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This paper is divided in three parts. The first part deals with the equivalence of finite determination on the right and finite relative determination (with respect to S) under some conditions on S. The second part deals with infinite determinacy (with respect to S, a germ of a closed set of $\mathbb{R}^n$). Both generalize results of P. Porto [P] for a big family of closed subsets S of $\mathbb{R}^n$. The third part is a special case which is quite interesting, when S coincides with the closure of its interior.

Introduction.

This paper continues the work done in [**K**]. In that paper there were proven results of finite relative determination for particular algebraic subsets of $\mathbb{R}^n$. Here we continue in this direction. In the first part we prove the equivalence of finite determination on the right and finite relative determination for a big family of algebraic subsets, generalizing the results of [**P-L**]. In the second part we continue with the concept of infinite determinacy and remarking the importance of quasihomogeneous polynomials. In the third part we generalize the results on relative stability in [**P-L**] and [**P**] for a broader family of closed subsets of $\mathbb{R}^n$, such as good semianalytic subsets.

Notation.

We shall work in $\mathcal{E}(n)$, the local algebra of C^∞ function germs of $\mathbb{R}^n$ to $\mathbb{R}$ around the origin with maximal ideal $m(n)$. The powers of $m(n)$ will be denoted by $m(n)^k$ and $m(n)^\infty = \cap_{k=1}^{\infty} m(n)^k$. For $I = (i_1, \dots, i_n)$ a multi-index of natural numbers and $x = (x_1, \dots, x_n)$ we shall write $x^I = x_1^{i_1} \dots x_n^{i_n}$ and $|I| = i_1 + \cdots + i_n$, also for a germ f, $\frac{\partial^{|I|} f}{\partial x^I} = \frac{\partial^{|I|} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$.

For S a subset of $\mathbb{R}^n$, $\overline{0} \in S$, $cl(S)$ and $\operatorname{int}(S)$ will denote the closure and the interior of S respectively and G_S will be the group of germs of diffeomophisms ϕ at $\overline{0}$, such that $\phi(x) = x \;\forall\; x \in S$. Also $d(x, S)$ will denote the usual distance from the point x to the subset S.

Finally if f is a germ, $\partial f / \partial x_i$ will be the partial derivatives of f and $\left\langle \frac{\partial f}{\partial x_i} \right\rangle$ will be the ideal of $\mathcal{E}(n)$ generated by them. Also for a germ f,

$j^k f(x)$ will be the Taylor expansion of f at the point x up to degree k and it is called the $k-$jet of f at x. We will denote by $J^k(n,1)$ the $\mathbb{R}$-vector space of all polynomials in n-coordinates up to degree k. In the case $k=\infty$ we understand $j^\infty f(x)$ the Taylor series of f at x. Also $J^\infty(n,1)$, the set of all these jets will be identified with the formal power series ring $\mathbb{R}\,[[x_1,\ldots,x_n]]$.

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1. Finite Relative Determination and Finite Determination on the Right.

Definition 1.

(a) Let S be a germ of a subset of $\mathbb{R}^n$, f be a germ with $f(\overline{0})=0$ and let $k\le\infty$. We shall say that f is $k-$ determined relative to G_S if whenever g is a germ such that $j^k g(\overline{0}) = j^k f(\overline{0})$ and $g-f$ vanishes at S, there exists h in G_S with $g = f\circ h$. In the case $S=\left\{\overline{0}\right\}$ we say that f is finitely or infinitely determined on the right according to k is finite or not. In general if k is finite, then we say that f is finitely determined relative to G_S.

(b) Let I be an ideal of $\mathcal{E}\,(n)$ and $S = z\,(I)$ the germ of the common zeroes of I (we suppose $\overline{0}\in S$). We denote by $\operatorname{rad} I$ the ideal of $\mathcal{E}\,(n)$ consisting of all germs vanishing at S, and we say that I is radical if $I=\operatorname{rad} I$.

Lemma 2 (Artin-Rees). *If I is an ideal of $\mathbb{R}\,[[x]] = \mathbb{R}\,[[x_1,\ldots,x_n]]$, there exists k such that $I\cap M^m = M^{m-k}(I\cap M^k)$ $(\forall\, m\ge k)$.*

We shall denote $\mathcal{A}(I)$ the minimum k satisfying the equality of Lemma 2. Consider $\mathbb{R}\,[[x]]$ the algebra of formal power series, M its maximal ideal, and the canonical projection $\pi:\mathcal{E}\,(n)\longrightarrow\mathbb{R}\,[[x]]$ which sends a germ to its Taylor infinite series and J an ideal of $\mathcal{E}\,(n)$, we will get by Artin-Rees lemma for $l=\mathcal{A}(\pi(J))$, $M^m\cap\pi(J) = M^{m-l}(M^l\cap\pi(J))$, $\forall\, m\ge l$. Hence applying π^{-1} to the above equality and intersecting each member with J we get

$$(*)\qquad J\cap m(n)^m = m(n)^{m-l}(J\cap m(n)^l) + J\cap m(n)^\infty \quad (\forall\, m\ge l).$$

We shall denote $\mathcal{A}(J)$ the minimum l satisfying this equality, therefore $\mathcal{A}(J)\le\mathcal{A}(\pi(J))$.

Since $m(n)^k\supseteq\ker\pi$, then $\pi(J\cap m(n)^k) = \pi(J)\cap\pi(m(n)^k) = \pi(J)\cap M^k$. If we apply the epimorphism π to the equality $(*)$, we get $M^m\cap\pi(J) = M^{m-l}(M^l\cap\pi(J))$, $\forall\, m\ge l$. Therefore $\mathcal{A}(\pi(J))\le\mathcal{A}(J)$ and hence $\mathcal{A}(J) = \mathcal{A}(\pi(J))$.

In case I is a radical ideal of $\mathcal{E}\,(n)$, we get in fact $I\cap m(n)^m = m(n)^{m-k}(I\cap m(n)^k)$ $\forall m\ge k$.

