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Using the formal reduction by a method of deformation of orbits under the adjoint representation of $GL(n, \mathbf{C})$, we have proved the existence and uniqueness (up to equivalence under $GL(n, \mathbf{C})$) of a formal canonical form of systems of singular linear difference equations. In this paper we study the stability of the irregular part of the canonical form under perturbation of the matrix coefficients.

1. Introduction and Notations.

The formal reduction of singular linear difference systems or of difference equations is studied in many ways: Formal classification, canonical forms or formal solutions (see [9], [4], [8], [2], [3]). One of the approaches is the reduction to canonical forms given in [3] by using the method of Babbitt and Varadarajan [1] for singular differential systems. We study in this paper the stability of the canonical forms of singular linear difference systems. Similar results for singular differential systems can be found in [1] or [7].

We shall use the following notations.

Let $K = \mathbf{C}((1/x))$ be the field of formal power series with coefficients in $\mathbf{C}$. ϕ is the $\mathbf{C}$-automorphism of K defined by $\phi(x) = x + 1$. For $q \in \mathbf{N}^*$, $x^{1/q}$ is a fixed root of $y^q = x$, $\mathcal{O}_q = \mathbf{C}[[x^{-1/q}]]$, $K_q = \mathbf{C}((x^{1/q}))$ and $\overline{K} = \bigcup_{q\in\mathbf{N}^*} K_q$ is the field of formal Puisieux power series over $\mathbf{C}$. ϕ can be extended to $\overline{K}$ by $\phi(x^{1/q}) = x^{1/q}(1 + x^{-1})^{1/q}$. Let $a \in K_q$ be nonzero, then it can be written in the form

$$a = a(x) = x^{-k/q} \sum_{j=0}^{+\infty} a_j x^{-j/q}, \quad a_0 \neq 0$$

where k is an integer. We write $\mathrm{ord}(a)$ for k/q, ($\mathrm{ord}(0) = +\infty$). For $A \in \mathrm{gl}(n, K_q)$, $A \neq 0$, we define

$$\mathrm{ord}(A) = \max\left\{\frac{r}{q} \mid r \in \mathbf{Z}, A \in x^{-r/q}\mathrm{gl}(n, \mathcal{O}_q)\right\}$$

and $\mathrm{ord}(0) = +\infty$.

We consider systems of linear difference equations of the following type

$$\phi(u) = Au \tag{1}$$

where $A \in \mathrm{GL}(n, K_q)$, $q \in \mathbf{N}^*$. One can write

$$A = \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \mathrm{GL}(n, K_q) \tag{2}$$

where $r \in \mathbf{Z}$, $A_{r+j} \in \mathrm{gl}(n, \mathbf{C})$ and $A_r \neq 0$.

Recall ([**3**]) that a matrix A or its associate system is said of level 0 if

$$A = I + \sum_{m=q}^{\infty} \frac{A_m}{x^{m/q}};$$

of level ≤ 1 if

$$A = I + \sum_{m=0}^{\infty} \frac{A_{r+m}}{x^{(r+m)/q}},\ r \in \mathbf{N}^*,\ 1 \leq r < q,\ A_r \neq 0, \tag{3}$$

where I denotes the $n \times n$ identity matrix.

Let $T \in \mathrm{GL}(n, K_q)$. The change $\tilde{u} = Tu$ transforms the system (1) to $\phi(\tilde{u}) = \tilde{A}\tilde{u}$ where

$$\tilde{A} = T[A] \stackrel{\text{def}}{=} \phi(T)AT^{-1}.$$

We shall say that the matrices $A, \tilde{A}$ (or the corresponding difference systems) are equivalent (under $\mathrm{GL}(n, K_q)$).

We recall (cf. [**3**]) the definition of a canonical form for a matrix or its associate linear difference system.

Definition 1.1. Let $p \in \mathbf{N}^*$. We shall say that a matrix $B \in \mathrm{GL}(n, K_p)$ is in canonical form if $B = \dfrac{1}{x^{r/p}} \displaystyle\bigoplus_{i=1}^{s} \frac{B_i}{x^{\ell_i}}$ with

$$r \in \mathbf{Z}, \quad \ell_i \in \frac{1}{p}\mathbf{N}, \quad \ell_1 < \ell_2 < \cdots < \ell_s,$$

$$B_i \in \mathrm{GL}(n^{(i)}, \mathcal{O}_p), \quad n^{(i)} \in \mathbf{N}^*,$$

$$\sum_{i=1}^{s} n^{(i)} = n, \quad B_i = \bigoplus_{\alpha=1}^{t_i} \lambda_\alpha^{(i)} \left(B_\alpha^{(i)} + \frac{C_\alpha^{(i)}}{x} \right)$$

where

$$B_\alpha^{(i)} = I_\alpha^{(i)} + \frac{D_{\alpha,1}^{(i)}}{x^{r_{\alpha,1}^{(i)}}} + \cdots + \frac{D_{\alpha,j_\alpha^{(i)}}^{(i)}}{x^{r_{\alpha,j_\alpha^{(i)}}^{(i)}}}$$

and

- $\lambda_\alpha^{(i)} \in \mathbf{C}^*$, $\lambda_\alpha^{(i)} \neq \lambda_\beta^{(i)}$ for $\alpha \neq \beta$,

- $I_\alpha^{(i)}$ is the $n_\alpha^{(i)} \times n_\alpha^{(i)}$ identity matrix, $n_\alpha^{(i)} \in \mathbf{N}^*$, $\sum_{\alpha=1}^{t_i} n_\alpha^{(i)} = n^{(i)}$,
- $r_{\alpha,j}^{(i)} \in \frac{1}{p}\mathbf{N}^*, r_{\alpha,1}^{(i)} < r_{\alpha,2}^{(i)} < \cdots < r_{\alpha,j_\alpha^{(i)}}^{(i)} < 1$, $D_{\alpha,j}^{(i)} \in \mathrm{gl}\Big(n_\alpha^{(i)}, \mathbf{C}\Big)\Big(1 \leq j \leq j_\alpha^{(i)}\Big)$ are nonzero diagonal matrices,
- $C_\alpha^{(i)} \in \mathrm{gl}\left(n_\alpha^{(i)}, \mathbf{C}\right)$ commutes with the $D_{\alpha,j}^{(i)}$ for $1 \leq j \leq j_\alpha^{(i)}$.

We make the convention that for $j_\alpha^{(i)} = 0$, $B_\alpha^{(i)} = I_\alpha^{(i)}$.

We will call $\dfrac{1}{x^{r/p}} \bigoplus_{i=1}^{s} \dfrac{\bigoplus_{\alpha=1}^{t_i} \lambda_\alpha^{(i)} B_\alpha^{(i)}}{x^{\ell_i}}$ the irregular part of the canonical form. The aim of this paper is to study the dependency of the irregular part in the canonical form of a singular linear difference system on the matrix coefficients A_{r+j}.

In [**3**] we have proved that for any matrix $A \in \mathrm{GL}(n, K_q)$ there exist some $p \in q\mathbf{N}^*$ and $T \in \mathrm{GL}(n, K_p)$ such that $T[A] \in \mathrm{GL}(n, K_p)$ is in a canonical form and its irregular part is unique up to equivalence in $\mathrm{GL}(n, \mathbf{C})$. It is based on the formal reduction using the method of Babbitt and Varadarajan [**1**], i.e., the method of deformation of orbits under the adjoint representation of $\mathrm{GL}(n, \mathbf{C})$ in the nilpotent case of the leading matrix.

