = g_k(p_k)$, $(g_k^* H)_{p_k}^3 = H_p^3 \cong \Lambda^2 T_p \mathbb{CP}^2$ so the weight of $(g_k^* H)_{p_k}^2$ is $\frac{2}{3}(\Phi_1 + \Phi_2)$. Since p_k is on the intersection of invariant curves $T_{p_k} D_k$ has weight $(n_1 \Phi_1 + n_2 \Phi_2, m_1 \Phi_1 + m_2 \Phi_2)$. Then $(K_D^{-1})_{p_k}$ has weight $m\Phi_1 + n\Phi_2$ for some integers m, n. From diagram (6.4), after the curves joining q_1 to q_2, and q_2 to q_∞ are blown down, we see that $\Phi_1 = 0$ and $\Phi_2 = 1$. Therefore, the representation on $[D_k]_{p_k}$ is non-trivial. This gives $([D_k]_{p_k})_{S^1} = 0$, $0 \le k \le n - 1$, and then by induction (6.7) gives $H^1(D, [D])_{S^1} = 0$.   □

Theorem 6.1 should be compared with LeBrun's examples [**12**] of S^1-symmetric anti-self-dual Hermitian metrics on blow-up of Hopf surfaces. Note that the S^1-action in his examples is semi-free.

References

[1] M.F. Atiyah, N.J. Hitchin and I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London, **A362** (1978), 425-461.

[2] S.K. Donaldson and R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity, **2** (1989), 197-239.

[3] P. Gauduchon, *Structures de Weyl et théorèmes d'annulation sur une variété conforme autoduale*, Norm. Sup. Pisa, **18** (1991), 563-629.

[4] R. Hartshorne, *Algebraic Geometry*, GTM 52, 1977, Springer-Verlag, New York.

[5] N.J. Hitchin, *Linear field equations on self-dual spaces*, Proc. Roy. Soc. London, Sér A, **370** (1980), 173-191.

[6] D. Joyce, *Explicit construction of self-dual 4-manifolds*, Duke Math. J., **77** (1995), 519-552.

[7] J. Kim, *On the scalar curvature of self-dual manifolds*, Math. Ann., **297** (1993), 235-251.

[8] J. Kim and M. Pontecorvo, *A new method of constructing scalar-flat Kähler surfaces*, J. Differential Geometry, **41** (1995), 449-477.

[9] K. Kodaira and D. Spencer, *On deformations of complex analytic structure*, I, II, III, Ann. of Math., **67** (1958), 328-466; **71** (1960), 43-76.

[10] C. LeBrun, *On the topology of self-dual 4-manifolds*, Proc. Amer. Math. Soc., **98** (1986), 637–640.

[11] ———, *Explicit self-dual metrics on $\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2$*, J. Differential Geometry, **34** (1991), 223-253.

[12] ———, *Anti-self-dual Hermitian metrics on blown-up Hopf surfaces*, Math. Ann., **289** (1991), 383-392.

[13] H. Pedersen and Y.S. Poon, *Equivariant connected sums of compact self-dual manifolds*, Math. Ann., **301** (1995), 717-749.

[14] ———, *Self-duality and differentiable structures on the connected sum of complex projective planes*, Proc. Amer. Math. Soc., **121** (1994), 859-864.

[15] Y.S. Poon, *Compact self-dual manifolds with positive scalar curvature*, J. Differential Geometry, **24** (1986), 97-132.

[16] ———, *Conformal transformations of compact self-dual manifolds*, International J. Math., **5** (1994), 125-140.

[17] ———, *On the algebraic structure of twistor spaces*, J. Differential Geometry, **36** (1992), 451-491.

[18] Z. Ran, *Stability of certain holomorphic maps*, J. Differential Geometry, **34** (1991), 37–47.

[19] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geometry, **20** (1984), 478-495.

ODENSE UNIVERSITY
DK-5230 ODENSE M
DENMARK
E-mail address: henrik@imada.ou.dk

UNIVERSITY OF CALIFORNIA, RIVERSIDE
RIVERSIDE, CA 92521
E-mail address: ypoon@math.ucr.edu