Theorem 3. *Consider I a finitely generated ideal of $\mathcal{E}(n)$. Then for any $k<\infty$, $I\cap m(n)^k$ is also finitely generated.*

Proof. Consider $g_1,\dots,g_s$ generators of I and let $f=\sum_{i=1}^s h_i g_i$. Then we have

$$f=\sum_{i=1}^s h_i^{(k)} g_i+\sum_{i=1}^s h_i^{[k]} g_i,$$
$$\text{where } h_i^{(k)}\text{is the } (k-1)-\text{jet of } h_i \text{ and } h_i^{[k]}=h_i-h_i^{(k)}.$$

Hence as vector spaces $I=V+m(n)^kI$, where V is the vector space generated by $\{x^I g_i\}$ with $|I|\le k-1$. Therefore $I\cap m(n)^k=V\cap m(n)^k+m(n)^kI$. It is clear that a basis of the subspace $V\cap m(n)^k$ of V and the generators of $m(n)^kI$ generate $I\cap m(n)^k$ as an ideal of $\mathcal{E}(n)$. ☐

Theorem 4. *Suppose I is a radical ideal of $\mathcal{E}(n)$, $I\cap m(n)^k$ a finitely generated ideal and $I\cap m(n)^k\subseteq Im(n)\left\langle\frac{\partial f}{\partial x_i}\right\rangle$ with $k\ge\mathcal{A}(I)$. Then f is k-determined relative to G_S, where $S=z(I)$.*

Theorem 5. *Let f be a k-determined germ relative to G_S, $S=z(I)$ and I a radical finitely generated ideal. Then $I\cap m(n)^{k+1}\subseteq I\left\langle\frac{\partial f}{\partial x_i}\right\rangle+m(n)^{k+2}\cap I$.*

Joining Theorems 4 and 5 we get for I a finitely generated ideal, the following:

Theorem 6. *Let f be a germ, I a finitely generated radical ideal, $S=z(I)$, and $k\ge\mathcal{A}(I)$. Then f is finitely determined relative to G_S if and only if there exists a number l greater or equal than k such that $m(n)^l\cap I\subseteq I\left\langle\frac{\partial f}{\partial x_i}\right\rangle$.*

The proofs of the above theorems can be found in [**K**], Theorems 11 and 15.

We can change Theorem 4 in the following way.

Theorem 7. *Let I be a radical ideal, $k=\mathcal{A}(I)$ and suppose that $I\cap m(n)^k$ is finitely generated. Let l be a natural number such that $m(n)^lI\subseteq m(n)I\left\langle\frac{\partial f}{\partial x_i}\right\rangle$. Then f is $(k+l-1)$ determined relative to G_S, where $S=z(I)$.*

Proof. Let g be a germ with $g\equiv f$ on S and $j^{k+l-1}g(0)=j^{k+l-1}f(0)$.

If we define the trivial homotopy $F(x,t)=(1-t)\,f(x)+tg(x)$ we get $\frac{\partial F}{\partial t}=g-f\in m(n)^{k+l}\cap I$ and $\frac{\partial f}{\partial x_i}=\frac{\partial F}{\partial x_i}+t\left(\frac{\partial f}{\partial x_i}-\frac{\partial g}{\partial x_i}\right)$.

Since $m(n)^{k+l} \cap I \subseteq m(n)^l I$ and $Im\,(n)^{p-1} \subseteq I \cap m(n)^p \;\forall\, p$, we get

$$\left(m(n)^{k+l} \cap I\right) \mathcal{E}\,(n+1) \subseteq m(n) I \left\langle \frac{\partial F}{\partial x_i} \right\rangle \mathcal{E}\,(n+1) + m(n) \left(m(n)^{k+l} \cap I\right) \mathcal{E}\,(n+1)\,.$$

By Nakayama's lemma we arrive to the inclusion:

$$(m(n)^{k+l} \cap I) \mathcal{E}\,(n+1) \subseteq m(n) I \left\langle \frac{\partial F}{\partial x_i} \right\rangle \mathcal{E}\,(n+1)\,.$$

Hence $\frac{\partial F}{\partial t} = \sum h_i\,(x,t) \frac{\partial F}{\partial x_i}$ with $h_i(x,t) \equiv 0$ for $x \in S$, t near t_0 (t_0 fixed). We now proceed in the usual way. □

Remark 1.

(a) If $I = m(n)$ then $k = 1$ and we get that $m(n)^{l+1} \subseteq m(n)^2 \langle \frac{\partial f}{\partial x_i} \rangle$ implies f is $l-$ determined on the right ([**M**]).

(b) If $I = \langle x_1, \dots, x_s \rangle$ then $k = 1$ and we get that $m(n)^l I \subseteq m(n) I\ \langle \frac{\partial f}{\partial x_i} \rangle$ implies f is $l-$ determined relative to G_S, $S = \{0\} \times \mathbb{R}^{n-s}$ ([**P-L**]).

Corollary 8. *Let f be a germ, I a radical ideal, $k = \mathcal{A}\ (I)$ and $I \cap m(n)^k$ be finitely generated. Suppose that $m(n)^l \subseteq m(n) \langle \frac{\partial f}{\partial x_i} \rangle$. Then f is $(k+l-1)$-determined relative to G_S. Hence finite determination on the right implies finite relative determination.*

Proof. Since $m(n)^l \subseteq m(n) \langle \frac{\partial f}{\partial x_i} \rangle$ then $m(n)^l I \subseteq m(n) I\ \langle \frac{\partial f}{\partial x_i} \rangle$. We now use Theorem 7. □

We are now interested in determining for which ideals I we have the converse of Corollary 8. For this purpose we need the following:

Theorem 9. *Let A be a commutative ring, I, J, K ideals of A with $I = \langle g_1, \dots, g_k \rangle$. Suppose that $ag_i = 0$ for all i and $a \in J^k + K$ implies $a = 0$. Then if $IJ \subset IK$ hence $J^k \subseteq K$.*