Recall that a canonical form for a matrix (or the associate difference system) of level ≤ 1 is in the form:

$$I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}} + \frac{C}{x}$$

where the $D_j (1 \leq j \leq k)$ are nonzero diagonal matrices, $0 < r_1 < \cdots < r_k$ are rational numbers and the matrix C commutes with the matrices $D_j (1 \leq j \leq k)$. According to the convention of Definition 1.1, for $k = 0$ the canonical form is reduced to $I + Cx^{-1}$. The canonical form of level ≤ 1 is similar as in the differential case (see [**1**]). But for general difference systems the canonical form is more complicated.

We study in this paper the stability of the irregular part of the canonical form of a matrix or its associate linear difference system under perturbation of the matrix coefficients. A perturbed system of (1) is

$$\phi(u) = \left(A + \frac{P}{x^{(r+N)/q}}\right) u \tag{4}$$

with $N \in \mathbf{N}^*$ and $P \in \mathrm{gl}(n, \mathcal{O}_q)$, i.e., $\mathrm{ord}(P) \geq 0$.

Note that in [**2**] the first author of this paper has studied similar problems for formal solutions. More precisely it is proved that the irregular part in a fundamental matrix of formal solutions of difference systems associated to a matrix of level ≤ 1 (*resp.* of general systems) depends only on $A_r, A_{r+1}, \ldots, A_{r+n(q-r)-1}$ (*resp.* $A_r, A_{r+1}, \ldots, A_{r+\nu+nq-1}$) where ν denotes the integer such that $\frac{\nu}{q} = \mathrm{ord}(\det x^{r/q} A)$.

We shall prove, by using the method of [**1**], similar results on canonical forms for these two difference systems. More precisely, we will prove in Section 3 that for systems of level ≤ 1, if $N \geq n(q-r)$, the two systems (1) and (4) have the same irregular part in their canonical forms. This result is similar to the differential case [**1**]. In the general case the situation is more complicated and is considered in Section 4. Basing on the method of [**1**] for differential systems, we use also frequently the formal reduction procedure of linear difference systems presented in [**3**]. We state some of the results of [**1**] and [**3**] in Section 2 for the use in the sequel.

2. Preliminaries.

We present now some preliminary results which will be used in the next sections (see also [**1**] and [**3**]).

For $T \in \mathrm{GL}(n, K_q)$ we define the lag (see also [**1**]) of T as

$$\sigma_q(T) = \min \left\{ \frac{m}{q} \mid m \in \mathbf{N}, A \equiv 0 (\mathrm{mod}\ x^{-m/q}) \Longrightarrow TAT^{-1} \in \mathrm{gl}(n, \mathcal{O}_q) \right\}.$$

It is clear that if $\sigma_q(T) \leq m/q$ then

$$A \equiv B (\mathrm{mod}\ x^{-m'/q}) \Longrightarrow T[A] \equiv T[B] (\mathrm{mod}\ x^{-(m'-m)/q}). \tag{5}$$

Therefore if one controls the lag of a transformation matrix T, then one controls the first terms in the transformed system $T[A]$.

One has immediately the following properties (see also [**1**], p. 10-11):

(i) If q' is a multiple of q then for $T \in \mathrm{GL}(n, K_q) \subset \mathrm{GL}(n, K_{q'})$, $\sigma_{q'}(T) = \sigma_q(T)$. We will write $\sigma(T)$ for $\sigma_q(T)$ in the sequel.

(ii) $\sigma(T) = 0$ for $T \in \mathrm{GL}(n, \mathcal{O}_q) \cdot \mathbf{Z}_q$ where $\mathbf{Z}_q$ is the group of elements of the form $x^{-k/q} \cdot 1$ for $k \in \mathbf{Z}$.

(iii) $\sigma(T_1 T_2) \leq \sigma(T_1) + \sigma(T_2)$ for $T_1, T_2 \in \mathrm{GL}(n, K_q)$.

(iv) $\sigma(QT\tilde{Q}) = \sigma(T)$, $Q, \tilde{Q} \in \mathrm{GL}(n, \mathcal{O}_q) \cdot \mathbf{Z}_q$ and $T \in \mathrm{GL}(n, K_q)$.

(v) If $T = x^H$ for some semi-simple matrix H in $\mathrm{gl}(n, \mathbf{C})$ with eigenvalues $\lambda_i \in \frac{1}{q}\mathbf{Z}$ $(i = 1, \ldots, n)$ then

$$\sigma(T) = \max_{1 \leq i,j \leq n} \{|\lambda_i - \lambda_j|\}.$$

(vi) $\sigma(T) = \sigma(T^{-1})$ for $T \in \mathrm{GL}(n, K_q)$.

Let $\overline{\mathcal{O}}^{\times}$ be the group of units of $\overline{\mathcal{O}} = \bigcup_{q \in \mathbf{N}^*} \mathcal{O}_q$. We define

$$^{\circ}\mathrm{GL}(n, F) = \{T \in \mathrm{GL}(n, F) | \det T \in \overline{\mathcal{O}}^{\times}\}$$

where F may represent $\overline{K}, K_q, \mathcal{O}_q$ *etc.* If H is semi-simple in $\mathrm{gl}(n, \mathbf{C})$ with eigenvalues in $\mathbf{Q}$, it is immediate that

$$x^H \in {}^{\circ}\mathrm{GL}(n, \overline{K}) \Longleftrightarrow \mathrm{tr}(H) = 0.$$

We then have (cf. [**1**], Proposition 1.2).

(vii) Let $T = \bigoplus_{i=1}^{m} T_i$ where $T_i \in {}^\circ\mathrm{GL}(n_i, K_q)$ and $n = \sum_{i=1}^{m} n_i$. Then $T \in {}^\circ\mathrm{GL}(n, K_q)$ and

$$\sigma(T) \leq \sigma(T_1) + \cdots + \sigma(T_m).$$

Let $\mathcal{G} = \mathrm{gl}(n, \mathbf{C})$. For $M \in \mathcal{G}$, $\mathcal{G}_M$ and $[\mathcal{G}, M]$ denote respectively the kernel and the image of the adjoint homomorphism $\mathrm{ad}(M)$. $d(M)$ is the dimension of the $\mathrm{GL}(n, \mathbf{C})$-orbit of M with respect to the adjoint representation of $\mathcal{G}$.

Proposition 2.1 ([6], [1]). *Let Y be a nonzero nilpotent in $\mathcal{G}$; then we can find $H, X \in \mathrm{sl}(n, \mathbf{C})$ such that H is semi-simple, X is nilpotent and*

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

(Y, H, X) is called a standard triple.

Proposition 2.2 ([1]). *Let Y be a nonzero nilpotent and (Y, H, X) a standard triple. Let $Z \in \mathcal{G}_X, Z \neq 0$. Suppose that $Y + Z$ is nilpotent. Then $d(Y + Z) > d(Y)$.*

For a standard triple (Y, H, X), we have $\mathcal{G} = \mathcal{G}_X \oplus [\mathcal{G}, Y]$. Moreover there exists a basis $\{Z_1, \ldots, Z_\ell\}$ of $\mathcal{G}_X$ such that $Z_1 = I, Z_j \in \mathrm{sl}(n, \mathbf{C})$ for $j \geq 2$ (see [1], p. 15) and

$$[H, Z_j] = \lambda_j Z_j, \quad \lambda_j \in \mathbf{N} \ \text{ for } \ 1 \leq j \leq \ell.$$

In particular $\lambda_1 = 0$. Define $\Lambda = \max\limits_{1 \leq j \leq \ell} \left(\dfrac{\lambda_j}{2} + 1 \right)$, then $1 \leq \Lambda \leq n$.

