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DEHORNOY'S ORDERING OF THE BRAID GROUPS  
EXTENDS THE SUBWORD ORDERING

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## DEHORNOY'S ORDERING OF THE BRAID GROUPS EXTENDS THE SUBWORD ORDERING

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**We give a simple proof of the result of Laver and Burckel that inserting a conjugate of a positive standard generator of the braid group anywhere in a given braid yields a braid which is larger in the sense of Dehornoy.**

P. Dehornoy has defined a right invariant total ordering of the braid group  $B_n$  for all  $n \in \mathbb{N}$  (see [3], [4]), and this was reinterpreted in [6] in more geometrical terms. Here we are using this interpretation to give a quick proof of the following:

**Theorem 1.** *For any braids  $\alpha, \beta_1, \beta_2 \in B_n$  and any  $i \in \{1, \dots, n-1\}$  we have  $\beta_1(\alpha\sigma_i\alpha^{-1})\beta_2 > \beta_1\beta_2$ .*

Since the ordering is right-invariant this is equivalent to the following, with  $\gamma = \beta_1\alpha$ :

**Theorem 1'.** *For any braid  $\gamma \in B_n$  and any  $i \in \{1, \dots, n-1\}$  we have  $\gamma\sigma_i > \gamma$ .*

It follows that Dehornoy's ordering extends the (partial) subword ordering defined in [5]. Theorem 1 was first proved by Laver [8] and Burckel [2] using very different methods. As explained in [8], it can be combined with a theorem of Higman [7] to prove that the restriction of the ordering to the positive braid monoid  $B_n^+$  is a well-ordering.

We briefly recall the definition of the ordering of  $B_n$  given in [6]. Let  $D^2$  be the unit disk in  $\mathbb{C}$ , and let  $D_n$  be equal to  $D^2$  with  $n$  holes in the real line, labelled 1 to  $n$ . The holes divide the real interval  $[-1, 1]$  into  $n+1$  line segments which we label 1 to  $n+1$ . Now any braid  $\gamma$  determines a way of sliding the holes about in  $D^2$ . Extending this movement to an isotopy of  $D^2$  which is fixed on  $\partial D^2$ , we obtain, at the end of the isotopy, a self homeomorphism of  $D_n$ ; this self homeomorphism maps the  $n+1$  line segments to  $n+1$  disjoint simple curves, again numbered 1 to  $n+1$ , and the image of the whole interval  $[-1, 1]$  under the self homeomorphism is called a *curve diagram*.

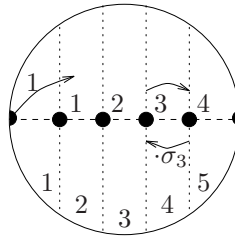
If  $\Gamma$  is a curve-diagram in  $D_n$  of some braid  $\gamma$ , and  $\Delta$  is another curve diagram of some braid  $\delta$ , then we can reduce  $\Gamma$  and  $\Delta$  with respect to each other, i.e. we can 'pull the diagrams tight'. Then the braid  $\gamma$  is called  *$j$ -larger* than  $\delta$  if the curves number  $1, \dots, j-1$  of  $\Gamma$  and  $\Delta$  coincide and

an initial segment of the  $j$ th curve of  $\Gamma$  lies in the upper component of  $D_n - \Delta$ . It is proved in [6] that this is equivalent to the braid  $\gamma\delta^{-1}$  being representable by a word  $w_1\sigma_j w_2 \dots w_{l-1}\sigma_j w_l$ , where  $w_1, \dots, w_l$  are words in the letters  $\sigma_{j+1}^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$ . If  $\gamma$  is  $j$ -larger than  $\delta$  for some  $j$ , then we say  $\gamma$  is larger than  $\delta$ . If the curves number  $1, \dots, j - 1$  of  $\Gamma$  and  $\Delta$  coincide (with no further restrictions on the subsequent curves), then we say  $\gamma$  and  $\delta$  are  $j - 1$ -parallel ( $j \in \{1, \dots, n\}$ ); equivalently,  $\gamma\delta^{-1}$  can be represented by a word not containing the letters  $\sigma_1^{\pm 1}, \dots, \sigma_{j-1}^{\pm 1}$ .