Proof. Let $m_1, \dots, m_k$ be arbitrary elements of J, then $g_i m_i = \sum_{j=i}^k g_j d_{ij}$ $\forall\ i$ with $1 \leq i \leq k$ $(d_{ij} \in K)$. In matricial notation we can write

$$(*) \qquad C \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{where } C = (\delta_{ij} m_i - d_{ij}).$$

If we multiply $(*)$ by the adjoint of C we get $(\det C) g_i = 0\ \forall\, i$, but $\det\ C = m_1 \cdots m_k + b$ with $b \in K$. Hence by hypothesis $\det\ C = 0$ and then $m_1 \cdots m_k \in K$, therefore $J^k \subseteq K$. □

Corollary 10. *Let* $A = \mathcal{E}(n)$, $J = m(n)^l$, (*or* $J = m(n)^\infty$), $K = \langle \frac{\partial f}{\partial x_i} \rangle$ *and* I *ideal with* $I = \langle g_1, \dots, g_k \rangle$. *Suppose that* $hg_i = 0$ *for all* i *and* $h \in m(n)^{lk} + \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ (*or* $h \in m(n)^\infty + \langle \frac{\partial f}{\partial x_i} \rangle$) *implies* $h \equiv 0$. *Then if* $I\, m(n)^l \subseteq I \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ (*or* $I\, m(n)^\infty \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$) *hence* $m(n)^{lk} \subseteq \langle \frac{\partial f}{\partial x_i} \rangle$ (*or* $m(n)^\infty \subseteq \langle \frac{\partial f}{\partial x_i} \rangle$) *and* f *is* $(lk+1)$*-determined on the right* (∞ *- determined on the right*).

This result motivates us to find examples where I is a finitely generated radical ideal satisfying the hypothesis of the above corollary.

Example 11. Let I be a radical ideal generated by a non trivial analytic germ g. If $hg = 0$ then $h \equiv 0$ and we will have finite relative determination implies determination on the right ($hg \equiv 0 \Longrightarrow h^{-1}(0) \cup g^{-1}(0) = \mathbb{R}^n$ but $g^{-1}(0)$ is a closed set with empty interior, therefore $h^{-1}(0) = \mathbb{R}^n$).

Example 12. Consider in $\mathcal{E}(3)$ the ideal I generated by $\{x_1x_2, x_1x_3, x_2x_3\}$. It is easy to see that I is radical and $\mathcal{A}(I) = 2$. Moreover if we denote $P_1 = x_1x_2$, $P_2 = x_1x_3$ and $P_3 = x_2x_3$,we get for $i \neq j$, $i \neq k$, $j \neq k$ that the closure of $z(P_i) \cap z(P_j) - z(P_k)$ is a plane and does not contain $z(I)$, which is the union of the three axes, hence it does not satisfy the hypothesis of Theorem 20 [**K**], but the conclusion is still true. We give a proof since it is important for the converse of Corollary 8.

Proposition 13. *With the above notation, if* f *is* m*-determined relative to* G_S, *where* $S = z(I)$ *are the coordinate axes, then* f *is* $(2m-2)$*-determined on the right.*

Proof. By Theorem 15 ([**K**]) we know that $m(3)^{m+1} \cap I \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$ which in this case is equivalent to $I m(3)^{m-1} \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$. We shall show that $m(3)^{2m-1} \subseteq \langle \frac{\partial f}{\partial x_i} \rangle m(3)^2$ and hence f is $(2m-2)$-determined on the right. Any mixed monomial of $m(3)^{2m-1}$ has a factor of I times a monomial of degree $2m-3$, hence for $m \geq 2$ it is contained in the Jacobian ideal. We now give the proof for x_1^{2m-1}, the other two are similar,

$$(*) \quad x_1^{m-1}(x_1x_2) = x_1x_2 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j} + x_1x_3 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{2j} + x_2x_3 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{3j}.$$

If we denote $\phi = x_1^{m-1} - \sum_{i=1}^{3} \frac{\partial f}{\partial x_j} h_{1j}$ we get that the zeroes of ϕ contain $\{x_3 = 0\}$ and the zeroes of $x_1\phi$ contain $\{x_3 = 0\} \cup \{x_1 = 0\}$, hence $x_1\phi \in \hat{I} = I$. From $(*)$ and the definition of ϕ, $x_1^m = x_1 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j} + x_1\phi$, therefore $x_1^{2m-1} \in m(3)^2 \langle \frac{\partial f}{\partial x_i} \rangle$ and f is $(2m-2)$-determined. $\square$

Remark 2. By Corollary 10, since $I = \hat{I} = \langle x_1x_2, x_1x_3, x_2x_3\rangle$ then $Im\,(3)^{m-1} \subseteq I\langle\frac{\partial f}{\partial x_i}\rangle$ implies $m(3)^{3(m-1)} \subseteq \langle\frac{\partial f}{\partial x_i}\rangle$ and f is $(3m-2)$-determined on the right.

Definition 14. Let $f_1, \dots, f_r$ be germs in $m(n)$. We say that they are linearly independent if their gradients denoted by $\nabla f_1, \dots, \nabla f_r$ are linearly independent at the origin.

