PARTIAL REPRESENTATIONS AND AMENABLE
FELL BUNDLES OVER FREE GROUPS

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We show that a Fell bundle $\mathbb{B} = \{B_t\}_{t \in \mathbb{F}}$, over an arbitrary free group $\mathbb{F}$, is amenable, whenever it is orthogonal (in the sense that $B_x^* B_y = 0$, if x and y are distinct generators of $\mathbb{F}$) and semi-saturated (in the sense that B_{ts} coincides with the closed linear span of $B_t B_s$, when the multiplication "ts" involves no cancelation).

In this work we continue the study of the phenomena of amenability for Fell bundles over discrete groups, initiated in [**E3**]. By definition, a Fell bundle is said to be amenable if the left regular representation of its cross-sectional C^*-algebra is faithful. This property is also equivalent to the faithfulness of the standard conditional expectation. The reader is referred to [**E3**] for more information, but we also offer a very brief survey containing some of the most relevant definitions, in our section on preliminaries below.

The starting point for our work is Theorem 6.7 of [**E3**], where it is shown that a certain grading of the Cuntz–Krieger algebra gives rise to an amenable Fell bundle over a free group. Our main goal is to further pursue the argument leading to this result, in order to obtain a large class of amenable Fell bundles. We find that the crucial properties implying the amenability of a Fell bundle, over a free group $\mathbb{F}$, are *orthogonality* and *semi-saturatedness*. A Fell bundle $\mathbb{B} = \{B_t\}_{t \in \mathbb{F}}$ is said to be *orthogonal* if the fibers B_x and B_y, corresponding to two distinct generators x and y of $\mathbb{F}$, are orthogonal in the sense that $B_x^* B_y = 0$. On the other hand, $\mathbb{B}$ is said to be *semi-saturated* when each fiber B_t is "built up" from the fibers corresponding to the generators appearing in the reduced decomposition of t. More precisely, if $t = x_1 x_2 \cdots x_n$ is in reduced form, then one requires that $B_t = B_{x_1} B_{x_2} \cdots B_{x_n}$ (meaning closed linear span). This property makes sense for any group G, which, like the free group, is equipped with a length function $|\cdot|$. A Fell bundle over such a group is said to be semi-saturated if $B_{ts} = B_t B_s$ (closed linear span), whenever t and s satisfy $|ts| = |t| + |s|$.

Our main result, Theorem 6.3, states, precisely, that any Fell bundle over $\mathbb{F}$, which is orthogonal, semi-saturated, and has separable fibers, must be amenable.

To arrive at this conclusion we first restrict ourselves to a very special case of Fell bundles, namely those which are associated to a partial representation of $\mathbb{F}$ (see below for definitions). For these bundles, we prove an even stronger result, which is that they satisfy the approximation property of [**E3**]. This property implies amenability and also some other interesting facts related to induced ideals of the cross-sectional C^*-algebra (see [**E3**, 4.10]).

The proof of the approximation property for these restricted bundles is a direct generalization of [**E3**, 6.6], where we proved that, for every semi-saturated partial representation σ of $\mathbb{F}_n$ (see below for definitions), such that $\sum_{i=1}^{n} \sigma(g_i)\sigma(g_i)^* = 1$, the associated Fell bundle satisfies the approximation property. Here, $\{g_1, \ldots, g_n\}$ are the generators of the free group $\mathbb{F}_n$.

Our generalization of this result, namely Theorem 3.7, below, amounts to replacing the hypothesis that $\sum_{i=1}^{n} \sigma(g_i)\sigma(g_i)^* = 1$, by the weaker requirement that this sum is no larger than 1, or, equivalently, that the $\sigma(g_i)\sigma(g_i)^*$ are pairwise orthogonal projections.

Arriving at this generalization turns out to require a considerable understanding of the various idempotents accompanying a partial representation of $\mathbb{F}$, and underlines the richness of ideas surrounding the concept of partial representations. In addition, the new hypothesis, that is, the orthogonality of these projections, is easily generalizable to free groups with infinitely many generators. With the same ease, based on a simple inductive limit argument, we extend Theorem 3.7 to the infinitely generated case, obtaining Theorem 4.1, below.

In Section 5, armed with this partial result, we study Fell bundles which satisfy, in addition to the hypothesis of our main theorem, a stability property. Employing a fundamental result of Brown, Green and Rieffel [**BGR**], we are able to show that, for such bundles, there is a hidden partial representation of $\mathbb{F}$ which sends us back to the previously studied situation. We finally remove the extra stability hypothesis by means of a simple stabilization argument.

It does not seem outlandish to expect that all amenable Fell bundles satisfy some form of the approximation property. However, having no definite evidence that this is so, we must be cautious in distinguishing these properties. Accordingly, we must stress that our main result falls short of proving the approximation property for the most general situation treated, that is, of orthogonal semi-saturated bundles. In this case, all we obtain is amenability, leaving that stronger property as an open question.

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1. Preliminaries.

For the reader's convenience, and also to fix our notation, we shall begin by briefly discussing some basic facts about partial group representations, Fell bundles and the rich way in which these concepts are interrelated. The reader is referred to [**FD**], [**E2**], and [**E3**] for more information on these subjects.

Let G be a group, fixed throughout this section. Also, let H be a Hilbert space and denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on H.

Definition 1.1. A *partial representation* of G on H is, by definition, a map $\sigma : G \to \mathcal{B}(H)$ such that

i) $\sigma(t)\sigma(s)\sigma(s^{-1}) = \sigma(ts)\sigma(s^{-1})$,

ii) $\sigma(t^{-1}) = \sigma(t)^*$,

iii) $\sigma(e) = I$,

for all $t, s \in G$, where e denotes the unit group element and I is the identity operator on H.

Let σ be a partial representation of G on H. It is an easy consequence of the definition that each $\sigma(t)$ is a partially isometric operator and hence that

$$\mathrm{e}(t) := \sigma(t)\sigma(t)^*$$

is a projection (that is, a self-adjoint idempotent). It is not hard to show (see [**E2**]) that these projections commute among themselves, and satisfy the commutation relation

(1.2) $$\sigma(t)\mathrm{e}(s) = \mathrm{e}(ts)\sigma(t),$$

for all $t, s \in G$.

There is a special kind of partial representations worth considering, whenever G is equipped with a "length" function, that is, a non-negative real valued function $|\cdot| : G \to \mathbf{R}_+$ satisfying $|e| = 0$ and the triangular inequality $|ts| \le |t| + |s|$.

Definition 1.3. A partial representation σ of G is said to be *semi-saturated* (with respect to a given length function $|\cdot|$ on G) if $\sigma(t)\sigma(s) = \sigma(ts)$ whenever t and s satisfy $|ts| = |t| + |s|$.

The concept of partial representations is closely related to that of Fell bundles (also known as C^*-algebraic bundles [**FD**]) as we shall now see.

Definition 1.4. Given a partial representation σ of G, for each t in G, let B_t^σ be the closed linear subspace of $\mathcal{B}(H)$ spanned by the set of operators of the form

$$\mathrm{e}(r_1)\mathrm{e}(r_2) \cdots \mathrm{e}(r_k)\sigma(t),$$

where $k \in \mathbf{N}$, and $r_1, r_2, \dots, r_k$ are arbitrary elements of G.

Using the axioms of partial representations and 1.2, it is an easy exercise to show (see [**E3**, Section 6]) that, for all $t, s \in G$, we have

$$B_t^\sigma B_s^\sigma \subseteq B_{ts}^\sigma \quad \text{and} \quad (B_t^\sigma)^* = B_{t^{-1}}^\sigma.$$

Therefore, the collection

$$\mathbb{B}^\sigma := \{B_t^\sigma\}_{t \in G}$$

is seen to form a Fell bundle over G.

Definition 1.5. A Fell bundle over a discrete group G is a collection $\mathbb{B} = \{B_t\}_{t \in G}$ of closed subspaces of $\mathcal{B}(H)$, such that $B_t B_s \subseteq B_{ts}$ and $(B_t)^* = B_{t^{-1}}$, for all t and s in G.

As is the case with C^*-algebras, which can be defined, concretely, as a norm-closed *-subalgebra of $\mathcal{B}(H)$, as well as a certain abstract mathematical object, defined via a set of axioms, Fell bundles may also be seen under a dual point of view, specially if one restricts attention to the case of discrete groups. The above definition of Fell bundles is the one we adopt here, referring the reader to [**FD**, VIII.16.2] for the abstract version and to [**FD**, VIII.16.4] for the equivalence of these. Nevertheless, it should be said that the point of view one usually adopts in the study of Fell bundles stresses that each B_t should be viewed as a Banach space in its own, and that for each t and s in G, one has certain algebraic operations

$$\cdot : B_t \times B_s \to B_{ts}$$

and

$$* : B_t \to B_{t^{-1}},$$

which, in our case, are induced by the multiplication and involution on $\mathcal{B}(H)$, respectively. If G is not discrete, then one should also take into account a topology on the disjoint union $\dot{\bigcup}_{t \in G} B_t$, which is compatible with the other ingredients present in the situation. See [**FD**] for details. Since we will only deal with Fell bundles over discrete groups, we need not worry about this topology.

Definition 1.6. A Fell bundle $\mathbb{B}$, over of G, is said to be *semi-saturated* (with respect to a given length function $|\cdot|$ on G) if $B_{ts} = B_t B_s$ (closed linear span), whenever t and s satisfy $|ts| = |t| + |s|$.

