Pacific Journal of Mathematics

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Volume 192 No. 2 February 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 192, No. 2, 2000

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We introduce braided Lie bialgebras as the infinitesimal version of braided groups. They are Lie algebras and Lie coalgebras with the coboundary of the Lie cobracket an infinitesimal braiding. We provide theorems of transmutation, Lie biproduct, bosonisation and double-bosonisation relating braided Lie bialgebras to usual Lie bialgebras. Among the results, the kernel of any split projection of Lie bialgebras is a braided-Lie bialgebra. The Kirillov-Kostant Lie cobracket provides a natural braided-Lie bialgebra on any complex simple Lie algebra, as the transmutation of the Drinfeld-Sklyanin Lie cobracket. Other nontrivial braided-Lie bialgebras are associated to the inductive construction of simple Lie bialgebras along the C and exceptional series.

1. Introduction.

Braided geometry has been developed in recent years as a natural generalisation of super-geometry with the role of $\mathbb{Z}/2\mathbb{Z}$ grading played by braid statistics. It is also the kind of noncommutative geometry appropriate to quantum group symmetry because the modules over a strict quantum group (a quasitriangular Hopf algebra [**3**]) form a braided category, hence any object covariant under the quantum group is naturally braided. In particular, one has braided groups [**6**] as generalisations of super-groups or super-Hopf algebras. The famous quantum-braided plane with relations $yx = qxy$ is a braided group with additive coproduct [**7**]. We refer to [**8**], [**10**] for introductions to the 50-60 papers in which the theory of braided groups is developed.

In a different direction, Drinfeld [**2**] has introduced Lie bialgebras as an infinitesimalisation of the theory quantum groups. This concept has led (on exponentiation) to an extensive theory of Poisson-Lie groups, as well as to a Yang-Baxter theoretic approach to classical results of Lie theory, such as a new proof of the Iwasawa decomposition and the structure of Bruhat cells; see for example [**9**], [**5**]. For an introduction to quantum groups and Lie bialgebras, see [**10**].

We now combine these ideas for the first time by introducing the infinitesimal theory of braided groups. All computations and results will be in the setting of Lie algebras, although motivated from the theory of braided groups. In fact, there are several different concepts of precisely what one may mean by the infinitesimal theory of braided groups. Firstly, one may keep the braided category in which one works fixed and look at algebras which depart infinitesimally from being commutative. In the category of vector spaces this leads to Drinfeld's notion of Poisson-Lie group. Then one can consider the coalgebra also in an infinitesimal form, which leads in the category of vector spaces to Drinfeld's notion of Lie bialgebra. In the case of a braided category one already has the notion of braided-Lie algebra [**11**] and, adding to this, one could similarly consider a Lie bialgebra in a braided category. By contrast, we now go further and let the braiding also depart infinitesimally from the usual vector space transposition. In principle, the degree of braiding is independent of the degree of algebra commutativity or coalgebra cocommutativity. Thus one could have infinitesimally braided algebras, coalgebras and Hopf algebras as well. However, the case which appears to be of most interest, on which we concentrate, is the case in which all three aspects are made infinitesimal simultaneously, which we call a braided-Lie bialgebra. The formal definition appears in Section 2. It consists of a Lie algebra $\mathfrak{b}$ equipped with further structure.

In Section 3 we provide the Lie version of the basic theorems from the theory of braided groups. These basic theorems connect braided groups and quantum groups by transmutation [**12**], [**6**] and bosonisation [**13**], [**14**] procedures, thereby establishing (for example) the existence of braided groups associated to all simple Lie algebras. The theorems in Section 3 likewise connect braided-Lie bialgebras with quasitriangular Lie bialgebras and establish the existence of the former. The Lie versions of biproducts [**20**] and of the more recent double-bosonisation theorem [**15**] are covered as well. For example, the Lie version of the theory of biproducts states that the kernel of any split Lie bialgebra projection $\mathfrak{g} \to \mathfrak{f}$ is a braided-Lie bialgebra $\mathfrak{b}$, and $\mathfrak{g} = \mathfrak{b} \rtimes\!\!\!\triangleleft \mathfrak{f}$.

In Section 4 we study some concrete examples of braided-Lie bialgebras, including ones not obtained by transmutation. The simplest are ones with zero braided-Lie cobracket as the infinitesimal versions of the q-affine plane braided groups in [**7**]. As an application of braided-Lie bialgebras, their bosonisations provide maximal parabolic or inhomogeneous Lie bialgebras. Meanwhile, double-bosonisation allows the formulation in a basis-free way of the notion of adjoining a node to a Dynkin diagram. For every simple Lie bialgebra $\mathfrak{g}$ and braided-Lie bialgebra $\mathfrak{b}$ in its category of modules we obtain a new simple Lie bialgebra $\mathfrak{b} \rtimes\!\!\!\triangleleft \mathfrak{g} \triangleright\!\!\!\ltimes \mathfrak{b}^{*\mathrm{op}}$ as its double-bosonisation. This provides the inductive construction of all complex simple Lie algebras, complete with their Drinfeld-Sklyanin quasitriangular Lie bialgebra structure (which is built up inductively at the same time). Some concrete examples are given in detail.

These results have been briefly announced in [16, Sec. 3], of which the present paper is the extended text. We work over a general ground field k of characteristic not 2.

Acknowledgements. The results were obtained during a visit in June 1996 to the Mathematics Dept., Basel, Suisse. I thank the chairman for access to the facilities.

2. Braided-Lie bialgebras.

We will be concerned throughout with the Lie version of braided categories obtained as module categories over quantum groups. In principle one could also formulate an abstract notion of 'infinitesimal braiding' as a Lie version of a general braided category, but since no examples other than the ones related to quantum groups are known we limit ourselves essentially to this concrete setting. Some slight extensions (such as to Lie crossed modules) will be considered as well, later on.

As a Lie version of a strict quantum group we use Drinfeld's notion of a quasitriangular Lie bialgebra [2], [3]. We recall that a Lie bialgebra is a Lie algebra $\mathfrak{g}$ equipped with linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ forming a Lie coalgebra (in the finite dimensional case this is equivalent to a Lie bracket on $\mathfrak{g}^*$) and being a 1-cocycle with values in $\mathfrak{g} \otimes \mathfrak{g}$ as a $\mathfrak{g}$-module by the natural extension of ad. It is quasitriangular if there exists $r \in \mathfrak{g} \otimes \mathfrak{g}$ obeying $\mathrm{d}r = \delta$ in the Lie algebra complex, obeying the Classical Yang-Baxter Equation (CYBE)

$$[r^{(1)}, r'^{(1)}] \otimes r^{(2)} \otimes r'^{(2)} + r^{(1)} \otimes [r^{(2)}, r'^{(1)}] \otimes r'^{(2)} + r^{(1)} \otimes r'^{(1)} \otimes [r^{(2)}, r'^{(2)}] = 0 \tag{1}$$

and having ad-invariant symmetric part $2r_+ = r + \tau(r)$, where τ is transposition. We use the conventions and notation similar to [10, Ch. 8], with $r = r^{(1)} \otimes r^{(2)}$ denoting an element of $g \otimes g$ (summation understood) and r' denoting another distinct copy of r. We also use $\delta\xi = \xi_{(1)} \otimes \xi_{(2)}$ to denote the output in $\mathfrak{g} \otimes \mathfrak{g}$ for $\xi \in \mathfrak{g}$ (summation understood). A quasitriangular Lie bialgebra is called factorisable if $2r_+$ is surjective when viewed as a map $\mathfrak{g}^* \to \mathfrak{g}$.

In view of the discussion above, we are interested in Lie-algebraic objects living in the category ${}_{\mathfrak{g}}\mathcal{M}$ of modules over a quasitriangular Lie bialgebra $\mathfrak{g}$. If V is a $\mathfrak{g}$-module, we define its *infinitesimal braiding* to be the operator

$$\psi : V \otimes V \to V \otimes V, \quad \psi(v \otimes w) = 2r_+ \triangleright (v \otimes w - w \otimes v)$$

where $\triangleright$ denotes the left action of $\mathfrak{g}$.