Lemma 15. *Let $f_1, \dots, f_r$ be linearly independent germs. Then the ideal I generated by them is radical.*

Proof. Let H be the germ of the common zeroes of I and $P_{r+1}, \dots, P_n$ linear polynomials such that $\nabla f_1(\bar{0}), \dots, \nabla f_r(\bar{0}), \nabla P_{r+1}(\bar{0}), \dots, \nabla P_n(\bar{0})$ is a basis of $\mathbb{R}^n$. Thus $\phi = (f_1, \dots, f_r, P_{r+1}, \dots, P_n)$ is a germ of diffeomorphism. Let $f \ \in \widehat{I}$ hence $f \equiv 0$ on H if and only if $f \circ \phi^{-1} \equiv 0$ on $\{0\} \times \mathbb{R}^{n-r}$. By Hadamard's lemma we get $f \circ \phi^{-1}(x_1, \dots, x_n) = \sum_{i=1}^r x_i g_i$, therefore $f = \sum_{i=1}^r f_i(g_i \circ \phi)$, and hence f belongs to the ideal I. □

Lemma 16. *Let $I_1, \dots, I_r$ be radical ideals in $\mathcal{E}\,(n)$. Then their intersection is also a radical ideal.*

Proof. In general $\mathrm{rad}(\cap_{i=1}^r I_i) \subseteq\ \cap_{i=1}^r \mathrm{rad}\ I_i$, hence we get

$$\cap_{i=1}^r I_i \subseteq \mathrm{rad} \cap_{i=1}^r I_i \subseteq \cap_{i=1}^r \mathrm{rad}\, I_i = \cap_{i=1}^r I_i.$$

Therefore the equality $\cap_{i=1}^r I_i = \mathrm{rad} \cap_{i=1}^r I_i$. □

Lemmas 15 and 16 generate a special collection of algebraic sets. They are called bouquets of subspaces.

Example 17. Consider $I \subseteq \mathcal{E}\,(3)$ the ideal generated by x and yz, hence $z(I)$ is the union of the $y-$axis and $z-$axis, they are not in general position (in $\mathbb{R}^3$). By Lemma 16, I is clearly a radical ideal since $I = I_1 \cap I_2$ where $I_1 = \langle x, y\rangle$ and $I_2 = \langle x, z\rangle$.

Definition 18. Let I be a finitely generated ideal of $\mathcal{E}\,(n)$, $I = \langle g_1, \dots, g_k\rangle$. We say that I is integral if $S = z(I)$ is nowhere dense.

We now arrive at the main theorem of this section.

Theorem 19. *Let I be a finitely generated ideal of $\mathcal{E}\,(n)$ which is radical. Then if f is finitely determined relative to G_S, $S = z(I)$, hence f is finitely determined on the right.*

Proof. Suppose $I = \langle g_1, \dots, g_k\rangle$ and that $hg_i \equiv 0\ \forall\ i$. Therefore $z(h) \cup z(g_i) = \mathbb{R}^n\ \forall\ i$ and hence $z(h) \cup z(I) = \mathbb{R}^n$. Since I is an integral ideal, see [**R**], $z(h) = \mathbb{R}^n$ and hence $h \equiv 0$. On the other side there exists a natural number p such that $m(n)^p I \ \subset \langle\frac{\partial f}{\partial x_i}\rangle I$. By Corollary 10, we get $m(n)^{pk} \subset \langle\frac{\partial f}{\partial x_i}\rangle$ and therefore f is $(pk\ +1)$-determined on the right. □

Corollary 20. *Consider $I_1, \dots, I_r$ ideals each of them generated by linearly independent linear polynomials and S the union of their common zeroes (bouquet of subspaces). Then a germ f is finitely determined on the right if and only if f is finitely determined relative to G_S.*

We finish this section with an observation about homogeneous polynomials.

Proposition 21. *Let $h_1, \dots, h_k$ be homogeneous polynomials of degree $s_1, \dots, s_k$ respectively and let s be the maximum of these degrees. Hence if I is the ideal generated by $h_1, \dots, h_k$ we get $\mathcal{A}(I) \leq s$.*

Proof. We have to show that $(I \cap m(n)^s)m(n)^r = I \cap m(n)^{s+r}$ $\forall\, r \geq 0$. Let $f \in I \cap m(n)^{s+r}$, hence we have the following equalities.

$$(*) \qquad f = h_1 g_1 + \ldots + h_k g_k$$

$$(**) \qquad 0 = j^{s+r-1} f(0) = h_1 j^{r+s-1-s_1}\, g_1(0) + \cdots + h_k j^{r+s-1-s_k} g_k(0).$$

Sustracting $(**)$ from $(*)$ we get $f = h_1\ \widetilde{g}_1 + \cdots + h_k\ \widetilde{g}_k$, where $\widetilde{g}_i \in m(n)^{r+s-s_i}$.

Hence each $\widetilde{g}_i$ is a sum of elements of the form $\tilde{h}{}_i^j \overset{\approx}{h}{}_i^j$, with $\tilde{h}{}_i^j \in m(n)^r$ and $\overset{\approx}{h}{}_i^j$ is a homogeneous monomial of degree $s - s_i$.

Therefore f is a sum of elements of the form $(h_i \overset{\approx}{h}{}_i^j)\ \tilde{h}{}_i^j$,with $(h_i \overset{\approx}{h}{}_i^j) \in I \cap m(n)^s$, so $f \in (I \cap m(n)^s)m(n)^r$. We have shown that $I \cap m(n)^{s+r} \subseteq (I \cap m(n)^s)\ m(n)^r$ $\forall\, r \geq 0$. The other inclusion is obvious. $\square$

2. Infinite determinacy on germs of closed subsets of $\mathbb{R}^n$.

In this setction we will assume that S is a germ of a closed subset of $\mathbb{R}^n$ such that the origin is an accumulation point of S.

Definition 22. Let $S \subseteq \mathbb{R}^n$ be a germ of a closed set such that $\bar{0}$ is an accumulation point of S. We say that a germ f in $\mathcal{E}\,(n)$ is $S-$infinitely determined if given a germ g such that $j^\infty g(x) = j^\infty f(x)$ $\forall\, x \in S$ there exists a germ of a diffeomorphism ϕ such that $g = f \circ \phi$.

