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# CUBIC MODULAR EQUATIONS AND NEW RAMANUJAN-TYPE SERIES FOR $1/\pi$

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*Dedicated to our advisor, Professor Bruce C. Berndt on his $60^{th}$ birthday*

**In this paper, we derive new Ramanujan-type series for $1/\pi$ which belong to "*Ramanujan's theory of elliptic functions to alternative base* 3" developed recently by B.C. Berndt, S. Bhargava, and F.G. Garvan.**

## 1. Introduction.

Let $(a)_0 = 1$ and, for a positive integer $m$,

$$(a)_m := a(a+1)(a+2)\cdots(a+m-1),$$

and

$$_2F_1(a,b;c;z) := \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}, \quad |z|<1.$$

In his famous paper "Modular equations and approximations to $\pi$" [**10**], S. Ramanujan offered 17 beautiful series representations for $1/\pi$. He then remarked that two of these series

$$\frac{27}{4\pi} = \sum_{m=0}^{\infty}(2+15m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{2}{27}\right)^m \tag{1.1}$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{m=0}^{\infty}(4+33m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{4}{125}\right)^m \tag{1.2}$$

"belong to the theory of $q_2$," where

$$q_2 = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-k^2)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;k^2)}\right).$$

Ramanujan did not elaborate on his "theory of $q_2$," neither did he provide details for his proofs of (1.1) and (1.2).

Ramanujan's formulas (1.1) and (1.2) were first proved by J.M. Borwein and P.B. Borwein in 1987. Motivated by their study of Ramanujan's series for $1/\pi$ associated with the classical theory of elliptic functions, they established the following result:

**Theorem 1.1** ([**3**, p. 186])**.** *Let*

$$K(x) := {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; x), \quad \textit{and} \quad \dot{K}(x) := \frac{dK(x)}{dx}.$$

*For $n \in \mathbb{Q}^+$, define the cubic singular modulus to be the unique number $\alpha_n$ satisfying*

$$\frac{K(1-\alpha_n)}{K(\alpha_n)} = \sqrt{n}. \tag{1.3}$$

*Set*

$$\epsilon(n) = \frac{3\sqrt{3}}{8\pi}(K(\alpha_n))^{-2} - \sqrt{n}\left(\frac{3}{2}\alpha_n(1-\alpha_n)\frac{\dot{K}(\alpha_n)}{K(\alpha_n)} - \alpha_n\right), \tag{1.4}$$

$$a_n := \frac{8\sqrt{3}}{9}\left(\epsilon(n) - \sqrt{n}\alpha_n\right), \tag{1.5}$$

*and*

$$b_n := \frac{2\sqrt{3n}}{3}\sqrt{1-H_n}, \tag{1.6}$$

*where*

$$H_n := 4\alpha_n(1-\alpha_n). \tag{1.7}$$

*Then*

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(a_n + b_n m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}H_n^m. \tag{1.8}$$

**Remark.** We state this theorem with a different definition of $\epsilon(n)$ than that given in [**3**]. We have avoided using elliptic integrals of the second kind and Legendre's relation.

The Borweins' theorem indicates that for each positive rational number $n$, we can easily derive a series for $1/\pi$ belonging to the "theory of $q_2$" if the values of $\alpha_n$ and $\epsilon(n)$ (the rest of the constants can be computed from these) are known. The computation of these constants for any given $n$, however, is far from trivial.

The Borweins' method of evaluating $\alpha_n$ involves solving a quartic equation. More precisely, they show that when $n$ is an odd positive integer, $\alpha_n$ is the smaller of the two real solutions of the equation

$$\frac{(9-8\alpha_n)^3}{64\alpha_n^3(1-\alpha_n)} = \frac{(4G_{3n}^{24}-1)^3}{27G_{3n}^{24}}, \tag{1.9}$$

where $G_n$ is the classical Ramanujan-Weber class invariant defined by

$$G_n := 2^{-1/4}e^{\pi\sqrt{n}/24}\prod_{m=1}^{\infty}(1+e^{-\pi\sqrt{n}(2m-1)}).$$

Using known values for $G_{3n}$, they derive $\alpha_n$ for $n=3$ and 5 from (1.9). For example, from (see [**1**, p. 190])

$$G_{15}^{12} = 8\left(\frac{\sqrt{5}+1}{2}\right)^4,$$

they deduce that

$$\alpha_5 = \frac{1}{2} - \frac{11\sqrt{5}}{50}.$$

When $n$ is an even positive integer, the corresponding formula between $\alpha_n$ and $g_{3n}$ is

$$\frac{(9-8\alpha_n)^3}{64\alpha_n^3(1-\alpha_n)} = \frac{(4g_{3n}^{24}+1)^3}{27g_{3n}^{24}}, \tag{1.10}$$

where $g_n$ is the other Ramanujan-Weber class invariant defined by

$$g_n := 2^{-1/4}e^{\pi\sqrt{n}/24}\prod_{m=1}^{\infty}(1-e^{-\pi\sqrt{n}(2m-1)}).$$

Using (1.10) and known values of $g_{3n}$, they compute $\alpha_n$ for $n=2$, 4, and 6. Together with the values of $\epsilon(n)$ for $n=2$, 3, 4, 5, and 6 [**3**, p. 190, Problem 20], they obtained five series for $1/\pi$. Ramanujan's series (1.1) and (1.2) then correspond to $n=4$ and 5, respectively. At the end of [**3**, Chapter 5, Section 5], the Borweins remark that their explanation of Ramanujan's series (1.1) and (1.2) is "a bit disappointing" as they only have "well-concealed analogues of the original theory for $K$."

In a recent paper, B. C. Berndt, S. Bhargava, and F. G. Garvan [**2**] succeeded in developing Ramanujan's "corresponding theories" mentioned in [**10**]. One of these theories is Ramanujan's "theory of $q_2$" and its discovery has motivated us to revisit Ramanujan's series (1.1) and (1.2). This theory is now known as "*Ramanujan's theory of elliptic functions to alternative base* 3" or "*Ramanujan's elliptic functions in the theory of signature* 3."

In this article, we derive some new formulas from the "theory of $q_2$" which will facilitate the computations of $\alpha_n$ and $\epsilon(n)$. With the aid of cubic

Russell-type modular equations (see [6]) and Kronecker's Limit Formula, we discover new Ramanujan-type series for $1/\pi$ belonging to the "theory of $q_2$." An example of these series, which corresponds to $n = 59$, is

(1.11)
$$\frac{2153559\sqrt{3}}{\pi} = \sum_{m=0}^{\infty}(a+bm)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{73-40\sqrt{3}}{2^{1/3}\cdot 23^2(4+5\sqrt{3})}\right)^{3m},$$

where

$$a := 1028358\sqrt{3} - 593849 \quad \text{and} \quad b := 19101285\sqrt{3} - 795.$$

Each term in this series gives approximately 10 decimal places of $\pi$.

