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We consider a weighted Cheeger's constant for a graph and we examine the gap between the first two eigenvalues of Laplacian. We establish several isoperimetric inequalities concerning the unweighted Cheeger's constant, weighted Cheeger's constants and eigenvalues for Neumann and Dirichlet conditions.

1. Introduction.

The study of eigenvalue ratios and gaps has a long and prolific history. The motivation stems not only from their physical relevance but also from the significance of their geometric content. The early seminal work of Polyá and S. Szegö [**29**] lay the foundation for the geometric study of eigenvalues. One of the main techniques involves deriving isoperimetric inequalities, in one form or another, to associate geometric constraints with analytic invariants of a given manifold. The isoperimetic methods have been developed by Cheeger [**13**], among others, to bound the first eigenvalue of a compact manifold by the isoperimetric constant. Further generalizations of Cheeger's constant can be attributed to Yau [**32**], Croke [**20**], Brooks [**6**], and others. The question of the extent to which the eigenvalues of the Laplace operator characterize a compact manifold has been investigated by Yau [**30**], Sunada [**31**], Brooks [**7**, **8**, **9**], Gordon, Webb and Wolpert [**24**], just to name a few. Numerous related results can be found in [**3**, **5**, **11**, **14**, **22**, **23**, **27**].

The ideas developed in this paper have their roots in results of the continuous setting which have been contributed by numerous people. For example, the early work of Payne, Polya and Weinberger [**28**] used geomtric arguments to develop quite general bounds on eigenvalue gaps. Hile and Protter [**25**] and later Ashbaugh and Benguria [**1**] have obtained sharp upper bounds on the ratio of the first two Dirichlet eigenvalues of a compact manifold.

Davies [**21**] first transformed the problem to a weighted L^2 space with weighted operator in the continous setting and he considered eigenvalue gaps for the weighted cases. In this paper, we introduce the weighted Cheeger constant of a graph which is a discrete analogue of the results of Cheng and Oden [**15**].

For an induced subgraph S of a graph G, the weighted Cheeger constant arises quite naturally by considering a weighted Laplacian (using the first Dirichlet eigenfunction u). The following study parallels in many respects the study in [**15**] for the continuous cases. We establish a weighted Cheeger's inequality for the first eigenvalue λ_u of the weighted Laplacian of a graph. We derive several inequalities involving the unweighted eigenvalue λ and weighted eigenvalues λ_u as well as the Dirichlet eigenvalues $\lambda_{D,i}$ and Neumann eigenvalues $\lambda_{N,i}$. (The detailed definition will be given in Section 2.) For example, we show that the following relation between the weighted eigenvalue λ_u, and the spectral gap of the Dirichlet eigenvalues:

$$\lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{2 - \lambda_{D,1} - \lambda_{D,2}} h_u^2. \tag{1}$$

We prove the following eigenvalue inequality involving the unweighted and weighted eigenvalues, the Neumann eigenvalues and the Dirichlet eigenvalues.

$$\lambda_u - \lambda_{D,1} \geq \lambda_{N,1}. \tag{2}$$

We also prove the following inequality involving the Dirichlet eigenvalues, the unweighted Cheeger's constant h and the weighted constant h_u.

$$h \leq 2h_u + \lambda_{D,1} + 2\sqrt{\lambda_{D,1} \cdot h_u}. \tag{3}$$

For a strongly convex subgraph S of an abelian homogeneous graph Γ, we show that

$$\lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{8kD^2}$$

where D denotes the diameter of S and k is the degree of Γ (which is regular). For undefined terminology in graph theory and spectral geometry, the reader is referred to [**4**, **16**] and [**12**, **30**], respectively.

The organizaton of this paper is as follows: In §2 we give basic definitions and describe basic properties for the Laplacian of graphs. In §3 we define a weighted graph Laplacian and its associated first eigenvalue and the weighted Cheeger's constant. In §4 we prove the weighted Cheeger's inequalities. In §5, we establish several isoperimetric inequalities concerning Neumann and Dirichlet eigenvalues.