We denote by $\mathcal{E}\,(S,n)$ the ideal of $\mathcal{E}\,(n)$ consisting of the germs f such that $j^\infty f(x) = 0$ for all $x \in S$. If f is a germ in this ideal, we can write $f = gh$ where $\{g,h\} \subseteq m(n)^\infty$ and $h(x) > 0$ for $x \neq 0$. Then $j^\infty g(x) = 0$ $\forall$ $x \neq 0$, $x \in S$ and therefore $\mathcal{E}\,(S,n) \subseteq \mathcal{E}\,(S,n)\, m(n)^\infty$. We get in fact the equality.

Remark 3. If $f \in \mathcal{E}\,(S,n)$ then for all multi-index I, $\frac{\partial^{|I|} f}{\partial x^I} \in \mathcal{E}\,(S,n)$.

Definition 23. A germ f is S-infinitesimally stable if $\mathcal{E}(S,n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S,n)$.

Theorem 24. *Let S be a germ of a closed subset of $\mathbb{R}^n$ such that the origin is an accumulation point of S. If f is S-infinitesimally stable then f is S-infinitely determined.*

Proof. Let $g(x)$ be a germ such that $j^\infty g(x) = j^\infty f(x)\ \forall\ x \in S$. We define the homotopy $F(x,t) = tg(x) + (1-t)\ f(x)$. Consider the following $\mathcal{E}(n+1)-$ modules $N = \mathcal{E}(n+1)\langle \frac{\partial f}{\partial x_i} \rangle$ and $K = \mathcal{E}(n+1)\langle \frac{\partial F}{\partial x_i} \rangle$. If $h \in N$, we have $h(x,t) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)\ h_i(x,t) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}\ (x,t)\ h_i(x,t) +$ $t\sum_{i=1}^n \frac{\partial (f-g)}{\partial x_i}(x)\ h_i(x,t)$. Since $\frac{\partial (f-g)}{\partial x_i}(x) \in \mathcal{E}(S,n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S,n)$, we get $N \subseteq K + \mathcal{E}(S,n)N$, and by Nakayama's lemma, $N \subseteq K$ and hence $\mathcal{E}(S \times \mathbb{R}, n+1)\ \langle \frac{\partial f}{\partial x_i} \rangle\ \mathcal{E}(n+1) \subseteq \mathcal{E}(S \times \mathbb{R}, n+1)\langle \frac{\partial F}{\partial x_i} \rangle \mathcal{E}(n+1)$.

Since $g - f \in \mathcal{E}(S \times \mathbb{R},\ n+1)\ \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(n+1)$, hence $\frac{\partial F}{\partial t} \in \mathcal{E}(S \times \mathbb{R}, n+1)\langle \frac{\partial F}{\partial x_i} \rangle \mathcal{E}(n+1)$. We now proceed in the usual way. □

Proposition 25. *If f is a germ, finitely (infinitely) determined on the right, it is S-infinitesimally stable and therefore S-infinitely determined.*

Proof. By our hypothesis we have

$$m(n)^\infty \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \text{ and } \mathcal{E}(S,n) \subseteq \mathcal{E}(S,n)m(n)^\infty \subseteq \mathcal{E}(S,n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle.$$

□

Example 26.

(a) Let f be a germ and k a natural number, denote by I_k the ideal generated by f^k. Suppose $\mathcal{E}(S,n) \subseteq I_k$, $f^k \in \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ and $j^\infty f(x) \neq 0\ \forall\ x \in T$, where the closure of T is S. If $h \in \mathcal{E}(S,n)$, $h = f^k g$ where $g \in \mathcal{E}(S,n)$. Therefore $\mathcal{E}(S,n) \subseteq \mathcal{E}(S,n)\left\langle f^k \right\rangle \subseteq \mathcal{E}(S,n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle$, and thus f is S-infinitely determined.

(b) In particular let $S = \{(x_1, \ldots, x_n) | x_1 \leq 0\}$. Then for the germs $f_1(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2$, and $f_2(x_1, \ldots, x_n) = x_1$, we get $j^\infty f_1(x) \neq 0$ and $j^\infty f_2(x) \neq 0\ \ \forall\ x \in \text{int}\ S$. Since $S = cl(\text{int}\ S)$ and $\mathcal{E}(S,n) \subseteq \left\langle f_i^k \right\rangle$ for $i = 1,2$ (Proposition 5.4 of Chapter V, [**T**]), and clearly $f_i^k \in \left\langle \frac{\partial f_i}{\partial x_j} \right\rangle$ for $i = 1,2$, we get that f_1^k, f_2^k are S-infinitely determined. ([**P-L**]).

Definition 27.

(a) Let S be a closed subset of $\mathbb{R}^n$ (containing the origin) and f a germ with $f\left(\bar{0}\right) = 0$. We say that f satisfies a Lojasiewicz inequality for S if for any K, a germ of a compact set with $\bar{0} \in K$, there exist constants $c > 0$ and $\alpha \geq 0$ such that $|f(x)| \geq cd(x,S)^\alpha$ for all $x \in K$.

(b) Let I be a finitely generated ideal of $\mathcal{E}(n)$ and S the germ of its common zeroes. We say that I is a Lojasiewicz ideal if there exists f in I satisfying a Lojasiewicz inequality for S.

(c) Let $f \in m(n)$ and S a closed subset of $\mathbb{R}^n$, we say that f satisfies a Jacobi-Lojasiewicz condition for S if $|\nabla f|$ satisfies a Lojasiewicz inequality for S.

Remark 4. If $\{f_1, \ldots, f_s\}$ is a set of generators of a Lojasiewicz ideal I, then $\sum_{i=1}^{s} f_i^2$, $\sum_{i=1}^{s} |f_i|$ and max $\left\{f_1^2, \ldots, f_s^2\right\}$ also satisfy a Lojasiewicz inequality for S.

