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We give a simple proof of the result of Laver and Burckel that inserting a conjugate of a positive standard generator of the braid group anywhere in a given braid yields a braid which is larger in the sense of Dehornoy.

P. Dehornoy has defined a right invariant total ordering of the braid group B_n for all $n \in \mathbb{N}$ (see [**3**], [**4**]), and this was reinterpreted in [**6**] in more geometrical terms. Here we are using this interpretation to give a quick proof of the following:

Theorem 1. *For any braids α, β_1, $\beta_2 \in B_n$ and any $i \in \{1, \dots, n-1\}$ we have $\beta_1(\alpha\sigma_i\alpha^{-1})\beta_2 > \beta_1\beta_2$.*

Since the ordering is right-invariant this is equivalent to the following, with $\gamma = \beta_1\alpha$:

Theorem 1′. *For any braid $\gamma \in B_n$ and any $i \in \{1, \dots, n-1\}$ we have $\gamma\sigma_i > \gamma$.*

It follows that Dehornoy's ordering extends the (partial) subword ordering defined in [**5**]. Theorem 1 was first proved by Laver [**8**] and Burckel [**2**] using very different methods. As explained in [**8**], it can be combined with a theorem of Higman [**7**] to prove that the restriction of the ordering to the positive braid monoid B_n^+ is a well-ordering.

We briefly recall the definition of the ordering of B_n given in [**6**]. Let D^2 be the unit disk in $\mathbb{C}$, and let D_n be equal to D^2 with n holes in the real line, labelled 1 to n. The holes divide the real interval $[-1, 1]$ into $n+1$ line segments which we label 1 to $n+1$. Now any braid γ determines a way of sliding the holes about in D^2. Extending this movement to an isotopy of D^2 which is fixed on ∂D^2, we obtain, at the end of the isotopy, a self homeomorphism of D_n; this self homeomorphism maps the $n+1$ line segments to $n+1$ disjoint simple curves, again numbered 1 to $n+1$, and the image of the whole interval $[-1, 1]$ under the self homeomorphism is called a *curve diagram.*

If Γ is a curve-diagram in D_n of some braid γ, and Δ is another curve diagram of some braid δ, then we can reduce Γ and Δ with respect to each other, i.e. we can 'pull the diagrams tight'. Then the braid γ is called *j-larger* than δ if the curves number $1, \dots, j-1$ of Γ and Δ coincide and an initial segment of the jth curve of Γ lies in the upper component of $D_n - \Delta$. It is proved in [**6**] that this is equivalent to the braid $\gamma\delta^{-1}$ being representable by a word $w_1\sigma_j w_2 \ldots w_{l-1}\sigma_j w_l$, where $w_1, \ldots, w_l$ are words in the letters $\sigma_{j+1}^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$. If γ is j-larger than δ for some j, then we say γ is *larger* than δ. If the curves number $1,\ldots,j-1$ of Γ and Δ coincide (with no further restrictions on the subsequent curves), then we say γ and δ are $j-1$-parallel ($j \in \{1, \ldots, n\}$); equivalently, $\gamma\delta^{-1}$ can be represented by a word not containing the letters $\sigma_1^{\pm 1},\ldots,\sigma_{j-1}^{\pm 1}$.

The proof of Theorem $1'$ is in two steps. We shall deduce Theorem $1'$ from the following, seemingly weaker, result:

Proposition 2. *For any braid $\gamma \in B_n$ and any $i \in \{1, \ldots n-1\}$ the braid $\gamma\sigma_i$ is not* 1*-smaller than the braid γ; equivalently, $\gamma\sigma_i$ is either* 1*-larger than, or* 1*-parallel to γ.*

Proof of Proposition 2. We consider the intersections of a curve diagram Γ with the vertical lines indicated in Figure 1, which divide D_n into $n+1$ regions, labelled 1 to $n+1$. We say Γ is *v-reduced* if there are no disks in D_n bounded by precisely one segment of some curve of Γ and one segment of vertical line. By a sequence of isotopies across such disks we can turn Γ into a v-reduced diagram.

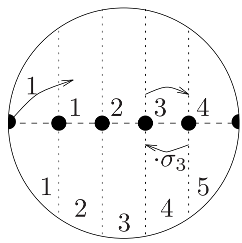

Figure 1. Vertical lines dividing D_n into a number of regions.

We say an isotopy of Γ is a *v-equivalence*, if it leaves Γ transverse to the vertical lines at all times. In particular, a v-equivalence does not change the number of intersections of Γ with the vertical lines.