In Section 2, we recall some important results proved in [2] and establish new formulas satisfied by $\epsilon(n)$ which lead to a new formula for $a_n$. In Section 3, we describe our strategy for computing $a_n$. In Section 4, we indicate that if $3n$ is an *Euler convenient number*, then $\alpha_n$, as well as other related cubic singular moduli, can be computed explicitly via Kronecker's Limit Formula. These values are used to derive the constants $a_n$, $b_n$, and $H_n$ listed in our final section.

## 2. Ramanujan's elliptic functions in the theory of signature 3 (Ramanujan's "theory of $\mathbf{q_2}$").

Define

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

**Theorem 2.1.** *If*

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{K(1-\alpha)}{K(\alpha)}\right), \tag{2.1}$$

*then*

$$\alpha = \frac{c^3(q)}{a^3(q)}. \tag{2.2}$$

**Theorem 2.2** (Borweins' Inversion Formula)**.** *We have*

$$a(q) = K\left(\frac{c^3(q)}{a^3(q)}\right) = K(\alpha), \tag{2.3}$$

*where $K(\cdot)$ is defined in Theorem 1.1.*

Theorem 2.1 and Theorem 2.2 are important results in Ramanujan's theory of elliptic functions in the signature 3 which can be found in [**2**] as Lemma 2.9 and Lemma 2.6, respectively.

Let $\alpha$ be given as in (2.2). Then it is known that (see [**2**, (4.4)] and [**5**, (4.7)])

$$q\frac{d\alpha}{dq} = K^2(\alpha)\alpha(1-\alpha). \tag{2.4}$$

The *modulus* $\beta$ is said to have *degree* $n$ over the *modulus* $\alpha$ when there is a relation

$$\frac{K(1-\beta)}{K(\beta)} = n\frac{K(1-\alpha)}{K(\alpha)}. \tag{2.5}$$

Hence, when $q$ satisfies (2.1),

$$q^n = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{K(1-\beta)}{K(\beta)}\right),$$

and applying (2.4) with $q$ and $\alpha$ replaced by $q^n$ and $\beta$, respectively, we deduce that

$$q\frac{d\beta}{dq} = nK^2(\beta)\beta(1-\beta). \tag{2.6}$$

Combining (2.6) and (2.4), we arrive at:

**Theorem 2.3.** *If $\beta$ has degree $n$ over $\alpha$, then*

$$m^2(\alpha,\beta)\frac{d\beta}{d\alpha} = n\frac{\beta(1-\beta)}{\alpha(1-\alpha)}, \tag{2.7}$$

*where*

$$m(\alpha,\beta) := \frac{K(\alpha)}{K(\beta)}. \tag{2.8}$$

We call the quantity $m(\alpha,\beta)$ the *multiplier of degree $n$ in the theory of signature* 3. We are now ready to derive new formulas satisfied by $\epsilon(n)$.

**Theorem 2.4.** *Let $\epsilon(r)$ be defined as in (1.4). Then*

$$\epsilon\left(\frac{1}{r}\right) = \frac{\sqrt{r}-\epsilon(r)}{r}.$$

*Proof.* Set

$$\tau = \frac{K(1-\alpha)}{K(\alpha)}.$$

Then

$$\frac{d\tau}{d\alpha}K(\alpha) + \dot{K}(\alpha)\tau = -\dot{K}(1-\alpha).$$

From (2.1) and (2.4), we deduce that

$$\frac{d\alpha}{d\tau} = \frac{-2\pi}{\sqrt{3}} K^2(\alpha)\alpha(1-\alpha).$$

Hence,

$$\dot{K}(1-\alpha) = \frac{\sqrt{3}}{2\pi} \frac{1}{K(\alpha)\alpha(1-\alpha)} - \dot{K}(\alpha)\tau. \tag{2.9}$$

Next, note that from (1.3)

$$\alpha_{1/r} = 1-\alpha_r. \tag{2.10}$$

Therefore, by (1.4) and (2.9) with $\tau = \sqrt{r}$,

$$\begin{aligned}\epsilon\left(\frac{1}{r}\right) &= \frac{3\sqrt{3}}{8\pi} K^{-2}(1-\alpha_r) - \sqrt{\frac{1}{r}} \left(\frac{3\alpha_r(1-\alpha_r)\dot{K}(1-\alpha_r)}{2K(1-\alpha_r)} - (1-\alpha_r)\right) \\ &= \frac{3\sqrt{3}}{8\pi}\frac{1}{K^2(1-\alpha_r)} - \frac{3\sqrt{3}}{4\pi r K^2(\alpha_r)} + \frac{3\alpha_r(1-\alpha_r)\dot{K}(\alpha_r)}{2\sqrt{r}K(\alpha_r)} + \frac{1}{\sqrt{r}} - \frac{\alpha_r}{\sqrt{r}} \\ &= \frac{\sqrt{r}-\epsilon(r)}{r}.\end{aligned}$$

□

**Theorem 2.5.** *Let*

$$m^* := m(\alpha_r, \alpha_{n^2r}) \quad \textit{and} \quad \dot{m}^* := \frac{dm}{d\alpha}(\alpha_r, \alpha_{n^2r}).$$

*Then*

$$\epsilon(n^2 r) = m^{*\,2}\left(\epsilon(r) - \sqrt{r}\left(\alpha_r - \frac{3}{2}m^{*\,-1}\alpha_r(1-\alpha_r)\dot{m}^* - \frac{n\alpha_{n^2r}}{m^{*\,2}}\right)\right). \tag{2.11}$$

*Proof.* Suppose $\beta$ has degree $n$ over $\alpha$. Then from (2.8), we deduce that

$$m\frac{dK(\beta)}{d\alpha} + K(\beta)\frac{dm}{d\alpha} = \frac{dK(\alpha)}{d\alpha}. \tag{2.12}$$

Using (2.7), we may rewrite (2.12) as

$$\frac{n\beta(1-\beta)}{K(\beta)}\frac{dK(\beta)}{d\beta} = \frac{m^2\alpha(1-\alpha)}{K(\alpha)}\frac{dK(\alpha)}{d\alpha} - m\alpha(1-\alpha)\frac{dm}{d\alpha}. \tag{2.13}$$