Definition 28. Let (b_i) be a sequence of positive real numbers converging to zero. We say that a sequence of real numbers (a_i) is flat along (b_i) if given $r > 0$ there exists a natural number $N = N(r)$ such that $|a_i| \leq b_i^r$ for $i \geq N$. Sequences of vectors, matrices, jets are flat along a sequence (b_i) if each entry is flat along (b_i). A sequence is flat along a sequence (x_i) of nonzero vectors in $\mathbb{R}^n$ converging to the vector $\bar{0}$ if it is flat along the sequence $(|x_i|)$. In the case of $\infty-$jets, we ask for a uniform $N = N(r)$ for all entries. Here we are identifying $\sum_{\alpha} a_{\alpha} \frac{(x-x_0)^{\alpha}}{\alpha!}$ with (a_{α}).

Remark 5. We can change $r > 0$ for $r = n$, n a natural number since for $n > r$, we get $b_i^n \leq b_i^r (0 \leq b_i \leq 1)$.

We state an interesting equivalence.

Lemma 29. *A germ g does not satisfy a Lojasiewicz inequality for a closed subset S if and only if there exists a sequence of vectors $x_i \in \mathbb{R}^n - S$ converging to the vector $\bar{0}$ such that $(g(x_i))$ is flat along $(d(x_i, S))$.*

Remark 6. For a germ g not identically zero we can choose $g(x_i) \neq 0 \; \forall \, i$.

Definition 30. Let S be a closed subset of $\mathbb{R}^n$. Then $M(S, n)$ is the set of maps $\phi : \mathbb{R}^n - S \longrightarrow \mathbb{R}$ such that if K is a germ of a compact set and I is a multi-index of natural numbers, there exist constants $c > 0$ and $\alpha > 0$ such that $\left| \frac{\partial^{|I|} \phi}{\partial x^I}(x) \right| \leq c d(x, S)^{-\alpha}$ for all $x \in K - S$.

We state the following proposition (Chapter IV, Proposition 4.2 of [T]).

Proposition 31. *Let $\phi \in M(S, n)$ and $f \in \mathcal{E}(S, n)$. Then we can extend ϕf in a unique way to a germ in $\mathcal{E}(S, n)$, denoted also by ϕf.*

Theorem 32. *Let f be a germ, S a germ of a closed subset of $\mathbb{R}^n$ such that $\bar{0}$ is an accumulation point of S. Suppose that f satisfies a Jacobi-Lojasiewicz condition for S. Then f is S-infinitesimally stable and therefore S-infinitely determined.*

Proof. Consider $g = |\nabla f|^2$, we shall show that $\mathcal{E}(S, n) \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \mathcal{E}(S, n)$. Let K be a germ of a compact subset and g_1 be a representative of g; for each I multi-index there exists C_I constant such that $\left|\frac{\partial^{|I|}(\frac{1}{g_1})}{\partial x^I}\right| \leq \frac{C_I}{|g_1(x)|^{(|I|+1)}}$ $\forall x \in K$. Since g_1 satisfies a Lojasiewicz inequality for S, there exist $c > 0$ and $\alpha \geq 0$ such that $|g_1(x)| \geq cd(x,S)^\alpha \; \forall x \in K - S$ and therefore $\left|\frac{\partial^{|I|}(\frac{1}{g_1})}{\partial x^I}\right| \leq \frac{C_I}{c^{|I|+1}\; d(x,S)^{\alpha(|I|+1)}} \forall x \in K - S$, hence $\frac{1}{g_1} \in M(S,n)$. Now for $h \in \mathcal{E}(S,n)$ and $x \notin S$ we have $h(x) = \frac{h(x)}{g_1(x)} g_1(x)$, extend $\frac{h(x)}{g_1(x)}$ to a germ H in $\mathcal{E}(S,n)$ and $h = Hg_1$ in $\mathcal{E}(S,n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle$. Therefore we get $\mathcal{E}(S,n) \subseteq \mathcal{E}(S,n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle$ and f is S-infinitesimally stable. □

Lemma 33 ([**W**, Lemma 3.3])**.** *Suppose there exist a sequence (w_i) in $J^k(n,1)$, $k \leq \infty$, a sequence (x_i) in $\mathbb{R}^n - \{\bar{0}\}$ converging to the origin and a germ f such that $q_i = w_i - j^k f(x_i)$ is flat along (x_i). Then there exists a germ g such that $j^k g(x_{i_j}) = w_{i_j}$ holds for (x_{i_j}) subsequence of (x_i), and $j^\infty g(\overline{0}) = j^\infty f(\overline{0})$.*

Lemma 34. *Suppose there exist a sequence (w_i) in $J^k(n,1)$, $k \leq \infty$, a sequence (x_i) in $\mathbb{R}^n - S$ converging to zero and a germ $f \in \mathcal{E}(n)$ such that $(q_i) = (w_i - j^k f(x_i))$ is flat along $(d(x_i,S))$, where S is a closed subset of $\mathbb{R}^n (\overline{0} \in S)$. Then there exists a germ $g \in \mathcal{E}(n)$, such that $j^\infty g(x) = j^\infty f(x) \, \forall x \in S$ and $j^k g(x_i) = w_i$ holds for a subsequence of (x_i).*

Proof. If k is finite, then we transform q_i into an ∞-jet in such a way that all the terms of order greater than k of q_i are zero. Thus we will assume $k = \infty$.

We define Q, a Taylor field, by q_i at x_i and by the zero series on S. We want to show that Q is a C^∞ Whitney field. It is enough to show (Proposition 1.5 of Chapter IV, [**T**]) for each m and each multi-index I with $|I| \leq m$, that $(R_y^m Q)^I \; (x) = o(|x-y|^{m-|I|})$, where $(R_y^m Q)^I(x) = Q^I(x) - \sum_{|L| \leq m-|I|} Q^{I+L}(y) \frac{(x-y)^L}{L!}$.