2. Preliminaries.

We consider a graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The value of a function $f : V(G) \to \mathbb{R}$ at a vertex y is denoted by f_y. For $y \in V(G)$, we let d_y denote the degree of y (which is the number of vertices adjacent to y). We define the normalized Laplacian of G to be the following matrix:

$$\mathcal{L}(x,y) = \begin{cases} 1 & \text{if } x = y, \text{ and } d_x \neq 0, \\ -\frac{1}{\sqrt{d_x d_y}} & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $\mathcal{L}$ are denoted by $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ and when G is k-regular, it is easy to see that

$$\mathcal{L} = I - \frac{1}{k}A,$$

where A is the adjacency matrix of G. It is often convenient to write $\mathcal{L}$ as a product of simpler matrices for a connected graph.

$$\mathcal{L} = T^{-1/2} L T^{-1/2},$$

where L denotes the combinatorial Laplacian defined as follows:

$$L(x,y) = \begin{cases} d_x & \text{if } x = y, \\ -1 & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

and T is the diagonal matrix with the (x,x)-th entry having value d_x.

For a regular graph of degree k, L is just a multiple k of $\mathcal{L}$. For a general graph, our definition of the normalized Laplacian leads to a clean version of the Cheeger inequality for graphs [**17**] (also see (4)), while the Cheeger inequality using the combinatorial Laplacian involves additional complications concerning scaling. The advantages of the normalized Laplacian are perhaps due to the fact that it is consistent with the formulation in spectral geometry and in stochastic processes. In the rest of the paper, we will call $\mathcal{L}$ the Laplacian, for short.

Associated with $\mathcal{L}$ is the positive definite quadratic form $Q(f) = \langle f, \mathcal{L} f \rangle$. For any real-valued function f we have

$$\begin{aligned} \frac{Q(g)}{\langle g, g \rangle} &= \frac{\langle g, T^{-1/2} L T^{-1/2} g \rangle}{\langle g, g \rangle} \\ &= \frac{\langle f, Lf \rangle}{\langle T^{1/2} f, T^{1/2} f \rangle} \\ &= \frac{\displaystyle\sum_{x \sim y} (f_x - f_y)^2}{\displaystyle\sum_x f_x^2 d_x} \end{aligned}$$

where $f = T^{-1/2} g$, $x \sim y$ denotes x is adjacent to y, and the sum $\displaystyle\sum_{x \sim y}$ ranges over all unordered adjacent pairs x and y.

Let λ denote the least nontrivial eigenvalue of $\mathcal{L}$ of a graph G. The eigenvalue λ is closely related to the isoperimetric invariant, so called the Cheeger constant, defined as follows:

In a graph G, the volume of a subset X of the vertex set V, denoted by vol (X) , is defined to be $\sum_{x \in X} d_x$.

Definition. The Cheeger constant of a graph G with vertex set V is defined to be

$$h = \min_{X \subseteq V(G)} \frac{e(X, V \setminus X)}{\min\{\text{vol (X)}, \text{vol (V} \setminus \text{X)}\}}$$

where $e(X, V \setminus X)$ denotes the number of edges between X and $V \setminus X$.

The Cheeger inequality for a graph G states [17] that

$$2h \geq \lambda \geq \frac{h^2}{2}. \tag{4}$$

We will establish several variations of the Cheeger inequality by considering eigenvalues of induced subgraphs of a graph.

In a graph G with vertex set V, an induced subgraph on a subset S of V has vertex set S and edge set consisting of all edges with both endpoints in S. We denote the induced subgraph determined by S also by S if there is no confusion. The extension S' of S consists of all the edges $\{x, y\}$ with at least one endpoint in S. The boundary of S, denoted by δS, is defined to be $\{x \in V(G) \setminus S : x \text{ is adjacent to some } y \in S\}$. We now define various eigenvalues associated with the induced subgraph S that we shall study:

Definition. The Neumann Eigenvalue $\lambda_{N,1}$ of the induced subgraph S is defined to be

$$\lambda_{N,1} = \inf_{f \not\equiv 0} \frac{\displaystyle\sum_{\{x,y\} \in S'} (f_x - f_y)^2}{\displaystyle\sum_{x \in S} f_x^2 d_x} \tag{5}$$

where the infimum is taken over all nontrivial functions $f : S \bigcup \delta S \to \mathbb{R}$ satisfying, for each $x \in \delta S$,

$$\sum_{\substack{y \in S \\ y \sim x}} (f_x - f_y) = 0. \tag{6}$$

We remark that (6) is called the Neumann boundary condition for a function $f : V \to \mathbb{R}$. It corresponds to the Neumann boundary condition in the continuous setting. That is,

$$\frac{\partial f}{\partial \nu}(x) = 0$$

for $x \in \delta S$ where ν is the normal direction orthogonal to the tangent hyperplane at x.