Next, suppose $\alpha = \alpha_r$. Then $\beta = \alpha_{n^2r}$, and by (1.4), (2.8), and (2.13),

$$\begin{aligned}\epsilon(n^2r) &= \frac{3\sqrt{3}}{8\pi K^2(\alpha_{n^2r})} - n\sqrt{r}\left(\frac{3\alpha_{n^2r}(1-\alpha_{n^2r})}{2K(\alpha_{n^2r})}\dot{K}(\alpha_{n^2r}) - \alpha_{n^2r}\right) \\ &= \frac{3\sqrt{3}m^{*\,2}}{8\pi K^2(\alpha_r)} \\ &\quad - \sqrt{r}\left(\frac{3m^{*\,2}\alpha_r(1-\alpha_r)}{2K(\alpha_r)}\dot{K}(\alpha_r) - \frac{3}{2}m^*\alpha_r(1-\alpha_r)\dot{m}^* - n\alpha_{n^2r}\right)\end{aligned}$$

$$= {m^*}^2 \left(\epsilon(r) - \sqrt{r}\left(\alpha_r - \frac{3}{2}{m^*}^{-1}\alpha_r(1-\alpha_r)\dot{m}^* - \frac{n\alpha_{n^2 r}}{{m^*}^2}\right)\right).$$

□

If we set $r = 1/n$ in (2.11) and use (2.10), we find that

$$\epsilon(n) = n\left(\epsilon\left(\frac{1}{n}\right) - \sqrt{\frac{1}{n}}\left(1-\alpha_n - \frac{3\alpha_n(1-\alpha_n)}{2\sqrt{n}}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n) - \alpha_n\right)\right)$$

$$= -\epsilon(n) + 2\alpha_n\sqrt{n} + \frac{3\alpha_n(1-\alpha_n)}{2}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n).$$

Hence, we have:

**Theorem 2.6.**

$$\epsilon(n) = \sqrt{n}\alpha_n + \frac{3\alpha_n(1-\alpha_n)}{4}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n).$$

**Corollary 2.7.** *With $a_n$ and $H_n$ defined in Theorem 1.1, we have*

$$a_n = \frac{H_n}{2\sqrt{3}}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n).$$

Theorems 2.4, 2.5, and 2.6 are the respective cubic analogues of [**3**, (5.1.5), Theorem 5.2, and (5.2.5)].

## 3. Computations of $\mathbf{a_n}$.

It is clear from Corollary 2.7 that in order to compute $a_n$ it suffices to compute $\alpha_n$ and $dm/d\alpha$, where $m$ is the multiplier of degree $n$. We will discuss the computation of the latter in this section. Suppose there is a relation between $\alpha$ and $\beta$, where $\beta$ has degree $n$ over $\alpha$. Then we can determine $d\beta/d\alpha$ by implicitly differentiating the relation with respect to $\alpha$. Substituting $d\beta/d\alpha$ into (2.7), we conclude that $m$ can be expressed in terms of $\alpha$ and $\beta$. This implies that $dm/d\alpha$ is a function of $\alpha$ and $\beta$.

A relation between $\alpha$ and $\beta$ induced by (2.5) (i.e., when $\beta$ has degree $n$ over $\alpha$) is known as a *modular equation of degree $n$ in the theory of signature* 3. (We sometimes call these *cubic modular equations*.) Our discussion in the previous paragraph indicates that our computations of $dm/d\alpha$ depend on the existence of such modular equations.

The first few modular equations in the theory of signature 3 are given by Ramanujan in his notebooks. One of these is the following modular equation of degree 2:

(3.1) $$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} = 1.$$

Proofs of Ramanujan's modular equations in the theory of signature 3 are now available in [**2**] and [**6**].

Recently, we showed that [**6**] if $p$ is a prime, then there is a relation between $x := (\alpha\beta)^{s/6}$ and $y := \{(1-\alpha)(1-\beta)\}^{s/6}$, when $(p+1)/3 = N/s$ and $\gcd(N,s) = 1$. Moreover, we proved that the degree of the polynomial satisfied by $x$ and $y$ is $N$. This proves the existence of modular equations of prime degrees and we conclude that when $m$ is a multiplier of degree $p$,

$$\frac{dm}{d\alpha} = F_p(\alpha,\beta), \tag{3.2}$$

where $F_p$ is a certain function in $\alpha$ and $\beta$. If we know the value of $\alpha_p$, then the value of $a_p$ follows by substituting $\alpha = 1-\alpha_p$ and $\beta = \alpha_p$ into (3.2) and simplifying. Our simplification is done with the help of MAPLE V.

When $n$ is not a prime, except for modular equations of degrees 4 and 9, it is difficult to derive a modular equation of degree $n$. However, deriving such a modular equation is unnecessary. We illustrate our point with $n = pq$. If $\beta$ has degree $q$ over $\alpha$ then from a cubic modular equation of degree $q$ and (2.7), we can write

$$m_q = \frac{K(\alpha)}{K(\beta)} = G_q(\alpha,\beta), \tag{3.3}$$

where $m_q$ is the multiplier of degree $q$ and $G_q$ is a certain function of $\alpha$ and $\beta$. Similarly, we can deduce that if $\gamma$ has degree $p$ over $\beta$, then from a cubic modular equation of degree $p$, we may write

$$m_p = \frac{K(\beta)}{K(\gamma)} = G_p(\beta,\gamma), \tag{3.4}$$

where $m_p$ is the multiplier of degree $p$ and $G_p$ is a certain function of $\beta$ and $\gamma$. It follows that $\gamma$ has degree $pq$ over $\alpha$ and

$$m_{pq}(\alpha,\gamma) = \frac{K(\alpha)}{K(\gamma)} = \frac{K(\alpha)}{K(\beta)}\cdot\frac{K(\beta)}{K(\gamma)} = m_q(\alpha,\beta)\cdot m_p(\beta,\gamma).$$

Hence, differentiating with respect to $\alpha$ and substituting $\alpha = \alpha_{1/(pq)}$, we have

$$\begin{aligned}\frac{dm_{pq}}{d\alpha}(1-\alpha_{pq},\alpha_{pq}) &= m_p(\alpha_{q/p},\alpha_{pq})\frac{dm_q}{d\alpha}(1-\alpha_{pq},\alpha_{q/p})\\ &\qquad + m_q(1-\alpha_{pq},\alpha_{q/p})\frac{d\beta}{d\alpha}\cdot\frac{dm_p}{d\beta}(\alpha_{q/p},\alpha_{pq}).\end{aligned}$$

This allows us to compute $a_{pq}$ provided we know modular equations of degrees $p$ and $q$ and the singular moduli $\alpha_{pq}$ and $\alpha_{q/p}$.

When $n$ is a squarefree product of more than 2 primes, say $n = p_1p_2\cdots p_l$, then the above idea can be extended with the computation of $a_n$ reduced to that of finding modular equations of degrees $p_1$, $p_2$, ... , $p_{l-1}$, and $p_l$, and constants $\alpha_{n/(p_{i_1}^2\cdots p_{i_s}^2)}$, where $1 \le s \le l-1$ and $1 \le i_j \le l$.