The proof of Theorem 1' is in two steps. We shall deduce Theorem 1' from the following, seemingly weaker, result:

**Proposition 2.** *For any braid  $\gamma \in B_n$  and any  $i \in \{1, \dots, n - 1\}$  the braid  $\gamma\sigma_i$  is not 1-smaller than the braid  $\gamma$ ; equivalently,  $\gamma\sigma_i$  is either 1-larger than, or 1-parallel to  $\gamma$ .*

*Proof of Proposition 2.* We consider the intersections of a curve diagram  $\Gamma$  with the vertical lines indicated in Figure 1, which divide  $D_n$  into  $n + 1$  regions, labelled 1 to  $n + 1$ . We say  $\Gamma$  is  $v$ -reduced if there are no disks in  $D_n$  bounded by precisely one segment of some curve of  $\Gamma$  and one segment of vertical line. By a sequence of isotopies across such disks we can turn  $\Gamma$  into a  $v$ -reduced diagram.



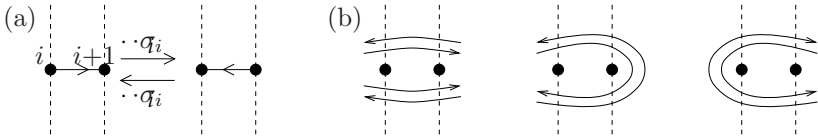
**Figure 1.** Vertical lines dividing  $D_n$  into a number of regions.

We say an isotopy of  $\Gamma$  is a  $v$ -equivalence, if it leaves  $\Gamma$  transverse to the vertical lines at all times. In particular, a  $v$ -equivalence does not change the number of intersections of  $\Gamma$  with the vertical lines.

We now give a recipe how to obtain a  $v$ -reduced curve diagram for the braid  $\gamma\sigma_i$  from a  $v$ -reduced curve diagram  $\Gamma$  for the braid  $\gamma$ . The crucial observation is that under right multiplication by  $\sigma_i$  a  $v$ -reduced curve diagram changes only in the regions labelled  $i, i + 1$ , and  $i + 2$ , according to a simple set of rules. We call the union of these three regions the *augmented  $i + 1$ -region*.

We equip all curves of  $\Gamma$  consistently with an orientation such that, when stuck together, they form an oriented curve in  $D^2$  starting at  $-1$ , through all holes of  $D_n$ , ending at  $1$ . For every curve of  $\Gamma$  we consider the segments in

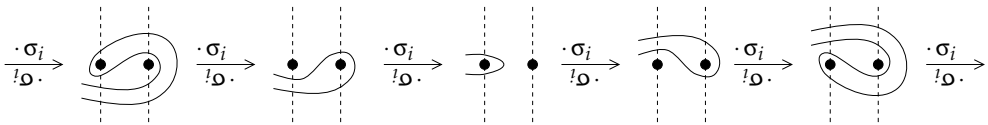
which it intersects the augmented  $i$ -region. If the whole curve is contained in the  $i + 1$ st region itself, connecting the  $i$ th and the  $i + 1$ st hole, then right multiplication by  $\sigma_i$  has simply the effect of reversing the orientation of the curve (see Figure 2(a)). If a segment of curve can, by a  $v$ -equivalence, be made disjoint from the straight line from the  $i$ th to the  $i + 1$ st hole, then it is unaffected by the right multiplication by  $\sigma_i$  (Figure 2(b)).



**Figure 2.** Unaffected by performing  $\sigma_i$ .

If a segment of curve cannot be made disjoint from the straight line from the  $i$ th to the  $i + 1$ st hole, then it is affected by the right multiplication by  $\sigma_i$ . We shall find that there are ten  $\mathbb{Z}$ -families of possibilities for what such a segment of curve can look like.

If both ends of the segment of curve lie in the  $i - 1$ st vertical line (i.e. the left boundary of the augmented  $i + 1$ -region), then it is easy to check that in a neighbourhood of the  $i + 1$ st region it looks like one of the curves in Figure 3, and under right multiplication by  $\sigma_i$  the diagram changes as indicated.

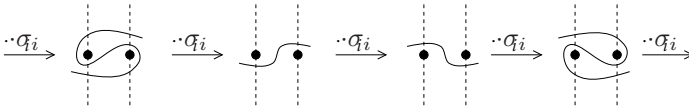


**Figure 3.** Two  $\mathbb{Z}$ -families of possibilities ((1) and (2)).

Note that all curves in Figure 3 can have two different orientations so Figure 3 represents two different  $\mathbb{Z}$ -families of curve segments. Also note that we can assume that the right multiplication by  $\sigma_i$  leaves the curve diagram fixed in a neighbourhood of the boundary of the augmented  $i + 1$ -region, and Figure 3 shows its effect only ‘well inside’ this region.