If $\{x,y\} \subseteq S$ then the proof is obvious. In the case $\{x,y\} \subseteq \{x_i\} \cup \{\overline{0}\}$ we proceed as in the proof of Lemma 3.3 of [**W**]. If $\{x,y\} = \{x_j, s\}$, $s \in S$, we use the flatness of (q_i) along $(d(x_i,S))$ to obtain for each natural number l another $N(l)$ such that $|(R_s^m)^I \; (x_j)| = |q_j^I| \leq d(x_j,S)^l \leq d(x_j,s)^l$ and $|(R_{x_j}^m)^I \; (s)| \leq \left|\sum_{|L| \leq m-|I|} q_j^{(I+L)} \frac{(s-x_j)^L}{L!}\right| \leq Cd(x_j,s)^l$ for $j \geq N(l)$, where C is a positive real number depending only on m and I. Let $l = m+1$.

Hence, using Whitney Extension Theorem (Theorem 3.1 of Chapter IV, [**T**]), there exists a smooth germ q such that $j^\infty q(x) = 0 \, \forall x \in S$ and $j^\infty q(x_i) = q_i$. If $g = f + q$, we see that g has the desired properties.

Theorem 35. *Let f be a germ, S a closed subset of $\mathbb{R}^n$ and $\bar{0}$ an accumulation point of S. Hence if f is S-infinitely determined, then f satisfies a Jacobi-Lojasiewicz condition for S.*

Proof. We shall prove the theorem by contradiction. Then there is a sequence (x_j) in $\mathbb{R}^n - S$ converging to the origin such that $(|\nabla f(x_j)|)$ is flat along $(d(x_j, S))$. Choose (y_j) a sequence of regular values of f converging to zero and such that $(f(x_j) - y_j)$ is flat along $(d(x_j, S))$. It clearly follows that $(y_j, 0) - (f(x_j), \nabla f(x_j))$ is flat along $(d(x_j, S))$.

If we denote $q_j = (y_j, 0) - (f(x_j), \nabla f(x_j))$ and setting $k = 1$ in the previous lemma, there exists a germ g such that $j^1 g(x_{_j}) = (y_j, 0)$ and $g - f \in \mathcal{E}(S, n)$. Now since f is S-infinitely determined, f and g must have the same critical and regular values, which is not the case, since the points y_j are regular values for f but critical values for g. □

As a consequence of Theorems 24, 32 and 35 we get the main theorem of part II:

Theorem 36. *Let $f \in \mathcal{E}(n)$. The concepts of S-infinitesimally stability, S-infinite determinacy and the Jacobi-Lojasiewicz condition at S are equivalent for the germ f and S a germ of a closed subset of $\mathbb{R}^n$ with $\bar{0}$ an accumulation point of S.*

3. A special case.

Definition 37. Let S be a germ of a closed subset of $\mathbb{R}^n$ such that $\bar{0} \in cl(\mathrm{int}\,)$. We say that a germ f is S-stable, if given a germ g such that $g(x) = f(x)\ \forall\ x \in S$, there exists a germ of a diffeomorphism $\phi \in G_S$ such that $g = f \circ \phi$.

Note that if $cl(\mathrm{int}) = S$, the previous definition is apparently much stronger than Definition 23. In this case $f(x) = g(x)\ \forall\ x \in S$ and $j^\infty g(x) = j^\infty f(x)\ \forall\ x \in S$ are equivalent but now we restrict ourselves to the group G_S, hence the diffeomorphism must be the identity on S.

Example 38.

(a) Let $S = \{(x, y) \in \mathbb{R}^2 | x \leq 0 \text{ and } y = 0\}$, then S is closed but $\bar{0} \notin cl(\mathrm{int}\, S)$.
(b) Let $S = \{(x, y) \in \mathbb{R}^2 | x^4 - x^3 - xy^2 \geq 0\}$, in this case $S = cl(\mathrm{int}\, S)$.
(c) Let $S = \{(x, y, z) \in \mathbb{R}^3 | z(x^2 + y^2) - x^3 \leq 0\}$, in this case $\bar{0} \in cl(\mathrm{int} S)$ but clearly $cl(\mathrm{int}\, S) \neq S$.

For S any germ of subset of $\mathbb{R}^n$containing the origin, we let $C_S(\mathbb{R}^n)$ be the $\mathbb{R}$-algebra of germs constant at S. It is a local algebra with maximal ideal $m(S)$ consisting of germs of $C_S(\mathbb{R}^n)$ vanishing at S. In fact $m(S)$ is an ideal of $\mathcal{E}(n)$.

Remark 7. If $f \in m(S)$ and $S = cl(\mathrm{int}\, S)$ we have $f \in m(n)^\infty$ and $\frac{\partial^{|I|} f}{\partial x^I} \in m(S)$ for all multi-index I. We also get in this case the equality $m(S) = m(n)^\infty m(S)$.

Lemma 39. *Let S a subset of $\mathbb{R}^n$. Suppose S_0 is a nonempty open subset of S. Then $cl(S_0) = cl(\operatorname{int} S)$ if and only if $\operatorname{int}(S - S_0) \subseteq cl(S_0)$.*

Proof. We decompose $\operatorname{int} S$ in the following way: $\operatorname{int} S = S_0 \cup \operatorname{int}(S - S_0) \cup T$, where $\operatorname{int} T = \emptyset$. Then $cl(\operatorname{int} S) = cl(S_0) \cup cl(\operatorname{int}(S - S_0))$, since $cl(T) \subseteq cl(S_0 \cup \operatorname{int}(S - S_0))$. Hence $cl(\operatorname{int} S) = cl(S_0)$ if and only if $cl(\operatorname{int}(S - S_0)) \subseteq cl(S_0)$ and this is equivalent to $\operatorname{int}(S - S_0) \subseteq cl(S_0)$. $\square$

Definition 40. Let A be a closed subset of $\mathbb{R}^n$. We say that A is good if there exists a locally finite partition $\mathcal{P}$ of A into C^0-submanifolds of $\mathbb{R}^n$, called strata, such that if $\mathcal{X} \in \mathcal{P}$ and $\dim \mathcal{X} < n$, then there exists a non void open stratum $\mathcal{Y} \in \mathcal{P}$ such that $\mathcal{X} \subset cl(\mathcal{Y})$.