When σ is semi-saturated, one can prove, as in [**E3**, 6.2], that the Fell bundle $\mathbb{B}^\sigma$ is also semi-saturated.

Definition 1.7. Given any Fell bundle $\mathbb{B} = \{B_t\}_{t \in G}$, with G discrete, one defines its l_1 cross-sectional algebra [**FD**, VIII.5], denoted $l_1(\mathbb{B})$, to be the Banach *-algebra consisting of the l_1 cross-sections of $\mathbb{B}$, under the multiplication

$$fg(t) = \sum_{s \in G} f(s)g(s^{-1}t), \quad \text{for } t \in G, \ f, g \in l_1(\mathbb{B}),$$

involution

$$f^*(t) = (f(t^{-1}))^*, \quad \text{for } t \in G, \ f \in l_1(\mathbb{B}),$$

and norm

$$\|f\| = \sum_{s \in G} \|f(s)\|, \quad \text{for } f \in l_1(\mathbb{B}).$$

The cross-sectional C^*-algebra of $\mathbb{B}$ [**FD**, VIII.17.2], denoted $C^*(\mathbb{B})$, is defined to be the enveloping C^*-algebra of $l_1(\mathbb{B})$.

There is also a *reduced* cross-sectional C^*-algebra, indicated by $C_r^*(\mathbb{B})$, which is defined to be the closure of $l_1(\mathbb{B})$ in a certain *regular representation* (acting on the right-B_e-Hilbert-bimodule formed by the l_2 cross-sections). See [**E3**, 2.3] for a precise definition.

Both $C^*(\mathbb{B})$ and $C_r^*(\mathbb{B})$ contain a copy of the algebraic direct sum $\bigoplus_{t \in G} B_t$, as a dense subalgebra, making them into G-graded C^*-algebras in the sense of [**FD**, VIII.16.11] (see also [**E3**, 3.1]). In both cases, the projections onto the factors extend to bounded linear maps on the whole algebra, and, in particular, for B_e, that projection gives a conditional expectation [**E3**, 3.3].

This conditional expectation, say E, is faithful in the case of $C_r^*(\mathbb{B})$, in the sense that

$$E(x^*x) = 0 \ \Rightarrow \ x = 0$$

for every $x \in C_r^*(\mathbb{B})$ [**E3**, 2.12]. However, the same cannot be said with respect to $C^*(\mathbb{B})$. In fact, there always exists an epimorphism

$$\Lambda : C^*(\mathbb{B}) \to C_r^*(\mathbb{B})$$

which restricts to the identity map on $\bigoplus_{t \in G} B_t$ (see the discussion following [**E3**, 2.2] as well as [**E3**, 3.3]). The kernel of Λ coincides with the *degeneracy* ideal for E, namely

$$\mathcal{D} = \{x \in C^*(\mathbb{B}) : E(x^*x) = 0\},$$

where we are also denoting the conditional expectation for $C^*(\mathbb{B})$ by E, by abuse of language (see [**E3**, 3.6]). This can be used to give an alternate definition of $C_r^*(\mathbb{B})$, as the quotient of $C^*(\mathbb{B})$ by that ideal.

The crucial property of Fell bundles with which we will be concerned throughout this work is that of amenability. This property is inspired, first of all, in the corresponding concept for groups [**G**], but also in the work of Anantharaman-Delaroche [**A**] and Nica [**Ni**]. In the context we are interested, that is, for Fell bundles, it first appeared in [**E3**], for discrete groups, and was subsequently generalized by Ng for the non-discrete case [**Ng**]. See also [**EN**].

Definition 1.8. A Fell bundle $\mathbb{B}$, over a discrete group G, is said to be amenable, if Λ is an isomorphism.

According to the characterization of the kernel of Λ, as in our discussion above, we see that:

Proposition 1.9. *A necessary and sufficient condition for $\mathbb{B}$ to be amenable is that the conditional expectation E of $C^*(\mathbb{B})$ be faithful.*

Any bundle is amenable when the base group G is amenable [**E3**, 4.7], whereas the typical example of non-amenable bundle is the *group bundle* [**FD**, VIII.2.7] over a non-amenable group G. That is, the bundle $\mathbf{C} \times G$, with the operations (abstractly) defined by

$$(z,t)(w,s) = (zw, ts) \quad \text{and} \quad (z,t)^* = (\bar{z}, t^{-1})$$

for $t, s \in G$ and $z, w \in \mathbf{C}$. In this case $C^*(\mathbb{B})$ is the full group C^*-algebra of G, while $C_r^*(\mathbb{B})$ is its reduced algebra. It is well known [**P**] that Λ is an isomorphism, in this case, if and only if G is amenable.

A Fell bundle may be amenable even if its base group G is not. One example of this situation is given by [**E3**, 6.7]. It consists of a Fell bundle over the non-amenable free group which is, itself, amenable. This example is particularly interesting, since its cross-sectional C^*-algebra is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_A$.

Definition 1.10. We say that $\mathbb{B}$ has the *approximation property* [**E3**, 4.5] if there exists a net $\{a_i\}_{i \in I}$ of finitely supported functions $a_i : G \to B_e$, which is uniformly bounded in the sense that there exists a constant $M > 0$ such that

$$\left\| \sum_{t \in G} a_i(t)^* a_i(t) \right\| \leq M,$$

for all i, and such that for all b_t in each B_t one has that

$$b_t = \lim_{i \to \infty} \sum_{r \in G} a_i(tr)^* b_t a_i(r).$$

The relevance of the approximation property is that:

Theorem 1.11. *If a Fell bundle $\mathbb{B}$ has the approximation property, then it is amenable.*

Proof. See [**E3**, 4.6]. $\hfill\square$

For later use, it will be convenient to have certain equivalent forms of the approximation property, which we now study.

Lemma 1.12. *Let $\mathbb{B}$ be a Fell bundle over G, and let $a : G \to B_e$ be a finitely supported function. Then, for each t in G, the map*

$$b_t \in B_t \mapsto \sum_{r \in G} a(tr)^* b_t a(r) \in B_t$$

is bounded, with norm no bigger than $\left\| \sum_{r \in G} a(r)^* a(r) \right\|$.

Proof. Recall that $\left\|\sum_{i=1}^n x_i^* y_i\right\| \leq \left\|\sum_{i=1}^n x_i^* x_i\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^n y_i^* y_i\right\|^{\frac{1}{2}}$, whenever $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are elements of a C^*-algebra. Therefore, letting $M = \left\|\sum_{r \in G} a(r)^* a(r)\right\|$, we have, for all b_t in B_t, that

$$\left\|\sum_{r \in G} a(tr)^* b_t a(r)\right\| \leq \left\|\sum_{r \in G} a(tr)^* a(tr)\right\|^{\frac{1}{2}} \left\|\sum_{r \in G} a(r)^* b_t^* b_t a(r)\right\|^{\frac{1}{2}}$$

$$\leq M^{\frac{1}{2}} \|b_t\| \left\|\sum_{r \in G} a(r)^* a(r)\right\|^{\frac{1}{2}} = M\|b_t\|.$$

$\square$

Proposition 1.13. *Let $\mathbb{B} = \{B_t\}_{t \in G}$ be a Fell bundle over the discrete group G. Also, suppose we are given a dense subset D_t of B_t, for each t in G. Then the following are equivalent:*

i) *$\mathbb{B}$ satisfies the approximation property.*

ii) *There exists a net $\{a_i\}_{i \in I}$ satisfying all of the properties of 1.10, except that the condition involving the limit is only assumed for b_t belonging to D_t.*

iii) *There exists a constant $M > 0$ such that, for all finite sets $\{b_{t_1}, b_{t_2}, \ldots, b_{t_n}\}$, with $b_{t_k} \in D_{t_k}$, and any $\varepsilon > 0$, there exists a finitely supported function $a : G \to B_e$ such that*

$$\left\|\sum_{t \in G} a(t)^* a(t)\right\| \leq M$$

and

$$\left\|b_{t_k} - \sum_{r \in G} a(t_k r)^* b_{t_k} a(r)\right\| < \varepsilon,$$

for $k = 1, \ldots, n$.

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious. We shall than prove that (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). With respect to our first task, consider the set of pairs (X, ε), where X is any finite subset of the disjoint union $\dot{\bigcup}_{t \in G} D_t$, and ε is a positive real. If these pairs are ordered by saying that $(X_1, \varepsilon_1) \leq (X_2, \varepsilon_2)$ if and only if $X_1 \subseteq X_2$ and $\varepsilon_1 \geq \varepsilon_2$, we clearly get a directed set. For each such (X, ε), let $a_{X,\varepsilon}$ be chosen such as to satisfy the conditions of (iii) with respect to the set X and ε. It is then clear that the net $\{a_{X,\varepsilon}\}_{(X,\varepsilon)}$ provides the required net.