## 4. Euler's convenient numbers, Kronecker's Limit Formula, and cubic singular moduli.

An *Euler convenient number* is a number $c$ satisfying the following criterion:

> *Let $l > 1$ be an odd number relatively prime to $c$ which is properly represented by $x^2 + cy^2$. If the equation $l = x^2 + cy^2$ has only one solution with $x, y \geq 0$, then $l$ is a prime number.*

Euler was interested in these numbers because they helped him to generate large primes. The above criterion, however, is not very useful for finding these numbers.

Let $d$ be squarefree, $K = \mathbb{Q}(\sqrt{-d})$, $C_K$ denote the class group of $K$ and $C_K^2$ be the subgroup of squares in $C_K$. A genus group $G_K$ is defined as the quotient group $C_K/C_K^2$. Gauss observed that $G_K \simeq C_K$ if and only if $d$ is a convenient number. (Some convenient numbers are not squarefree but Gauss' criterion is also true for class groups of orders in $K$.) Using this new criterion, Gauss determined 65 Euler convenient numbers [8], [7, p. 60]. We reproduce here those $c$'s ($\neq 3$) which are squarefree and divisible by 3.

| $h(-4c) := \left\lvert C_{\mathbb{Q}(\sqrt{-4c})} \right\rvert$ | Euler's convenient number $c$ |
|---|---|
| 2 | 6, 15 |
| 4 | 21, 30, 33, 42, 57, 78, 93, 102, 177 |
| 8 | 105, 165, 210, 273, 330, 345, 357, 462 |
| 16 | 1365 |

**Table 1.** Convenient numbers in Gauss' table which are squarefree and divisible by 3 (except 3).

For each $c$ in Table 1, we will deduce the corresponding values $a_{c/3}$, $b_{c/3}$, and $H_{c/3}$, which in turn yield new series for $1/\pi$.

A group homomorphism $\chi : G_K \longrightarrow \{\pm 1\}$ is known as a genus character. One can show that a genus character arises from a certain decomposition of $D_K$, where $D_K$ is the discriminant of $K$. More precisely, if $\chi$ is a genus character, then there exist $d_1$ and $d_2$ satisfying $D_K = d_1 d_2$, $d_1 > 0$, and

$d_i \equiv 0$ or $1 \pmod 4$, such that for any prime ideal $\mathfrak{p}$ in $K$,

$$\chi([\mathfrak{p}]) = \begin{cases} \left(\frac{d_1}{N(\mathfrak{p})}\right), & \text{if} \quad N(\mathfrak{p}) \nmid d_1, \\ \\ \left(\frac{d_2}{N(\mathfrak{p})}\right), & \text{if} \quad N(\mathfrak{p}) \mid d_1, \end{cases} \tag{4.1}$$

where $N(\mathfrak{p})$ is the norm of the ideal $\mathfrak{p}$ and $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. If $[\mathfrak{a}]$ is an ideal class in $C_K$ and $\mathfrak{a} = \prod \mathfrak{p}^{\alpha_\mathfrak{p}}$, then we define

$$\chi([\mathfrak{a}]) = \prod \chi([\mathfrak{p}])^{\alpha_\mathfrak{p}}.$$

**Theorem 4.1.** *Let $\chi$ be a genus character arising from the decomposition $D_K = d_{1,\chi}d_{2,\chi}$. Let $h_{i,\chi}$ be the class number of the field $\mathbb{Q}(\sqrt{d_{i,\chi}})$, $w_{2,\chi}$ be the number of roots of unity in $\mathbb{Q}(\sqrt{d_{2,\chi}})$, and $\epsilon_\chi$ be the fundamental unit of $\mathbb{Q}(\sqrt{d_{1,\chi}})$. Let*

$$F([\mathfrak{a}]) = \sqrt{N([1,\tau])}|\eta(\tau)|^2,$$

*where $N(\cdot)$ denotes the norm of a fractional ideal, $\eta(z)$ denotes the Dedekind eta-function defined by*

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \qquad \operatorname{Im} z > 0,$$

*and*

$$\tau = \frac{\tau_2}{\tau_1}, \quad \operatorname{Im} \tau > 0, \quad \textit{where} \quad \mathfrak{a} = [\tau_1, \tau_2].$$

*Then*

$$\epsilon_\chi^{2h_{1,\chi}h_{2,\chi}/w_{2,\chi}} = \prod_{[\mathfrak{a}] \in C_K} F([\mathfrak{a}])^{-\chi([\mathfrak{a}])}. \tag{4.2}$$

Theorem 4.1 follows from Kronecker's First Limit Formula [**11**, p. 72, Theorem 6]. In [**9**], K.G. Ramanathan applied Theorem 4.1 to compute products of the form

$$t_n = \frac{1}{5\sqrt{5}} \left( \frac{\eta(\frac{1+\sqrt{-n/5}}{2})}{\eta(\frac{1+\sqrt{-5n}}{2})} \right)^6$$

when $5n$ is a convenient number. These products are then used to deduce special values of the Rogers-Ramanujan continued fraction. In the same article, he defined [**9**, Eq. (51)]

$$\mu_n = \frac{1}{3\sqrt{3}} \left( \frac{\eta(\sqrt{-n/3})}{\eta(\sqrt{-3n})} \right)^6, \tag{4.3}$$

and remarked that $\mu_n$ can be evaluated when $3n$ is one of the convenient numbers listed in Table 1 (15 is missing from his list). Ramanathan's result can be stated as follows:

**Theorem 4.2** ([**9**, Theorem 4]). *Let $c$ be a convenient number listed in Table 1 and let $K = \mathbb{Q}(\sqrt{-c})$. Let $[\mathfrak{t}]$ be the ideal class containing $\mathfrak{t}$ such that $\mathfrak{t}^2 = (3)$. Then with the same notation as in Theorem 4.1*

$$\mu_{c/3} = \prod_{\chi([\mathfrak{t}])=-1} \epsilon_\chi^{3e_\chi},$$

*where the exponents are given by*

$$e_\chi = \frac{2wh_{1,\chi}h_{2,\chi}}{w_{2,\chi}h},$$

*with $h$ being the class number of $K$ and $w$ the number of roots of unity in $K$.*

It turns out that Ramanathan's $\mu_n$ is related to $\alpha_n$, namely [**5**, (2.7)],

$$\frac{1}{\alpha_n} = \mu_n^2 + 1. \tag{4.4}$$

Hence, from Theorem 4.2, (4.1), and (4.4), we can determine $\alpha_n$ explicitly. Using the same technique as given in the proof of Theorem 4.2, one can compute $\alpha_{n/(p_{i_1}^2 \cdots p_{i_s}^2)}$, $1 \le s \le l-1$ and $1 \le i_j \le l$, which will be needed in the evaluations of $a_n$.