Lemma 2.1. *Let $\mathfrak{b} \in {}_{\mathfrak{g}}\mathcal{M}$ be a $\mathfrak{g}$-covariant Lie algebra. Then the associated $\psi : \mathfrak{b} \otimes \mathfrak{b} \to \mathfrak{b} \otimes \mathfrak{b}$ is a 2-cocycle $\psi \in Z^2_{\mathrm{ad}}(\mathfrak{b}, \mathfrak{b} \otimes \mathfrak{b})$.*

Proof. The proof that $\mathrm{d}\psi = 0$ is a straightforward computation in Lie algebra cohomology. We use covariance of $\mathfrak{b}$ in the form: $\xi\triangleright[x,y] = [\xi\triangleright x, y] + [x, \xi\triangleright y]$ for all $\xi \in \mathfrak{g}$. Then,

$$\begin{aligned}
&(\mathrm{d}\psi)(x,y,z)\\
&= -\psi([x,y],z) + \psi([x,z],y) - \psi([y,z],x) + \mathrm{ad}_x\psi(y,z)\\
&\quad - \mathrm{ad}_y\psi(x,z) + \mathrm{ad}_z\psi(x,y)\\
&= 2r_+\triangleright(-[x,y]\otimes z + z\otimes[x,y]) + [x, 2{r_+}^{(1)}\triangleright y]\otimes {r_+}^{(2)}\triangleright z\\
&\quad + 2{r_+}^{(1)}\triangleright y\otimes[x, {r_+}^{(2)}\triangleright z]\\
&\quad - [x, 2{r_+}^{(1)}\triangleright z]\otimes {r_+}^{(2)}\triangleright y - 2{r_+}^{(1)}\triangleright z\otimes[x, {r_+}^{(2)}\triangleright y] + \text{cyclic}\\
&= -[2{r_+}^{(1)}\triangleright x, y]\otimes {r_+}^{(2)}\triangleright z + 2{r_+}^{(1)}\triangleright y\otimes[x, {r_+}^{(2)}\triangleright z] - [x, 2{r_+}^{(1)}\triangleright z]\otimes {r_+}^{(2)}\triangleright y\\
&\quad + 2{r_+}^{(1)}\triangleright z\otimes[{r_+}^{(2)}\triangleright x, y] + \text{cyclic} = 0
\end{aligned}$$

on using the cyclic invariance in x, y, z and antisymmetry of the Lie bracket. Note that this works for any element $2r_+ \in \mathfrak{g}\otimes\mathfrak{g}$ in the definition of ψ. $\square$

Definition 2.2. A *braided-Lie bialgebra* $\mathfrak{b} \in {}_{\mathfrak{g}}\mathcal{M}$ is a $\mathfrak{g}$-covariant Lie algebra and $\mathfrak{g}$-covariant Lie coalgebra with cobracket $\underline{\delta} : \mathfrak{b} \to \mathfrak{b}\otimes\mathfrak{b}$ obeying $\forall x, y \in \mathfrak{b}$,

$$\underline{\delta}([x,y]) = \mathrm{ad}_x\underline{\delta}y - \mathrm{ad}_y\underline{\delta}x - \psi(x\otimes y); \quad \psi = 2r_+\triangleright(\mathrm{id} - \tau),$$

i.e., $\underline{\delta}$ obeys the coJacobi identity and $\mathrm{d}\underline{\delta} = \psi$.

The definition is motivated from that of a braided group, where the coproduct fails to be multiplicative up to a braiding Ψ [**6**]. The results in the next section serve to justify it further.

3. Lie versions of braided group theorems.

The existence of nontrivial quasitriangular Lie bialgebra structures is known [**3**] for all simple $\mathfrak{g}$ at least over $\mathbb{C}$. Our first theorem ensures likewise the existence of braided-Lie bialgebras.

Theorem 3.1. *Let $i : \mathfrak{g} \to \mathfrak{f}$ be a map of Lie bialgebras, with $\mathfrak{g}$ quasitriangular. There is a braided-Lie bialgebra $\mathfrak{b}(\mathfrak{g}, \mathfrak{f})$, the* transmutation *of $\mathfrak{f}$, living in ${}_{\mathfrak{g}}\mathcal{M}$. It has the Lie algebra of $\mathfrak{f}$ and for all $x \in \mathfrak{f}$, $\xi \in \mathfrak{g}$,*

$$\underline{\delta}x = \delta x + r^{(1)}\triangleright x\otimes i(r^{(2)}) - i(r^{(2)})\otimes r^{(1)}\triangleright x, \quad \xi\triangleright x = [i(\xi), x].$$

Proof. We first verify that $\underline{\delta}$ as stated is indeed a $\mathfrak{g}$-module map. Thus

$$\begin{aligned}
&\underline{\delta}(\xi\triangleright x)\\
&= \delta[i(\xi), x] + r^{(1)}\xi\triangleright x\otimes i(r^{(2)}) - i(r^{(2)})\otimes r^{(1)}\xi\triangleright x\\
&= \xi\triangleright\delta x - [x, i(\xi_{(1)})]\otimes i(\xi_{(2)}) - i(\xi_{(1)})\otimes[x, i(\xi_{(2)})]\\
&\quad + r^{(1)}\xi\triangleright x\otimes i(r^{(2)}) - i(r^{(2)})\otimes r^{(1)}\xi\triangleright x\\
&= \xi\triangleright\delta x - [x, [i(\xi), i(r^{(1)})]]\otimes i(r^{(2)}) - [x, i(r^{(1)})]\otimes[i(\xi), i(r^{(2)})]
\end{aligned}$$

$$
\begin{aligned}
&\quad - [i(\xi), i(r^{(1)})] \otimes [x, i(r^{(2)})] - i(r^{(1)}) \otimes [x, [i(\xi), i(r^{(2)})]] \\
&\quad + r^{(1)}\xi{\triangleright}x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)}\xi{\triangleright}x \\
&= \xi{\triangleright}\delta x + [\xi, r^{(1)}]{\triangleright}x \otimes i(r^{(2)}) - [x, i(r^{(1)})] \otimes [i(\xi), i(r^{(2)})] \\
&\quad - [i(\xi), i(r^{(1)})] \otimes [x, i(r^{(2)})] + i(r^{(1)}) \otimes [\xi, r^{(2)}]{\triangleright}x \\
&\quad + r^{(1)}\xi{\triangleright}x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)}\xi{\triangleright}x \\
&= \xi{\triangleright}\delta x + r^{(1)}{\triangleright}x \otimes [i(\xi), i(r^{(2)})] + [i(\xi), i(r^{(1)})] \otimes r^{(2)}{\triangleright}x \\
&\quad + \xi r^{(1)}{\triangleright}x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes \xi r^{(1)}{\triangleright}x \\
&= \xi{\triangleright}\underline{\delta}x
\end{aligned}
$$

where we used the definitions of $\triangleright$ and $\underline{\delta}$ and the fact that $\mathfrak{g}$ is quasitriangular, so that $\delta\xi = [\xi, r^{(1)}] \otimes r^{(2)} + r^{(1)} \otimes [\xi, r^{(2)}]$.

Antisymmetry of the output of $\underline{\delta}$ is clear. Next we verify the coJacobi identity,

$$
\begin{aligned}
&(\mathrm{id} \otimes \underline{\delta})\underline{\delta}x \\
&\qquad + \text{cyclic} = (\mathrm{id} \otimes \underline{\delta})\delta x + r^{(1)}{\triangleright}x \otimes \underline{\delta} i(r^{(2)}) - i(r^{(2)}) \otimes \underline{\delta}(r^{(1)}{\triangleright}x) + \text{cyclic} \\
&= x_{(1)} \otimes r^{(1)}{\triangleright}x_{(2)} \otimes i(r^{(2)}) - x_{(1)} \otimes i(r^{(2)}) \otimes r^{(1)}{\triangleright}x_{(2)} + r^{(1)}{\triangleright}x \otimes \delta i(r^{(2)}) \\
&\quad + r^{(1)}{\triangleright}x \otimes i([r'^{(1)}, r^{(2)}]) \otimes i(r'^{(2)}) - r^{(1)}{\triangleright}x \otimes i(r'^{(2)}) \otimes i([r'^{(1)}, r^{(2)}]) \\
&\quad - i(r^{(2)}) \otimes r^{(1)}{\triangleright}x_{(1)} \otimes x_{(2)} - i(r^{(2)}) \otimes x_{(1)} \otimes r^{(1)}{\triangleright}x_{(2)} \\
&\quad - i(r^{(2)}) \otimes r^{(1)}r'^{(1)}{\triangleright}x \otimes i(r'^{(2)}) - i(r^{(2)}) \otimes r'^{(1)}{\triangleright}x \otimes i([r^{(1)}, r'^{(2)}]) \\
&\quad + i(r^{(2)}) \otimes i([r^{(1)}, r'^{(2)}]) \otimes r'^{(1)}{\triangleright}x + i(r^{(2)}) \otimes i(r'^{(2)}) \otimes r^{(1)}r'^{(1)}{\triangleright}x + \text{cyclic}
\end{aligned}
$$

using the definition of $\underline{\delta}$ and the previous covariance result. Several of the resulting terms cancel immediately. Using the quasitriangular form of δ on $r^{(2)}$ and the further freedom to cyclically rotate all tensor products so that x appears in the first factor, our expression becomes

$$
\begin{aligned}
&= r^{(1)}{\triangleright}x \otimes i([r^{(2)}, r'^{(1)}]) \otimes i(r'^{(2)}) + r^{(1)}{\triangleright}x \otimes i(r'^{(1)}) \otimes i([r^{(2)}, r'^{(2)}]) \\
&\quad + r^{(1)}{\triangleright}x \otimes i([r'^{(1)}, r^{(2)}]) \otimes i(r'^{(2)}) - r^{(1)}{\triangleright}x \otimes i(r'^{(2)}) \otimes i([r'^{(1)}, r^{(2)}]) \\
&\quad + [r^{(1)}, r'^{(1)}]{\triangleright}x \otimes i(r^{(2)}) \otimes i(r'^{(2)}) \\
&\quad + r'^{(1)}{\triangleright}x \otimes i(r^{(2)}) \otimes i([r^{(1)}, r'^{(2)}]) - r'^{(1)}{\triangleright}x \otimes i([r^{(1)}, r'^{(2)}]) \otimes i(r^{(2)}) + \text{cyclic} \\
&= ((\ ){\triangleright}x \otimes i \otimes i)\Big([r^{(1)}, r'^{(1)}] \otimes r^{(2)} \otimes r'^{(2)} \\
&\qquad + r^{(1)} \otimes [r^{(2)}, r'^{(1)}] \otimes r'^{(2)} + r^{(1)} \otimes r'^{(1)} \otimes [r^{(2)}, r'^{(2)}]\Big) + \text{cyclic} \\
&= 0
\end{aligned}
$$

by the CYBE (1).