As for (ii) $\Rightarrow$ (i), since the maps of 1.12 are bounded, the convergence referred to in (ii) is easily seen to hold throughout B_t. $\square$

2. Free groups and orthogonal partial representations.

This section is devoted to introducing a certain class of partial representa-
tions of the free group. Let $\mathbb{F}$ denote the free group on a possibly infinite
set $\mathcal{S}$ (whose elements we call the generators). Each t in $\mathbb{F}$ has a unique
decomposition (called its reduced decomposition, or reduced form)

$$t = x_1 x_2 \cdots x_k,$$

where $x_i \in \mathcal{S} \cup \mathcal{S}^{-1}$ and $x_{i+1} \neq x_i^{-1}$ for all i. In this case, we set $|t| = k$
and it is not hard to see that this gives, in fact, a length function for $\mathbb{F}$. It
is with respect to this length function that we will speak of semi-saturated
partial representations of $\mathbb{F}$.

Definition 2.1. A partial representation σ of $\mathbb{F}$ is said to be *orthogonal* if
$\sigma(x)^*\sigma(y) = 0$ whenever $x, y \in \mathcal{S}$ are generators with $x \neq y$.

A partial representation of a group may not be determined by its values
on a set of generators. For example, if we set

$$\sigma(t) = \begin{cases} 1 & \text{if } |t| \text{ is even} \\ 0 & \text{if } |t| \text{ is odd} \end{cases}$$

then σ is a partial representation of $\mathbb{F}$ (on a one dimensional Hilbert space),
which coincides, on the generators, with the partial representation

$$\sigma'(t) = \begin{cases} 1 & \text{if } t = e \\ 0 & \text{otherwise.} \end{cases}$$

However, if σ is semi-saturated, then

$$\sigma(t) = \sigma(x_1)\sigma(x_2)\cdots\sigma(x_k),$$

whenever $t = x_1 x_2 \cdots x_k$ is in reduced form. Therefore, the values of σ
on the generators end up characterizing σ completely. For this reason the
fact that σ is orthogonal often says little, unless one supposes that σ is
semi-saturated as well.

We shall denote by $\mathcal{W}$ the sub-semigroup of $\mathbb{F}$ generated by $\mathcal{S}$, that is,
the set of all products of elements from $\mathcal{S}$ (as opposed to $\mathcal{S} \cup \mathcal{S}^{-1}$). By
convention, $\mathcal{W}$ also includes the identity group element. The elements of $\mathcal{W}$
are called the *positive* elements and will usually be denoted by letters taken
from the beginning of the Greek alphabet. For each natural number k we
will denote by $\mathcal{W}_k$ the set of positive elements of length k.

Note that, if σ is a semi-saturated partial representation of $\mathbb{F}$, and $\alpha, \beta \in$
$\mathcal{W}$, then $\sigma(\alpha)\sigma(\beta) = \sigma(\alpha\beta)$, since $|\alpha\beta| = |\alpha| + |\beta|$. This property will be
useful in many situations, below.

Proposition 2.2. *Let σ be an orthogonal, semi-saturated partial representation of $\mathbb{F}$. Then $\sigma(t) = 0$ for all elements t in $\mathbb{F}$ which are not of the form $\mu\nu^{-1}$, with μ and ν positive.*

Proof. Let $t = x_1 x_2 \cdots x_k$, with $x_i \in \mathcal{S} \cup \mathcal{S}^{-1}$, be in reduced form. Then, since σ is semi-saturated, $\sigma(t) = \sigma(x_1)\sigma(x_2)\cdots\sigma(x_k)$. Now, because σ is orthogonal, if $x_i \in \mathcal{S}^{-1}$ and $x_{i+1} \in \mathcal{S}$ then $\sigma(x_i)\sigma(x_{i+1}) = 0$. So, in order to have $\sigma(t)$ nonzero, all elements from $\mathcal{S}$ must be to the left of the elements from $\mathcal{S}^{-1}$ in the decomposition of t. That is, t is of the form described in the statement. $\square$

Proposition 2.3. *Let σ be an orthogonal, semi-saturated partial representation of $\mathbb{F}$, and let $\alpha, \beta \in \mathcal{W}$. If $|\alpha| = |\beta|$, but $\alpha \neq \beta$, then $\sigma(\alpha)^*\sigma(\beta) = 0$.*

Proof. Let $m = |\alpha| = |\beta|$. If $m = 1$ then α and β are in $\mathcal{S}$ and the conclusion is a consequence of the orthogonality assumption. If $m > 1$ write $\alpha = x\tilde{\alpha}$ and $\beta = y\tilde{\beta}$ with $\tilde{\alpha}, \tilde{\beta} \in \mathcal{W}$ and $x, y \in \mathcal{S}$.

Assume, by way of contradiction, that $\sigma(\alpha)^*\sigma(\beta) \neq 0$. Then

$$0 \neq \sigma(x\tilde{\alpha})^*\sigma(y\tilde{\beta}) = \sigma(\tilde{\alpha})^*\sigma(x)^*\sigma(y)\sigma(\tilde{\beta}).$$

So, in particular, $\sigma(x)^*\sigma(y) \neq 0$, which implies that $x = y$.

We therefore have, using 1.2,

$$0 \neq \sigma(\tilde{\alpha})^*\sigma(x)^*\sigma(x)\sigma(\tilde{\beta}) = \sigma(\tilde{\alpha})^*\mathrm{e}(x^{-1})\sigma(\tilde{\beta}) = \sigma(\tilde{\alpha})^*\sigma(\tilde{\beta})\mathrm{e}(\tilde{\beta}^{-1}x^{-1}),$$

which implies that $\sigma(\tilde{\alpha})^*\sigma(\tilde{\beta}) \neq 0$, and hence, by induction, that $\tilde{\alpha} = \tilde{\beta}$. So $\alpha = \beta$. $\square$

The concept of orthogonality also applies to Fell bundles over free groups.

Definition 2.4. A Fell bundle $\mathbb{B} = \{B_t\}_{t\in\mathbb{F}}$ over $\mathbb{F}$ is said to be *orthogonal* if $B_x^* B_y = \{0\}$ whenever $x, y \in \mathcal{S}$ are generators with $x \neq y$.

The parallel between this concept and its homonym 2.1 is illustrated by our next:

Proposition 2.5. *If σ is an orthogonal partial representation of $\mathbb{F}$, then $\mathbb{B}^\sigma$ is an orthogonal Fell bundle.*

Proof. Left to the reader. $\square$

3. The finitely generated case.

We now start the main technical section of the present work. Here we shall prove the approximation property for Fell bundles arising from certain partial representations of free groups. Even though our long range objective is to treat arbitrary free groups, we shall temporarily restrict our attention to finitely generated free groups. So we make the following:

Standing hypothesis 3.1. For the duration of this section, the set $\mathcal{S}$, of generators of $\mathbb{F}$ will be assumed to be finite and σ will be a fixed orthogonal, semi-saturated partial representation of $\mathbb{F}$.

Recall that $\mathrm{e}(t)$ denotes the final projection $\sigma(t)\sigma(t)^*$ of the partial isometry $\sigma(t)$. In addition to $\mathrm{e}(t)$, the following operators will play a crucial role:

$$P_k = \sum_{\alpha \in \mathcal{W}_k} \mathrm{e}(\alpha), \quad k \geq 1$$

$$Q_0 = 1 - P_1,$$

$$f(t) = \sigma(t)Q_0\sigma(t)^*, \quad t \in \mathbb{F}$$

$$Q_k = \sum_{\alpha \in \mathcal{W}_k} f(\alpha), \quad k \geq 1.$$

The only place where the finiteness hypothesis in 3.1 will be explicitly used is in the observation that these sums are finite sums.

Proposition 3.2. *The following relations hold among the operators defined above:*

   i) *P_1 and Q_0 are projections.*
   ii) *Each $f(t)$ is a projection.*
  iii) *If $t, s \in \mathbb{F}$ then $\sigma(t)f(s) = f(ts)\sigma(t)$.*
  iv) *For every t one has $f(t) \leq \mathrm{e}(t)$.*
   v) *If $\alpha, \beta \in \mathcal{W}$ are such that $|\alpha| = |\beta|$ but $\alpha \neq \beta$, then $\mathrm{e}(\alpha) \perp \mathrm{e}(\beta)$, $f(\alpha) \perp f(\beta)$, and $\mathrm{e}(\alpha) \perp f(\beta)$.*
  vi) *For all $k \geq 1$, both P_k and Q_k are projections and $Q_k = P_k - P_{k+1}$.*
 vii) *For every n, we have that $Q_0 + Q_1 + \cdots + Q_{n-1} + P_n = 1$.*
viii) *If α and β are distinct positive elements of $\mathbb{F}$ then, regardless of their length, we have that $f(\alpha) \perp f(\beta)$.*

Proof. For $x, y \in \mathcal{W}_1 = \mathcal{S}$, with $x \neq y$, we have, by the orthogonality assumption, that $\mathrm{e}(x)\mathrm{e}(y) = \sigma(x)\sigma(x)^*\sigma(y)\sigma(y)^* = 0$. Hence P_1 is a sum of pairwise orthogonal projections, and thus, itself a projection. Therefore Q_0 is also a projection.