Definition. The Dirichlet Eigenvalues of S is defined as follows:

$$\lambda_{D,1} = \inf_{f|\delta S=0} \frac{\sum\limits_{\{x,y\}\in S'} (f_x - f_y)^2}{\sum\limits_{x\in S} \left(f_x - f'_x\right)^2 d_x} \tag{7}$$

where $f|\delta S = 0$ means $f(x) = 0$ for $x \in \delta S$. In general, we define

$$\lambda_{D,i} = \inf_{f|\delta S=0} \sup_{f'\in C_{i-1}} \frac{\sum\limits_{\{x,y\}\in S'} (f_x - f_y)^2}{\sum\limits_{x\in S} \left(f_x - f'_x\right)^2 d_x}$$

where C_i is the subspace spanned by the $j-$th eigenfunctions ϕ_j with eigenvalue $\lambda_{D,j}$ for $j \leq i$.

3. Weighted Graph Laplacian.

Let u be the first eigenfunction for Dirichlet condtions of the induced subgraph S achieving $\lambda_{D,1}$ in (7). Here are some useful facts about u and $\lambda_{D,1}$ which follows from the definitions (see [**16**]):

Fact 1: $u \geq 0$ on S and $u = 0$ on δS.

Fact 2: $\lambda_{D,1} < 1$.

Fact 3: For $x \in S$, we have

$$\sum_{\substack{y \\ \{x,y\}\in S'}} (u_x - u_y) = \lambda_{D,1} u_x d_x.$$

Fact 4:

$$\sum_x \sum_{\substack{y \\ \{x,y\}\in S'}} (u_x - u_y) = \sum_{\{x,y\}\in S'} (u_x - u_y)^2.$$

We will use u to define a weighted Laplacian $\mathcal{L}_u$ as follows:

$$\mathcal{L}_u(x,y) = \begin{cases} u_x^2 & \text{if } x = y \text{ and } d_x \neq 0, \\ -\frac{u_x u_y}{\sqrt{d_x d_y}} & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

where d_x is the degree of x in G.

The first eigenvalue λ_u of $\mathcal{L}_u$ satisfies

$$\lambda_u = \inf_{\substack{f \not\equiv 0 \\ \sum_{x \in S} f_x u_x^2 d_x = 0}} \frac{\displaystyle\sum_{\{x,y\} \in S'} (f_x - f_y)^2 u_x u_y}{\displaystyle\sum_{x \in S} f_x^2 u_x^2 d_x}. \tag{8}$$

We define the weighted Cheeger's constant h_u to be

$$h_u = \min \frac{\displaystyle\sum_{\substack{x \in X,\, y \in S \setminus X \\ x \sim y}} u_x u_y}{\displaystyle\sum_{x \in X} u_x^2 d_x}$$

where the minimum ranges over all $X \subseteq S$ and $\displaystyle\sum_{x \in X} u_x^2 d_x \le \sum_{x \in S \setminus X} u_x^2 d_x$.

4. Weighted Cheeger's inequalities.

We will first give a functional formulation of the weighted Cheeger's constant which will be used later. As in the continuous setting [**30**], this shows the connection between the functional properties of a graph and its spectral properties.

Theorem 1. *Suppose u is a nonnegative vector in $\mathbb{R}^n$ (i.e. $u_i \ge 0$ for all i) where $n = V(S)$. Then*

$$h_u = \inf_{f \not\equiv 0} \sup_{C \in \mathbb{R}} \frac{\displaystyle\sum_{x \sim y} |f_x - f_y| u_x u_y}{\displaystyle\sum_{x \in S} |f_x - C| u_x^2 d_x}.$$

Proof. We choose C to satisfy:

1) For $\sigma < 0$ we have $\displaystyle\sum_{f_x - C < \sigma} u_x^2 d_x \le \sum_{f_x - C \ge \sigma} u_x^2 d_x$.