$\{Z_1, \ldots, Z_\ell\}$ can be extended to a basis $\{Z_1, \ldots, Z_\ell, Z_{\ell+1}, \ldots, Z_{n^2}\}$ of $\mathcal{G}$ with the following properties:

$$\text{For all } j > \ell, \quad [H, Z_j] = \lambda_j Z_j, \quad \lambda_j \in \mathbf{Z}, \quad |\lambda_j| \leq \max_{1 \leq i \leq \ell} \lambda_i.$$

If $M \in \mathrm{gl}(n, \mathbf{C})$ is such that $[H, M] = cM$ for some $c \in \mathbf{Z}$ then

$$x^{\alpha H} M x^{-\alpha H} = x^{c\alpha} M, \quad \text{for} \quad \alpha \in \mathbf{Q}.$$

In particular

$$x^{\alpha H} Y x^{-\alpha H} = x^{-2\alpha} Y; \quad x^{\alpha H} Z_j x^{-\alpha H} = x^{\lambda_j \alpha} Z_j. \tag{6}$$

One has for $\alpha \in \mathbf{Q}$

$$\sigma(x^{\alpha H}) \leq |\alpha| \max_{1 \leq j \leq n^2} \{|\lambda_j|\} \leq 2(\Lambda - 1)|\alpha|. \tag{7}$$

We need also the following lemmas.

Lemma 2.1 ([3]). *Let a matrix $A \in \mathrm{GL}(n, K_q)$ be in one of the following forms,*

$$\text{(I)} \qquad I + \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}}, \quad 1 \leq r < q.$$

or

$$\text{(II)} \qquad x^{-r/q}\sum_{j=0}^{\infty}\frac{A_{r+j}}{x^{j/q}}, \quad r\in\mathbf{Z};$$

where $A_{r+j}\in\mathcal{G}, A_r\neq 0$. *Let* $\mathcal{L}\subset\mathcal{G}$ *be a linear subspace such that* $\mathcal{G}=\mathcal{L}+[\mathcal{G},A_r]$. *Then there exist sequences* $(T_j)_{j\geq 1}$ *in* $\mathcal{G}, (A'_{r+j})_{j\geq 1}$ *in* $\mathcal{L}$,

$$T=\prod_{j=\infty}^{1}\left(I+\frac{T_j}{x^{j/q}}\right)=\lim_{J\to\infty}\prod_{j=J}^{1}\left(I+\frac{T_j}{x^{j/q}}\right)$$

such that $A'_r=A_r$ *and*

$$T[A]=I+\sum_{j=0}^{\infty}\frac{A'_{r+j}}{x^{(r+j)/q}}, \text{ in the case (I)},$$

or

$$T[A]=x^{-r/q}\sum_{j=0}^{\infty}\frac{A'_{r+j}}{x^{j/q}}, \text{ in the case (II)}.$$

Moreover A'_{r+j} *only depends on* $A_r, A_{r+1},\ldots,A_{r+j}$.

Corollary 2.1 ([3]; Splitting lemma)**.** *Let notations be as in the above lemma. Let* Σ *be the set of eigenvalues of* A_r, P_λ *be the matrix of the projection of* $\mathbf{C}^n$ *on the eigenspace corresponding to* λ *in* Σ. *Let* S *be the semi-simple part of* A_r. *Choose* $\mathcal{L}=\mathcal{G}_S$. *Then* A'_{r+j} *commutes with* P_λ *for* $j\geq 1$; *moreover* $T[A]=\bigoplus_{\lambda\in\Sigma}A'_\lambda$ *where*

$$\begin{aligned} A'_\lambda &= I+\sum_{j=0}^{\infty}\frac{P_\lambda A'_{r+j}}{x^{(r+j)/q}} \quad \text{in the case (I)},\\ A'_\lambda &= x^{-r/q}\sum_{j=0}^{\infty}\frac{P_\lambda A'_{r+j}}{x^{j/q}} \quad \text{in the case (II)}.\end{aligned}$$

Corollary 2.2 ([3])**.** *Let notations be as in the above lemma. Assume that* A_r *is nilpotent and* (A_r,H,X) *a standard triple. Let* $\mathcal{L}=\mathcal{G}_X$ *and let* $m\geq 2$ *be an integer. There exists*

$$T=\left(I+\frac{T_{m-1}}{x^{(m-1)/q}}\right)\cdots\left(I+\frac{T_1}{x^{1/q}}\right)\in \mathrm{GL}(n,K_q)$$

such that

$$T[A]=I+\sum_{j=0}^{\infty}\frac{A'_{r+j}}{x^{(r+j)/q}}, \text{ in the case (I)}$$

$$T[A]=x^{-r/q}\sum_{j=0}^{\infty}\frac{A'_{r+j}}{x^{j/q}}, \text{ in the case (II)}$$

with $A'_r = A_r$, $A'_{r+j} \in \mathcal{G}_X$ *for* $1 \leq j < m$. *Furthermore for* $j \in \mathbf{N}^*, A'_{r+j}$ *only depends on* $A_r, \dots, A_{r+j}$.

Lemma 2.2 ([3]). *Let a matrix* $B \in \mathrm{GL}(n, K_p)$ *be in the form*

$$B = I + \frac{D_1}{x^{1/p}} + \cdots + \frac{D_{p-1}}{x^{(p-1)/p}} + \frac{C}{x} + R_B$$

where the $D_j \in \mathrm{gl}(n, \mathbf{C})$ *are diagonal matrices,* $C \in \mathrm{gl}(n, \mathbf{C})$, $\mathrm{ord}(R_B) > 1$. *Then* B *is equivalent to a canonical matrix of the form* $I + \frac{D_1}{x^{1/p}} + \cdots + \frac{D_{p-1}}{x^{(p-1)/p}} + \frac{C'}{x}$ *for some* $C' \in \mathrm{gl}(n, \mathbf{C})$.

3. Difference Systems of Level ≤ 1.

We consider at first, as in [3], difference systems of level ≤ 1, i.e., systems (1) with matrix A in the special form (3):

$$A = I + \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \mathrm{GL}(n, K_q)$$

with $A_r \neq 0, 1 \leq r < q$.

We will prove, using the method of [1], that the irregular part of a canonical form of difference systems of level ≤ 1 is determined by the matrices $A_r, A_{r+1}, \dots, A_{r+n(q-r)-1}$. Similar result for formal solutions has been proved in [2] by a different method.

Recall (cf. [3]) that a canonical form for matrices of level ≤ 1 is in the form:

$$A_{\text{cano}} = I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}} + \frac{C_A}{x} \in \mathrm{GL}(n, K_p) \tag{8}$$

for some $p \in \mathbf{N}^*$ and the irregular part of this canonical form is $I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}}$. If A is of level 0, the irregular part in its canonical form is reduced to I.