Speaking of (ii) we have

$$f(t)^2 = \sigma(t)Q_0\sigma(t)^*\sigma(t)Q_0\sigma(t)^* = \sigma(t)Q_0\mathrm{e}(t^{-1})Q_0\sigma(t)^*.$$

Taking into account that the final and initial projections associated to the partial isometries in a partial representation all commute with each other [**E2**], we see that the above equals

$$\sigma(t)Q_0\sigma(t)^*\sigma(t)\sigma(t)^* = \sigma(t)Q_0\sigma(t)^* = f(t).$$

To prove (iii), let $t, s \in \mathbb{F}$. Then

$$\sigma(t)f(s) = \sigma(t)\sigma(s)Q_0\sigma(s)^* = \sigma(t)\sigma(s)\sigma(s)^*\sigma(s)Q_0\sigma(s)^*$$
$$= \sigma(ts)\mathrm{e}(s^{-1})Q_0\sigma(s)^* = \sigma(ts)\sigma(ts)^*\sigma(ts)Q_0\mathrm{e}(s^{-1})\sigma(s)^*$$
$$= \sigma(ts)Q_0\sigma(ts)^*\sigma(ts)\sigma(s)^* = \sigma(ts)Q_0\sigma(ts)^*\sigma(t) = f(ts)\sigma(t).$$

As for (iv)

$$\mathrm{e}(t)f(t) = \sigma(t)\sigma(t)^*\sigma(t)Q_0\sigma(t)^* = f(t).$$

Given α and β as in (v) we have, by 2.3, that

$$\mathrm{e}(\alpha)\mathrm{e}(\beta) = \sigma(\alpha)\sigma(\alpha)^*\sigma(\beta)\sigma(\beta)^* = 0$$

which, when combined with (iv) above, yields the other statements of (v).

That P_k is a projection follows from the fact that the summands in its definition are pairwise orthogonal projections. The same reasoning applies to Q_k. Now

$$Q_k = \sum_{\alpha \in \mathcal{W}_k} \sigma(\alpha)(1 - P_1)\sigma(\alpha)^*$$
$$= \sum_{\alpha \in \mathcal{W}_k} \sigma(\alpha)\sigma(\alpha)^* - \sum_{\alpha \in \mathcal{W}_k}\sum_{x \in \mathcal{S}} \sigma(\alpha)\sigma(x)\sigma(x)^*\sigma(\alpha)^*$$
$$= P_k - \sum_{\alpha \in \mathcal{W}_k}\sum_{x \in \mathcal{S}} \sigma(\alpha x)\sigma(\alpha x)^*$$
$$= P_k - \sum_{\beta \in \mathcal{W}_{k+1}} \sigma(\beta)\sigma(\beta)^* = P_k - P_{k+1}.$$

To prove (vii), we just note that

$$Q_0 + Q_1 + \cdots + Q_{n-1} + P_n$$
$$= 1 - P_1 + P_1 - P_2 + \cdots + P_{n-1} - P_n + P_n = 1.$$

Finally, let $\alpha \neq \beta$ be positive and let $k = |\alpha|$ and $l = |\beta|$. If $k = l$ then we have already seen in (v), that $f(\alpha) \perp f(\beta)$. On the other hand, if $k \neq l$, then

$$f(\alpha) \leq Q_k \perp Q_l \geq f(\beta),$$

where the orthogonality of Q_k and Q_l follows from (vii). This implies, again, that $f(\alpha) \perp f(\beta)$. $\qquad\square$

Recall that $\mathbb{B}^\sigma = \{B_t^\sigma\}_{t \in G}$ denotes the Fell bundle associated to σ, and consider, for each integer $n \geq 1$, the map $b_n : \mathcal{W} \to B_e^\sigma$ given by

$$b_n(\alpha) = \begin{cases} f(\alpha) & \text{if } |\alpha| < n \\ \mathrm{e}(\alpha) & \text{if } |\alpha| = n \\ 0 & \text{if } |\alpha| > n. \end{cases}$$

Lemma 3.3. *For every $n \geq 1$ we have $\sum_{\alpha \in \mathcal{W}} b_n(\alpha) = 1$.*

Proof. We have

$$\sum_{\alpha \in \mathcal{W}} b_n(\alpha) = \sum_{k=0}^{n} \sum_{\alpha \in \mathcal{W}_k} b_n(\alpha) = \sum_{k=0}^{n-1} \sum_{\alpha \in \mathcal{W}_k} f(\alpha) + \sum_{\alpha \in \mathcal{W}_n} e(\alpha) = \sum_{k=0}^{n-1} Q_k + P_n = 1.$$

$\square$

The last relevant definition is that of another sequence of maps $a_n : \mathcal{W} \to B_e^\sigma$, this time given by

$$a_n(\alpha) = \left(\frac{1}{n} \sum_{k=1}^{n} b_k(\alpha) \right)^{\frac{1}{2}}.$$

Note that $a_n(\alpha) = 0$ for $|\alpha| > n$. We shall also think of the a_n as functions defined on the whole of $\mathbb{F}$, by setting $a_n(t) = 0$ when t is not positive.

Lemma 3.4. *For every $n \geq 1$ we have $\sum_{t \in \mathbb{F}} a_n(t)^* a_n(t) = 1$.*

Proof. As already observed, $a_n(t)$ vanishes unless t is positive. In addition, $a_n(\alpha)$ is self-adjoint, so we must compute

$$\sum_{\alpha \in \mathcal{W}} a_n(\alpha)^2 = \sum_{\alpha \in \mathcal{W}} \frac{1}{n} \sum_{k=1}^{n} b_k(\alpha) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\alpha \in \mathcal{W}} b_k(\alpha) = 1,$$

where we have used 3.3 in order to conclude the last step above. $\square$

The square root appearing in the definition of a_n can be explicitly computed if we note that, for $|\alpha| \leq n$, we have

$$\sum_{k=1}^{n} b_k(\alpha) = \left(\sum_{k=|\alpha|+1}^{n} f(\alpha) \right) + e(\alpha) = (n - |\alpha|) f(\alpha) + e(\alpha)$$

$$= (n - |\alpha| + 1) f(\alpha) + \Big(e(\alpha) - f(\alpha) \Big),$$

and that the expression above consists of a linear combination of orthogonal projections, namely $f(\alpha)$ and $e(\alpha) - f(\alpha)$ (see 3.2.iv). It follows that $a_n(\alpha)$ is given, explicitly, by

$$(3.5) \qquad a_n(\alpha) = \left(\frac{n - |\alpha| + 1}{n} \right)^{\frac{1}{2}} f(\alpha) + \left(\frac{1}{n} \right)^{\frac{1}{2}} \Big(e(\alpha) - f(\alpha) \Big).$$

The following is the main technical point in showing the approximation property for $\mathbb{B}^\sigma$:

Lemma 3.6. *For every t in $\mathbb{F}$ we have $\sigma(t) = \lim_{n \to \infty} \sum_{r \in \mathbb{F}} a_n(tr)^* \sigma(t) a_n(r)$.*

Proof. By 2.2 we may assume that $t = \mu\nu^{-1}$, where μ and ν are in $\mathcal{W}$. We may also suppose that $|t| = |\mu| + |\nu|$, that is, no cancellation takes place when μ and ν^{-1} are multiplied together.

In addition, since $a_n(r) = 0$ unless r is positive and $|r| \leq n$, each sum above is actually a finite sum. In fact, the nonzero summands in it are among those for which both r and tr are positive of length no bigger than n.

Since $tr = \mu\nu^{-1}r$, if both r and tr are to be positive, we must have $r = \nu\beta$, for some $\beta \in \mathcal{W}$, and then $|tr| = |\mu\nu^{-1}\nu\beta| = |\mu| + |\beta|$.

Also, in order to have $|r|$ and $|tr|$ no larger than n, we will need $|r| = |\nu| + |\beta| \leq n$, as well as $|\mu| + |\beta| \leq n$, which are equivalent to $|\beta| \leq m$, where

$$m = \min\{n - |\nu|, n - |\mu|\}.$$

Summarizing, for every n, we have

$$\sum_{r \in \mathbb{F}} a_n(tr)^* \sigma(t) a_n(r) = \sum_{|\beta| \leq m} a_n(\mu\beta) \sigma(t) a_n(\nu\beta),$$

where we have also taken into account that each $a_n(t)$ is self-adjoint.

Substituting the expression for a_n, obtained in 3.5, in the above sum, we conclude that each individual summand equals

$$\left[\left(\frac{n - |\mu\beta| + 1}{n}\right)^{\frac{1}{2}} f(\mu\beta) + \left(\frac{1}{n}\right)^{\frac{1}{2}} \left(\mathrm{e}(\mu\beta) - f(\mu\beta)\right)\right]$$

$$\cdot \sigma(t) \left[\left(\frac{n - |\nu\beta| + 1}{n}\right)^{\frac{1}{2}} f(\nu\beta) + \left(\frac{1}{n}\right)^{\frac{1}{2}} \left(\mathrm{e}(\nu\beta) - f(\nu\beta)\right)\right]$$

$$= \left(\frac{n - |\mu\beta| + 1}{n}\right)^{\frac{1}{2}} \left(\frac{n - |\nu\beta| + 1}{n}\right)^{\frac{1}{2}} f(\mu\beta) \ \sigma(t) \ f(\nu\beta)$$

$$+ \left(\frac{n - |\mu\beta| + 1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{n}\right)^{\frac{1}{2}} f(\mu\beta) \ \sigma(t) \ \left(\mathrm{e}(\nu\beta) - f(\nu\beta)\right)$$

$$+ \left(\frac{1}{n}\right)^{\frac{1}{2}} \left(\frac{n - |\nu\beta| + 1}{n}\right)^{\frac{1}{2}} \left(\mathrm{e}(\mu\beta) - f(\mu\beta)\right) \ \sigma(t) \ f(\nu\beta)$$

$$+ \left(\frac{1}{n}\right)^{\frac{1}{2}} \left(\frac{1}{n}\right)^{\frac{1}{2}} \left(\mathrm{e}(\mu\beta) - f(\mu\beta)\right) \ \sigma(t) \ \left(\mathrm{e}(\nu\beta) - f(\nu\beta)\right).$$