2) For $\sigma > 0$ we have $\displaystyle\sum_{f_x - C < \sigma} u_x^2 d_x \ge \sum_{f_x - C \ge \sigma} u_x^2 d_x$.

Let $g(\sigma) = \displaystyle\sum_{\substack{\{x,y\} \in E \\ f_x \le \sigma + C \le f_y}} u_x u_y$. Then we have

$$\sum_{x \sim y} |f_x - f_y| u_x u_y = \int_{-\infty}^{\infty} g(\sigma) d\sigma$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \frac{g(\sigma)}{\sum\limits_{f_x - C \le \sigma} u_x^2 d_x} \cdot \sum_{f_x - C \le \sigma} u_x^2 d_x \, d\sigma \\
&\quad + \int_{0}^{\infty} \frac{g(\sigma)}{\sum\limits_{f_x - C \ge \sigma} u_x^2 d_x} \cdot \sum_{f_x - C \ge \sigma} u_x^2 d_x \, d\sigma \\
&\ge h_u \int_{-\infty}^{0} \sum_{f_x - C \le \sigma} u_x^2 d_x \, d\sigma + h_u \int_{0}^{\infty} \sum_{f_x - C \ge \sigma} u_x^2 d_x \, d\sigma \\
&= h_u \sum_{x \in S} |f_x - C| u_x^2 d_x .
\end{aligned}
$$

Conversely, suppose $X_0 \subset S$ is a subset such that

$$
h_u = \frac{\sum\limits_{\substack{x \in X_0,\, y \in S \setminus X_0 \\ x \sim y}} u_x u_y}{\sum\limits_{x \in X_0} u_x^2 d_x}.
$$

Define f as follows:

$$
f_x = \begin{cases} 1 & x \in X_0 \\ -1 & x \in S \setminus X_0. \end{cases}
$$

Then

$$
\inf_f \sup_{C \in \mathbb{R}} \frac{\sum\limits_{x \sim y} |f_x - f_y| u_x u_y}{\sum\limits_{x \in S} |f_x - C| u_x^2 d_x} \quad \le \quad \sup_C \frac{\sum\limits_{\substack{x \in X_0, y \in S \setminus X_0 \\ x \sim y}} 2 u_x u_y}{\sum\limits_{x \in X_0} |1 - C| u_x^2 d_x + \sum\limits_{x \in S \setminus X_0} |1 + C| u_x^2 d_x}.
$$

We consider $\inf_C \left(\sum_{x \in X_0} |1 - C| u_x^2 + \sum_{x \in S \setminus X_0} |1 + C| \, u_x^2 \right)$, for $-1 \le C \le 1$. Define

$$
f(c) = \left(\sum_{x \in S \setminus X_0} u_x^2 d_x - \sum_{x \in X_0} u_x^2 d_x \right) \cdot C + \left(\sum_{x \in S \setminus X_0} u_x^2 d_x + \sum_{x \in X_0} u_x^2 d_x \right)
$$

on the interval $-1 \le C \le 1$. Since $\sum_{x \in S \setminus X_0} u_x^2 d_x - \sum_{x \in X_0} u_x^2 d_x \ge 0$, f has a minimum at $C = -1$ by elemetary calculation. Therefore,

$$
\inf_f \sup_{C \in \mathbb{R}} \frac{\sum\limits_{x \sim y} |f_x - f_y| u_x u_y}{\sum\limits_{x \in S} |f_x - C| u_x^2 d_x} \le \frac{\sum\limits_{\substack{x \in X_0, y \in S \setminus X_0 \\ x \sim y}} 2 u_x u_y}{\sum\limits_{x \in X_0} 2 u_x^2 d_x} \le h_u
$$

which completes the proof of Theorem 1. □

The above theorem leads to several Cheeger-type inequalities concerning eigenvalue gaps. We will show that the eigenvalue gap $\lambda_{D,2} - \lambda_{D,1}$ is, in fact, the first eigenvalue of the weighted Laplacian defined in §3.