We conclude this section with a list of singular moduli which will be needed in the evaluations of $a_n$, $b_n$, and $H_n$ with $n = c/3$.

| $n$ | Cubic singular moduli |
|---|---|
| 2 | $\alpha_2 = \frac{1}{2} - \frac{\sqrt{2}}{4}$ |
| 5 | $\alpha_5 = \frac{1}{2} - \frac{11\sqrt{5}}{50}$ |

**Table 2.** Cubic singular moduli for $h(-12n) = 2$.

| $n$ | Cubic singular moduli |
|---|---|
| 7 | $\alpha_7 = \frac{1}{2} - \frac{13-\sqrt{7}}{12\sqrt{3}}$ |
| 10 | $\alpha_{10} = \frac{1}{2} - \frac{35\sqrt{2}+2\sqrt{5}}{108}, \quad \alpha_{5/2} = \frac{1}{2} - \frac{35\sqrt{2}-2\sqrt{5}}{108}$ |
| 11 | $\alpha_{11} = \frac{1}{2} - \frac{45\sqrt{3}-5}{44\sqrt{11}}$ |
| 14 | $\alpha_{14} = \frac{1}{2} - \frac{99\sqrt{6}+2\sqrt{14}}{500}, \quad \alpha_{7/2} = \frac{1}{2} - \frac{99\sqrt{6}-2\sqrt{14}}{500}$ |
| 19 | $\alpha_{19} = \frac{1}{2} - \frac{301\sqrt{51}-13\sqrt{3}}{4500}$ |
| 26 | $\alpha_{26} = \frac{1}{2} - \frac{6930\sqrt{2}+5\sqrt{26}}{19652}, \quad \alpha_{13/2} = \frac{1}{2} - \frac{6930\sqrt{2}-5\sqrt{26}}{19652}$ |
| 31 | $\alpha_{31} = \frac{1}{2} - \frac{35113\sqrt{3}-7\sqrt{93}}{121500}$ |
| 34 | $\alpha_{34} = \frac{1}{2} - \frac{17420\sqrt{17}+35\sqrt{2}}{143748}, \quad \alpha_{17/2} = \frac{1}{2} - \frac{17420\sqrt{17}-35\sqrt{2}}{143748}$ |
| 59 | $\alpha_{59} = \frac{1}{2} - \frac{6367095\sqrt{177}-265\sqrt{59}}{169413308}$ |

**Table 3.** Cubic singular moduli for $h(-12n) = 4$.

| $n$ | Cubic singular moduli |
|---|---|
| 35 | $\alpha_{35} = \frac{1}{2} + \frac{549}{50}\sqrt{5} + \frac{63}{10}\sqrt{15} - \frac{83}{20}\sqrt{35} - \frac{243}{100}\sqrt{105}$ <br> $\alpha_{7/5} = \frac{1}{2} + \frac{549}{50}\sqrt{5} - \frac{63}{10}\sqrt{15} + \frac{83}{20}\sqrt{35} - \frac{243}{100}\sqrt{105}$ |
| 55 | $\alpha_{55} = \frac{1}{2} + \frac{92995}{88434}\sqrt{3} - \frac{8259}{9826}\sqrt{5} - \frac{13755}{19652}\sqrt{11} + \frac{25879}{176868}\sqrt{165}$ <br> $\alpha_{11/5} = \frac{1}{2} - \frac{92995}{88434}\sqrt{3} + \frac{8259}{9826}\sqrt{5} - \frac{13755}{19652}\sqrt{11} + \frac{25879}{176868}\sqrt{165}$ |
| 70 | $\alpha_{70} = \frac{1}{2} + \frac{43971}{608350}\sqrt{5} - \frac{10023}{121670}\sqrt{14} - \frac{38621}{1095030}\sqrt{21} - \frac{383383}{10950300}\sqrt{30}$ <br> $\alpha_{35/2} = \frac{1}{2} - \frac{43971}{608350}\sqrt{5} - \frac{10023}{121670}\sqrt{14} + \frac{38621}{1095030}\sqrt{21} - \frac{383383}{10950300}\sqrt{30}$ <br> $\alpha_{7/10} = \frac{1}{2} + \frac{43971}{608350}\sqrt{5} - \frac{10023}{121670}\sqrt{14} + \frac{38621}{1095030}\sqrt{21} + \frac{383383}{10950300}\sqrt{30}$ |
| 91 | $\alpha_{91} = \frac{1}{2} - \frac{53880905}{23706108}\sqrt{3} + \frac{10182535}{11853054}\sqrt{21} - \frac{8256325}{11853054}\sqrt{39} + \frac{5523815}{23706108}\sqrt{273}$ <br> $\alpha_{13/7} = \frac{1}{2} + \frac{53880905}{23706108}\sqrt{3} + \frac{10182535}{11853054}\sqrt{21} - \frac{8256325}{11853054}\sqrt{39} - \frac{5523815}{23706108}\sqrt{273}$ |
| 110 | $\alpha_{110} = \frac{1}{2} - \frac{1204702947}{25128011089}\sqrt{5} - \frac{7922677455}{50256022178}\sqrt{6}$ <br> $- \frac{995983605}{50256022178}\sqrt{33} + \frac{1027373417}{100512044356}\sqrt{110}$ <br> $\alpha_{55/2} = \frac{1}{2} + \frac{1204702947}{25128011089}\sqrt{5} - \frac{7922677455}{50256022178}\sqrt{6}$ <br> $- \frac{995983605}{50256022178}\sqrt{33} - \frac{1027373417}{100512044356}\sqrt{110}$ <br> $\alpha_{11/10} = \frac{1}{2} + \frac{1204702947}{25128011089}\sqrt{5} - \frac{7922677455}{50256022178}\sqrt{6}$ <br> $+ \frac{995983605}{50256022178}\sqrt{33} + \frac{1027373417}{100512044356}\sqrt{110}$ |
| 115 | $\alpha_{115} = \frac{1}{2} - \frac{132769793}{1177056540}\sqrt{3} + \frac{14283759}{163480075}\sqrt{5} + \frac{1293747}{32696015}\sqrt{23} - \frac{218554427}{5885282700}\sqrt{345}$ <br> $\alpha_{23/5} = \frac{1}{2} + \frac{132769793}{1177056540}\sqrt{3} + \frac{14283759}{163480075}\sqrt{5} - \frac{1293747}{32696015}\sqrt{23} - \frac{218554427}{5885282700}\sqrt{345}$ |
| 119 | $\alpha_{119} = \frac{1}{2} + \frac{50276655}{165246443}\sqrt{17} - \frac{162387225}{660985772}\sqrt{51} + \frac{19683565}{165246443}\sqrt{119} - \frac{45457245}{660985772}\sqrt{357}$ <br> $\alpha_{17/7} = \frac{1}{2} - \frac{50276655}{165246443}\sqrt{17} - \frac{162387225}{660985772}\sqrt{51} + \frac{19683565}{165246443}\sqrt{119} + \frac{45457245}{660985772}\sqrt{357}$ |
| 154 | $\alpha_{154} = \frac{1}{2} - \frac{1255233}{5404300}\sqrt{2} + \frac{3988079}{121596750}\sqrt{21} - \frac{14120327}{668782125}\sqrt{66} - \frac{254553}{14861825}\sqrt{77}$ <br> $\alpha_{77/2} = \frac{1}{2} - \frac{1255233}{5404300}\sqrt{2} - \frac{3988079}{121596750}\sqrt{21} - \frac{14120327}{668782125}\sqrt{66} + \frac{254553}{14861825}\sqrt{77}$ <br> $\alpha_{11/14} = \frac{1}{2} - \frac{1255233}{5404300}\sqrt{2} + \frac{3988079}{121596750}\sqrt{21} + \frac{14120327}{668782125}\sqrt{66} + \frac{254553}{14861825}\sqrt{77}$ |