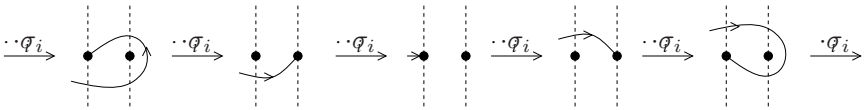
If both ends of the segment of curve lie in the  $i + 2$ nd vertical line (i.e. the right boundary of the augmented  $i + 1$ -region), then we have another two  $\mathbb{Z}$ -families of possibilities ((3) and (4)). Figure 3, turned by  $180^\circ$ , illustrates these two families.

In the case that the two ends of the segment of curve lie on opposite sides of the augmented  $i + 1$ -region (i.e. one in the  $i - 1$ st and one in the  $i + 2$ nd vertical line), we obtain two more  $\mathbb{Z}$ -families of possibilities ((5) and (6)), which differ only in the orientation of the curve segment (Figure 4). Again, it is easy to see that in this case these are the only possibilities.



**Figure 4.** Two  $\mathbb{Z}$ -families of possibilities ((5) and (6)).

Finally, for the case that one of the ends of a curve segment lies in the boundary of the augmented  $i$ -region and the other end is in a hole of  $D_n$ , we have four more  $\mathbb{Z}$ -families: The one in Figure 5 (Family (7)), the one represented by Figure 5 turned by  $180^\circ$  (Family (8)), and the same two families with the orientation of the curve segments reversed (Families (9) and (10)).



**Figure 5.** One more  $\mathbb{Z}$ -family of possibilities (7).

This completes our construction of a  $v$ -reduced curve diagram for  $\gamma\sigma_i$  from the diagram  $\Gamma$ .

Our next aim is to compare the  $v$ -reduced curve diagrams of  $\gamma$  and  $\gamma\sigma_i$ , in order to decide which one is larger. The proof of the following lemma is similar to the proof in [6] that three curve diagrams can be reduced with respect to each other.

**Lemma 3.** *If  $\Gamma$  and  $\Delta$  are two  $v$ -reduced curve diagrams, then  $\Delta$  can be reduced with respect to  $\Gamma$  by an isotopy which is a  $v$ -equivalence.*

Given a  $v$ -reduced curve diagram  $\Gamma$ , we use the above recipe to determine a  $v$ -reduced curve diagram  $\Delta$  of  $\gamma\sigma_i$ . Then we reduce  $\Delta$  with respect to  $\Gamma$  to obtain a curve diagram  $\Delta'$ , and Lemma 3 tells us that this can be done by an isotopy of  $\Delta$  which is a  $v$ -equivalence; that means, the intersections of  $\Delta$  with the vertical lines can slide up and down without crossing the holes of  $D_n$ , and no intersections are created or cancelled in the course of this isotopy.

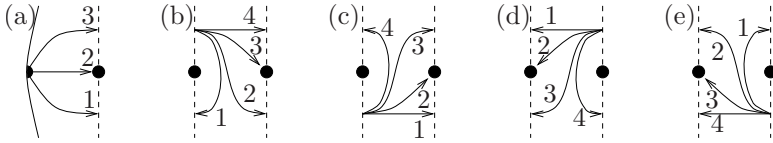
There are now two possibilities:

(1)  $\gamma$  and  $\gamma\sigma_i$  are 1-parallel, i.e. the first curves  $\Gamma_1$  and  $\Delta'_1$  of  $\Gamma$  and  $\Delta'$  coincide.

(2)  $\Gamma_1$  and  $\Delta'_1$  do not coincide.

In case (1) we are done. In case (2) we walk along the curve  $\Gamma_1$ , starting at  $-1$ , reading off a finite sequence of numbers according to Figure 6. The first number lies in  $\{1, 2, 3\}$ , the subsequent ones in  $\{1, \dots, 4\}$ . The sequence ends with a 2 or a 3. We do the same for  $\Delta'_1$ , and obtain a different sequence

of numbers (for if the sequences were the same, then the curves  $\Gamma_1$  and  $\Delta'_1$  would coincide). Since the reduction of  $\Delta$  with respect to  $\Gamma$  was a  $v$ -equivalence, the number sequences obtained by reading along  $\Delta_1$  and  $\Delta'_1$  agree.