Proposition 1.1. *Suppose u is the first Dirichlet eigenfunction of the Laplacian on the induced subgraph S of G. Let λ_u be the first eigenvalue of the u-weighted Laplacian, $\mathcal{L}_u$. Then,*

$$\lambda_u = \lambda_{D,2} - \lambda_{D,1}.$$

Proof. For any function $f : S \cup \delta S \to \mathbb{R}^+$, by using Fact 3 in Section 3 we have

$$\begin{aligned}
&\lambda_{D,1} \sum_x f_x^2 u_x^2 d_x \\
&= \sum_x f_x^2 u_x \cdot \sum_{y \sim x} (u_x - u_y) \\
&= \sum_{x \sim y} (u_x - u_y)(f_x^2 u_x - f_y^2 u_y) \\
&= \sum_{x \sim y} \left(f_x^2 u_x^2 - f_x^2 u_x u_y + f_y^2 u_y^2 - f_y^2 u_y u_x + 2 f_x f_y u_x u_y - 2 f_x f_y u_x u_y\right) \\
&= \sum_{x \sim y} (f_x u_x - f_y u_y)^2 - \left(f_y^2 u_x u_y + f_x^2 u_x u_y - 2 f_x f_y u_x u_y\right) \\
&= \sum_{x \sim y} (f_x u_x - f_y u_y)^2 - \sum_{x \sim y} (f_x - f_y)^2 u_x u_y.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_u &= \inf_{\substack{f \not\equiv 0 \\ \sum_{x \in S} f_x u_x^2 = 0}} \frac{\sum_{x \sim y} (f_x - f_y)^2 u_x u_y}{\sum_x f_x^2 u_x^2 d_x} \\
&= \inf_{\substack{f \not\equiv 0 \\ \sum_{x \in S} f_x u_x^2 = 0}} \frac{\sum_{x \sim y} (f_x u_x - f_y u_y)^2}{\sum_x f_x^2 u_x^2 d_x} - \lambda_{D,1} \\
&= \inf_{\substack{g \not\equiv 0 \\ \sum_{x \in S} g_x u_x = 0}} \frac{\sum_{x \sim y} (g_x - g_y)^2}{\sum_x g_x^2 d_x} - \lambda_{D,1} \\
&= \lambda_{D,2} - \lambda_{D,1}.
\end{aligned}$$

□

In the preceding proof of the proposition we have also shown the following:

Corollary 1.1.

$$\inf_{\substack{f\not\equiv 0\\ \sum_{x\in S} f_x u_x^2 d_x=0}} \frac{\sum\limits_{x\sim y} (f_x u_x - f_y u_y)^2}{\sum\limits_{x} f_x^2 u_x^2 d_x} = \inf_{\substack{g\not\equiv 0\\ \sum_{x\in S} g_x u_x d_x=0}} \frac{\sum\limits_{x\sim y} (g_x - g_y)^2}{\sum\limits_{x} g_x^2 d_x} = \lambda_{D,2}.$$

Proposition 1.2.

$$2(1-\lambda_{D,1}) \geq \lambda_u.$$

Proof. Let f denote the eigenfunction achieving the Dirichlet eigenvalue $\lambda_{D,2}$. We consider

$$\begin{aligned} 2\sum_x f_x^2 u_x \sum_{y\sim x} u_y &\geq \sum_{x\sim y} 2(f_x^2+f_y^2)u_x u_y \\ &\geq \sum_{x\sim y} (f_x - f_y)^2 u_x u_y. \end{aligned}$$

Using the fact that

$$\sum_{y\sim x} u_y = (1-\lambda_{D,1}) u_x d_x$$

we then have

$$2(1-\lambda_{D,1})\sum_x f_x^2 u_x^2 d_x \geq \sum_{x\sim y} (f_x - f_y)^2 u_x u_y.$$

This implies

$$\begin{aligned} 2(1-\lambda_{D,1}) &\geq \frac{\sum\limits_{x\sim y} (f_x - f_y)^2 u_x u_y}{\sum\limits_x f_x^2 u_x^2 d_x} \\ &= \lambda_{D,2}. \end{aligned}$$

□

We will use the above facts to prove several versions of weighted Cheeger's inequalities. The following proof is somewhat similar to the unweighted case in [15].