Since the irregular part is the first terms of a canonical form, from (5) one needs to make normalizations by matrices with convenient lags. The following proposition shows that, by a transformation matrix with a lag not exceeding a certain number, a difference system of level ≤ 1 with nilpotent leading matrix can be converted to a new one with non nilpotent leading matrix.

Proposition 3.1. *Let a matrix* $A = I + \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \mathrm{GL}(n, K_q)$ *with* $A_r \neq 0$ *be of level* ≤ 1, *i.e.,* $1 \leq r < q$. *Then we can find a matrix* $U \in {}^\circ\mathrm{GL}(n, K_p)$ *for some* $p \in q\mathbf{N}^*$ *and* $1 \leq s \leq p$, *such that:*

(1) $U[A] = I + \sum_{j=0}^{\infty} \tilde{A}_{s+j} x^{-(s+j)/p} \in \mathrm{GL}(n, K_p)$ *where either* $U[A]$ *is of level 0 in which case* $s = p$ *or* $\tilde{A}_s$ *is not nilpotent.*

(2) $\sigma(U) \leq (n-1)(\frac{s}{p} - \frac{r}{q})$.

Proof. If A_r is not nilpotent then the proposition is true with $s = r$, $p = q$ and $U = I$, $\sigma(U) = 0$.

If A_r is nilpotent we prove it by downward induction on $d(A)$, the dimension of the $\mathrm{GL}(n, \mathbf{C})$-orbit of A_r.

Let (Y, H, X) be a standard triple with $Y = A_r$. We apply at first the Corollary 2.2 (for the case (I)) with $m = \Lambda(q-r)$. Recall that

$$T = \left(I + \frac{T_{\Lambda(q-r)-1}}{x^{(r+\Lambda(q-r)-1)/q}}\right) \cdots \left(I + \frac{T_1}{x^{r/q}}\right)$$

and

$$A' = T[A] = I + \frac{A_r}{x^{r/q}} + \frac{A'_{r+1}}{x^{(r+1)/q}} + \cdots$$

where $A'_{r+j} \in \mathcal{G}_X$ for $1 \leq j < \Lambda(q-r)$. Then $T \in {}^\circ\mathrm{GL}(n, K_q)$ and $\sigma(T) = 0$ according to the property (ii) of Section 2.

We can write

$$\begin{aligned} A'_{r+j} &= \sum_{i=1}^{\ell} a_{r+j,i} Z_i, \quad 1 \leq j < \Lambda(q-r), \\ A'_{r+j} &= \sum_{i=1}^{n^2} a_{r+j,i} Z_i, \quad j \geq \Lambda(q-r). \end{aligned}$$

Define

$$E = \left\{ \frac{j}{\frac{\lambda_i}{2} + 1} \middle| 1 \leq j < \Lambda(q-r), 1 \leq i \leq \ell, a_{r+j,i} \neq 0 \right\}.$$

Let

$$\beta = \begin{cases} \inf E & \text{if } E \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

Define $\alpha = \min\{q-r, \beta\}$ and $S = x^{\alpha H/(2q)}$. It is clear that $\beta > 0, \alpha > 0$. According to (7), $\sigma(S) \leq (\Lambda - 1)\alpha/q \leq (n-1)\alpha/q$. Since $\mathrm{tr}(H) = 0$, $S \in {}^\circ\mathrm{GL}(n, K_p)$. According to (6) we have,

$$\begin{aligned} S[Y] &= \left(1 + x^{-1}\right)^{\alpha H/(2q)} x^{-\alpha/q} Y, \\ S[Z_j] &= \left(1 + x^{-1}\right)^{\alpha H/(2q)} x^{\alpha\lambda_j/(2q)} Z_j, \ 1 \leq j \leq n^2. \end{aligned}$$

Therefore

$$A'' \overset{\text{def}}{=} S[A'] = \left(1+x^{-1}\right)^{\alpha H/(2q)} \left[I + x^{-(r+\alpha)/q} \left(Y + \sum_{\substack{1\le j<\Lambda(q-r)\\ 1\le i\le \ell}} \frac{a_{r+j,i} Z_i}{x^{\frac{1}{q}[j-(\frac{\lambda_i}{2}+1)\alpha]}} + \sum_{\substack{j\ge\Lambda(q-r)\\ 1\le i\le n^2}} \frac{a_{r+j,i} Z_i}{x^{\frac{1}{q}[j-(\frac{\lambda_i}{2}+1)\alpha]}} \right) \right].$$

If $\alpha = q-r$ then $s=p=q$ and $\sigma(S) \le (n-1)\left(\frac{s}{p}-\frac{r}{q}\right)$. A'' is of level 0. If $0<\alpha<q-r$, write $\alpha = r'/q'$. Then

$$A'' = S[A'] = ST[A] = I + Y' x^{-\tilde{r}/\tilde{q}} + \cdots \in \mathrm{GL}(n, \mathcal{O}_{\tilde{q}})$$

where $\tilde{r} = 2(q'r+r')$, $\tilde{q} = 2q'q$ and $Y' = Y + \sum_{(j,i)\in\Omega} a_{r+j,i} Z_i$ with

$$\Omega = \left\{ (j,i) \middle| 1 \le j < \Lambda(q-r), 1 \le i \le \ell, a_{r+j,i} \neq 0, \alpha = \frac{j}{\frac{\lambda_i}{2}+1} \right\}.$$

Moreover $Y' = Y + Z \neq Y$ with $Z \in \mathcal{G}_X$. We have $\sigma(S) \le (n-1)\frac{\alpha}{q} = (n-1)\left(\frac{\tilde{r}}{\tilde{q}} - \frac{r}{q}\right)$. There are two distinct cases.

(a) If Y' is not nilpotent (this case occurs when $d(Y)$ has the maximal dimension, i.e., when Y is a principal nilpotent) then take $p=\tilde{q}$, $s=\tilde{r}$, and

$$\sigma(S) \le (n-1)\left(\frac{s}{p}-\frac{r}{q}\right).$$

Hence $U = ST \in {}^\circ\mathrm{GL}(n, K_p)$ has the claimed properties.

(b) If Y' is nilpotent, then $d(Y') > d(Y)$ according to Proposition 2.2. The induction hypothesis is applicable to A''. One deduces the existence of a $S' \in {}^\circ\mathrm{GL}(n, K_p)$ for some $p \in \tilde{q}\mathbf{N}^*$ and $1 \le s \le p$ such that $\sigma(S') \le (n-1)\left(\frac{s}{p} - \frac{\tilde{r}}{\tilde{q}}\right)$ and $S'[A']$ has the property (1). Let $U = S'ST$ then $U \in {}^\circ\mathrm{GL}(n, K_p)$ and

$$\begin{aligned}\sigma(U) \le \sigma(S') + \sigma(S) &\le (n-1)\left(\frac{s}{p}-\frac{\tilde{r}}{\tilde{q}}\right) + (n-1)\left(\frac{\tilde{r}}{\tilde{q}}-\frac{r}{q}\right)\\ &= (n-1)\left(\frac{s}{p}-\frac{r}{q}\right).\end{aligned}$$

□

The next proposition proves that for a system of level ≤ 1 one can obtain the irregular part of a canonical form with a transformation matrix whose lag is not greater than the number $(n-1)\left(1-\frac{r}{q}\right)$.

