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We introduce a new geometric invariant Λ to measure the convexity of the boundary of a riemannian manifold with non-negative Ricci curvature in the interior. Based on a theorem of Perelman, we are able to show that this new invariant has topological implications. More specifically, we show that if Λ is close to 1 and the sectional curvature is positive on the boundary, then the manifold is contractible.

1. Introduction.

Due to a theorem of Gromov [2], without any control on the boundary, every compact manifold with non-empty boundary admits a riemannian metric of positive sectional curvature. However, there are topological obstructions if one further imposes convexity restrictions on the boundary. For example, it is a result of Gromoll [1] that a compact riemannian manifold of positive sectional curvature with non-empty convex boundary is diffeomorphic to the standard disc. In this paper, we study the case of positive Ricci curvature.

Throughout this paper, M^n denotes an n-dimensional compact connected riemannian manifold with non-empty boundary ∂M. Let $S_{\partial M} : X \mapsto \nabla_X \eta$ be the second fundamental form of ∂M with respect to the unit outward normal η. Furthermore, let λ be the infimum of eigenvalues of $S_{\partial M}$ taken over all points on ∂M. Then ∂M is convex if and only if $\lambda > 0$. In dimension 3, similar to the case of positive sectional curvature, a Ricci-positively curved manifold which further satisfies the condition $\lambda > 0$ has to be diffeomorphic to the 3-disc. It is not difficult to show this using Hamilton's theorem on closed 3-manifolds [3]. We shall prove it in the Appendix. In the case of dimension 4 and above, however, there are manifolds with positive Ricci curvature and convex boundary that are not even contractiable [9, 10]. Thus, a stronger convexity condition in higher dimensional cases is necessary.

We define a new convexity invariant Λ by

$$\Lambda(M^n) = \lambda \left(\frac{\mathrm{vol}\,(\partial M)}{\omega_{n-1}} \right)^{\frac{1}{n-1}},$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere $S^{n-1}(1)$. Note that Λ is scaling-invariant. The magnitude of Λ, for example, $\Lambda = 1$

for any geodesic ball in the euclidean space, and $\Lambda = \cos r$ for a geodesic ball of radius r in the unit sphere, does describe different degrees of convexity. Based on a theorem of Perelman [**5**], we are able to show that this new invariant has topological implications. Let K_M and Ric_M denote the sectional curvature and Ricci curvature of M^n respectively. Our main result is the following.

Theorem 1. *For any integer $n \geq 4$, there exists $1 > \delta_n > 0$ with the following property. If M^n satisfies:*

(i) $\mathrm{Ric}_M > 0$;
(ii) $\Lambda(M^n) > 1 - \delta_n$;
(iii) $K_M \geq 0$ at ∂M,

then M^n is contractible.

Since $\cos r \to 1$ as $r \to 0$, Theorem 1 implies that one cannot replace a geodesic ball in the standard sphere by anything non-contractible, while keeping the Ricci curvature positive and the metric on the rest of the sphere unchanged, if the ball is too small. Consequently, one cannot construct positively Ricci-curved manifolds with arbitrary topological complexity simply by this kind of surgery on round spheres.

This paper represents part of author's Ph.D. thesis. I would like to take this opportunity to thank my advisor, Professor Ji-Ping Sha, for his constant inspiration and guidance. The paper was completed while the author was visiting Oberlin College.

2. Proof of Theorem 1.

We now prove a slightly stronger version of Theorem 1 (condition (ii) and (iii) in Theorem 1 imply (iii) in Theorem 1′).

Theorem 1′. *For any integer $n \geq 4$, there exists $1 > \delta_n > 0$ with the following property. If M^n satisfies:*

(i) $\mathrm{Ric}_M > 0$;
(ii) $\Lambda(M^n) > 1 - \delta_n$;
(iii) $\left(\frac{\mathrm{Ric}_{\partial M}}{n-2}\right)^{\frac{1}{2}} \cdot \left(\frac{\mathrm{vol}(\partial M)}{\omega_{n-1}}\right)^{\frac{1}{n-1}} > 1 - \delta_n$, *where* $\mathrm{Ric}_{\partial M}$ *denotes the intrinsic Ricci curvature of ∂M,*

then M^n is contractible.

By the Bishop-Gromov volume comparison theorem, the following hypotheses:

(I) $\mathrm{Ric}_{N^n} \geq 0$, and
(II) $\mathrm{vol}(B_p(r)) > (1 - c_n)\mathrm{vol}(B_0(r))$, where $1 > c_n > 0$ for some $p \in N^n$ and for all $r > 0$, imply

$(*)\quad \mathrm{vol}(B_q(\rho)) > (1-c_n)\mathrm{vol}(B_0(\rho))$ for all $q \in N^n$ and for all $\rho > 0$.