Let us indicate the four summands after the last equal sign above by (i), (ii), (iii), and (iv), in that order. In regards to (i), note that, employing 3.2.iii, we have

$$f(\mu\beta)\sigma(t)f(\nu\beta) = f(\mu\beta)\sigma(\mu)\sigma(\nu^{-1})f(\nu\beta)$$
$$= \sigma(\mu)f(\beta)f(\beta)\sigma(\nu^{-1}) = \sigma(\mu)f(\beta)\sigma(\nu)^*.$$

Referring to (ii) we have

$$f(\mu\beta)\sigma(t)\Big(\mathrm{e}(\nu\beta) - f(\nu\beta)\Big) = f(\mu\beta)\Big(\mathrm{e}(\mu\beta) - f(\mu\beta)\Big)\sigma(t) = 0,$$

because of 3.2.iv. Similarly one proves that (iii) vanishes as well. As for (iv)

$$\Big(\mathrm{e}(\mu\beta) - f(\mu\beta)\Big)\sigma(t)\Big(\mathrm{e}(\nu\beta) - f(\nu\beta)\Big)$$

$$= \Big(\mathrm{e}(\mu\beta) - f(\mu\beta)\Big)\sigma(\mu)\sigma(\nu^{-1})\Big(\mathrm{e}(\nu\beta) - f(\nu\beta)\Big)$$

$$= \sigma(\mu)\Big(\mathrm{e}(\beta) - f(\beta)\Big)\sigma(\nu)^*.$$

So,

$$a_n(\mu\beta)\sigma(t)a_n(\nu\beta)$$

$$= \sigma(\mu)\Bigg[\left(\frac{n - |\mu\beta| + 1}{n}\right)^{\frac{1}{2}}\left(\frac{n - |\nu\beta| + 1}{n}\right)^{\frac{1}{2}} f(\beta)$$

$$+ \frac{1}{n}\Big(\mathrm{e}(\beta) - f(\beta)\Big)\Bigg]\sigma(\nu)^*,$$

and the conclusion will follow once we prove that the term between brackets above, summed over $|\beta| \leq m$, converges, in norm, to the identity operator, as $n \to \infty$. We now set to do precisely this. Speaking of the identity operator, recall from 3.4 and 3.5 that,

$$1 = \sum_{t \in \mathbb{F}} a_m(t)^2 = \sum_{|\beta| \leq m} \frac{m - |\beta| + 1}{m} f(\beta) + \frac{1}{m}\Big(\mathrm{e}(\beta) - f(\beta)\Big).$$

Using this expression for the identity operator, we must then prove the vanishing of the following limit

$$\lim_{n \to \infty} \Bigg\| \sum_{|\beta| \leq m} \left(\frac{n - |\mu\beta| + 1}{n}\right)^{\frac{1}{2}}\left(\frac{n - |\nu\beta| + 1}{n}\right)^{\frac{1}{2}} f(\beta)$$

$$+ \frac{1}{n}\Big(\mathrm{e}(\beta) - f(\beta)\Big) - \frac{m - |\beta| + 1}{m} f(\beta) - \frac{1}{m}\Big(\mathrm{e}(\beta) - f(\beta)\Big)\Bigg\|$$

$$= \lim_{n \to \infty} \Bigg\| \sum_{|\beta| \leq m} \left(\frac{1}{n}(n - |\mu\beta| + 1)^{\frac{1}{2}}(n - |\nu\beta| + 1)^{\frac{1}{2}} - \frac{m - |\beta| + 1}{m}\right) f(\beta)$$

$$+ \left(\frac{1}{n} - \frac{1}{m}\right)\Big(\mathrm{e}(\beta) - f(\beta)\Big)\Bigg\|$$

$$\leq \lim_{n \to \infty} \Bigg\| \sum_{|\beta| \leq m} \left(\frac{1}{n}(n - |\mu\beta| + 1)^{\frac{1}{2}}(n - |\nu\beta| + 1)^{\frac{1}{2}} - \frac{m - |\beta| + 1}{m}\right) f(\beta)\Bigg\|$$

$$+ \lim_{n\to\infty} \left\| \sum_{|\beta|\leq m} \left(\frac{1}{n} - \frac{1}{m} \right) \left(e(\beta) - f(\beta) \right) \right\|.$$

The two limits will now be shown to equal zero. We should point out that, with respect to the the first one, we are facing a linear combination of pairwise orthogonal projections by 3.2.viii. The same, however, is not true for the second.

Using this observation, we see that the norm, in the first case, equals

$$\max_{|\beta|\leq m} \left| \frac{1}{n}(n - |\mu\beta| + 1)^{\frac{1}{2}}(n - |\nu\beta| + 1)^{\frac{1}{2}} - \frac{m - |\beta| + 1}{m} \right|.$$

In order to show that this goes to zero as $n \to \infty$, let us assume, without loss of generality, that $|\mu| \geq |\nu|$, and hence that $m = n - |\mu|$.

In addition, it is easy to see that, for every pair of positive reals x and y, one has that $|x - y| \leq |x^2 - y^2|^{\frac{1}{2}}$. So, the task facing us can be replaced by

$$\lim_{n\to\infty} \max_{|\beta|\leq m} \left| \frac{(n - |\mu\beta| + 1)(n - |\nu\beta| + 1)}{n^2} - \frac{(n - |\mu\beta| + 1)^2}{(n - |\mu|)^2} \right| \overset{?}{=} 0.$$

The term between the single bars is no bigger than

$$\left| \frac{(n - |\mu\beta| + 1)(n - |\nu\beta| + 1)}{n^2} - \frac{(n - |\mu\beta| + 1)^2}{n^2} \right|$$

$$+ (n - |\mu\beta| + 1)^2 \left| \frac{1}{n^2} - \frac{1}{(n - |\mu|)^2} \right|$$

$$\leq \frac{n - |\mu\beta| + 1}{n^2} \Big| |\mu| - |\nu| \Big| + (n - |\mu\beta| + 1)^2 \left| \frac{-2n|\mu| + |\mu|^2}{n^2(n - |\mu|)^2} \right|$$

$$\leq \frac{n + 1}{n^2} \Big| |\mu| - |\nu| \Big| + (n + 1)^2 \left| \frac{-2n|\mu| + |\mu|^2}{n^2(n - |\mu|)^2} \right|,$$

which is now easily seen to go to zero, uniformly on β, as $n \to \infty$.

To conclude, we need only show the vanishing of

$$\lim_{n\to\infty} \left\| \sum_{|\beta|\leq m} \left(\frac{1}{n} - \frac{1}{m} \right) \left(e(\beta) - f(\beta) \right) \right\|$$

$$= \lim_{n\to\infty} \left| \frac{1}{n} - \frac{1}{m} \right| \left\| \sum_{k=0}^{m} \sum_{\beta\in\mathcal{W}_k} e(\beta) - f(\beta) \right\|$$

$$= \lim_{n\to\infty} \left| \frac{1}{n} - \frac{1}{m} \right| \left\| \sum_{k=0}^{m} P_k - Q_k \right\|$$

$$= \lim_{n\to\infty} \left| \frac{1}{n} - \frac{1}{m} \right| \left\| \sum_{k=0}^{m} P_{k+1} \right\|$$

$$\leq \lim_{n\to\infty} \left| \frac{1}{n} - \frac{1}{m} \right| (m+1).$$

Now, recalling our assumption that $m = n - |\mu|$, the limit above equals

$$\lim_{n\to\infty} \left| \frac{1}{n} - \frac{1}{n-|\mu|} \right| (n - |\mu| + 1) = \lim_{n\to\infty} \frac{|\mu|(n - |\mu| + 1)}{n(n - |\mu|)} = 0.$$

$\square$

We are now prepared to face one of our main goals.

Theorem 3.7. *Let σ be an orthogonal, semi-saturated partial representation of a finitely generated free group $\mathbb{F}$. Then the Fell bundle $\mathbb{B}^\sigma$ satisfies the approximation property and hence is amenable. Moreover, the constant M referred to in* 1.10 *may be taken to be 1.*

Proof. Let $\{a_n\}_{n\in\mathbf{N}}$ be defined as above. Then $\left\| \sum_{t\in\mathbb{F}} a_n(t)^* a_n(t) \right\| = 1$, for all n, by 3.4, and employing 1.13.ii, it is now enough to show that

$$b_t = \lim_{n\to\infty} \sum_{r\in\mathbb{F}} a_n(tr)^* b_t a_n(r),$$

for all b_t of the form $b_t = \mathrm{e}(r_1)\mathrm{e}(r_2)\cdots\mathrm{e}(r_k)\sigma(t)$, where $k \in \mathbf{N}$, and r_1, $r_2, \ldots, r_k$ are arbitrary elements of $\mathbb{F}$. This is because the linear combinations of the elements of this form are dense in B_t^σ, by Definition 1.4.