Proposition 3.2. *Let A be as in the above proposition. Then we can find a matrix $U \in {}^\circ\mathrm{GL}(n, K_p)$ for some $p \in q\mathbf{N}^*$, such that*

(1) *there exists a canonical form*

$$A_{\text{cano}} = I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}} + \frac{C_A}{x} \in \mathrm{GL}(n, K_p)$$

such that $U[A] = A_{\text{cano}} + R_A \in \mathrm{GL}(n, K_p)$ with $\operatorname{ord}(R_A) > 1$.

(2) $\sigma(U) \leq (n-1)(1 - \frac{r}{q})$.

Remark. With the convention of Definition 1.1, if $k > 0$ then $D_j (1 \leq j \leq k)$ are nonzero diagonal matrices and for $k = 0$, $A_{\text{cano}} = I + C_A x^{-1}$ is of level 0.

Proof. We prove the proposition by induction on n. For $n = 1$ one can take $U = 1$. Suppose $n > 1$. We assume the assertion in dimension $< n$.

Assume at first that A_r has at least two distinct eigenvalues. By applying the Corollary 2.1 we obtain a matrix $T = \prod_{j=\infty}^{1}(I + T_j x^{-j/q})$. Take

$$T_A = \prod_{j=n(q-r)-1}^{1} \left(I + \frac{T_j}{x^{j/q}}\right) \quad \text{and} \quad A' = T_A[A].$$

Let A'' be the matrix obtained from A' by omitting all terms of $x^{-j/q}$ with $j \geq r + n(q-r)$. Then $A' = A'' + E$ where $\operatorname{ord}(E) \geq (r + n(q-r))/q$ and A'' commutes with the spectral projections of A_r.

If $A_r = \bigoplus_\lambda A_\lambda^{(r)}$ with $n_\lambda = \dim\left(A_\lambda^{(r)}\right)$ then $A'' = \bigoplus_\lambda A''_\lambda$. By induction we can find matrices $U_\lambda \in {}^\circ\mathrm{GL}(n_\lambda, K_p)$ verifying the condition (1) and $\sigma(U_\lambda) \leq (n_\lambda - 1)\left(1 - \frac{r}{q}\right)$. We now use the property (vii) (cf. Section 2) to conclude that if $U' = \bigoplus_\lambda U_\lambda$, then $U' \in {}^\circ\mathrm{GL}(n, K_p)$ and

$$\sigma(U') \leq \sum_\lambda \sigma(U_\lambda) \leq \sum_\lambda (n_\lambda - 1)\left(1 - \frac{r}{q}\right) \leq (n-2)\left(1 - \frac{r}{q}\right)$$

and $U'[A'']$ verifies the condition (1). We now check that $\operatorname{ord}(U'[E]) > 1$ by using (5):

$$\operatorname{ord}(U'[E]) \geq \frac{r + n(q-r)}{q} - \frac{(n-2)(q-r)}{q} > 1.$$

Then $U = U' T_A$ has the claimed properties.

We now consider the case where A_r has a unique eigenvalue, $A_r = \omega I + Y$ where Y is nilpotent. We proceed by induction on the number $k = k(A)$ in the canonical form. If this number is 0 then by the above proposition one can find a matrix $T \in {}^\circ\mathrm{GL}(n, K_p)$ for some $p \in q\mathbf{N}^*$ such that $\sigma(T) \leq (n-1)\left(1 - \frac{r}{q}\right)$ and $T[A]$ is of level 0.

We may thus suppose that $k \geq 1$. If $\omega \neq 0$ then let $A = (1 + \omega x^{-r/q})\tilde{A}$. We have $k(\tilde{A}) < k(A)$. Therefore the induction hypothesis is applicable to $\tilde{A}$ and proves the proposition for A.

Suppose now that $\omega = 0$ so that $A_r = Y$ is a nonzero nilpotent matrix. Let U_1 be chosen to satisfy the conditions of Proposition 3.1 for some $p^* \in q\mathbf{N}^*$. Then

$$A^* = U_1[A] = I + A_s^* x^{-s/p^*} + \cdots$$

and $\sigma(U_1) \leq (n-1)\left(\frac{s}{p^*} - \frac{r}{q}\right)$. Either A^* is of level 0 in which case $s = p$ the proof is thus finished or A_s^* is not nilpotent which we consider in the following.

If A_s^* has at least two distinct eigenvalues, the earlier result allows us to find a matrix $U^* \in {}^\circ\mathrm{GL}(n, K_p)$ for some $p \in p^*\mathbf{N}^*$ such that $\sigma(U^*) \leq (n-1)\left(1 - \frac{s}{p^*}\right)$ and $U^*[A^*]$ has the property (1). If $U = U^* U_1$, then one has immediately the second assertion:

$$\sigma(U) \leq (n-1)\left(1 - \frac{s}{p^*}\right) + (n-1)\left(\frac{s}{p^*} - \frac{r}{q}\right) = (n-1)\left(1 - \frac{r}{q}\right).$$

If A_s^* has a single eigenvalue ω^* (which should be nonzero), one can write

$$A^* = (1 + \omega^* x^{-s/p^*})A^{**}$$

where $k(A^{**}) < k(A^*) = k(A)$. The induction hypothesis applied to the matrix

$$A^{**} = I + A_{s'}^{**} x^{-s'/p*} + \cdots \text{ (with } s' \geq s)$$

gives a matrix U^{**} having properties (1) and

$$\sigma(U^{**}) \leq (n-1)\left(1 - \frac{s'}{p^*}\right) \leq (n-1)\left(1 - \frac{s}{p^*}\right).$$

As before we take $U = U^{**}U_1$ and note that $\sigma(U) \leq (n-1)\left(1 - \frac{r}{q}\right)$. The proof is thus complete. □

Let $A \in \mathrm{GL}(n, K_q)$ be a matrix of level ≤ 1 as in the above propositions. We denote by $\Omega(A, m)$ the set of matrices $B \in \mathrm{GL}(n, K_q)$ of the same form as A with $B_{r+j} = A_{r+j}$ for all $0 \leq j < m$, i.e., $B \equiv A (\mathrm{mod}\ x^{-(r+m)/q})$.

Corollary 3.1. *Let notations be as in the proposition, $m = n(q - r)$. If $B \in \Omega(A, m)$ then*

$$U[B] = I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}} + \frac{C_B}{x} + R_B \in \mathrm{GL}(n, K_p)$$

where $\mathrm{ord}(R_B) > 1$. *If further $B \equiv A (\mathrm{mod}\ x^{-(r+m')/q})$ for some $m' > m$ then $C_B = C_A$.*

Proof. $\sigma(U) \leq (n-1)\left(1-\frac{r}{q}\right)$. From $B \equiv A(\bmod\ x^{-(r+m)/q})$ and (5) we have

$$U[A] \equiv U[B] \left(\bmod\ x^{-\left[\frac{r+m}{q}-(n-1)\left(1-\frac{r}{q}\right)\right]}\right)$$

and $\frac{r+m}{q} - (n-1)\left(1-\frac{r}{q}\right) = 1$, proving the first assertion.

If $B \equiv A \left(\bmod\ x^{-\frac{r+m'}{q}}\right)$ then $U[A] \equiv U[B] \left(\bmod\ x^{-[1+\frac{m'-m}{q}]}\right)$, proving the second statement. □

According to Lemma 2.2, the canonical form of $A_{\text{cano}} + R_A$ is $I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}} + \frac{C'}{x}$ where only the matrix C' may be different from C_A of A_{cano}. The following theorem is now immediate.