Finally, we prove that $\mathrm{d}\underline{\delta} = \psi$. Thus,

$$
\underline{\delta}([x, y]) = \delta[x, y] + r^{(1)}{\triangleright}[x, y] \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)}{\triangleright}[x, y]
$$

$$
\begin{aligned}
&= \mathrm{ad}_x\delta y - \mathrm{ad}_y\delta x + [r^{(1)}\triangleright x, y]\otimes i(r^{(2)}) + [x, r^{(1)}\triangleright y]\otimes i(r^{(2)}) \\
&\quad - i(r^{(2)})\otimes[r^{(1)}\triangleright x, y] - i(r^{(2)})\otimes[x, r^{(1)}\triangleright y] \\
&= \mathrm{ad}_x\underline{\delta} y - \mathrm{ad}_y\underline{\delta} x - r^{(1)}\triangleright y\otimes[x, i(r^{(2)})] + r^{(1)}\triangleright x\otimes[y, i(r^{(2)})] \\
&\quad + [x, i(r^{(2)})]\otimes r^{(1)}\triangleright y - [y, i(r^{(2)})]\otimes r^{(1)}\triangleright x \\
&= \mathrm{ad}_x\underline{\delta} y - \mathrm{ad}_y\underline{\delta} x - 2r_+\triangleright(x\otimes y - y\otimes x)
\end{aligned}
$$

as required. We used the definitions of $\underline{\delta}$ and $\triangleright$. □

Corollary 3.2. *Every quasitriangular Lie bialgebra* $\mathfrak{g}$ *has a braided version* $\underline{\mathfrak{g}}\in{}_{\mathfrak{g}}\mathcal{M}$ *by* ad*, the same Lie bracket, and*

$$\underline{\delta} x = 2{r_+}^{(1)}\otimes[x, {r_+}^{(2)}]. \tag{2}$$

Proof. We take the identity map $i = \mathrm{id} : \mathfrak{g}\to\mathfrak{g}$ and $\underline{\mathfrak{g}} = \mathfrak{b}(\mathfrak{g},\mathfrak{g})$. Its braided-Lie cobracket from Theorem 3.1 is $\underline{\delta} x = [x, r^{(1)}]\otimes r^{(2)} + r^{(1)}\otimes[x, r^{(2)}] + r^{(1)}\triangleright x\otimes r^{(2)} - r^{(2)}\otimes r^{(1)}\triangleright x$ using the quasitriangular form of δ. □

The corollary ensures the existence of non-trivial braided-Lie bialgebras since nontrivial quasitriangular Lie bialgebras are certainly known.

Example 3.3. Let $\mathfrak{g}$ be a finite-dimensional factorisable Lie bialgebra. Then $\underline{\delta}$ in Corollary 3.2 is equivalent under the isomorphism $2r_+ : \mathfrak{g}^*{\cong}\mathfrak{g}$ to the Kirillov-Kostant Lie cobracket on $\mathfrak{g}^*$ (defined as the dualisation of the Lie bracket $\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$). The braided-Lie bialgebra $\underline{\mathfrak{g}}$ is self-dual.

Proof. It is well known that for any Lie algebra the vector space $\mathfrak{g}^*$ acquires a natural Poisson bracket structure. Considering $\mathfrak{g}$ as a subset of the functions on $\mathfrak{g}^*$, this Kirillov-Kostant Poisson bracket is $\{\xi,\eta\}(\phi) = \langle\phi, [\xi,\eta]\rangle$ where $\langle\ ,\ \rangle$ denotes evaluation and $\xi,\eta\in\mathfrak{g}$, $\phi\in\mathfrak{g}^*$. The associated Lie coalgebra structure $\underline{\delta} : \mathfrak{g}^*\to\mathfrak{g}^*\otimes\mathfrak{g}^*$ is defined by $\{\xi,\eta\}(\phi) = \langle\xi\otimes\eta, \underline{\delta}\phi\rangle$ and is therefore the dualisation of the Lie bracket of $\mathfrak{g}$. We call it the Kirillov-Kostant Lie coalgebra structure on $\mathfrak{g}^*$.

Let $K(\phi) = 2{r_+}^{(1)}\langle\phi, {r_+}^{(2)}\rangle$ denote the isomorphism $K : \mathfrak{g}^*{\cong}\mathfrak{g}$ resulting from our factorisability assumption. Then

$$
\begin{aligned}
&\langle\xi\otimes\eta, (K^{-1}\otimes K^{-1})\underline{\delta} K(\phi)\rangle \\
&= \langle\xi, K^{-1}(2{r_+}^{(1)})\rangle\langle\eta, K^{-1}([K(\phi), {r_+}^{(2)}])\rangle \\
&= \langle K^{-1}(\xi), 2{r_+}^{(1)}\rangle\langle[\eta, K(\phi)], K^{-1}({r_+}^{(2)})\rangle \\
&= \langle[\eta, K(\phi)], K^{-1}(\xi)\rangle = \langle K(\phi), K^{-1}([\xi,\eta])\rangle = \langle\phi, [\xi,\eta]\rangle.
\end{aligned}
$$

We used symmetry and ad-invariance of K as an element of $\mathfrak{g}\otimes\mathfrak{g}$, with its corresponding property $\langle\eta, K^{-1}([\xi,\zeta])\rangle = \langle[\eta,\xi], K^{-1}(\zeta)\rangle\ \forall\xi,\eta,\zeta\in\mathfrak{g}$, for the map $K : \mathfrak{g}^*\to\mathfrak{g}$.

Next, we give $\mathfrak{g}^*$ with the above Kirillov-Kostant Lie cobracket $\delta\phi = \phi_{(1)} \otimes \phi_{(2)}$ (dual to the Lie algebra of $\mathfrak{g}$) a Lie bracket and $\mathfrak{g}$-module structure

$$(3) \qquad [\phi, \chi] = \chi_{(1)} 2r_+(\phi, \chi_{(2)}), \quad \xi \triangleright \phi = \phi_{(1)} \langle \phi_{(2)}, \xi \rangle$$

for all $\xi \in \mathfrak{g}$, $\phi, \chi \in \mathfrak{g}^*$ and with $2r^+$ viewed as a map $\mathfrak{g}^* \otimes \mathfrak{g}^* \to k$. Then $\mathfrak{g}^*$ becomes a braided-Lie bialgebra in ${}_{\mathfrak{g}}\mathcal{M}$, which we denote $\underline{\mathfrak{g}}^*$. Its Lie cobracket is $\underline{\delta} = \delta$ the dual of the Lie bracket of $\underline{\mathfrak{g}}$ (since this is the same as that of $\mathfrak{g}$), and its Lie bracket is dual to the Lie cobracket of $\mathfrak{g}$ in Corollary 3.2 since

$$\begin{aligned} \langle \xi, [\phi, \chi] \rangle &= \langle \xi, \chi_{(1)} \rangle \langle 2r_+(\phi), \chi_{(2)} \rangle = \langle [\xi, 2r_+(\phi)], \chi \rangle \\ &= \langle 2r_+{}^{(2)} \otimes [\xi, r_+{}^{(1)}], \phi \otimes \chi \rangle = \langle \underline{\delta}\xi, \phi \otimes \chi \rangle \end{aligned}$$

for all $\xi \in \mathfrak{g}$ and $\phi, \chi \in \mathfrak{g}^*$. On the other hand,

$$\langle \xi, [\phi, \chi] \rangle = \langle \xi, \chi_{(1)} \rangle \langle \chi_{(2)}, K(\phi) \rangle = \langle [\xi, K(\phi)], \chi \rangle = \langle \xi, K^{-1}([K(\phi), K(\chi)]) \rangle$$

hence $2r_+ : \underline{\mathfrak{g}}^* \to \underline{\mathfrak{g}}$ is an isomorphism of braided-Lie bialgebras. □

This is the Lie analogue of the theorem that braided groups obtained by full transmutation of factorisable quantum groups are self-dual via the quantum Killing form [**14**]. Also, the fact that the data corresponding to the original Lie cobracket on $\mathfrak{g}$ does not enter into $\underline{\mathfrak{g}}$ corresponds in braided group theory to transmutation rendering a quasitriangular Hopf algebra braided-cocommutative. There is also a theory of quasitriangular braided-Lie bialgebras of which the more general $\mathfrak{b}(\mathfrak{g}, \mathfrak{f})$ are examples when $\mathfrak{f}$ is itself quasitriangular. The braided-quasitriangular structure is the difference of the quasitriangular structures on $\mathfrak{f}, \mathfrak{g}$ as the Lie version of results in [**12**].