We have already observed that the projections associated to the partial isometries in a partial representation form a commutative set. Since $a_n(\alpha)$ is given by a linear combination of such projections, by 3.5 (and $a_n(t) = 0$ when $t \notin \mathcal{W}$), it is clear that $a_n(t)$ commutes with the $\mathrm{e}(r_j)$. Therefore, by 3.6,

$$\lim_{n\to\infty} \sum_{r\in\mathbb{F}} a_n(tr)^* b_t a_n(r)$$

$$= \mathrm{e}(r_1)\mathrm{e}(r_2)\cdots\mathrm{e}(r_k) \lim_{n\to\infty} \sum_{r\in\mathbb{F}} a_n(tr)^* \sigma(t) a_n(r)$$

$$= \mathrm{e}(r_1)\mathrm{e}(r_2)\cdots\mathrm{e}(r_k)\sigma(t) = b_t.$$

This concludes the proof of the approximation property and hence of the amenability of $\mathbb{B}^\sigma$, by 1.11. $\square$

4. Arbitrary free groups.

We will now extend the results of the previous section, by dropping the finiteness hypothesis of 3.1, and hence including infinitely generated free groups in our study. The strategy will be to adapt the work done above to the general case using an inductive limit argument.

Let, therefore, $\mathbb{F}$ be the free group on a set $\mathcal{S}$, no longer assumed to be finite, or even countable. Also let σ be an orthogonal, semi-saturated partial representation of $\mathbb{F}$, considered fixed throughout this section.

For each finite subset X of $\mathcal{S}$, let $\mathbb{F}_X$ denote the subgroup of $\mathbb{F}$ generated by X. It is quite obvious that $\mathbb{F}_X$ is again a free group, and that $\mathbb{F}$ is the union of the increasing net $\{\mathbb{F}_X\}_X$. The length functions we've been considering are compatible in the sense that the one for $\mathbb{F}$ restricts to the one for $\mathbb{F}_X$. Therefore the restriction of σ to $\mathbb{F}_X$ is also semi-saturated, and obviously also orthogonal. Let $\mathbb{B}^X$ denote the Fell bundle for $\sigma|_{\mathbb{F}_X}$, as in 1.4.

It is clear that, for each t in $\mathbb{F}$, one has that B_t^σ is the closure of the union of the B_t^X, as X ranges in the collection of finite subsets $X \subseteq \mathcal{S}$, such that $t \in \mathbb{F}_X$.

Theorem 4.1. *Let σ be an orthogonal, semi-saturated partial representation of an arbitrary free group $\mathbb{F}$. Then the Fell bundle $\mathbb{B}^\sigma$ satisfies the approximation property and hence is amenable.*

Proof. For each t in $\mathbb{F}$, let D_t be the union of the B_t^X, as described above, which is dense in B_t^σ. We will now prove 1.13.iii, with respect to this choice of D_t. Let $M = 1$. Then, given a finite set $\{b_{t_1}, b_{t_2}, \ldots, b_{t_n}\}$, with $b_{t_k} \in D_{t_k}$, and any $\varepsilon > 0$, there clearly exists a single finite $X \subseteq \mathcal{S}$, such that every $b_{t_k} \in B_{t_k}^X$. Now, by 3.7 we conclude that a finitely supported map $a : \mathbb{F}_X \to B_e^X \subseteq B_e^\sigma$ exists, satisfying $\left\| \sum_{t \in \mathbb{F}_X} a(t)^* a(t) \right\| \leq 1$, and $\left\| b_{t_k} - \sum_{r \in \mathbb{F}_X} a(t_k r)^* b_{t_k} a(r) \right\| < \varepsilon$, for all $k = 1, \ldots, n$. If we extend a to the whole of $\mathbb{F}$ by declaring it zero outside $\mathbb{F}_X$, then these two sums may be taken for $t \in \mathbb{F}$, as opposed to $t \in \mathbb{F}_X$, without changing the result. We conclude that 1.13.iii holds, and hence that $\mathbb{B}^\sigma$ satisfies the approximation property . $\qquad\square$

5. Stable Fell bundles.

We shall now treat Fell bundles over arbitrary free groups, which are not necessarily presented in terms of a partial representation. This section does not yet contain our strongest result, because we shall be working under the assumption that the unit fiber algebra of $\mathbb{B}$, that is B_e, is a stable C^*-algebra, in the sense that it is isomorphic to $B_e \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable, infinite dimensional Hilbert space. In the next and final section we will then remove this stability hypothesis.

The method we shall adopt here will be to construct a partial representation which is closely related to $\mathbb{B}$. In fact this method is, essentially, the one we have used in [**E1**] to obtain the classification of Fell bundles in terms of twisted partial actions (see [**E1**], 7.3]). However, we will refrain from utilizing the full machinery of [**E1**] for two reasons. First, by proceeding more

or less from scratch, and using the special features of the free group, we will be able to make the presentation somewhat more elementary and self contained. Secondly, the classification theorem mentioned includes a 2-cocycle, which will not show up here, again because of the special properties of the group we are dealing with.

We begin with some simple facts about partial isometries and projections on a Hilbert space H.

Lemma 5.1. *Let p be an operator on H, such that $p^2 = p$ and $\|p\| \leq 1$. Then $p = p^*$.*

Proof. Let $\xi \in p(H)^{\perp}$. Then, for every $\lambda \in \mathbf{R}$ we have $\|p(\xi + \lambda p(\xi))\| \leq \|\xi + \lambda p(\xi)\|$, which implies, after a short calculation, that $(1 + 2\lambda)\|p(\xi)\|^2 \leq \|\xi\|^2$, and hence that $p(\xi) = 0$. This says that p vanishes on $p(H)^{\perp}$ and, since p is the identity on $p(H)$, then it must be the orthogonal projection onto $p(H)$. Hence $p = p^*$. $\qquad\qquad\square$

Lemma 5.2. *Let p and q be projections (self-adjoint idempotents) in $\mathcal{B}(H)$. Then pq is idempotent if and only if p and q commute.*

Proof. If pq is idempotent, then, since $\|pq\| \leq 1$, we have, by 5.1, that $pq = (pq)^* = qp$. The converse is trivial. $\qquad\qquad\square$

Lemma 5.3. *Let u and v be partial isometries in $\mathcal{B}(H)$. Then uv is a partial isometry if and only if u^*u and vv^* commute (see also [**S**]).*

Proof. We have that uv is a partial isometry, if and only if

$$uv(uv)^*uv = uv \iff uvv^*u^*uv = uv$$
$$\iff u^*uvv^*u^*uvv^* = u^*uvv^* \iff (u^*uvv^*)^2 = u^*uvv^*,$$

which, by 5.2, is equivalent to the commutativity of u^*u and vv^*. $\qquad\square$

Proposition 5.4. *Let $U = \{u_x\}_{x \in \mathcal{S}}$ be a family of partial isometries on a Hilbert space H and let $\mathcal{I}$ be the multiplicative sub-semigroup of $\mathcal{B}(H)$ generated by $U \cup U^*$. Denote by $\mathbb{F}$ the free group on $\mathcal{S}$. Then, the following are equivalent:*

  i) *There exists a semi-saturated partial representation σ of $\mathbb{F}$ such that $\sigma(x) = u_x$ for every $x \in \mathcal{S}$.*
  ii) *There exists a partial representation σ of $\mathbb{F}$ such that $\sigma(x) = u_x$ for every $x \in \mathcal{S}$.*
  iii) *Every u in $\mathcal{I}$ is a partial isometry.*
  iv) *For any $u, v \in \mathcal{I}$ we have that uu^* and vv^* commute.*

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Recall that an operator u is a partial isometry, if and only if $uu^*u = u$. So, let $u = u_1 \cdots u_n$, with $u_i \in U \cup U^*$, and take $t_i \in \mathcal{S} \cup \mathcal{S}^{-1}$

such that $\sigma(t_i) = u_i$. Then $u = \sigma(t_i) \cdots \sigma(t_n)$ and, by induction on n,

$$uu^*u = \sigma(t_1) \cdots \sigma(t_{n-1})\mathrm{e}(t_n)\sigma(t_{n-1})^* \cdots \sigma(t_1)^*\sigma(t_1) \cdots \sigma(t_n)$$
$$= \mathrm{e}(t_1 \cdots t_n)\sigma(t_1) \cdots \sigma(t_{n-1})\sigma(t_{n-1})^* \cdots \sigma(t_1)^*\sigma(t_1) \cdots \sigma(t_n)$$
$$= \mathrm{e}(t_1 \cdots t_n)\sigma(t_1) \cdots \sigma(t_{n-1})\sigma(t_n)$$
$$= \sigma(t_1) \cdots \sigma(t_{n-1})\mathrm{e}(t_n)\sigma(t_n) = \sigma(t_1) \cdots \sigma(t_{n-1})\sigma(t_n) = u.$$

(iii) $\Rightarrow$ (iv): If $u, v \in \mathcal{I}$, then $u^*v \in \mathcal{I}$ and, by (iii), it is a partial isometry. Hence, using 5.3, we have that uu^* and vv^* commute.

(iv) $\Rightarrow$ (i): Define, for all $x \in \mathcal{S}$, $\sigma(x) = u_x$ and $\sigma(x^{-1}) = u_x^*$. Now, if $t = x_1 \cdots x_n$, with $x_i \in \mathcal{S} \cup \mathcal{S}^{-1}$, is in reduced form, put $\sigma(t) = \sigma(x_1) \cdots \sigma(x_n)$. It is then obvious that $\sigma(t)\sigma(s) = \sigma(ts)$ whenever t and s satisfy $|ts| = |t| + |s|$.