We clearly have the next:

Proposition 41. *Suppose that S is a good subset of $\mathbb{R}^n$. Then $cl(S) = cl(\operatorname{int} S)$.*

Joining Lemma 39 and Proposition 41 we get the following:

Proposition 42. *Let $P_1, \dots, P_s$ be real continous functions on $\mathbb{R}^n$ such that $S = \{x | P_i(x) \leq 0 \ \forall\, i\}$ is good and define $S_0 = \{x | P_i(x) < 0 \ \forall\, i\}$. Suppose $\operatorname{int}(S - S_0) \subseteq cl(S_0)$. Then $cl(S_0) = S$.*

Remark 8. If $P_1, \dots, P_s$ are real analytic functions on $\mathbb{R}^n$ then $S = \{x | P_i(x) \leq 0 \ \forall\, i\}$ will be good if we have for a decomposition of S, that whenever T is a stratum of lower dimension, then there exists a nonempty open stratum T' such that $T \subset cl(T')$. Obviously there are more examples of good sets than the semianalytical ones. For this purpose see for instance Sections 1 and 2 of [**V-M**].

We remind here the following:

Definition 43. Suppose that S is a closed subset of $\mathbb{R}^n$ containing the origin and such that $S = cl\,(\operatorname{int} S)$. We say that f is $S-$infinitesimally stable if $m(S) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle m(S)$.

Theorem 44. *Suppose S is a closed subset of $\mathbb{R}^n$ such that $\overline{0} \in S$ and $S = cl(\operatorname{int} S)$. If f is $S-$infinitesimally stable then f is S-stable.*

Proof. Following the proof of Theorem 24 we start with g a germ such that $g(x) = f(x) \ \forall\ x \in S$, therefore $\frac{\partial^{|I|} f}{\partial x^I}(x) = \frac{\partial^{|I|} g}{\partial x^I}(x) \ \forall\ x \in S$, and we arrive to the inclusion $m(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x_i} \rangle C_{S\times\mathbb{R}}(\mathbb{R}^n \times \mathbb{R}) \subseteq m(S \times \mathbb{R}) \langle \frac{\partial F}{\partial x_i} \rangle C_{S\times\mathbb{R}}(\mathbb{R}^n \times \mathbb{R})$.

Since $\frac{\partial F}{\partial t} = g - f \in m(S \times \mathbb{R}) \subseteq m(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x_i} \rangle C_{S\times\mathbb{R}}(\mathbb{R}^n \times \mathbb{R})$, then $\frac{\partial F}{\partial t}(x,t) = \sum_{i=1}^n h_i(x,t) \frac{\partial F}{\partial x_i}(x,t)$, with $h_i(x,t) \in m(S \times \mathbb{R})$, hence $h_i(x,t) = 0$ $\forall\ (x,t) \in S \times \mathbb{R}$. When we integrate, the required diffeomorphism will belong to G_S. $\square$

Proposition 45. *If $f \in \mathcal{E}(n)$ is a finitely (infinitely) determined on the right, then f is S-infinitesimally stable and therefore S-stable for $S = cl(\operatorname{int} S)$.*

Proof. Since $m(S) = m(n)^{\infty} m(S)$ and $m(n)^k \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ for some $k \leq \infty$, we get that $m(S) \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle m(S)$. We now use Theorem 44. □

Definition 46. Let P be a polynomial in variables $x_1, \dots, x_n$. We say that P is quasihomogeneous of degree l and weights $k_1, \dots, k_n$ if $P(t^{k_1}x_1, \dots, t^{k_n}x_n) = t^l P(x_1, \dots, x_n)$.

For P quasihomogeneous we get $\frac{\partial P}{\partial x_j}(t^{k_1}x_1, \dots, t^{k_n}x_n) = t^{l-kj} \frac{\partial P}{\partial x_j}(x_1, \dots, x_n)$.

Also if we write $P = \sum a_I x^I$, for a quasihomogeneous polynomial we obtain for any multi-index $I = (i_1, \dots, i_n)$, $i_1 k_1 + \dots + i_n k_n = l$ $(a_I \neq 0)$.

Theorem 47. *Let $P(x)$ be a quasihomogeneous polynomial and S a closed subset of $\mathbb{R}^n$ containing the origin and such that $S = cl(\operatorname{int} S)$. Suppose that $m(S) \subseteq \langle P \rangle$ and that $z(P) \cap \operatorname{int} S = \phi$. Then P is S-infinitesimally stable. In the case $S = \{x | P(x) \leq 0\}$ is a good semialgebraic set, we can skip the equality $z(P) \cap \operatorname{int} S = \phi$.*

Proof. By hypothesis we get $m(S) \subseteq \langle P \rangle$ and $P \in \left\langle \frac{\partial P}{\partial x_i} \right\rangle$, this together with $z(P) \cap \operatorname{int} S = \phi$ give the result using Example 26. For the second part it is obvious that $z(P) \cap \operatorname{int} S = \phi$ since S is a good semialgebraic set. □

As in the previous section, we get the following:

Theorem 48. *Let $f \in \mathcal{E}(n)$, S be a closed subset of $\mathbb{R}^n$ such that the origin is an accumulation point of S and $S = cl(\operatorname{int} S)$. Then the concepts for f of S-infinitesimally stability, S-stability and the Jacobi-Lojasiewicz condition for S are equivalent.*

Proof. Our Theorem 44 shows that S-infinitesimally stability implies S-stability. Now as in Theorem 32, we show that the Jacobi-Lojasiewicz condition at S implies S-infinitesimally stability. Since Lemma 34 is true for any closed subset of $\mathbb{R}^n$, the proof of Theorem 35 will be true in the case $S = cl(\operatorname{int} S)$, and hence S-stability implies the Jacobi-Lojasiewicz condition of f for S. □

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