**Table 4.** Cubic singular moduli for $h(-12n) = 8$.

| $n$ | Cubic singular moduli |
|---|---|
| 455 | $\alpha_{455} = \frac{1}{2} - \frac{52602592750172050462677}{248717948742554175611950}\sqrt{5} - \frac{56682141893433349857381}{49743589748510835122390}\sqrt{15}$ |
| | $- \frac{5696597990275946071461}{49743589748510835122390}\sqrt{35} + \frac{538462633924678371678}{24871794874255417561195}\sqrt{65}$ |
| | $- \frac{682503637304416627557}{9948717949702167024478}\sqrt{105} + \frac{25146509927196138320763}{497435897485108351223900}\sqrt{195}$ |
| | $+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} + \frac{1196360473602901817979}{99487179497021670244780}\sqrt{1365}$ |
| | $\alpha_{91/5} = \frac{1}{2} - \frac{52602592750172050462677}{248717948742554175611950}\sqrt{5} + \frac{56682141893433349857381}{49743589748510835122390}\sqrt{15}$ |
| | $+ \frac{5696597990275946071461}{49743589748510835122390}\sqrt{35} - \frac{538462633924678371678}{24871794874255417561195}\sqrt{65}$ |
| | $- \frac{682503637304416627557}{9948717949702167024478}\sqrt{105} + \frac{25146509927196138320763}{497435897485108351223900}\sqrt{195}$ |
| | $+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} - \frac{1196360473602901817979}{99487179497021670244780}\sqrt{1365}$ |
| | $\alpha_{65/7} = \frac{1}{2} + \frac{52602592750172050462677}{248717948742554175611950}\sqrt{5} - \frac{56682141893433349857381}{49743589748510835122390}\sqrt{15}$ |
| | $+ \frac{5696597990275946071461}{49743589748510835122390}\sqrt{35} + \frac{538462633924678371678}{24871794874255417561195}\sqrt{65}$ |
| | $- \frac{682503637304416627557}{9948717949702167024478}\sqrt{105} - \frac{25146509927196138320763}{497435897485108351223900}\sqrt{195}$ |
| | $+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} - \frac{1196360473602901817979}{99487179497021670244780}\sqrt{1365}$ |
| | $\alpha_{35/13} = \frac{1}{2} + \frac{52602592750172050462677}{248717948742554175611950}\sqrt{5} + \frac{56682141893433349857381}{49743589748510835122390}\sqrt{15}$ |
| | $- \frac{5696597990275946071461}{49743589748510835122390}\sqrt{35} - \frac{538462633924678371678}{24871794874255417561195}\sqrt{65}$ |
| | $- \frac{682503637304416627557}{9948717949702167024478}\sqrt{105} - \frac{25146509927196138320763}{497435897485108351223900}\sqrt{195}$ |
| | $+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} + \frac{1196360473602901817979}{99487179497021670244780}\sqrt{1365}$ |

**Table 5.** Cubic singular moduli for $h(-12n) = 16$.

## 5. Values of $\mathbf{a_n}$, $\mathbf{b_n}$, and $\mathbf{H_n}$.

The values of $H_n$ follow immediately from the values of $\alpha_n$ by (1.7). From (1.6), it appears that we need to denest the expression $\sqrt{1-H_n}$ in order to determine $b_n$. The next simple lemma shows that this is not necessary.

**Lemma 5.1.** *Let $\mu_n$ be defined as in (4.3). Then*

$$b_n = \frac{2\sqrt{3n}}{3}\frac{\mu_n^2-1}{\mu_n^2+1}.$$

*Proof.* From (4.4), we deduce that

$$\frac{1}{1-\alpha_n} = \frac{1}{\mu_n^2}+1. \tag{5.1}$$

Hence, by (1.7), (4.4), and (5.1), we conclude that

$$\frac{4}{H_n} = \mu_n^2 + \frac{1}{\mu_n^2} + 2.$$

Hence,

$$\begin{aligned}\sqrt{1-H_n} &= \sqrt{1-\frac{4}{\mu_n^2+\mu_n^{-2}+2}} \\ &= \frac{\mu_n-\mu_n^{-1}}{\mu_n+\mu_n^{-1}}.\end{aligned} \tag{5.2}$$

Substituting (5.2) into (1.6) completes our proof of the lemma. □

Finally, to compute $a_n$, we use the method outlined in Section 3, together with the singular moduli given in Section 4. Our final results are shown in the following tables, grouped once again according to class numbers.

| $n$ | $a_n$ | $b_n$ | $H_n$ |
|---|---|---|---|
| 2 | $\frac{1}{3\sqrt{3}}$ | $\frac{2}{\sqrt{3}}$ | $\frac{1}{2}$ |
| 5 | $\frac{8}{15\sqrt{3}}$ | $\frac{22}{5\sqrt{3}}$ | $\frac{4}{125}$ |