For use later on, the general duality for braided-Lie bialgebras relevant to Example 3.3 is given by

Lemma 3.4. *If $\mathfrak{b} \in {}_{\mathfrak{g}}\mathcal{M}$ is a finite-dimensional braided-Lie bialgebra then there is a dual braided-Lie bialgebra $\mathfrak{b}^* \in {}_{\mathfrak{g}}\mathcal{M}$. It is built on the vector space $\mathfrak{b}^*$ with action $(\xi \triangleright \phi)(x) = -\phi(\xi \triangleright x)$ and Lie* (*co*)*bracket structure maps given by dualisation.*

Proof. The Jacobi and coJacobi (and antisymmetry) axioms are clear by dualisation, as is the specified left action on $\mathfrak{b}^*$. The induced infinitesimal braiding on the dual is the usual dual:

$$\langle x \otimes y, \psi_{\mathfrak{b}^*}(\phi, \chi) \rangle = \langle x \otimes y, 2r_+ \triangleright (\phi \otimes \chi - \chi \otimes \phi) \rangle = \langle \psi(x \otimes y), \phi \otimes \chi \rangle$$

for all $x, y \in \mathfrak{b}$ and $\phi, \chi \in \mathfrak{b}^*$. Moreover, the map $\mathrm{d}\underline{\delta}$ for $\mathfrak{b}$ dualises to $\mathrm{d}\underline{\delta}$ for $\mathfrak{b}^*$. The proof is identical to the proof that the dual of a usual Lie bialgebra is a Lie bialgebra (see [**10**] for details). Hence $\mathrm{d}\underline{\delta} = \psi$ for $\mathfrak{b}^*$ by dualisation of this relation for $\mathfrak{b}$. □

Note also that if $\mathfrak{b}\in {}_{\mathfrak{g}}\mathcal{M}$ is any braided-Lie bialgebra then so is $\mathfrak{b}^{\mathrm{op/cop}}$ with opposite bracket and cobracket, in the same category. This is because the covariance conditions on the Lie bracket and cobracket are each linear in those structures and hence valid even with the additional minus signs in either case. Meanwhile, $\mathrm{d}\underline{\delta}$ is linear in $\underline{\delta}$ and linear in the Lie bracket, hence invariant when both are changed by a minus sign. The infinitesimal braiding does not involve either the bracket or cobracket and is invariant. Applying this observation to $\mathfrak{b}^*$ gives us another dual, $\mathfrak{b}^\star$. This is the Lie analogue of the more braided-categorical dual which is more natural in the theory of braided groups. In the Lie setting, however, we have $\mathfrak{b}^\star\cong\mathfrak{b}^*$ by $x\mapsto -x$, so can work entirely with $\mathfrak{b}^*$. We also conclude, in passing, that $\mathfrak{b}^{\mathrm{op}}$ and $\mathfrak{b}^{\mathrm{cop}}$ are braided-Lie bialgebras in the category of modules over the opposite quasitriangular Lie bialgebra (i.e., with quasitriangular structure $-r_{21}$ in place of r).

We consider now the adjoint direction to Theorem 3.1, to associate to a braided-Lie bialgebra an ordinary Lie bialgebra. The quantum group version of this result [**13**] has been used to construct inhomogeneous quantum groups [**7**].

Theorem 3.5. *Let $\mathfrak{b}\in {}_{\mathfrak{g}}\mathcal{M}$ be a braided-Lie bialgebra. Its* bosonisation *is the Lie bialgebra $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}$ with $\mathfrak{g}$ as sub-Lie bialgebra, $\mathfrak{b}$ as sub-Lie algebra and*

$$(4)\quad [\xi,x]=\xi{\triangleright}x,\quad \delta x=\underline{\delta}x+r^{(2)}\otimes r^{(1)}{\triangleright}x-r^{(1)}{\triangleright}x\otimes r^{(2)},\quad \forall\xi\in\mathfrak{g},\ x\in\mathfrak{b}.$$

Proof. The Lie algebra structure of $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}$ is constructed as a semidirect sum by the given action of $\mathfrak{g}$ on $\mathfrak{b}$. The coassociativity of the Lie cobracket may be verified directly from the CYBE along the lines of the proof of coassociativity in Theorem 3.1. The line of deduction is reversed but the formulae are similar. That the result is a Lie bialgebra has three cases. For $\xi,\eta\in\mathfrak{g}$ we have $\delta([\xi,\eta])$ as required since $\mathfrak{g}$ is a Lie bialgebra. For the mixed case we have

$$\begin{aligned}\delta([\xi,x])&=\delta(\xi{\triangleright}x)=\underline{\delta}(\xi{\triangleright}x)+r^{(2)}\otimes r^{(1)}\xi{\triangleright}x-r^{(1)}\xi{\triangleright}x\otimes r^{(2)}\\ -\mathrm{ad}_\xi\delta x&=-\xi{\triangleright}\underline{\delta}x-[\xi,r^{(2)}]\otimes r^{(1)}{\triangleright}x-r^{(2)}\otimes\xi r^{(1)}{\triangleright}x+\xi r^{(1)}{\triangleright}x\otimes r^{(2)}\\ &\quad+r^{(1)}{\triangleright}x\otimes[\xi,r^{(2)}]\\ \mathrm{ad}_x\delta\xi&=-\xi_{(1)}{\triangleright}x\otimes\xi_{(2)}-\xi_{(1)}\otimes\xi_{(2)}{\triangleright}x\\ &=[r^{(1)},\xi]{\triangleright}x\otimes r^{(2)}+r^{(1)}{\triangleright}x\otimes[r^{(2)},\xi]+[r^{(1)},\xi]\otimes r^{(2)}{\triangleright}x\\ &\quad+r^{(1)}\otimes[r^{(2)},\xi]{\triangleright}x.\end{aligned}$$

We used the definitions and, in the last line, the form of $\delta\xi$ as a quasitriangular Lie bialgebra. Adding these expressions, we obtain

$$\begin{aligned}&\delta([\xi,x])-\mathrm{ad}_\xi\delta x+\mathrm{ad}_x\delta\xi\\ &=\underline{\delta}\xi{\triangleright}x-\xi{\triangleright}\underline{\delta}x+[2r_+{}^{(2)},\xi]\otimes r_+{}^{(1)}{\triangleright}x+2r_+{}^{(2)}\otimes[r_+{}^{(1)},\xi]{\triangleright}x=0\end{aligned}$$

by covariance of $\underline{\delta}$ and ad-invariance of $2r_+$. The remaining case is

$$\begin{aligned}\delta([x,y]) &= \underline{\delta}([x,y]) + r^{(2)}\otimes r^{(1)}\triangleright[x,y] - r^{(1)}\triangleright[x,y]\otimes r^{(2)}\\ &= (\mathrm{ad}_x\underline{\delta}y + r^{(2)}\otimes \mathrm{ad}_x(r^{(1)}\triangleright y) - \mathrm{ad}_x(r^{(1)}\triangleright y)\otimes r^{(2)} - (x\leftrightarrow y))\\ &\quad - \psi(x\otimes y)\\ &= (\mathrm{ad}_x\delta y - \mathrm{ad}_x(r^{(2)})\otimes r^{(1)}\triangleright y + r^{(1)}\triangleright y\otimes \mathrm{ad}_x(r^{(2)})\\ &\quad - (x\leftrightarrow y)) - \psi(x\otimes y)\\ &= \mathrm{ad}_x\delta y - \mathrm{ad}_y\delta x\end{aligned}$$

on writing $\mathrm{ad}_x(r^{(2)}) = -r^{(2)}\triangleright x$ and comparing with the definition of ψ. We used the braided-Lie bialgebra property of $\underline{\delta}$. □

The construction in the bosonisation theorem can also be viewed as a special case of a more general construction for Lie bialgebras which are semidirect sums as Lie algebras and Lie coalgebras by a simultaneous Lie action and Lie coaction. We call such Lie algebras *bisum* Lie algebras. They are the analogue of *biproduct* Hopf algebras in [**20**]. In the general case one only needs covariance under a Lie bialgebra, not necessarily quasitriangular. However, any Lie bialgebra has a Drinfeld double [**2**] which is quasitriangular. In order to explain these topics we need quite a bit more formalism. Firstly, if $\mathfrak{f}$ is any Lie coalgebra, we have a notion of left *Lie coaction* on a vector space V. This is a map $\beta: V\to \mathfrak{f}\otimes V$ such that [**10**],

$$(\delta\otimes\mathrm{id})\circ\beta = ((\mathrm{id}-\tau)\otimes\mathrm{id})\circ(\mathrm{id}\otimes\beta)\circ\beta. \tag{5}$$

The category of left Lie comodules is denoted ${}^{\mathfrak{g}}\mathcal{M}$ and is monoidal in the obvious derivation-like way. Morphisms are defined as linear maps intertwining the Lie coactions, again in the obvious way.