In [5], Perelman shows, using $(*)$ in an essential way, that for every $n \geq 2$, there exists a constant $c_n > 0$ such that the following true: If N^n is a complete non-compact riemannian manifold satisfying (I) and (II), then N^n is contractible. A priori, one does not have estimate like $(*)$ for manifolds with boundary. However, as we shall show in the proof of Theorem $1'$, under suitable curvature conditions on M^n, the assumption $\Lambda(M^n) > 1-\delta_n$, where $1-\delta_n = (1-c_n)^{\frac{1}{n-1}}$, implies that one can attach an "open end" to M^n such that the resulting open manifold satisfies (I) and (II).

Proof of Theorem $1'$. Let $g_{\partial M}$ denote the intrinsic metric of ∂M. We shall construct a riemannian manifold N such that N is diffeomorphic to $\partial M \times [0,\infty)$ with the following properties:

(a) ∂N is isometric to ∂M, let ϕ be such a fixed isometry;
(b) N has positive Ricci curvature;
(c) $S_{\partial N}(p) + S_{\partial M}(\phi(p))$ is positive definite.

Furthermore, N satisfies a volume condition which will be specified later.

We now consider the warped metrics $f^2(t)\, g_{\partial M} + dt^2$ on $\partial M \times [0,\infty)$, where $f : [0,\infty) \to (0,\infty)$ is a smooth function with $f(0) = 1$. We denote such a riemannian manifold with respect to the fixed f by N_f. Then, obviously, ∂N_f is isometric to ∂M. Also note that $S_{\partial N_f} = -f'(0)\cdot I$. Moreover, the Ricci curvature of N_f can be expressed explicitly in terms of $\mathrm{Ric}_{\partial M}$ and f as follows.

Let p be a point in the interior of N_f, then $p = \exp_q(\rho\eta_q)$ for some q on ∂N_f and $\rho > 0$, where η_q is the unit inward normal of ∂N_f at q. We may choose an orthonormal basis of $\mathrm{T}_q\partial N_f$, denoted by $\{e_i\}_{i=1}^{n-1}$, such that

$$\mathrm{Ric}_{\partial M}(e_i, e_j) = 0\ ,\ \text{ whenever } i \neq j.$$

Let E_i be the parallel extension of e_i along the ray $\gamma : t \mapsto \exp_q(t\eta_q)$. Straightforward computations show that the Ricci tensor of N_f at p can be expressed as follows:

$$\begin{cases} \mathrm{Ric}_{N_f}(X_i, X_i) = \dfrac{1}{f^2}\mathrm{Ric}_{\partial M}(e_i, e_i) - \dfrac{f''}{f} - (n-2)\dfrac{f'^2}{f^2}, \quad i = 1, \dots, n-1; \\ \mathrm{Ric}_{N_f}(X_n, X_n) = -(n-1)\dfrac{f''}{f}; \\ \mathrm{Ric}_{N_f}(X_i, X_j) = 0, \quad \text{whenever } i \neq j, \end{cases}$$

where X_i and X_n denote the vector fields $\frac{1}{f(t)}E_i(t)$ and $\frac{\partial}{\partial t}$ respectively. Therefore, $\mathrm{Ric}_{N_f} > 0$ if f satisfies

$$f'' < 0 \text{ and } ff'' + (n-2)f'^2 < (n-2)k,$$

where $k := \frac{1}{n-2}\inf_x \mathrm{Ric}_{\partial M}(x,x)$ over all unit vectors x tangent to ∂M.

By assumption (ii) and (iii) in Theorem $1'$, we can choose positive constants c and α such that

$$(1-\delta_n)\left(\frac{\omega_{n-1}}{\mathrm{vol}(\partial M)}\right)^{\frac{1}{n-1}} < c < \min\{\sqrt{k}, \lambda\} \text{ and } c < \alpha < \lambda \ .$$

We shall construct a function F of t on $[0,\infty)$, which satisfies $F' > c$, $F > ct$, $F'' < 0$, and $FF'' + (n-2){F'}^2 = (n-2)c^2 < (n-2)k$. It then follows that N_F satisfies (a), (b) and (c), which then imply that the metric on $M \cup N_F$ can be smoothed out in an ϵ-neighborhood of ∂M to have positive Ricci curvature [**6**], and the metric is obviously complete.