Theorem 3.1. *Let A be a matrix as in the above propositions. Let $m = n(q-r)$. If $B = I + B_r x^{-r/q} + \cdots$ and $A_{r+j} = B_{r+j}$ for $0 \leq j < m$, then A and B are either both of level 0 or both not, and have canonical forms with the same irregular part.*

As a consequence of this theorem, for systems of level ≤ 1, the irregular part in a fundamental matrix of formal solutions depends only on the matrix coefficients $A_{r+j}, 0 \leq j < n(q-r)$ (see also [**2**]).

4. General Difference Systems.

We now consider general difference systems of the form (1). We study at first as in the preceding section the nilpotent case, i.e., the case where the leading matrix A_r is nilpotent. We prove that for $N \geq 2\nu + nq$ the two difference systems (1) and (4) have the same irregular part in their canonical forms.

Definition 4.1. Let notations be as in Definition 1.1. We shall say that a matrix B' is in quasi-canonical form if $B' = \dfrac{1}{x^{r/p}} \bigoplus_{i=1}^{s} \dfrac{B'_i}{x^{\ell_i}}$ with

$$B'_i = \bigoplus_{\alpha=1}^{t_i} \lambda_\alpha^{(i)} \left(B_\alpha^{(i)} + \frac{C_\alpha^{(i)}}{x} + R_\alpha^{(i)} \right)$$

where $\operatorname{ord}(R_\alpha^{(i)}) > 1$.

Remark. A quasi-canonical form of a matrix of level ≤ 1 is simply a matrix of the form $A_{\text{cano}} + R_A$ where A_{cano} is a canonical matrix of the form (8) and $\operatorname{ord}(R_A) > 1$. It is clear that the matrices B and B' have the same irregular part according to Lemma 2.2.

At first we prove in the following proposition that one can always reach a non nilpotent leading matrix by a transformation with a convenient lag.

Proposition 4.1. *Let* $A = \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \mathrm{GL}(n, K_q)$ *with* $r \in \mathbf{Z}, A_r \neq 0$. *Let* ν *be the integer such that* $\frac{\nu}{q} = \mathrm{ord}(\det x^{r/q} A)$. *Then we can find a matrix* $U \in {}^\circ\mathrm{GL}(n, K_p)$ *for some* $p \in q\mathbf{N}^*$ *so that*

(1) $\tilde{A} = U[A] = \tilde{A}_s x^{-s/p} + \cdots \in \mathrm{GL}(n, K_p)$, *where* $\tilde{A}_s$ *is not nilpotent.*
(2) $\sigma(U) \leq \left(1 - \frac{1}{n}\right)\left(\frac{\nu}{q} - \frac{\tilde{\nu}}{p}\right)$ *where* $\frac{\tilde{\nu}}{p} = \mathrm{ord}(\det x^{s/p} \tilde{A})$.

Proof. If A_r is not nilpotent then $s = r$, $p = q$ and $U = I, \sigma(U) = 0$.

If A_r is nilpotent we prove it by induction on $d(A_r)$, the dimension of the $\mathrm{GL}(n, \mathbf{C})$-orbit of A_r. Let $Y = A_r$ and (Y, H, X) a standard triple. We apply at first Corollary 2.2 (for the case II) with $m = \nu\Lambda + 1$. We have

$$T = \prod_{j=\nu\Lambda}^{1} \left(I + \frac{T_j}{x^{j/q}}\right) \in {}^\circ\mathrm{GL}(n, K_q)$$

and $\sigma(T) = 0$ according to the property (ii) of the Section 2. Let $A' = T[A]$. Then

$$A' = x^{-r/q}\left(Y + \frac{A'_{r+1}}{x^{1/q}} + \cdots + \frac{A'_{r+\nu\Lambda}}{x^{(\nu\Lambda)/q}} + \cdots\right)$$

with $A'_{r+j} \in \mathcal{G}_X$ for $j = 1, \ldots, \nu\Lambda$. Furthermore for $j \in \mathbf{N}^*, A'_{r+j}$ only depends on $A_r, \ldots, A_{r+j}$.

Write

$$\begin{aligned} A'_{r+j} &= \sum_{i=1}^{\ell} a_{r+j,i} Z_i, \quad 1 \leq j \leq \nu\Lambda, \\ A'_{r+j} &= \sum_{i=1}^{n^2} a_{r+j,i} Z_i, \quad j > \nu\Lambda. \end{aligned}$$

Define

$$E = \left\{ \frac{j}{\frac{\lambda_i}{2} + 1} \middle| 1 \leq j \leq \nu\Lambda, 1 \leq i \leq \ell, a_{r+j,i} \neq 0 \right\}.$$

We claim that $E \neq \emptyset$ and $\inf E \leq \nu$ since $\det(\sum_{j=0}^{\nu} A_{r+j} x^{-j/q}) \neq 0$. Let $\beta = \inf E > 0$ and $S = x^{\beta H/(2q)}$. By (7), $\sigma(S) \leq (\Lambda - 1)\beta/q$. According to (6),

$$\begin{aligned} S[Y] &= (1 + x^{-1})^{\beta H/(2q)} x^{-\beta/q} Y, \\ S[Z_i] &= (1 + x^{-1})^{\beta H/(2q)} x^{\beta\lambda_i/(2q)} Z_i, \ 1 \leq i \leq n^2. \end{aligned}$$

Hence

$$S[A'] = (1+x^{-1})^{\beta H/(2q)} x^{-(r+\beta)/q} \Bigg[Y + \sum_{\substack{1\le j\le \nu\Lambda\\ 1\le i\le \ell}} \frac{a_{r+j,i} Z_i}{x^{\frac{1}{q}\left[j-\left(\frac{\lambda_i}{2}+1\right)\beta\right]}} + \sum_{\substack{j>\nu\Lambda\\ 1\le i\le n^2}} \frac{a_{r+j,i} Z_i}{x^{\frac{1}{q}\left[j-\left(\frac{\lambda_i}{2}+1\right)\beta\right]}} \Bigg].$$

Write $\beta = \frac{r'}{q'}$ with $r', q' \in \mathbf{N}^*$. Recall that $0 < \beta \le \nu$. For all $j > \nu\Lambda$ and $1 \le i \le n^2$, $j - \left(\frac{\lambda_i}{2}+1\right)\beta > 0$. Then $S[A'] \in \mathrm{GL}(n, \mathcal{O}_{2qq'})$. More precisely, with $r'' = 2(q'r + r')$, $q'' = 2q'q$,

$$A'' = S[A'] = x^{-r''/q''}\left[Y' + O(x^{-1/q''})\right]$$

where

$$Y' = Y + \sum_{(j,i)\in\Omega} a_{r+j,i} Z_i \ne Y.$$

The summation is over the (nonempty) set

$$\Omega = \left\{ (j,i) \middle| 1 \le j \le \nu\Lambda, 1 \le i \le \ell, a_{r+j,i} \ne 0, \beta = \frac{j}{\frac{\lambda_i}{2}+1} \right\}.$$

Let $\frac{\nu''}{q''} = \mathrm{ord}(\det x^{r''/q''} A'')$. Since H is semi-simple and $\mathrm{tr}(H) = 0$ then $S \in {}^\circ\mathrm{GL}(n, K_p)$ and we have also

$$\sigma(S) \le (\Lambda - 1)\frac{\beta}{q} \le (n-1)\left(\frac{r''}{q''} - \frac{r}{q}\right) = \left(1 - \frac{1}{n}\right)\left(\frac{\nu}{q} - \frac{\nu''}{q''}\right).$$

We distinguish two cases.