Lemma 3.6. *Let $\mathfrak{f}$ be a Lie bialgebra. There is a monoidal category of* Lie crossed modules ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ *having as objects vector spaces V which are are simultaneously $\mathfrak{f}$-modules $\triangleright: \mathfrak{f}\otimes V\to\otimes V$ and $\mathfrak{f}$-comodules $\beta: V\to\mathfrak{f}\otimes V$ obeying $\forall\xi\in\mathfrak{f}, v\in V$,*

$$\beta(\xi\triangleright v) = ([\xi,\,]\otimes\mathrm{id} + \mathrm{id}\otimes\xi\triangleright)\beta(v) + (\delta\xi)\triangleright v.$$

It can be identified when $\mathfrak{f}$ is finite-dimensional with the category ${}_{D(\mathfrak{f})}\mathcal{M}$ where $D(\mathfrak{f})$ is the Drinfeld double [**2**]*. Writing $\beta(v) = v^{(\bar 1)}\otimes v^{(\bar 2)}$, the corresponding infinitesimal braiding on any object $V\in{}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ is*

$$\psi(v\otimes w) = w^{(\bar 1)}\triangleright v\otimes w^{(\bar 2)} - v^{(\bar 1)}\triangleright w\otimes v^{(\bar 2)} - w^{(\bar 2)}\otimes w^{(\bar 1)}\triangleright v + v^{(\bar 2)}\otimes v^{(\bar 1)}\triangleright w.$$

Proof. Morphisms in ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ are maps intertwining both the Lie action and the Lie coaction. We start with $\mathfrak{f}$ finite-dimensional and use Drinfeld's formulae for $D(\mathfrak{f})$ in the conventions in [**10**], where it contains the Lie algebras $\mathfrak{f}$ and $\mathfrak{f}^{*\mathrm{op}}$ with the cross relations $[\xi,\phi] = \phi_{(1)}\langle\phi_{(2)},\xi\rangle + \xi_{(1)}\langle\phi,\xi_{(2)}\rangle$ for all $\xi\in\mathfrak{f}$ and

$\phi \in \mathfrak{f}^*$. A left module of $D(\mathfrak{f})$ therefore means a vector space which is a left $\mathfrak{f}$-module and a right $\mathfrak{f}^*$-module, obeying $\xi{\triangleright}(v{\triangleleft}\phi) - (\xi{\triangleright}v){\triangleleft}\phi = v{\triangleleft}\phi_{(1)}\langle\phi_{(2)}, \xi\rangle + \xi_{(1)}{\triangleright}v\langle\phi, \xi_{(2)}\rangle$. Next we view the right action of $\mathfrak{f}^*$ as, equivalently, a left coaction of $\mathfrak{f}$ by $v{\triangleleft}\phi = \langle\phi, v^{(\bar{1})}\rangle v^{(\bar{2})}$. Inserting this, we have the condition

$$\xi{\triangleright}v^{(\bar{2})}\langle\phi, v^{(\bar{1})}\rangle - \langle\phi, (\xi{\triangleright}v)^{(\bar{1})}\rangle(\xi{\triangleright}v)^{(\bar{2})} = \langle\phi, [v^{(\bar{1})}, \xi]\rangle v^{(\bar{2})} + \xi_{(1)}{\triangleright}v\langle\phi, \xi_{(2)}\rangle$$

for all ϕ. We wrote the Lie cobracket of $\mathfrak{f}^*$ in terms of the Lie algebra of $\mathfrak{f}$ here. This is the condition stated for β, which manifestly makes sense even for infinite-dimensional Lie algebras. It is easy to check that the category is well defined and monoidal even in this case. In the same spirit, $D(\mathfrak{f})$ has a quasitriangular structure given by the canonical element for the duality pairing [2]. Then $2r_+ = \sum_a f^a \otimes e_a + e_a \otimes f^a$ where $\{e_a\}$ is a basis of $\mathfrak{f}$ and $\{f^a\}$ is a dual basis. Hence the infinitesimal braiding in ${}_{D(\mathfrak{f})}\mathcal{M}$ is

$$\begin{aligned}\psi(v \otimes w) &= 2r_+{\triangleright}(v \otimes w - w \otimes v)\\ &= \langle f^a, v^{(\bar{1})}\rangle v^{(\bar{2})} \otimes e_a{\triangleright}w + e_a{\triangleright}v \otimes \langle f^a, w^{(\bar{1})}\rangle w^{(\bar{2})} - (v{\leftrightarrow}w)\end{aligned}$$

when the left action of $\mathfrak{f}^{*\mathrm{op}}$ is of the form given by a left coaction of $\mathfrak{f}$. This gives ψ as stated. Note that the use of a coaction to reformulate an action of the dual in the infinite dimensional case is a completely routine procedure in Hopf algebra theory; we have given the details here since the Lie version is less standard; the category of Lie crossed modules ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ should not be viewed as anything other than a *version* of the ideas behind Drinfeld's double construction. □

The resulting map ψ is well-defined even in the infinite dimensional case; we call it the infinitesimal braiding of the category ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ of crossed $\mathfrak{f}$-modules and define a braided-Lie bialgebra in ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ with respect to this.

Theorem 3.7. *Let $\mathfrak{f}$ be a Lie bialgebra and let $\mathfrak{b} \in {}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ be a braided-Lie bialgebra. The* bisum *Lie bialgebra $\mathfrak{b}{>\!\!\triangleleft}\mathfrak{f}$ has semidirect Lie bracket/cobracket and projects onto $\mathfrak{f}$. Conversely, any Lie bialgebra $\mathfrak{g}$ with a split Lie bialgebra projection $\mathfrak{g} \underset{i}{\overset{\pi}{\rightleftarrows}} \mathfrak{f}$ is of this form, with $\mathfrak{b} = \ker\pi$ and braided-Lie bialgebra structure given by $\mathfrak{b} \subset \mathfrak{g}$ as a Lie algebra and*

$$\xi{\triangleright} = \mathrm{ad}_{i(\xi)}, \quad \beta = (\pi \otimes \mathrm{id}) \circ \delta, \quad \underline{\delta} = (\mathrm{id} - i \circ \pi)^{\otimes 2} \circ \delta.$$

Proof. In the forward direction, since $\mathfrak{b}$ is covariant under an action of $\mathfrak{f}$ we can make, as usual, a semidirect sum $\mathfrak{b}{>\!\!\triangleleft}\mathfrak{f}$. The bracket on general elements of the direct sum vector space is $[x \oplus \xi, y \oplus \eta] = ([x, y] + \xi{\triangleright}y - \eta{\triangleright}x) \oplus [\xi, \eta]$ as usual. On the other hand, since the Lie coalgebra of $\mathfrak{b}$ is covariant under a Lie coaction of $\mathfrak{f}$, one may make a semidirect Lie coalgebra $\mathfrak{b}{>\!\!\blacktriangleleft}\mathfrak{f}$ with [10]

$$\delta(x \oplus \xi) = \delta\xi + \underline{\delta}x + (\mathrm{id} - \tau) \circ \beta(x) \tag{6}$$

where $\underline{\delta}$ is the Lie cobracket of $\mathfrak{b}$. The required covariance of the Lie coalgebra under the coaction here is

$$(\mathrm{id}\otimes\underline{\delta})\circ\beta=(\mathrm{id}\otimes(\mathrm{id}-\tau))\circ(\beta\otimes\mathrm{id})\circ\underline{\delta} \tag{7}$$

and ensures that δ on $\mathfrak{b}\blacktriangleright\!\!\blacktriangleleft\mathfrak{f}$ obeys the coJacobi identity.

The further covariance assumptions on $\mathfrak{b}$ are that its Lie bracket is covariant under the Lie coaction and its Lie cobracket is covariant under the Lie action. These assumptions are all needed to show that the semidirect Lie bracket and Lie cobracket form a Lie bialgebra $\mathfrak{b}\rtimes\!\!\!\lhd\mathfrak{f}$. The case $\delta([\xi,\eta])$ is clear since $\mathfrak{f}$ is a Lie subalgebra and Lie subcoalgebra. The mixed case is

$$\begin{aligned}\delta([\xi,x])&=\delta(\xi\triangleright x)=\underline{\delta}(\xi\triangleright x)+(\xi\triangleright x)^{(\bar1)}\otimes(\xi\triangleright x)^{(\bar2)}-(\xi\triangleright x)^{(\bar2)}\otimes(\xi\triangleright x)^{(\bar1)}\\&=\xi\triangleright\underline{\delta}x+\mathrm{ad}_\xi((\mathrm{id}-\tau)\circ\beta(x))+\xi_{(1)}\otimes\xi_{(2)}\triangleright x-\xi_{(2)}\triangleright x\otimes\xi_{(1)}\\&=\mathrm{ad}_\xi\delta x-\mathrm{ad}_x\delta\xi\end{aligned}$$

as required. We used the definition of δ for $\mathfrak{b}\blacktriangleright\!\!\blacktriangleleft\mathfrak{f}$, the covariance of $\underline{\delta}$ under ξ, the crossed module condition, antisymmetry of $\delta\xi$ and $\mathrm{ad}_\xi(x)=\xi\triangleright x=-\mathrm{ad}_x(\xi)$ when viewed in the Lie algebra $\mathfrak{b}\rtimes\!\!\!\lhd\mathfrak{f}$. The remaining case is