We now give a recipe how to obtain a v-reduced curve diagram for the braid $\gamma\sigma_i$ from a v-reduced curve diagram Γ for the braid γ. The crucial observation is that under right multiplication by σ_i a v-reduced curve diagram changes only in the regions labelled i, $i+1$, and $i+2$, according to a simple set of rules. We call the union of these three regions the *augmented $i+1$-region.*

We equip all curves of Γ consistently with an orientation such that, when stuck together, they form an oriented curve in D^2 starting at -1, through all holes of D_n, ending at 1. For every curve of Γ we consider the segments in which it intersects the augmented i-region. If the whole curve is contained in the $i+1$st region itself, connecting the ith and the $i+1$st hole, then right multiplication by σ_i has simply the effect of reversing the orientation of the curve (see Figure 2(a)). If a segment of curve can, by a v-equivalence, be made disjoint from the straight line from the ith to the $i+1$st hole, then it is unaffected by the right multiplication by σ_i (Figure 2(b)).

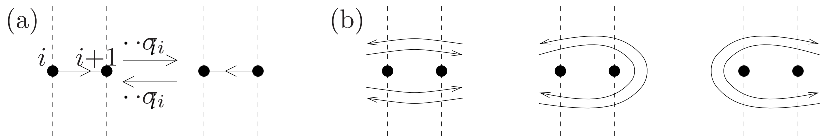

Figure 2. Unaffected by performing σ_i.

If a segment of curve cannot be made disjoint from the straight line from the ith to the $i+1$st hole, then it is affected by the right multiplication by σ_i. We shall find that there are ten $\mathbb{Z}$-families of possibilities for what such a segment of curve can look like.

If both ends of the segment of curve lie in the $i-1$st vertical line (i.e. the left boundary of the augmented $i+1$-region), then it is easy to check that in a neighbourhood of the $i+1$st region it looks like one of the curves in Figure 3, and under right multiplication by σ_i the diagram changes as indicated.

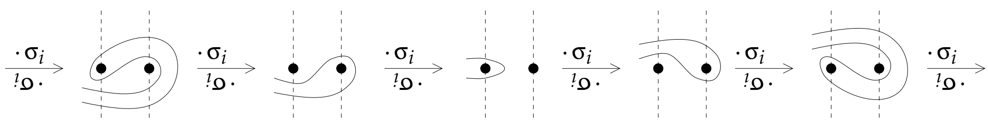

Figure 3. Two $\mathbb{Z}$-families of possibilities ((1) and (2)).

Note that all curves in Figure 3 can have two different orientations so Figure 3 represents two different $\mathbb{Z}$-families of curve segments. Also note that we can assume that the right multiplication by σ_i leaves the curve diagram fixed in a neighbourhood of the boundary of the augmented $i+1$-region, and Figure 3 shows its effect only 'well inside' this region.

If both ends of the segment of curve lie in the $i+2$nd vertical line (i.e. the right boundary of the augmented $i+1$-region), then we have another two $\mathbb{Z}$-families of possibilities ((3) and (4)). Figure 3, turned by 180°, illustrates these two families.

In the case that the two ends of the segment of curve lie on opposite sides of the augmented $i+1$-region (i.e. one in the $i-1$st and one in the $i+2$nd vertical line), we obtain two more $\mathbb{Z}$-families of possibilities ((5) and (6)), which differ only in the orientation of the curve segment (Figure 4). Again, it is easy to see that in this case these are the only possibilities.

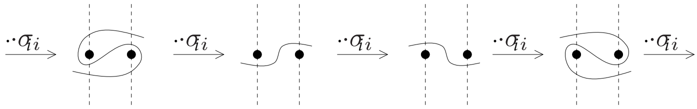

Figure 4. Two $\mathbb{Z}$-families of possibilities ((5) and (6)).

Finally, for the case that one of the ends of a curve segment lies in the boundary of the augmented i-region and the other end is in a hole of D_n, we have four more $\mathbb{Z}$-families: The one in Figure 5 (Family (7)), the one represented by Figure 5 turned by 180° (Family (8)), and the same two families with the orientation of the curve segments reversed (Families (9) and (10)).

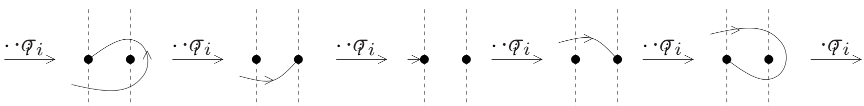

Figure 5. One more $\mathbb{Z}$-family of possibilities (7).

This completes our construction of a v-reduced curve diagram for $\gamma\sigma_i$ from the diagram Γ.

Our next aim is to compare the v-reduced curve diagrams of γ and $\gamma\sigma_i$, in order to decide which one is larger. The proof of the following lemma is similar to the proof in [**6**] that three curve diagrams can be reduced with respect to each other.

Lemma 3. *If Γ and Δ are two v-reduced curve diagrams, then Δ can be reduced with respect to Γ by an isotopy which is a v-equivalence.*

Given a v-reduced curve diagram Γ, we use the above recipe to determine a v-reduced curve diagram Δ of $\gamma\sigma_i$. Then we reduce Δ with respect to Γ to obtain a curve diagram Δ', and Lemma 3 tells us that this can be done by an isotopy of Δ which is a v-equivalence; that means, the intersections of Δ with the vertical lines can slide up and down without crossing the holes of D_n, and no intersections are created or cancelled in the course of this isotopy.