(a) Y' is not nilpotent (we have this case if $d(Y)$ is of the maximal dimension, i.e., if Y is a principal nilpotent). We take $s = r''$, $p = q''$, $U = ST$. Then $U \in {}^\circ\mathrm{GL}(n, K_p)$. $\tilde{A} = U[A] = A''$ verifies the assertion (1). With $\tilde{\nu} = \nu''$ the second assertion follows from

$$\sigma(U) \le \sigma(S) \le \left(1 - \frac{1}{n}\right)\left(\frac{\nu}{q} - \frac{\tilde{\nu}}{p}\right).$$

(b) Y' is nilpotent. We have $d(Y') > d(Y)$ by the Proposition 2.2. The induction hypothesis is applicable to A''. One deduces the existence of $U_1 \in {}^\circ\mathrm{GL}(n, K_p)$ for some $p \in q''\mathbf{N}^*$ such that $\sigma(U_1) \le \left(1 - \frac{1}{n}\right)\left(\frac{\nu''}{q''} - \frac{\tilde{\nu}}{p}\right)$ and

$$\tilde{A} = U_1[A''] = \tilde{Y} x^{-s/p} + \cdots \in \mathrm{GL}(n, K_p)$$

with $\tilde{Y}$ non nilpotent. Let $U = U_1ST$. Then $U \in {}^\circ\mathrm{GL}(n, K_p)$. $U[A]$ has the property (1) and $\sigma(U) \leq \sigma(U_1) + \sigma(S) \leq \left(1 - \frac{1}{n}\right)\left(\frac{\nu}{q} - \frac{\tilde{\nu}}{p}\right)$. $\square$

The next proposition shows that one can obtain the irregular part of a canonical form by a transformation matrix with a lag not exceeding a certain number that depends only on n, q and ν.

Proposition 4.2. *Let A and ν be as in the above proposition and m an integer $\geq \nu$. We can find a matrix $T \in {}^\circ\mathrm{GL}(n, K_p)$ for some $p \in q\mathbf{N}^*$ such that*

(1) $T[A] = A_{\text{q−cano}} + R_A$ *where $A_{\text{q−cano}}$ is a quasi-canonical matrix as in the Definition 4.1 and* $\mathrm{ord}(R_A) > \frac{r+m}{q} + 1$.
(2) $\sigma(T) \leq n - 1 + \frac{\nu}{q}$.

Proof. If $r \neq 0$ one considers $x^{r/q}A$ in the place of A. Then we can assume that $r = 0$. We prove the theorem by induction on n. It is trivial for $n = 1$. Suppose $n > 1$. We assume the assertion in dimension $< n$.

Assume that the leading matrix A_0 has at least two distinct eigenvalues. Then according to Corollary 2.1, there exists $\tilde{T} = \prod_{j=\infty}^{1}\left(I + \frac{T_j}{x^{j/q}}\right) \in {}^\circ\mathrm{GL}(n, \mathcal{O}_q)$ such that

$$\tilde{T}[A] = \sum_{j=0}^{\infty} A'_j x^{-j/q}$$

where $A'_0 = A_0$ and A'_j commutes with A_0 for all $j \geq 1$. Take $N = \max\{\nu\Lambda + 1, m + \nu + nq\}$ and

$$T_A = \prod_{j=N-1}^{1}\left(I + \frac{T_j}{x^{j/q}}\right) \in {}^\circ\mathrm{GL}(n, K_q).$$

Then $A' = T_A[A] = A'' + R$ where $A'' = \sum_{j=0}^{N-1} A'_j x^{-j/q}$ and $\mathrm{ord}(R) \geq N/q$. A'' commutes with the spectral projections of A_0.

If $A_0 = \bigoplus_\lambda A_\lambda^{(0)}$ with $n_\lambda = \dim\left(A_\lambda^{(0)}\right)$ then $A'' = \bigoplus_\lambda A''_\lambda$. Let

$$\frac{\nu_\lambda}{q} = \mathrm{ord}(\det x^{r_\lambda/q} A''_\lambda) \geq 0$$

with $r_\lambda/q = \mathrm{ord}(A''_\lambda) \geq 0$. We can find matrices $U_\lambda \in {}^\circ\mathrm{GL}(n_\lambda, K_p)$ such that $U_\lambda[A''_\lambda] = A^{(\lambda)}_{\text{q−cano}} + R_\lambda$ with $A^{(\lambda)}_{\text{q−cano}}$ in quasi-canonical form of dimension n_λ, $\mathrm{ord}(R_\lambda) > \frac{r_\lambda + m}{q} + 1 \geq \frac{m}{q} + 1$ and

$$\sigma(U_\lambda) \leq n_\lambda - 1 + \frac{\nu_\lambda}{q}.$$

Since $\nu \geq \sum_\lambda \nu_\lambda$, $n = \sum_\lambda n_\lambda$, we now use the property (vii) (cf. Section 2) to conclude that if $U' = \bigoplus_\lambda U_\lambda$, then $U' \in {}^\circ\mathrm{GL}(n, K_p)$ and

$$\sigma(U') \leq \sum_\lambda \sigma(U_\lambda) \leq n - 2 + \frac{\nu}{q}.$$

And $U'[A'']$ has the property (1). We now check that

$$\mathrm{ord}(U'[R]) \geq \frac{N}{q} - (n-2) - \frac{\nu}{q} > \frac{m}{q} + 1.$$

Then $T = U'T_A$ has the claimed properties.

We now consider the case where A_0 has a unique eigenvalue, $A_0 = \omega I + Y$ with Y nilpotent. Then either $\omega = 0$ or not.

Case 1. $\omega = 0$, then A_0 is nilpotent. According to Proposition 4.1, one can find a matrix $U \in {}^\circ\mathrm{GL}(n, K_{\tilde{p}}), \tilde{p} \in q\mathbf{N}^*$ with

$$\sigma(U) \leq \left(1 - \frac{1}{n}\right)\left(\frac{\nu}{q} - \frac{\tilde{\nu}}{\tilde{p}}\right) \leq \frac{\nu}{q} - \frac{\tilde{\nu}}{\tilde{p}}$$

such that

$$\tilde{A} = U[A] = \tilde{A}_s x^{-s/\tilde{p}} + \cdots \in \mathrm{GL}(n, K_{\tilde{p}})$$

where $\tilde{A}_s$ is not nilpotent and $\frac{\tilde{\nu}}{\tilde{p}} = \mathrm{ord}(\det x^{s/\tilde{p}}\tilde{A})$. Two cases may occur:

(a) $\tilde{A}_s$ has at least two distinct eigenvalues.