$$\begin{aligned}&\delta([x,y])\\&=\underline{\delta}([x,y])+(\mathrm{id}-\tau)\beta([x,y])\\&=\mathrm{ad}_x\underline{\delta}y+y^{(\bar2)}\otimes y^{(\bar1)}\triangleright x-y^{(\bar1)}\triangleright x\otimes y^{(\bar2)}+x^{(\bar1)}\otimes[x^{(\bar2)},y]\\&\quad-[x^{(\bar2)},y]\otimes x^{(\bar1)}-(x\leftrightarrow y)\\&=\mathrm{ad}_x\underline{\delta}y-y^{(\bar1)}\triangleright x\otimes y^{(\bar2)}+y^{(\bar1)}\otimes[x,y^{(\bar2)}]-[x,y^{(\bar2)}]\otimes y^{(\bar1)}\\&\quad+y^{(\bar2)}\otimes y^{(\bar1)}\triangleright x-(x\leftrightarrow y)\\&=\mathrm{ad}_x\underline{\delta}y+[x,y^{(\bar1)}]\otimes y^{(\bar2)}+y^{(\bar1)}\otimes[x,y^{(\bar2)}]-[x,y^{(\bar2)}]\otimes y^{(\bar1)}\\&\quad-y^{(\bar2)}\otimes[x,y^{(\bar1)}]-(x\leftrightarrow y)\\&=\mathrm{ad}_x\delta y-\mathrm{ad}_y\delta x\end{aligned}$$

where we used $\beta([x,y])=y^{(\bar1)}\otimes[x,y^{(\bar2)}]-x^{(\bar1)}\otimes[y,x^{(\bar2)}]$ (covariance of the Lie bracket of $\mathfrak{b}$ under the Lie coaction) and the assumption that $\underline{\delta}$ is a braided-Lie bialgebra in ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ with infinitesimal braiding from Lemma 3.6. We then used the crossed module compatibility condition also from Lemma 3.6 and $\xi\triangleright x=-\mathrm{ad}_x(\xi)$ to recognise the required result. It is easy to see that the projection $\mathfrak{b}\rtimes\!\!\!\lhd\mathfrak{f}\to\mathfrak{f}$ defined by setting elements of $\mathfrak{b}$ to zero is a Lie bialgebra map covering the inclusion $\mathfrak{f}\subset\mathfrak{b}\rtimes\!\!\!\lhd\mathfrak{f}$.

In the converse direction, we assume a split projection, i.e. a surjection $\pi:\mathfrak{g}\to\mathfrak{f}$ between Lie bialgebras covering an inclusion $i:\mathfrak{f}\to\mathfrak{g}$ of Lie bialgebras (so that $\pi\circ i=\mathrm{id}$). We define $\mathfrak{b}=\ker\pi$. Since this is a Lie ideal, it both forms a sub-Lie algebra of $\mathfrak{g}$ and is covariant under the action of $\mathfrak{f}$ given by pull-back along i of ad. Moreover, $\mathfrak{g}$ coacts on itself by its Lie cobracket δ (the adjoint coaction of any Lie bialgebra on itself) and hence push-out along π is an $\mathfrak{f}$-coaction $\beta = (\pi \otimes \mathrm{id}) \circ \delta$, which restricts to $\mathfrak{b}$ since $(\mathrm{id} \otimes \pi)\beta(x) = (\pi \otimes \pi)\delta x = \delta\pi(x) = 0$ for $x \in \ker \pi$. This Lie action and Lie coaction fit together to form a Lie crossed module,

$$\begin{aligned}\beta(\xi \triangleright x) &= (\pi \otimes \mathrm{id}) \circ \delta([i(\xi), x]) = (\pi \otimes \mathrm{id})(\mathrm{ad}_{i(\xi)}\delta x - \mathrm{ad}_x \delta i(\xi))\\ &= \pi([i(\xi), x_{(1)}]) \otimes x_{(2)} + \pi(x_{(1)}) \otimes [i(\xi), x_{(2)}] - \pi([x, i(\xi)_{(1)}]) \otimes i(\xi)_{(2)}\\ &\quad - \pi(i(\xi)_{(1)}) \otimes [x, i(\xi)_{(2)}]\\ &= [x, \pi(x_{(1)})] \otimes x_{(2)} + \pi(x_{(1)}) \otimes \xi \triangleright x_{(2)} + \xi_{(1)} \otimes \xi_{(2)} \triangleright x\\ &= ([\xi, \ ] \otimes \mathrm{id} + \mathrm{id} \otimes \xi \triangleright)\beta(x) + (\delta\xi) \triangleright x\end{aligned}$$

as required. We used that i is a Lie coalgebra map and π a Lie algebra map, along with $x \in \ker \pi$ to kill the term with $\pi([x, i(\xi)_{(1)}])$.

Finally, we give $\mathfrak{b}$ a Lie cobracket $\underline{\delta}$ as stated. Writing $p = \mathrm{id} - i \circ \pi$, we have

$$\begin{aligned}&(\mathrm{id} \otimes \underline{\delta}) \circ \underline{\delta} x\\ &= p(x_{(1)}) \otimes p(p(x_{(2)})_{(1)}) \otimes p(p(x_{(2)})_{(2)})\\ &= (p \otimes p \otimes p)(\mathrm{id} \otimes \delta)\delta x - p(x_{(1)}) \otimes p(i \circ \pi(x_{(2)})_{(1)}) \otimes p(i \circ \pi(x_{(2)})_{(2)})\\ &= (p \otimes p \otimes p)(\mathrm{id} \otimes \delta)\delta x\end{aligned}$$

since $i \circ \pi$ is a Lie coalgebra map and $p \circ i \circ \pi = 0$. Hence $\underline{\delta}$ obeys the coJacobi identity since δ does. Moreover, for all $x, y \in \ker \pi$,

$$\begin{aligned}\underline{\delta}([x, y]) &= (\mathrm{id} - i \circ \pi) \otimes (\mathrm{id} - i \circ \pi)\delta[x, y]\\ &= [x, y_{(1)}] \otimes y_{(2)} + y_{(1)} \otimes [x, y_{(2)}] - [x, y_{(1)}] \otimes i \circ \pi(y_{(2)})\\ &\quad - i \circ \pi(y_{(1)}) \otimes [x, y_{(2)}] - (x \leftrightarrow y)\end{aligned}$$

since $i \circ \pi([x, y_{(2)}]) = 0$ etc., as $i \circ \pi$ is a Lie algebra map. Also, from Lemma 3.6 and the form of β and antisymmetry of δ we have

$$\psi(x \otimes y) = [i \circ \pi(y_{(1)}), x] \otimes y_{(2)} + y_{(1)} \otimes [i \circ \pi(y_{(2)}), x] - (x \leftrightarrow y).$$

Then,

$$\begin{aligned}&\mathrm{ad}_x \underline{\delta} y - \mathrm{ad}_y \underline{\delta} x\\ &= [x, (\mathrm{id} - i \circ \pi)(y_{(1)})] \otimes (\mathrm{id} - i \circ \pi)(y_{(2)})\\ &\quad + (\mathrm{id} - i \circ \pi)(y_{(1)}) \otimes [x, (\mathrm{id} - i \circ \pi)(y_{(2)})] - (x \leftrightarrow y)\\ &= [x, y_{(1)}] \otimes y_{(2)} + y_{(1)} \otimes [x, y_{(2)}] - [x, i \circ \pi(y_{(1)})] \otimes y_{(2)} - [x, y_{(1)}] \otimes i \circ \pi(y_{(2)})\\ &\quad - i \circ \pi(y_{(1)}) \otimes [x, y_{(2)}] - y_{(1)} \otimes [x, i \circ \pi(y_{(2)})] - (x \leftrightarrow y)\\ &= \psi(x \otimes y) + \underline{\delta}([x, y])\end{aligned}$$

as required. The additional terms $i \circ \pi(y_{(1)}) \otimes [x, i \circ \pi(y_{(2)})]$ etc. vanish as $i \circ \pi$ is a Lie coalgebra map and $x, y \in \ker \pi$. Hence $\mathfrak{b} = \ker \pi$ becomes a braided-Lie bialgebra in ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$. One may then verify that the bisum Lie bialgebra $\mathfrak{b}{>\!\!\triangleleft}\mathfrak{f}$ coincides with $\mathfrak{g}$ viewed as a direct sum $\mathfrak{b}\oplus\mathfrak{f}$ of vector spaces according to the projection $i\circ\pi$. $\square$

This is the Lie analogue of the braided groups interpretation [**14**] of Radford's theorem [**20**]. It tells us that braided-Lie bialgebras are rather common as they arise whenever we have a projection of ordinary Lie bialgebras. Finally, we provide the Lie analogue of the functor [**17**] which connects biproducts and bosonisation.