There are now two possibilites:

(1) γ and $\gamma\sigma_i$ are 1-parallel, i.e. the first curves Γ_1 and Δ'_1 of Γ and Δ' coincide.

(2) Γ_1 and Δ'_1 do not coincide.

In case (1) we are done. In case (2) we walk along the curve Γ_1, starting at -1, reading off a finite sequence of numbers according to Figure 6. The first number lies in $\{1, 2, 3\}$, the subsequent ones in $\{1, \ldots, 4\}$. The sequence ends with a 2 or a 3. We do the same for Δ'_1, and obtain a different sequence of numbers (for if the sequences were the same, then the curves Γ_1 and Δ_1' would coincide). Since the reduction of Δ with respect to Γ was a v-equivalence, the number sequences obtained by reading along Δ_1 and Δ_1' agree.

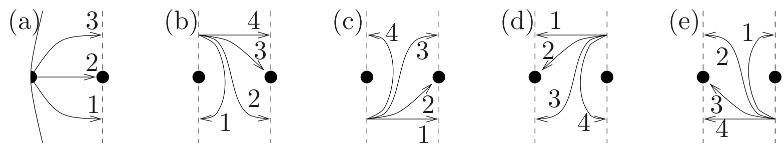

Figure 6. Reading a sequence of numbers off the curves Γ_1 and Δ_1'.

Now comes the key step of the whole proof. By carefully checking through all ten $\mathbb{Z}$-families described above it can be seen that the number sequence associated with Δ_1 (obtained from Γ_1 by applying σ_i) is always lexicographically larger than the number sequence of Γ_1; i.e. the first $k-1$ terms of the two sequences agree ($k \in \mathbb{N}$), but the kth number read off Δ_1 is larger than the kth number read off Γ_1. For instance, from the two leftmost diagrams in Figure 4 one reads $\dots 114314\dots$ and $\dots 13\dots$ respectively (independently of orientation), and the second sequence is lexicographically larger. Similarly, from the diagrams in Figure 5 one reads $\dots 1142$, $\dots 12$, $\dots 2$, $\dots 33$, and $\dots 3413$, which is lexicographically increasing. The other cases are similar and left to the reader.

Consider now the subarcs of the curves Γ_1 and Δ_1' consisting of the first k curve segments as in Figure 6. Since Γ and Δ_1' are reduced with respect to each other, these arcs do not intersect. It follows that an initial segment of Δ_1' lies in the upper component of $D_n - \Gamma$, ie that $\gamma\sigma_i$ is 1-larger than γ. This completes the proof of Proposition 2. $\square$

Proof of Theorem 1′. To see that Proposition 2 implies Theorem 1′ we assume, for a contradiction, that there exists a braid γ and an $i \in \{1, \dots, n\}$ such that $\gamma\sigma_i\gamma^{-1}$ is j-negative with $j > 1$. Then $\gamma\sigma_i\gamma^{-1}$ can be represented by a word not containing the letters $\sigma_1^{\pm 1}, \dots, \sigma_{j-1}^{\pm 1}$, and only negative powers of σ_j.

We consider the natural epimorphism $\pi\colon B_n \to S_n$, from the braid group to the symmetric group. We have $\pi(\sigma_i) = (i, i+1)$, $\pi(\gamma^{-1}) = (\pi(\gamma))^{-1}$, and $\pi(\gamma\sigma_i\gamma^{-1})(k) = k$ for $k = 1, \dots, j-1$. It follows that $\pi(\gamma)(k) \notin \{i, i+1\}$ for $k = 1, \dots, j-1$. Therefore the braid $\widetilde{\sigma}$ on $n-j+1$ strings which is obtained from the braid σ_i by removing the strings number $\pi(\gamma)(1), \dots, \pi(\gamma)(j-1)$ is again a positive standard generator of B_{n-j+1}.

Similarly, by removing from the braid γ the strings starting in the $j-1$ leftmost positions we obtain a braid $\widetilde{\gamma} \in B_{n-j+1}$.

Then the braid $\widetilde{\gamma}\widetilde{\sigma}\widetilde{\gamma}^{-1} \in B_{n-j+1}$ is a conjugate of the positive standard generator $\widetilde{\sigma}$. On the other hand, the braid $\widetilde{\gamma}\widetilde{\sigma}\widetilde{\gamma}^{-1}$ is obtained from $\gamma\sigma_i\gamma^{-1}$ by removing the $j-1$ leftmost strings. Therefore it can be represented by a word containing the letter σ_1^{-1}, but not σ_1, thus contradicting Proposition 2. □

Added in proof. Consider the monoid $\Pi = \{\pi \in B_n : \beta\pi > \beta \ \forall\ \beta \in B_n\}$ which is closed under conjugation by elements of B_n. I do not know a complete set of generators of Π. S. Orevkov has pointed out that, in addition to positive standard generators, Π also contains all braids of the form $\alpha\sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_{n-1}\ldots\sigma_1$, where α is any braid not containing $\sigma_1^{\pm1}$.

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