We claim that σ is a partial representation of $\mathbb{F}$. The crucial point is to prove that $\sigma(t)\sigma(s)\sigma(s)^* = \sigma(ts)\sigma(s)^*$ for $t, s \in \mathbb{F}$. To do this we use induction on $|t| + |s|$. If either $|t|$ or $|s|$ is zero, there is nothing to prove. So, write $t = \tilde{t}x$ and $s = y\tilde{s}$, where $x, y \in \mathcal{S} \cup \mathcal{S}^{-1}$ and, moreover, $|t| = |\tilde{t}| + 1$ and $|s| = |\tilde{s}| + 1$.

In case $x^{-1} \neq y$ we have $|ts| = |t| + |s|$ and hence $\sigma(ts) = \sigma(t)\sigma(s)$. If, on the other hand, $x^{-1} = y$, we have

$$\sigma(t)\sigma(s)\sigma(s)^* = \sigma(\tilde{t}x)\sigma(x^{-1}\tilde{s})\sigma(\tilde{s}^{-1}x) = \sigma(\tilde{t})\sigma(x)\sigma(x)^*\sigma(\tilde{s})\sigma(\tilde{s})^*\sigma(x).$$

By (iv) and the induction hypothesis we conclude that the above equals

$$\sigma(\tilde{t})\sigma(\tilde{s})\sigma(\tilde{s})^*\sigma(x)\sigma(x)^*\sigma(x) = \sigma(\tilde{t}\tilde{s})\sigma(\tilde{s})^*\sigma(x)$$
$$= \sigma(ts)\sigma(\tilde{s}^{-1}x) = \sigma(ts)\sigma(s)^*.$$

$\square$

It would be interesting to find a condition about a set U of partial isometries, which is equivalent to the ones above, but which refers exclusively to the u_x's, themselves, rather than to arbitrary products of them. A related observation is that the Cuntz-Krieger [**CK**] relations were shown to imply the conditions above [**E3**, 5.2].

In our next result, we will use two important concepts from [**E1**], which we briefly summarize here. By definition, a TRO (for ternary ring of operators), is a closed linear subspace $E \subseteq \mathcal{B}(H)$, such that $EE^*E \subseteq E$ (see also [**Z**]). We adopt the convention that the product of two or more sets, as above, is supposed to mean the *closed linear span* of the set of products.

Given a TRO E, we say that a partial isometry u is associated to E [**E1**, 5.4], and write $u \sim E$, if $u^*E = E^*E$ and $uE^* = EE^*$. If, in addition, the range of the final projection uu^* coincides with EH (equivalently, if the range of the initial projection u^*u coincides with E^*H), then we say that u is *strictly* associated to E [**E1**, 5.5], and write $u \overset{s}{\sim} E$.

It is a consequence of [**BGR**, 3.3 and 3.4], that whenever E is separable and stable, in the sense that E^*E and EE^* are stable C^*-algebras [**E1**,

4.11], then a partial isometry strictly associated to E always exists (see also [**E1**, 5.3 and 5.2]).

The reason why TROs are relevant here is that any fiber of a Fell bundle is a TRO, as one may easily see. But, because of the separability requirement, we shall now restrict to the case of separable bundles, according to the following:

Definition 5.5. A Fell bundle $\mathbb{B}$, over a discrete group G, is said to be *separable* if each B_t is a separable Banach space.

Theorem 5.6. *Let $\mathbb{B} = \{B_t\}_{t \in \mathbb{F}}$ be a semi-saturated, separable Fell bundle over the arbitrary free group $\mathbb{F}$, represented on a Hilbert space H. Suppose that B_e is stable. Then, there exists a semi-saturated partial representation σ of $\mathbb{F}$ on H, such that $\sigma(t) \overset{s}{\sim} B_t$. In addition, if $\mathbb{B}$ is orthogonal, then σ is necessarily orthogonal, as well.*

Proof. By [**E1**, 4.12], each B_t is stable and hence, by [**E1**, 5.2], there exists a partial isometry $u_t \overset{s}{\sim} B_t$. Let $U = \{u_x\}_{x \in \mathcal{S}}$, where, as before, $\mathcal{S}$ is the set of generators of $\mathbb{F}$. We claim that U satisfies 5.4.iii. In fact, we shall prove that, given $x_1, \ldots, x_n$ in $\mathcal{S} \cup \mathcal{S}^*$, then $B_{x_1} \cdots B_{x_n}$ is a TRO and that $u_{x_1} \cdots u_{x_n}$ is a partial isometry strictly associated to it. Proceeding by induction, we may assume that $E = B_{x_1} \cdots B_{x_{n-1}}$ is a TRO and that $u = u_{x_1} \cdots u_{x_{n-1}} \overset{s}{\sim} E$. Now, observe that E^*E and $B_{x_n} B_{x_n}^*$ are ideals in B_e, and hence they commute, as sets, that is, $E^*E B_{x_n} B_{x_n}^* = B_{x_n} B_{x_n}^* E^* E$. So, by [**E1**, 6.4], it follows that EB_{x_n} is a TRO, and that $u u_{x_n} \overset{s}{\sim} EB_{x_n}$. This proves our claim. So, let σ be a semi-saturated partial representation of $\mathbb{F}$ satisfying 5.4.i. It remains to show that $\sigma(t) \overset{s}{\sim} B_t$, but this follows from the conclusion just above, once we write $t = x_1 \ldots x_n$ in reduced form.

Assume, now, that $\mathbb{B}$ is orthogonal. Then, given $x \neq y$, in $\mathcal{S}$, we have, again by [**E1**, 6.4], that $\sigma(x)^* \sigma(y) \overset{s}{\sim} B_x^* B_y = \{0\}$. Therefore $\sigma(x)^* \sigma(y) = 0$. $\square$

Theorem 5.7. *Let $\mathbb{B}$ be an orthogonal, semi-saturated, separable Fell bundle over $\mathbb{F}$, and suppose that B_e is stable. Then $\mathbb{B}$ is amenable (see below for the non-stable case).*

Proof. Let H be the space where $\mathbb{B}$ acts, and let σ be the orthogonal, semi-saturated partial representation of $\mathbb{F}$ on H, provided by 5.6. As usual, we denote by $e(t)$ the final projection of $\sigma(t)$.

Let $\mathbb{B}^\sigma$ be the Fell bundle associated to σ as in 1.4. By 4.1, we know that $\mathbb{B}^\sigma$ satisfies the approximation property. Let, therefore, $\{a_i\}_{i \in I}$ be a net of maps satisfying the conditions of 1.10, with respect to $\mathbb{B}^\sigma$.

We claim that $e(t)$ commutes with B_e. In fact, because $\sigma(t) \overset{s}{\sim} B_t$, we have that $e(t)$ is the orthogonal projection onto $B_t H$. Now, observing that B_e leaves the latter invariant, we obtain the conclusion. It follows that

the C^*-algebra generated by all the e(t), namely B_e^σ, is contained in the commutant of B_e.

Let $t \in \mathbb{F}$ and pick $b_t \in B_t$. Define $c_t = b_t\sigma(t)^*$. Then $c_t \in B_t\sigma(t)^* = B_t B_t^* \subseteq B_e$. On the other hand, note that, since the range of b_t^* is contained in $B_{t^{-1}}H$, which is also the range of e(t^{-1}), we have that e(t^{-1})$b_t^* = b_t^*$, or simply b_te(t^{-1}) $= b_t$. This implies that $c_t\sigma(t) = b_t\sigma(t)^*\sigma(t) = b_t$e($t^{-1}$) $= b_t$.

Since each $a_i(r) \in B_e^\sigma$, we have that it commutes with c_t and hence

$$\lim_{i\to\infty} \sum_{r\in\mathbb{F}} a_i(tr)^* b_t a_i(r) = c_t \lim_{i\to\infty} \sum_{r\in\mathbb{F}} a_i(tr)^*\sigma(t)a_i(r) = c_t\sigma(t) = b_t.$$

We therefore see that the net $\{a_i\}$ satisfies the properties of 1.10 with respect to $\mathbb{B}$, except that there is no reason to expect that the values of the maps a_i lie in B_e. Therefore this falls short of proving the approximation property for $\mathbb{B}$ and hence we cannot use 1.11 to conclude the amenability of $\mathbb{B}$.

Fortunately, what we do have is enough to fit the hypothesis of a slight generalization of the results of [**E3**] leading to 1.11, which goes as follows: Let $C^*(\mathbb{B})$ be faithfully represented on a Hilbert space K. Since $C^*(\mathbb{B})$ contains $\bigoplus_{t\in G} B_t$, we may then identify each B_t with its image under that representation, and then think of B_t as a space of operators on K. In other words, this provides a faithful representation of $\mathbb{B}$ as operators on K, and hence we might as well assume that $H = K$, which we do, from now on. Under this assumption we have that the sub-C^*-algebra of $\mathcal{B}(H)$ generated by $\bigcup_t B_t$ is isomorphic to $C^*(\mathbb{B})$.

For each t, consider the space C_t of operators on $H \otimes l_2(\mathbb{F})$, given by $C_t = B_t \otimes \lambda_t$, where λ is the left regular representation of $\mathbb{F}$. It is easy to see that the C_t form a Fell bundle, which is again isomorphic to $\mathbb{B}$. However, the C^*-algebra generated by $\bigcup_t C_t$ is now isomorphic to $C_r^*(\mathbb{B})$, a fact that follows from [**E3**, 3.7].