Lemma 3.8. *Let $\mathfrak{g}$ be a quasitriangular Lie bialgebra. There is a monoidal functor ${}_{\mathfrak{g}}\mathcal{M}\to{}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$ respecting the infinitesimal braidings. It sends an action $\triangleright$ to a pair $(\triangleright,\beta)$ where $\beta=r_{21}\triangleright$, the* induced Lie coaction. *The bosonisation of $\mathfrak{b}\in{}_{\mathfrak{g}}\mathcal{M}$ in Theorem 3.5 can thereby be viewed as an example of a biproduct in Theorem 3.7.*

Proof. We first verify that $\beta(v)=r^{(2)}\otimes r^{(1)}\triangleright v$ defines a Lie coaction for any $\mathfrak{g}$-module $V\ni v$. This follows immediately from the identity $(\mathrm{id}\otimes\delta)r=[r^{(1)},r'^{(1)}]\otimes r'^{(2)}\otimes r^{(2)}$ holding for any quasitriangular Lie bialgebra (following from the CYBE and $\delta=\mathrm{d}r$). Thus,

$$\begin{aligned}(\mathrm{id}\otimes\delta)\beta(v)&=r^{(2)}\otimes r'^{(2)}\otimes r'^{(1)}r^{(1)}\triangleright v-r'^{(2)}\otimes r^{(2)}\otimes r'^{(1)}r^{(1)}\triangleright v\\&=((\mathrm{id}-\tau)\otimes\mathrm{id})\circ(\mathrm{id}\otimes\beta)\circ\beta(v)\end{aligned}$$

as required. This fits together with the given action to form a Lie crossed module as

$$\begin{aligned}\beta(\xi\triangleright v)&=r^{(2)}\otimes r^{(1)}\xi\triangleright v=r^{(2)}\otimes[r^{(1)},\xi]\triangleright v+r^{(2)}\otimes\xi r^{(1)}\triangleright v\\&=(\delta\xi)\triangleright v+[\xi,r^{(2)}]\otimes r^{(1)}\triangleright v+(\mathrm{id}\otimes\xi\triangleright)\beta(v)\end{aligned}$$

as required, using the quasitriangular form of $\delta\xi$. More trivially, a morphism $\phi:V\to W$ in ${}_{\mathfrak{g}}\mathcal{M}$ is automatically an intertwiner of the induced coactions (since $r^{(2)}\otimes r^{(1)}\triangleright\phi(v)=r^{(2)}\otimes\phi(r^{(1)}\triangleright v)$) and hence a morphism in ${}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$. It is also clear that the functor respects tensor products. In this way, ${}_{\mathfrak{g}}\mathcal{M}$ is a full monoidal subcategory.

Finally, we check that the infinitesimal braidings coincide. Computing ψ from Lemma 3.6 in the image of the functor, we have

$$\begin{aligned}\psi(v\otimes w)&=r^{(2)}\triangleright v\otimes r^{(1)}\triangleright w+r^{(1)}\triangleright v\otimes r^{(2)}\triangleright w-(v\leftrightarrow w)\\&=2r_+\triangleright(v\otimes w-w\otimes v)\end{aligned}$$

as required. From the form of the Lie cobracket in the bosonisation construction, it is clear that it can be viewed as a semidirect Lie coalgebra by the induced action, i.e., it can be viewed as a nontrivial construction for examples of bisum Lie algebras. $\square$

There is a dual theory of *dual quasitriangular* (or coquasitriangular) Lie bialgebras [**10**] where the Lie bracket has a special form

$$[\xi,\eta]=\xi_{(1)}r(\xi_{(2)},\eta)+\eta_{(1)}r(\xi,\eta_{(2)}),\quad\forall\xi,\eta\in\mathfrak{g},\tag{8}$$

defined by a dual quasitriangular structure $r:\mathfrak{g}\otimes\mathfrak{g}\to k$. This is required to obey the CYBE in a dual form

$$r(\xi,\eta_{(1)})r(\eta_{(2)},\zeta)+r(\xi_{(1)},\eta)r(\xi_{(2)},\zeta)+r(\xi,\zeta_{(1)})r(\eta,\zeta_{(2)})=0,\quad \forall\xi,\eta,\zeta\in\mathfrak{g} \tag{9}$$

and $2r_+$ is required to be invariant under the adjoint Lie coaction ($=\delta$, the Lie cobracket) according to $r_+(\xi,\eta_{(1)})\eta_{(2)}+r_+(\xi_{(1)},\eta)\xi_{(2)}=0$. All of the above theory goes through in this form. Thus, ${}^{\mathfrak{g}}\mathcal{M}$ has, by definition, an infinitesimal braiding defined by

$$\psi(v\otimes w)=r(v^{(\bar 1)},w^{(\bar 1)})(v^{(\bar 2)}\otimes w^{(\bar 2)}-w^{(\bar 2)}\otimes v^{(\bar 2)}) \tag{10}$$

with respect to which we define a braided-Lie bialgebra in ${}^{\mathfrak{g}}\mathcal{M}$. The Lie comodule transmutation theory associates to a map $\mathfrak{f}\to\mathfrak{g}$ of Lie bialgebras with $\mathfrak{g}$ dual quasitriangular, a braided-Lie bialgebra $\mathfrak{b}(\mathfrak{f},\mathfrak{g})\in{}^{\mathfrak{g}}\mathcal{M}$.

For example, the Lie comodule version of Corollary 3.2 is $\underline{\mathfrak{g}}\in{}^{\mathfrak{g}}\mathcal{M}$ with the same Lie cobracket as $\mathfrak{g}$, the adjoint coaction δ and

$$[\xi,\eta]=\eta_{(1)}2r_+(\xi,\eta_{(2)})\quad\forall\xi,\eta\in\mathfrak{g}. \tag{11}$$

A concrete example is provided by $\mathfrak{g}^*$ when $\mathfrak{g}$ is finite-dimensional quasitriangular. Then $\mathfrak{g}^*$ is dual quasitriangular and its transmutation $\underline{\mathfrak{g}^*}$ coincides with $(\underline{\mathfrak{g}})^*$ in (3) in Example 3.3.

Similarly, there is a functor ${}^{\mathfrak{g}}\mathcal{M}\to{}^{\mathfrak{g}}_{\mathfrak{g}}\mathcal{M}$ sending a Lie coaction by $\mathfrak{g}$ to a crossed module with an induced action $\xi\triangleright v=r(v^{(\bar 1)},\xi)v^{(\bar 2)}$ and respecting the infinitesimal braidings. A braided-Lie bialgebra in $\mathfrak{b}\in{}^{\mathfrak{g}}\mathcal{M}$ has a Lie bosonisation $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}$ given by a semidirect Lie cobracket by the given Lie coaction and semidirect Lie bracket given by the induced action. All of this dual theory follows rigorously and automatically by writing all constructions in terms of equalities of linear maps and then reversing all arrows. Such dualisation of theorems is completely routine in the theory of Hopf algebras, and similarly here. Hence we do not need to provide a separate proof of these assertions. Note that dualisation of theorems should not be confused with the dualisation of given algebras and coalgebras, which can be far from routine.

Example 3.9. Let $\mathfrak{g}$ be a finite-dimensional quasitriangular Lie bialgebra and $\underline{\mathfrak{g}}^*$ the dual of its transmutation. Its bosonisation $\underline{\mathfrak{g}}^*{>\!\!\!\triangleleft}\mathfrak{g}$ is isomorphic as a Lie bialgebra to the Drinfeld double $D(\mathfrak{g})$.

Proof. The required isomorphism $\theta:D(\mathfrak{g})\to\underline{\mathfrak{g}}^*{>\!\!\!\triangleleft}\mathfrak{g}$ is $\theta(\phi)=\phi-r^{(2)}\langle\phi,r^{(1)}\rangle$ and $\theta(\xi)=\xi$ for $\xi\in\mathfrak{g}$ and $\phi\in\mathfrak{g}^*$. We check first that it is a Lie algebra map. The $[\xi,\eta]$ case is automatic as $\mathfrak{g}$ is a sub-Lie algebra on both sides. The mixed case is

$$\begin{aligned}[\theta(\xi),\theta(\phi)]_{\rm bos}&=[\xi,\phi-r^{(2)}\langle\phi,r^{(1)}\rangle]_{\rm bos}=\xi\triangleright\phi-[\xi,r^{(2)}]\langle\phi,r^{(1)}\rangle\\&=\phi_{(1)}\langle\phi_{(2)},\xi\rangle+\xi_{(1)}\langle\phi,\xi_{(2)}\rangle+r^{(2)}\langle\phi,[\xi,r^{(1)}]\rangle\end{aligned}$$

$$= \theta(\phi_{(1)}\langle\phi_{(2)},\xi\rangle + \xi_{(1)}\langle\phi,\xi_{(2)}\rangle) = \theta([\xi,\phi])$$

where $[\ ,\ ]_{\rm bos}$ is the Lie bracket of $\underline{\mathfrak{g}}^*{>\!\!\!\triangleleft}\mathfrak{g}$. We use the definition of θ, the quasitriangular form of $\delta\xi$, the action $\xi{\triangleright}\phi = \phi_{(1)}\langle\phi_{(2)},\xi\rangle$ for $\underline{\mathfrak{g}}^*$ and the cross relations in $D(\mathfrak{g})$ (as recalled in Lemma 3.6) to recognise the result. The remaining case is