**Table 6.** $a_n$, $b_n$, and $H_n$ for $h(-12n)=2$.

| $n$ | $a_n$ | $b_n$ | $H_n$ |
|---|---|---|---|
| 7 | $-\frac{10}{27}+\frac{7}{27}\sqrt{7}$ | $\frac{13}{9}\sqrt{7}-\frac{7}{9}$ | $-\frac{17}{27}+\frac{13}{54}\sqrt{7}$ |
| 10 | $\frac{25}{243}\sqrt{15}-\frac{8}{243}\sqrt{6}$ | $\frac{70}{81}\sqrt{15}+\frac{10}{81}\sqrt{6}$ | $\frac{223}{1458}-\frac{35}{729}\sqrt{10}$ |
| 11 | $\frac{6}{11}-\frac{13}{99}\sqrt{3}$ | $\frac{45}{11}-\frac{5}{33}\sqrt{3}$ | $-\frac{194}{1331}+\frac{225}{2662}\sqrt{3}$ |
| 14 | $\frac{21}{125}\sqrt{7}-\frac{82}{1125}\sqrt{3}$ | $\frac{198}{125}\sqrt{7}+\frac{28}{375}\sqrt{3}$ | $\frac{1819}{31250}-\frac{198}{15625}\sqrt{21}$ |
| 19 | $\frac{1654}{3375}-\frac{133}{3375}\sqrt{19}$ | $\frac{5719}{1125}-\frac{13}{1125}\sqrt{19}$ | $-\frac{8522}{421875}+\frac{3913}{843750}\sqrt{19}$ |
| 26 | $\frac{1118}{14739}\sqrt{39}-\frac{3967}{44217}\sqrt{3}$ | $\frac{4620}{4913}\sqrt{39}+\frac{130}{14739}\sqrt{3}$ | $\frac{249913}{48275138}-\frac{34650}{24137569}\sqrt{13}$ |
| 31 | $-\frac{14662}{91125}+\frac{7843}{91125}\sqrt{31}$ | $-\frac{217}{30375}+\frac{35113}{30375}\sqrt{31}$ | $-\frac{684197}{307546875}+\frac{245791}{615093750}\sqrt{31}$ |
| 34 | $-\frac{7157}{323433}\sqrt{51}+\frac{62896}{323433}\sqrt{6}$ | $\frac{70}{107811}\sqrt{51}+\frac{296140}{107811}\sqrt{6}$ | $\frac{3555313}{2582935938}-\frac{304850}{1291467969}\sqrt{34}$ |
| 59 | $\frac{342786}{717853}-\frac{593849}{6460677}\sqrt{3}$ | $\frac{6367095}{717853}-\frac{265}{2153559}\sqrt{3}$ | $-\frac{1461224894}{30403462846931}+\frac{1687280175}{60806925693862}\sqrt{3}$ |

**Table 7.** $a_n$, $b_n$, and $H_n$ for $h(-12n)=4$.

| $n$ | $a_n$, $b_n$, and $H_n$ |
|---|---|
| 35 | $a_{35} = \frac{558}{5} - \frac{364}{15}\sqrt{21} - 42\sqrt{7} + \frac{577}{9}\sqrt{3}$ <br> $b_{35} = \frac{1701}{5} + \frac{581}{3}\sqrt{3} - 126\sqrt{7} - \frac{366}{5}\sqrt{21}$ <br> $H_{35} = -\frac{1210352}{125} - \frac{279531}{50}\sqrt{3} + \frac{91494}{25}\sqrt{7} + \frac{264132}{125}\sqrt{21}$ |
| 55 | $a_{55} = -\frac{1411054}{132651} + \frac{14375}{4913}\sqrt{15} + \frac{26884}{14739}\sqrt{33} - \frac{194150}{132651}\sqrt{55}$ <br> $b_{55} = -\frac{1423345}{44217} + \frac{50435}{4913}\sqrt{15} + \frac{27530}{4913}\sqrt{33} - \frac{185990}{44217}\sqrt{55}$ <br> $H_{55} = -\frac{40461639767}{651714363} + \frac{2329268305}{144825414}\sqrt{15} + \frac{782606510}{72412707}\sqrt{33} - \frac{5473886320}{651714363}\sqrt{55}$ |
| 70 | $a_{70} = \frac{57239}{1642545}\sqrt{7} + \frac{217912}{1642545}\sqrt{10} + \frac{18154}{182505}\sqrt{15} - \frac{5432}{60835}\sqrt{42}$ <br> $b_{70} = \frac{766766}{547515}\sqrt{7} + \frac{540694}{547515}\sqrt{10} + \frac{93548}{60835}\sqrt{15} - \frac{29314}{60835}\sqrt{42}$ <br> $H_{70} = \frac{263701974157}{999242250750} - \frac{3413048639}{55513458375}\sqrt{6} + \frac{8992317139}{499621125375}\sqrt{70} - \frac{1429629212}{55513458375}\sqrt{105}$ |
| 91 | $a_{91} = -\frac{513055226}{17779581} + \frac{197125250}{17779581}\sqrt{7} - \frac{140862644}{17779581}\sqrt{13} + \frac{52944437}{17779581}\sqrt{91}$ <br> $b_{91} = -\frac{502667165}{5926527} + \frac{214664450}{5926527}\sqrt{7} - \frac{142555490}{5926527}\sqrt{13} + \frac{53880905}{5926527}\sqrt{91}$ <br> $H_{91} = -\frac{3020198045742832}{11707907427243} + \frac{1141527555432550}{11707907427243}\sqrt{7} - \frac{838583339971300}{11707907427243}\sqrt{13} + \frac{633909424388075}{23415814854486}\sqrt{91}$ |
| 110 | $a_{110} = -\frac{51466456301}{226152099801}\sqrt{3} - \frac{1605347400}{25128011089}\sqrt{10} + \frac{2302296150}{25128011089}\sqrt{55} + \frac{2180745776}{75384033267}\sqrt{66}$ <br> $b_{110} = -\frac{113011075870}{75384033267}\sqrt{3} + \frac{21911639310}{25128011089}\sqrt{10} + \frac{31690709820}{25128011089}\sqrt{55} + \frac{8031352980}{25128011089}\sqrt{66}$ <br> $H_{110} = \frac{3280325101638066603637}{1262833882577813931842} - \frac{3496834328600515266 0}{631416941288906965921}\sqrt{22} - \frac{269221736376827284 05}{631416941288906965921}\sqrt{30} + \frac{114796388810356917 30}{631416941288906965921}\sqrt{165}$ |