Fix $p_o \in M$ and let d denote the diameter of M. Since we have chosen c so that $c^{n-1} > (1-\delta_n)^{n-1}\left(\frac{\omega_{n-1}}{\mathrm{vol}(\partial M)}\right)$, for sufficiently large r,

$$\begin{aligned}
\mathrm{vol}(B_{p_o}(r)) &\geq \mathrm{vol}(\partial M) \int_\epsilon^{r-d} F^{n-1}(t)dt \\
&> \frac{1}{n}\ \mathrm{vol}(\partial M)\cdot c^{n-1}\cdot\left((r-d)^n - \epsilon^n\right) \\
&> \frac{1}{n}\ (1-\delta_n)^{n-1}\cdot\omega_{n-1}\cdot r^n \\
&= (1-\delta_n)^{n-1}\cdot\mathrm{vol}(B_0(r))\ ,
\end{aligned}$$

where $B_0(r)$ is the ball of radius r in the euclidean space. Hence, by Perelman's theorem, $M \cup N_F$ is contractible, which implies that M is contractible.

The function F is constructed as follows. Consider the differential equation:

$$yz\frac{dz}{dy} + (n-2)z^2 = (n-2)c^2 \qquad (1)$$
$$z(1) = \alpha.$$

One has the following unique solution of (1) on $[0,\infty)$:

$$z = \sqrt{c^2 + (\alpha^2 - c^2)\ y^{-2(n-2)}}\ .$$

Let F be the unique solution of the following differential equation on $[0,\infty)$:

$$\frac{dy}{dx} = \sqrt{c^2 + (\alpha^2 - c^2)\ y^{-2(n-2)}}$$
$$y(0) = 1.$$

It is easy to see that F satisfies the above conditions. This completes the proof. □

Remarks.

1) We do not know whether condition (iii) in Theorem 1 is necessary or not.

2) It would be interesting to have a more explicit estimate for δ_n. Among all the examples we know of non-contractible manifolds with positive Ricci curvature, Λ is rather small ($< \frac{1}{2}$).
3) Using Bochner's formula, one can show that $\Lambda \le 1$ for every regular closed geodesic ball in a manifold with non-negative Ricci curvature. It would be interesting to know if $\Lambda \le 1$ is generally true for manifolds with non-empty boundary and non-negative Ricci curvature.

3. Appendix: The Case in Dimension 3.

Let (M, g) be a closed 3-dimensional riemannian manifold. Hamilton proved in [**3**] that if $\mathrm{Ric}_{_M} > 0$, then the metric g can be deformed to a metric with constant sectional curvature 1. We now prove the following using Hamilton's theorem.

Theorem 2. *Let M be a compact connected 3-dimensional riemannian manifold with non-empty boundary ∂M. If M satisfies:*

(i) $\mathrm{Ric}_{_M} > 0$;
(ii) *∂M is convex,*

then M is diffeomorphic to the 3-dimensional disc D^3.

Proof. Let N be the compact manifold obtained by gluing two copies of M along their boundaries with the identity map. Since ∂M is convex, $S_{\partial M}+S_{\partial M}$ is clearly positive definite. By the same reason as in the proof of Theorem $1'$, the metric near ∂M can be smoothed out so that the resulting new metric on N has positive Ricci curvature. Then Hamilton's theorem and the Synge Theorem imply that N (hence M) is orientable. Moreover, by the Meyers Theorem, both $\pi_1(N)$ and $\pi_1(M)$ are finite. We shall show that ∂M is diffeomorphic to the 2-dimensional sphere S^2, and N is diffeomorphic to the 3-dimensional sphere S^3. It then follows from the Schoenflies-Brown-Mazur Theorem that M is diffeomorphic to D^3.

Since $\pi_1(M)$ is finite, $H_1(M)$ is finite. This implies that

$$H_2(M, \partial M) \cong H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbf{Z}) = 0.$$

Therefore, the homomorphism $H_1(\partial M) \longrightarrow H_1(M)$ is injective in the following long exact sequence

$$\cdots \longrightarrow H_2(M, \partial M) \longrightarrow H_1(\partial M) \longrightarrow H_1(M) \longrightarrow \cdots .$$

Hence, $H_1(\partial M)$ is also finite. On the other hand, due to a theorem of Lawson [**4**], assumptions (i) and (ii) imply that ∂M is connected. Since S^2 is the only orientable closed surface with finite first homology group, ∂M is diffeomorphic to S^2. Therefore, $\pi_1(\partial M) = \pi_1(S^2) = 0$, which implies that $\pi_1(N)$ is the free product $\pi_1(M) * \pi_1(M)$ by the Van Kampen Theorem.

Since $\pi_1(N)$ is finite, $\pi_1(M)$ must be trivial, and hence $\pi_1(N)$ is also trivial. Hamilton's theorem then implies that N is diffeomorphic to S^3. This completes the proof. $\square$

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