Recall that the reasoning at the beginning of the present proof provided a net $\{a_i\}_{i\in I}$ of maps $a_i : G \to \mathcal{B}(H)$, such that $\sup_i \left\|\sum_{t\in G} a_i(t)^* a_i(t)\right\| < \infty$, and that $b_t = \lim_{i\to\infty} \sum_{r\in G} a_i(tr)^* b_t a_i(r)$, for all b_t in each B_t. Following the argument used in [**E3**, 4.3], let, for each i,

$$V_i : H \to H \otimes l_2(\mathbb{F}),$$

be given by the formula $V_i(\xi) = \sum_{t\in\mathbb{F}} a_i(t)\xi \otimes \delta_t$, where $\{\delta_t\}$ is the standard orthonormal base of $l_2(\mathbb{F})$. One then proves, as in [**E3**, 4.3], that $\|V_i\| \leq \|\sum_{t\in\mathbb{F}} a_i(t)^* a_i(t)\|^{1/2}$, and hence that the V_i are uniformly bounded.

Now, define the completely positive maps

$$\Psi_i : \mathcal{B}(H \otimes l_2(\mathbb{F})) \to \mathcal{B}(H)$$

by $\Psi_i(T) = V_i^* T V_i$, for each T in $\mathcal{B}(H \otimes l_2(\mathbb{F}))$. Again as in [**E3**, 4.3], one has that, for every b_t in B_t,

$$\Psi_i(b_t \otimes \lambda_t) = \sum_{r \in \mathbb{F}} a_i(tr)^* b_t a_i(r).$$

The somewhat annoying fact that $a_i(t)$ may not belong to B_e forbids us to say that Ψ_i is a map from $C_r^*(\mathbb{B})$ into $C^*(\mathbb{B})$, as stated in [**E3**, 4.3]. This, however, is not a cause for despair.

Consider the canonical map $\Lambda : C^*(\mathbb{B}) \to C_r^*(\mathbb{B})$, described in section 1. Under the current representation of $C_r^*(\mathbb{B})$ on $H \otimes l_2(\mathbb{F})$, we have that $\Lambda(b_t) = b_t \otimes \lambda_t$ for all b_t. Now, consider the composition of maps

$$C^*(\mathbb{B}) \xrightarrow{\Lambda} C_r^*(\mathbb{B}) \xrightarrow{\Psi_i} \mathcal{B}(H).$$

For b_t in B_t, we have

$$\lim_i \Psi_i(\Lambda(b_t)) = \lim_i \Psi_i(b_t \otimes \lambda_t) = \lim_i \sum_{r \in \mathbb{F}} a_i(tr)^* b_t a_i(r) = b_t.$$

Now, by the uniform boundedness of these maps we then conclude that $\lim_i \Psi_i(\Lambda(x)) = x$ for all $x \in C^*(\mathbb{B})$. This implies that Λ is injective and hence that $\mathbb{B}$ is amenable, as required. $\square$

6. The general case.

In this section we will prove our most general result, which is the amenability of orthogonal, semi-saturated, separable bundles. This amounts to dropping the stability hypothesis of the previous section, which we do by a "stabilization argument". Ideally one should develop the whole theory of tensor products for Fell bundles but we feel this is not the place to do it. Instead, we perform our tensor products in a way which is enough for our purposes, albeit in a somewhat crude manner.

Let $\mathbb{B}$ be any Fell bundle over a discrete group G, acting on the Hilbert space H. As in the proof of 5.7, we may assume that the sub-C^*-algebra of $\mathcal{B}(H)$ generated by $\bigcup_t B_t$ is isomorphic to $C^*(\mathbb{B})$.

For each $t \in G$, consider the the subset of $\mathcal{B}(H \otimes l_2)$ (where l_2 is the usual infinite dimensional separable Hilbert space), denoted by $B_t \otimes \mathcal{K}$, and defined by

$$B_t \otimes \mathcal{K} = \overline{\text{span}}\{b_t \otimes k : b_t \in B_t, k \in \mathcal{K}\}.$$

Here $\mathcal{K}$ is the algebra of compact operators on l_2. It is elementary to show that $\mathbb{B} \otimes \mathcal{K}$, as defined by $\mathbb{B} \otimes \mathcal{K} = \{B_t \otimes \mathcal{K}\}_{t \in G}$, is a Fell bundle in its own right. We shall say that $\mathbb{B} \otimes \mathcal{K}$ is the *stabilization* of $\mathbb{B}$.

Proposition 6.1. *Let $\mathbb{B}$ be a Fell bundle. Then $C^*(\mathbb{B} \otimes \mathcal{K})$ is isomorphic to $C^*(\mathbb{B}) \otimes \mathcal{K}$.*

Proof. Let us temporarily use the notation $\tilde{\mathbb{B}}$ for $\mathbb{B} \otimes \mathcal{K}$ and $\tilde{B}_t$ for $B_t \otimes \mathcal{K}$. Observe that $\mathcal{K}$ may be viewed, in a canonical way, as a subalgebra of the multiplier algebra $\mathcal{M}(\tilde{B}_e)$, which, in turn, may be viewed as a subalgebra of $\mathcal{M}(C^*(\tilde{\mathbb{B}}))$ [**FD**, VIII.5.8 and 1.15]. Since one clearly has $\mathcal{K}C^*(\tilde{\mathbb{B}}) = C^*(\tilde{\mathbb{B}})$, one may now show that $C^*(\tilde{\mathbb{B}})$ is isomorphic to $A \otimes \mathcal{K}$, where $A = pC^*(\tilde{\mathbb{B}})p$, and p is any minimal projection in $\mathcal{K}$.

Using the universal property [**FD**, VIII.16.12] of cross-sectional C^*-algebras, one may show that the assignment $b_t \in B_t \mapsto b_t \otimes p \in C^*(\tilde{\mathbb{B}})$ extends to a surjective *-homomorphism $\phi : C^*(\mathbb{B}) \to A$.

We claim that ϕ is injective. In fact, let H be the space where $\mathbb{B}$ acts, so that $\tilde{\mathbb{B}}$ sits inside of $\mathcal{B}(H \otimes l_2)$. The universal property, this time applied to $\tilde{\mathbb{B}}$, implies that there exists a *-representation

$$\pi : C^*(\tilde{\mathbb{B}}) \to \mathcal{B}(H \otimes l_2)$$

which, restricted to each $\tilde{B}_t$, coincides with the inclusion of $\tilde{B}_t$ in $\mathcal{B}(H \otimes l_2)$. It is then easy to see that $\pi\phi$ maps $C^*(\mathbb{B})$ onto the closed linear span of $\bigcup_t B_t \otimes p$, within $\mathcal{B}(H \otimes l_2)$, which is isomorphic to $C^*(\mathbb{B}) \otimes p$ (see the second paragraph of this section).

Since $\pi\phi$ sends each b_t to $b_t \otimes p$, we see that $\pi\phi$ is the canonical isomorphism between $C^*(\mathbb{B})$ and $C^*(\mathbb{B}) \otimes p$. This shows that ϕ is injective and hence an isomorphism onto A, concluding the proof. $\qquad\square$

Proposition 6.2. *If $\mathbb{B}$ is a Fell bundle such that $\mathbb{B} \otimes \mathcal{K}$ is amenable, then $\mathbb{B}$, itself, is amenable.*

Proof. Consider the diagram

$$
\begin{array}{ccc}
C^*(\mathbb{B}) & \overset{\phi}{\to} & C^*(\mathbb{B} \otimes \mathcal{K}) \\
\\
E \downarrow & & \downarrow \tilde{E} \\
\\
B_e & \to & B_e \otimes \mathcal{K}
\end{array}
$$

where ϕ is the injective map described in the proof of 6.1, the vertical arrows represent the corresponding conditional expectations, and the unlabeled horizontal arrow maps each b_e to $b_e \otimes p$. It is easy to see that this is a commutative diagram. Recall that a Fell bundle is amenable if and only if the conditional expectation on its cross-sectional algebra is faithful, as seen in 1.9. So, let us show that $\mathbb{B}$ possesses this property. If $x \in C^*(\mathbb{B})$ is such that $E(x^*x) = 0$, then we have that $\tilde{E}(\phi(x^*x)) = 0$, whence $\phi(x^*x) = 0$ and, finally, $x = 0$ because ϕ is injective. $\qquad\square$

The following is our main result:

Theorem 6.3. *Let $\mathbb{B}$ be an orthogonal, semi-saturated, separable Fell bundle over $\mathbb{F}$. Then $\mathbb{B}$ is amenable.*

Proof. All of the properties assumed on $\mathbb{B}$ are clearly inherited by $\mathbb{B} \otimes \mathcal{K}$. In addition, the latter possesses a stable unit fiber algebra and hence, by 5.7, it is amenable. The conclusion then follows from 6.2.   □

Some of the most useful facts about amenable Fell bundles (see e.g. [**E3**, 4.10]) require not only that the bundle be amenable, but that the approximation property holds. This leads one to ask whether the result above could be strengthened by replacing amenability with the approximation property. We do not have a satisfactory answer to this question but then, again, we do not know of any example of an amenable Fell bundle which does not satisfy the approximation property.

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