Theorem 2.

$$\lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{2-\lambda_{D,1}-\lambda_{D,2}} h_u^2.$$

Proof. Let f denote the function achieving λ_u in (8). We label the vertices of G, so that $f_i \equiv f_{v_i} \leq f_{i+1}$ and let p denote the least integer such that $f_p \geq 0$. For each i, $1 \leq i \leq |S|$ we consider the cut $C_i = \{\{v_j, v_k\} \in E(S) : 1 \leq j \leq i \leq k \leq n\}$. We define α to be

$$\alpha = \min_{1 \leq i \leq |S|} \frac{\displaystyle\sum_{\{j,k\} \in C_i} 2u_j u_k}{\min\left(\displaystyle\sum_{j \leq i} u_j^2 d_j, \sum_{j > i} u_j^2 d_j\right)}.$$

It is clear that $\alpha \geq h_u$. Without loss of generality, we may assume

$$\sum_{j \leq p} u_j^2 d_j \leq \sum_{j > p} u_j^2 d_j.$$

We define

$$\begin{aligned} V_+ &= \{x : f_x \geq 0\} \\ E_+ &= \{\{x, y\} \ : \ x \in V_+ \text{ or } y \in V_+\}. \end{aligned}$$

We define

$$g_x = \begin{cases} f_x & \text{if } x \in V_+ \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\lambda_u f_x u_x d_x = \sum_{y \sim x} (f_x - f_y) u_x u_y$$

we have

$$\begin{aligned} \lambda_u &= \frac{\displaystyle\sum_{x \in V_+} f_x \sum_{\{x,y\} \in E_+} (f_x - f_y) u_x u_y}{\displaystyle\sum_{x \in V_+} f_x^2 u_x^2 d_x} \\ &\geq \frac{\displaystyle\sum_{\{x,y\} \in E_+} (g_x - g_y)^2 u_x u_y}{\displaystyle\sum_{x \in V_+} f_x^2 u_x^2 d_x} \\ &= W. \end{aligned}$$

Then we have

$$W = \frac{\left(\sum_{\{x,y\}\in E_+} (g_x - g_y)^2 u_x u_y\right) \left(\sum_{\{x,y\}\in E_+} (g_x + g_y)^2 u_x u_y\right)}{\left(\sum_{x\in V_+} f_x^2 u_x^2 d_x\right) \cdot \left(\sum_{\{x,y\}\in E_+} (g_x + g_y)^2 u_x u_y\right)}$$
$$\geq \frac{\left(\sum_{\{x,y\}\in E_+} |g_x^2 - g_y^2|\ u_x u_y\right)^2}{\sum_{x\in V_+} f_x^2 u_x^2 d_x \left(2\sum_{x\in V_+} f_x^2 u_x \sum_{y\sim x} u_y - \sum_{\{x,y\}\in E_+} (g_x - g_y)^2 u_x u_y\right)}.$$

Since

$$\sum_{y\sim x} u_y = u_x d_x - \lambda_{D,1} u_x d_x$$

we have

$$W \geq \frac{\left(\sum_{\{x,y\}\in E_+} |g_x^2 - g_y^2|\ u_x u_y\right)^2}{\left(\sum_{x\in V_+} f_x^2 u_x^2 d_x\right)^2 \left(2(1-\lambda_{D,1}) - \frac{\sum_{\{x,y\}\in E_+} (g_x - g_y)^2 u_x u_y}{\sum_{x\in V_+} f_x^2 u_x^2 d_x}\right)}$$
$$\geq \frac{\left(\sum_{\{x,y\}\in E_+} |g_x^2 - g_y^2|\ u_x u_y\right)^2}{\left(\sum_{x\in V_+} f_x^2 u_x^2 d_x\right)^2 (2 - 2\lambda_{D,1} - W)}$$
$$\geq \frac{\left(\sum_{i\leq p} |f_{i+1}^2 - f_i^2| \sum_{C_i} u_x u_y\right)^2}{\left(\sum_{x\in V_+} f_x^2 u_x^2 d_x\right)^2 (2 - 2\lambda_{D,1} - \lambda_u)}$$