**Table 8.** $a_n$, $b_n$, and $H_n$ for $h(-12n) = 8$.

| $n$ | $a_n$, $b_n$, and $H_n$ |
|---|---|
| 115 | $a_{115} = \frac{2453452114}{882792405} - \frac{58294124}{98088045}\sqrt{15} - \frac{7317496}{32696015}\sqrt{69} + \frac{139937129}{882792405}\sqrt{115}$ <br> $b_{115} = \frac{5026751821}{294264135} - \frac{39674908}{32696015}\sqrt{15} - \frac{19045012}{32696015}\sqrt{69} + \frac{132769793}{294264135}\sqrt{115}$ <br> $H_{115} = -\frac{195193666847694106}{144318968578830375} + \frac{5599864542570116}{16035440953203375}\sqrt{15} + \frac{2653754247048632}{16035440953203375}\sqrt{69} - \frac{37000600425371947}{288637937157660750}\sqrt{115}$ |
| 119 | $a_{119} = \frac{103789302}{9720379} - \frac{547343732}{87483411}\sqrt{3} + \frac{43409625}{9720379}\sqrt{7} - \frac{72149336}{29161137}\sqrt{21}$ <br> $b_{119} = \frac{318200715}{9720379} - \frac{551139820}{29161137}\sqrt{3} + \frac{162387225}{9720379}\sqrt{7} - \frac{67035540}{9720379}\sqrt{21}$ <br> $H_{119} = -\frac{49978710596750603}{1606258054361897} + \frac{28855221888962700}{1606258054361897}\sqrt{3} - \frac{37979008521886575}{3212516108723794}\sqrt{7} + \frac{10963595445145200}{1606258054361897}\sqrt{21}$ |
| 154 | $a_{154} = \frac{965168}{4053225}\sqrt{6} - \frac{3910004}{182395125}\sqrt{7} - \frac{28870936}{182395125}\sqrt{22} + \frac{142457}{4053225}\sqrt{231}$ <br> $b_{154} = \frac{2375828}{1351075}\sqrt{6} + \frac{112962616}{60798375}\sqrt{7} - \frac{55833106}{60798375}\sqrt{22} + \frac{836822}{1351075}\sqrt{231}$ <br> $H_{154} = \frac{7319532242037247}{27107244286031250} - \frac{14157410807176}{301191603178125}\sqrt{33} + \frac{8770226416943}{301191603178125}\sqrt{42} - \frac{206104571568818}{13553622143015625}\sqrt{154}$ |

**Table 8 (continuation).** $a_n$, $b_n$, and $H_n$ for $h(-12n) = 8$.

$$a_{455} = -\frac{3595868681280484581654 6}{4974358974851083512239} - \frac{1996392418395099670880 08}{4476923077365975161015 1}\sqrt{3}$$

$$-\frac{9186631100963329536488 7}{2487179487425541756119 5}\sqrt{7} + \frac{1211428925150112749386 8}{4974358974851083512239}\sqrt{13}$$

$$-\frac{2117048987345310400144 0}{1492307692455325053671 7}\sqrt{21} + \frac{1904528892443548554957 8}{1492307692455325053671 7}\sqrt{39}$$

$$+\frac{3501400086019335742242}{4974358974851083512239}\sqrt{91} + \frac{3648283270713504351401 2}{7461538462276625268358 5}\sqrt{273}$$

$$b_{455} = -\frac{1088688030978640654360 89}{4974358974851083512239} - \frac{1994609401071469229902 40}{1492307692455325053671 7}\sqrt{3}$$

$$-\frac{3269046290535497981699 19}{2487179487425541756119 5}\sqrt{7} + \frac{4777525461130916392899 0}{4974358974851083512239}\sqrt{13}$$

$$-\frac{9333352321361091775752}{4974358974851083512239}\sqrt{21} + \frac{2658412395462108166681 8}{4974358974851083512239}\sqrt{39}$$

$$+\frac{1133642837868669971476 2}{4974358974851083512239}\sqrt{91} + \frac{3506839516678136697511 8}{2487179487425541756119 5}\sqrt{273}$$

$$H_{455} = -\frac{255932775752916780245309314978504441970015853 83}{123721236053407612450916800551885360313965605 00}$$

$$-\frac{726431859849607816583487985232610666030207597}{618606180267038062254584002759426801569828025}\sqrt{3}$$

$$-\frac{383934727653435883989974325837331066769920979 1}{494884944213630449803667202207541441255862420 0}\sqrt{7}$$

$$+\frac{351649374516601338100434337317872458470333402}{618606180267038062254584002759426801569828025}\sqrt{13}$$

$$-\frac{56196149792473665089405401522944398076147222}{123721236053407612450916800551885360313965605}\sqrt{21}$$

$$+\frac{103240262189601325316821581871878064155153629 4}{309303090133519031127292001379713400784914012 5}\sqrt{39}$$

$$+\frac{53580377184022523304261118862331927596113329}{247442472106815224901833601103770720627931210}\sqrt{91}$$

$$+\frac{152525837164039389925036504420966961190929163}{123721236053407612450916800551885360313965605 0}\sqrt{273}$$

**Table 9.** $a_n$, $b_n$, and $H_n$ for $h(-12n) = 16$.

## Concluding remarks.

The common feature of all our series computed here is that they involve only simple quadratic numbers. The series corresponding to $n = 455$ gives us approximately 33 additional digits per term and it is the fastest convergent series belonging to *the theory of* $q_2$ known so far. It might also be the fastest convergent series for $1/\pi$ which involves only real quadratic numbers. One should compare this with the spectacular series discovered by the Borweins [4] which gives "25 digits per term" using only real quadratics. The fastest convergent series known so far is that given by the Borweins [4] which gives roughly 50 additional digits per term.

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## References

[1] B.C. Berndt, *Ramanujan's Notebooks, Part* V, Springer-Verlag, New York, 1998.

[2] B.C. Berndt, S. Bhargava and F.G. Garvan, *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc., **347** (1995), 4163-4244.

[3] J.M. Borwein and P.B. Borwein, *Pi and the AGM*, Wiley, New York, 1987.

[4] ———, *Class number three Ramanujan type series for* $1/\pi$, J. Comput. Appl. Math., **46** (1993), 281-290.

[5] H.H. Chan, *On Ramanujan's cubic transformation formula for* ${}_2F_1(1/3, 2/3; 1; z)$, Math. Proc. Cambridge Philos. Soc., **124** (1998), 193-204.

[6] H.H. Chan and W.-C. Liaw, *On Russell-type modular equations*, Canad. J. Math., to appear.

[7] D.A. Cox, *Primes of the Form* $x^2 + ny^2$, Wiley, New York, 1989.

[8] C.F. Gauss, *Disquisitiones Arithmeticae*, Springer-Verlag, New York, 1986.

[9] K.G. Ramanathan, *Some applications of Kronecker's limit formula*, J. Indian Math. Soc., **52** (1987), 71-89.

[10] S. Ramanujan, *Modular equations and approximations to* $\pi$, Quart. J. Math. (Oxford), **45** (1914), 350-372.

[11] C.L. Siegel, Advanced Analytic Number Theory, Tata Institute of Fundamental Research, Bombay, 1980.

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