According to the above result one can find a matrix $T' \in {}^\circ\mathrm{GL}(n, K_p)$ with $p \in \tilde{p}\mathbf{N}$ such that $\sigma(T') \leq n - 1 + \frac{\tilde{\nu}}{\tilde{p}}$ and $T'[\tilde{A}]$ is in the desired form (1). Let $T = T'U$ then $T \in {}^\circ\mathrm{GL}(n, K_p)$ and

$$\sigma(T) \leq \sigma(T') + \sigma(U) \leq n - 1 + \frac{\nu}{q}.$$

(b) $\tilde{A}_s = wI + Y$ has only one nonzero eigenvalue w so that $\tilde{\nu} = 0$ and Y is nilpotent.

- If $Y = 0$ then $\tilde{A} = x^{-s/\tilde{p}} w A'$ where $A' = I + \sum_{j=0}^\infty A'_{r'+j} x^{-(r'+j)/p'} \in \mathrm{GL}(n, \mathcal{O}_{p'})$ is a matrix of level ≤ 1 with $r' \geq 2$ and $p' = 2\tilde{p}$.
- If $Y \neq 0$ then let (Y, H, X) be a standard triple. Let $p' = 2n\tilde{p}$ and $S = x^{H/(2n\tilde{p})}$. Then $S \in {}^\circ\mathrm{GL}(n, K_{p'})$ and $\sigma(S) \leq \frac{n-1}{n\tilde{p}}$. Write $\tilde{A} = x^{-s/\tilde{p}}(wI + B)$ with $B = Y + \sum_{j=1}^\infty \tilde{A}_{s+j} x^{-j/\tilde{p}}$.

  Write $\tilde{A}_{s+j} = \sum_{k=1}^{n^2} \tilde{a}_{s+j,k} Z_k$. According to (6), one has

  $$S[wI] = (1 + x^{-1})^{H/(2n\tilde{p})} wI, \quad S[Y] = (1 + x^{-1})^{H/(2n\tilde{p})} x^{-1/(n\tilde{p})} Y$$

  and

  $$S[Z_k] = (1 + x^{-1})^{H/(2n\tilde{p})} x^{\lambda_k/(2n\tilde{p})} Z_k \quad \text{for} \quad 1 \leq k \leq n^2.$$

Hence

$$S[B] = (1+x^{-1})^{H/(2n\tilde{p})} x^{-1/n\tilde{p}} \left(Y + \sum_{j\geq 1} \sum_{k=1}^{n^2} \frac{\tilde{a}_{s+j,k} Z_k}{x^{\frac{1}{\tilde{p}}\left[j - \left(\frac{\lambda_j}{2}+1\right)\frac{1}{n}\right]}} \right).$$

For all $j \geq 1$, since $\frac{\lambda_j}{2}+1 \leq n$, $j - \left(\frac{\lambda_j}{2}+1\right)\frac{1}{n} \geq 0$. Hence $\operatorname{ord}(S[B]) \geq \frac{1}{n\tilde{p}} = \frac{2}{p'}$. One has therefore $S[\tilde{A}] = x^{-s/\tilde{p}} w A'$ where

$$A' = I + \sum_{j=0}^{\infty} \frac{A'_{r'+j}}{x^{(r'+j)/p'}} \in \mathrm{GL}(n, \mathcal{O}_{p'}) \text{ with } r' \geq 2$$

is a matrix of level ≤ 1.

If $r' \geq p'$ then the matrix A' is of level 0. We are through.

If $r' < p'$ we apply the Proposition 3.2 to the matrix A' to obtain an integer $p \in p'\mathbf{N}^*$ and a matrix $U' \in {}^\circ\mathrm{GL}(n, K_p)$ such that $\sigma(U') \leq (n-1)\left(1 - \frac{r'}{p'}\right)$ and

$$U'[A'] = A'_{\text{cano}} + R_{A'} = A'_{\text{q−cano}} \in \mathrm{GL}(n, K_p)$$

where A'_{cano} is a canonical matrix of the form (8) of level ≤ 1 and $\operatorname{ord}(R_{A'}) > 1$. Let $T = U'SU$ then we have the assertion (1) and

$$\begin{aligned}\sigma(T) &\leq \sigma(U') + \sigma(S) + \sigma(U) \\ &\leq (n-1)\left(1 - \frac{r'}{p'} + \frac{2}{p'}\right) + \frac{\nu}{q} \leq n - 1 + \frac{\nu}{q}.\end{aligned}$$

Case 2. If $\omega \neq 0$ then $\nu = 0$ and the treatment is the same as in the case (b). ☐

Theorem 4.1. *Let notations be as above. Take $m = \nu$ and $N = 2\nu + nq$. Let B be a matrix in the same form as A such that $B_{r+j} = A_{r+j}$ for all $0 \leq j < N$. Let T be as in the above proposition. Then $T[B] = B_{\text{q−cano}} + R_B$ where $B_{\text{q−cano}}$ is a quasi-canonical matrix and* $\operatorname{ord}(R_B) \geq \frac{r+\nu}{q} + 1$. *The two matrices A and B have the same irregular part in their canonical forms.*

Proof. Since $\frac{\nu}{q} = \sum_{i=1}^{s} n^{(i)} \ell_i$ and $\ell_i \geq 0$, for all $i \in \{1, \dots, s\}$, one has $\ell_i \leq \frac{\nu}{q}$, where $n^{(i)}$ and ℓ_i are as in $A_{\text{q−cano}}$ (see Definition 4.1 and Definition 1.1).

From $\sigma(T) \leq n - 1 + \frac{\nu}{q}$, $B \equiv A(\operatorname{mod} x^{-(r+N)/q})$ and (5) we obtain

$$T[A] \equiv T[B] \left(\operatorname{mod} x^{-\left[\frac{r+N}{q} - (n-1) - \frac{\nu}{q}\right]}\right).$$

Since $\frac{r+N}{q} - (n-1) - \frac{\nu}{q} = \frac{r+\nu}{q} + 1$ and $T[A] = A_{\text{q−cano}} + R_A$ with $\operatorname{ord}(R_A) > \frac{r+\nu}{q} + 1$ the first assertion follows since $\ell_i \leq \frac{\nu}{q}$ for all $1 \leq i \leq s$. The second one follows from Lemma 2.2. ☐

References

[1] D.G. Babbitt and V.S. Varadarajan, *Formal reduction theory of meromorphic differential equations: A group theoretic view*, Pacific J. Math., **109** (1983), 1-80.

[2] G. Chen, *Forme normale d'Arnold et réduction formelle de systèmes d'équations linéaires aux différences*, Aequationes Math., **54** (1997), 264-288.

[3] G. Chen and A. Fahim, *Formal reduction of linear difference systems*, Pacific J. Math., **182** (1998), 37-54.

[4] A. Duval, *Lemme de Hessel et factorisation formelle pour les opérateurs aux différences*, Funkcial Ekvac., **26** (1983), 349-368.

[5] G. Immink, *Asymptotics of analytic difference equations*, Springer L. N. in Math., **1085** (1984).

[6] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math., **81** (1959), 973-1032.

[7] D.A. Lutz and R. Schäfke, *On the identification and stability of formal invariants for singular differential equations*, Linear Alg. and Appl., **72** (1985), 1-46.

[8] C. Praagman, *The formal classification of linear difference operators*, Proc. Kon. Ned. Ac. Wet. Ser A, **86** (1983), 249-261.

[9] H.L. Turrittin, *The formal theory of systems of irregular homogeneous linear difference and differential equations*, Bol. Soc. Math. Mexicana, **5** (1960), 255-264.

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