$$
\begin{aligned}
&\geq \frac{\left(\displaystyle\sum_{i\leq p} |f_i^2 - f_{i+1}^2| \, \alpha \sum_{j\leq i} u_j^2\right)^2}{\left(\displaystyle\sum_{x\in V_+} f_x^2 u_x^2 d_x\right)^2 (2 - 2\lambda_{D,1} - \lambda_u)} \\
&\geq \frac{\alpha^2 \left(\displaystyle\sum_{x\in V_+} f_x^2 u_x^2 dx\right)^2}{(2 - 2\lambda_{D,1} - \lambda_u)\left(\displaystyle\sum_{x\in V_+} f_x^2 u_x^2 d_x\right)^2} \\
&\geq \frac{\alpha^2}{2 - 2\lambda_{D,1} - \lambda_u} \\
&\geq \frac{h_u^2}{2 - \lambda_{D,1} - \lambda_{D,2}}
\end{aligned}
$$

by using Proposition 1.1. □

Here is another analogue of the Cheeger inequality relating the spectral gaps of Dirichlet eigenvalues to the weighted Cheeger's constant. Its proof follows immediately from Theorem 2 and Fact 2.

Corollary 2.1.

$$\lambda_{D,2} - \lambda_{D,1} \geq \frac{h_u^2}{2(1 - \lambda_{D,1})} \geq \frac{h_u^2}{2}.$$

A theorem of Payne, Polya and Weinberger [**28**] gives

$$\lambda_{D,k+1} - \lambda_{D,k} \leq \frac{4\displaystyle\sum_{1}^{k} \lambda_{D,i}}{n \cdot k}$$

for Dirichlet eigenvalues of a bounded domain $\Omega \subset \mathbb{R}^n$. It would be of interest to prove a similar inequality for graphs.

5. Several isoperimetric inequalities.

It was shown in [**15**] that the continous analogue of h_u was bounded below by $c\,h$, where c is a constant depending on the dimension of the manifold and its rolling sphere radius and h is the unweighted Neumann Cheeger's constant. In the discrete setting, similar relationships can be found between the various unweighted and weighted Cheeger's constants as well as the unweighted and weighted eigenvalues. The following results have their origins in the work of

Payne, Polýa and Weinberger [28] as well as the subsequent developments by Ashbaugh and Benguria [1], Hile and Protter [25], and Hile and Xu [26].

Theorem 3.

$$\lambda_u - \lambda_{D,1} \geq \lambda_{N,1}.$$

Proof. From the definiton, we have

$$\lambda_{N,1} = \inf_{\substack{f \not\equiv 0 \\ \sum_{x \in S} f_x d_x = 0}} \frac{\displaystyle\sum_{\{x,y\} \in S'} (f_x - f_y)^2}{\displaystyle\sum_{x \in S} f_x^2 d_x},$$

subject to the Neumann boundary condition $\displaystyle\sum_{\substack{y \\ y \sim x}} (f_x - f_y) = 0$ for any $x \in \delta S$. Using the Neumann boundary condition, we have

$$\sum_{\{x,y\} \in S'} (f_x - f_y)^2 = \sum_{x \in S} f_x \sum_{\substack{y \\ y \sim x}} (f_x - f_y).$$

Let h be the eigenfunction for the weighted Laplcian and set $f = h \cdot u$. Then we have

$$\begin{aligned}
\lambda_{N,1} &\leq \frac{\displaystyle\sum_{x \in S} f_x \cdot \sum_{y \sim x} (f_x - f_y)}{\displaystyle\sum_{x \in S} f_x^2 d_x} \\
&= \frac{\displaystyle\sum_{x} h_x u_x \cdot \sum_{y \sim x} (h_x u_x - h_y u_y)}{\displaystyle\sum_{x} h_x^2 u_x^2 d_x} \\
&= \frac{\displaystyle\sum_{\{x,y\} \in S'} (h_x - h_y)^2 u_x u_y - \sum_{x \in S} \sum_{y \sim x} h_x^2 u_x (u_x - u_y)}{\displaystyle\sum_{x} h_x^2 u_x^2 d_x} \\
&= \lambda_u - \frac{\lambda_{D,1} \displaystyle\sum_{x} h_x^2 u_x^2 \, d_x}{\displaystyle\sum_{x} h_x^2 u_x^2 d_x} \\
&= \lambda_u - \lambda_{D,1}.
\end{aligned}$$

□

Now by using Proposition 1.1, we have the following.