**Figure 6.** Reading a sequence of numbers off the curves  $\Gamma_1$  and  $\Delta'_1$ .

Now comes the key step of the whole proof. By carefully checking through all ten  $\mathbb{Z}$ -families described above it can be seen that the number sequence associated with  $\Delta_1$  (obtained from  $\Gamma_1$  by applying  $\sigma_i$ ) is always lexicographically larger than the number sequence of  $\Gamma_1$ ; i.e. the first  $k - 1$  terms of the two sequences agree ( $k \in \mathbb{N}$ ), but the  $k$ th number read off  $\Delta_1$  is larger than the  $k$ th number read off  $\Gamma_1$ . For instance, from the two leftmost diagrams in Figure 4 one reads  $\dots 114314 \dots$  and  $\dots 13 \dots$  respectively (independently of orientation), and the second sequence is lexicographically larger. Similarly, from the diagrams in Figure 5 one reads  $\dots 1142$ ,  $\dots 12$ ,  $\dots 2$ ,  $\dots 33$ , and  $\dots 3413$ , which is lexicographically increasing. The other cases are similar and left to the reader.

Consider now the subarcs of the curves  $\Gamma_1$  and  $\Delta'_1$  consisting of the first  $k$  curve segments as in Figure 6. Since  $\Gamma$  and  $\Delta'_1$  are reduced with respect to each other, these arcs do not intersect. It follows that an initial segment of  $\Delta'_1$  lies in the upper component of  $D_n - \Gamma$ , ie that  $\gamma\sigma_i$  is 1-larger than  $\gamma$ . This completes the proof of Proposition 2.  $\square$

*Proof of Theorem 1'.* To see that Proposition 2 implies Theorem 1' we assume, for a contradiction, that there exists a braid  $\gamma$  and an  $i \in \{1, \dots, n\}$  such that  $\gamma\sigma_i\gamma^{-1}$  is  $j$ -negative with  $j > 1$ . Then  $\gamma\sigma_i\gamma^{-1}$  can be represented by a word not containing the letters  $\sigma_1^{\pm 1}, \dots, \sigma_{j-1}^{\pm 1}$ , and only negative powers of  $\sigma_j$ .

We consider the natural epimorphism  $\pi: B_n \rightarrow S_n$ , from the braid group to the symmetric group. We have  $\pi(\sigma_i) = (i, i + 1)$ ,  $\pi(\gamma^{-1}) = (\pi(\gamma))^{-1}$ , and  $\pi(\gamma\sigma_i\gamma^{-1})(k) = k$  for  $k = 1, \dots, j - 1$ . It follows that  $\pi(\gamma)(k) \notin \{i, i + 1\}$  for  $k = 1, \dots, j - 1$ . Therefore the braid  $\tilde{\sigma}$  on  $n - j + 1$  strings which is obtained from the braid  $\sigma_i$  by removing the strings number  $\pi(\gamma)(1), \dots, \pi(\gamma)(j - 1)$  is again a positive standard generator of  $B_{n-j+1}$ .

Similarly, by removing from the braid  $\gamma$  the strings starting in the  $j - 1$  leftmost positions we obtain a braid  $\tilde{\gamma} \in B_{n-j+1}$ .

Then the braid  $\tilde{\gamma}\tilde{\sigma}\tilde{\gamma}^{-1} \in B_{n-j+1}$  is a conjugate of the positive standard generator  $\tilde{\sigma}$ . On the other hand, the braid  $\tilde{\gamma}\tilde{\sigma}\tilde{\gamma}^{-1}$  is obtained from  $\gamma\sigma_i\gamma^{-1}$  by removing the  $j-1$  leftmost strings. Therefore it can be represented by a word containing the letter  $\sigma_1^{-1}$ , but not  $\sigma_1$ , thus contradicting Proposition 2.  $\square$

**Added in proof.** Consider the monoid  $\Pi = \{\pi \in B_n : \beta\pi > \beta \forall \beta \in B_n\}$  which is closed under conjugation by elements of  $B_n$ . I do not know a complete set of generators of  $\Pi$ . S. Orevkov has pointed out that, in addition to positive standard generators,  $\Pi$  also contains all braids of the form  $\alpha\sigma_1\sigma_2 \dots \sigma_{n-1}\sigma_{n-1} \dots \sigma_1$ , where  $\alpha$  is any braid not containing  $\sigma_1^{\pm 1}$ .

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