Corollary 3.1.

$$\lambda_{D,2} \geq 2\lambda_{D,1} + \lambda_{N,1}.$$

In particular the theorem implies that $\lambda_{D,2} - \lambda_{D,1} \geq \lambda_{N,1}$. One of the authors and S.T. Yau [**18**] studied $\lambda_{N,1}$ on subgraphs of homogeneous graphs. When the induced subgraph S is strongly convex (i.e., all shortest paths in the host graph joining two vertices in S are contained in S, see [**19**] for more details) it was proved in [**18**] that

$$\lambda_{N,1} \geq \frac{1}{8kD^2},$$

where k is the degree of the S and D is the diameter. This immediately gives:

Corollary 3.2. *Suppose S is a strongly convex subgraph of an invariant abelian homogeneous graph Γ with edge generating set K consisting of k generators. Then*

$$\lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{8kD^2}$$

where D is the diameter of S.

The preceding results can be related to several bounds of $\lambda_{N,1}$ in terms of the heat kernel, as examined in [**19**].

The weighted Cheeger's constant incorporates more information about the graph in its definition. So in principle one would expect estimates involving the weighted Cheeger to be better than ones involving only the unweighted Cheeger's constants. In this respect, the following isoperimetric inequality is of interest, and can be contrasted with the upper bounds developed by Buser [**10**] in the continuous setting.

Theorem 4. *Suppose h is the Cheeger's constant of the induced subgraph S and h_u the weighted Cheeger's constant. Then,*

$$h \leq 2h_u + \lambda_{D,1} + 2\sqrt{\lambda_{D,1} \cdot h_u}.$$

Proof. From Theorem 1, we have

$$h = \inf_{f \not\equiv 0} \sup_C \frac{\sum_{x \sim y} |f_x - f_y|}{\sum_x |f_x - C| d_x}.$$

Since $V(S)$ and $E(S)$ are finite sets, there is some $X \subseteq S$ which achieves h. Using the first Dirichlet eigenfunction u, we define

$$f_x - C = \begin{cases} u_x^2 & \text{if } x \in X, \\ -u_x^2 & \text{if } x \in S \setminus X. \end{cases}$$

where C is as defined in the proof of Theorem 1. Therefore we have

$$h \le \frac{\displaystyle\sum_{\substack{x\sim y\\ x\in X, y\in S\setminus X}} (u_x^2+u_y^2) + \sum_{\substack{x\sim y\\ \{x,y\in X\} \text{ or } \{x,y\in S\setminus X\}}} |u_x^2-u_y^2|}{\displaystyle\sum_x u_x^2 d_x}$$

$$\le \frac{\displaystyle\sum_{x\sim y} \left((u_x-u_y)^2+2u_xu_y\right) + \sum_{\substack{x\sim y\\ \{x,y\in X\} \text{ or } \{x,y\in S\setminus X\}}} (|u_x^2-u_y^2|-(u_x-u_y)^2)}{\displaystyle\sum_x u_x^2 d_x}$$

$$= \lambda_{D,1} + 2h_u + \frac{\displaystyle\sum_{\substack{x\sim y\\ \{x,y\in X\} \text{ or } \{x,y\in S\setminus X\}}} \left(|u_x^2-u_y^2|-(u_x-u_y)^2\right)}{\displaystyle\sum_x u_x^2 d_x}.$$

We note that

$$\begin{aligned}
&\sum_{\substack{x\sim y\\ \{x,y\in X\} \text{ or } \{x,y\in S\setminus X\}}} \left(|u_x^2-u_y^2|-(u_x-u_y)^2\right)\\
&\le \sum_{x\sim y} |u_x-u_y|\; 2\min\{u_x,u_y\}\\
&\le \sum_{x\sim y} |u_x-u_y|\sqrt{u_xu_y}\\
&\le 2\left(\sum_{x\sim y}(u_x-u_y)^2\right)^{1/2}\cdot\left(\sum_{x\sim y}u_xu_y\right)^{1/2}\\
&\le 2\left(\sum_x u_x^2 d_x\right)\sqrt{\lambda_{D,1}\,h_u},
\end{aligned}$$

by using the definition of $\lambda_{D,1}$ and h_u. Therefore, we have

$$h \le 2h_u + \lambda_{D,1} + 2\sqrt{\lambda_{D,1}\,h_u}$$

as claimed. □

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