$$\begin{aligned}
&[\theta(\phi),\theta(\chi)]_{\rm bos}\\
&= [\phi - r^{(2)}\langle\phi,r^{(1)}\rangle, \chi - r'^{(2)}\langle\chi,r'^{(1)}\rangle]\\
&= [r^{(2)},r'^{(2)}]\langle\phi,r^{(1)}\rangle\langle\chi,r'^{(1)}\rangle - r^{(2)}{\triangleright}\chi\langle\phi,r^{(1)}\rangle + r^{(2)}{\triangleright}\phi\langle\chi,r^{(1)}\rangle\\
&\quad + \chi_{(1)}\langle 2r_+,\phi\otimes\chi_{(2)}\rangle\\
&= [r^{(2)},r'^{(2)}]\langle\phi,r^{(1)}\rangle\langle\chi,r'^{(1)}\rangle + \chi_{(1)}\langle r,\chi_{(2)}\otimes\phi\rangle + \phi_{(1)}\langle r,\chi\otimes\phi_{(2)}\rangle\\
&= [\chi,\phi] - r^{(2)}\langle\delta r^{(1)},\chi\otimes\phi\rangle = [\chi,\phi] - r^{(2)}\langle[\chi,\phi],r^{(1)}\rangle = \theta([\chi,\phi])
\end{aligned}$$

as required since $D(\mathfrak{g})$ contains $\mathfrak{g}^{*{\rm op}}$ as a sub-Lie algebra. We used the definition of θ and the Lie bracket (3) of $\underline{\mathfrak{g}}^*$ as a sub-Lie algebra of the bosonisation. We then used form of the action $r^{(1)}{\triangleright}\chi$ etc. and combined the result with the $2r_+$ term to recognise the Lie bracket $[\chi,\phi]$ (as in (8)) of the dual quasitriangular Lie bialgebra $\mathfrak{g}^*$. We also use the quasitriangular form of $\mathfrak{g}$ to recognise $\delta r^{(1)}$.

Next, we verify that θ is a Lie coalgebra map. This is automatic on $\xi\in\mathfrak{g}$ as a sub-Lie bialgebra on both sides. The remaining case is

$$\begin{aligned}
&\delta_{\rm bos}\theta(\phi)\\
&= \delta_{\rm bos}\phi - \delta r^{(2)}\langle\phi,r^{(1)}\rangle\\
&= \underline{\delta}\phi + r^{(2)}\otimes r^{(1)}{\triangleright}\phi - r^{(1)}{\triangleright}\phi\otimes r^{(2)} - r^{(2)}\otimes r'^{(2)}\langle\phi,[r^{(1)},r'^{(1)}]\rangle\\
&= \delta\phi + r^{(2)}\otimes\phi_{(1)}\langle\phi_{(2)},r^{(1)}\rangle - \phi_{(1)}\langle\phi_{(2)},r^{(1)}\rangle\otimes r^{(2)} - r^{(2)}\otimes r'^{(2)}\langle\phi,[r^{(1)},r'^{(1)}]\rangle\\
&= (\phi_{(1)} - r^{(2)}\langle\phi_{(1)},r^{(1)}\rangle)\otimes(\phi_{(2)} - r'^{(2)}\langle\phi_{(2)},r'^{(1)}\rangle) = (\theta\otimes\theta)\delta\phi
\end{aligned}$$

using the Lie cobracket $\delta_{\rm bos}$ on $\underline{\mathfrak{g}}^*{>\!\!\!\triangleleft}\mathfrak{g}$ from Theorem 3.5. The braided-Lie cobracket of $\underline{\mathfrak{g}}^*$ coincides with that of $\mathfrak{g}^*$, i.e., $\underline{\delta}\phi = \delta\phi$. We also use the quasitriangular form of $\mathfrak{g}$ to compute its Lie cobracket on $r^{(2)}$.

Note that another way to present the result is that $\pi(\xi)=\xi$ and $\pi(\phi) = -r^{(2)}\langle\phi,r^{(1)}\rangle$ is a Lie bialgebra projection $D(\mathfrak{g})\to\mathfrak{g}$ split by the inclusion of $\mathfrak{g}$, and recognise $\underline{\mathfrak{g}}^*$ as the image under θ of the braided-Lie bialgebra kernel of this according to Theorem 3.7. The computations involved are similar to the above proofs for θ. Similar formulae are obtained if one takes $\pi(\phi) = r^{(1)}\langle\phi,r^{(2)}\rangle$, corresponding to transmutation with respect to the conjugate quasitriangular structure. □

This is the Lie version of the result for the quantum double of a quasitriangular Hopf algebra in [**17**]. It completes the partial result in [**14**] where, in the absence of a theory of braided-Lie bialgebras we could only give the result $D(\mathfrak{g}){\cong}\underline{\mathfrak{g}}{>\!\!\!\triangleleft}\mathfrak{g}$ in the factorisable case (where $\underline{\mathfrak{g}}^*{\cong}\mathfrak{g}$) and only as a Lie algebra isomorphism. Since $\underline{\mathfrak{g}}{>\!\!\!\triangleleft}\mathfrak{g}$ by ad is easily seen to be isomorphic to a direct sum Lie algebra $\mathfrak{g}\oplus\mathfrak{g}$, one recovers the result that $D(\mathfrak{g})$ in the factorisable case is a Lie algebra direct sum, but now with a certain Lie bialgebra structure (namely the double cross cosum $\mathfrak{g}\blacktriangleright\!\!\blacktriangleleft\mathfrak{g}$ in [**10**]).

More recently, we have obtained a more general 'double bosonisation' theorem [**15**] which yields as output quasitriangular Hopf algebras. It provides an inductive construction for factorisable quasitriangular Hopf algebras such as $U_q(\mathfrak{g})$. The Lie version of this is as follows. We suppose $\mathfrak{c},\mathfrak{b}$ are dually paired in the sense of a morphism $\langle\ ,\ \rangle:\mathfrak{c}\otimes\mathfrak{b}\to k$ such that the Lie bracket of one is adjoint to the Lie cobracket of the other, and vice versa. The nicest case is where $\mathfrak{b}$ is finite-dimensional and $\mathfrak{c}=\mathfrak{b}^*$ as in Lemma 3.4, but we do not need to assume this for the main construction.

Theorem 3.10. *For dually paired braided Lie bialgebras $\mathfrak{b},\mathfrak{c}\in{}_{\mathfrak{g}}\mathcal{M}$ the vector space $\mathfrak{b}\oplus\mathfrak{g}\oplus\mathfrak{c}$ has a unique Lie bialgebra structure $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}{\triangleright\!\!\!<}\mathfrak{c}^{\rm op}$, the* double-bosonisation*, such that $\mathfrak{g}$ is a sub-Lie bialgebra, $\mathfrak{b},\mathfrak{c}^{\rm op}$ are sub-Lie algebras, and*

$$[\xi,x]=\xi{\triangleright}x,\quad [\xi,\phi]=\xi{\triangleright}\phi,$$
$$[x,\phi]=x_{\underline{(1)}}\langle\phi,x_{\underline{(2)}}\rangle+\phi_{\underline{(1)}}\langle\phi_{\underline{(2)}},x\rangle+2{r_+}^{(1)}\langle\phi,{r_+}^{(2)}{\triangleright}x\rangle$$
$$\delta x=\underline{\delta}x+r^{(2)}\otimes r^{(1)}{\triangleright}x-r^{(1)}{\triangleright}x\otimes r^{(2)},$$
$$\delta\phi=\underline{\delta}\phi+r^{(2)}{\triangleright}\phi\otimes r^{(1)}-r^{(1)}\otimes r^{(2)}{\triangleright}\phi$$

$\forall x\in\mathfrak{b},\xi\in\mathfrak{g}$ and $\phi\in\mathfrak{c}$. Here $\underline{\delta}x=x_{\underline{(1)}}\otimes x_{\underline{(2)}}$.

Proof. Here $\mathfrak{b},\mathfrak{g}$ clearly form the bosonisation Lie bialgebra $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}$ from Theorem 3.5. In the same way, we recognise $\mathfrak{g}{\triangleright\!\!\!<}\mathfrak{c}^{\rm op}$ as the bosonisation of $\mathfrak{c}^{\rm op}$ as a braided-Lie bialgebra in the category of $\mathfrak{g}$-modules with opposite infinitesimal braiding (see the remark below Lemma 3.4). Since these are already known to form Lie bialgebras, the coJacobi identity for the double-bosonisation holds, as well as the 1-cocycle axiom for all cases except $\delta([x,\phi])$ mixing $\mathfrak{b},\mathfrak{c}$. We outline the proof of this remaining case. From the definition of $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}{\triangleright\!\!\!<}\mathfrak{c}^{\rm op}$, we have

$$\begin{aligned}
&\delta([x,\phi])\\
&=\delta(x_{\underline{(1)}}\langle\phi,x_{\underline{(2)}}\rangle+\phi_{\underline{(1)}}\langle\phi_{\underline{(2)}},x\rangle+2{r_+}^{(1)}\langle\phi,{r_+}^{(2)}{\triangleright}x\rangle)\\
&=x_{\underline{(1)}\underline{(1)}}\otimes x_{\underline{(1)}\underline{(2)}}\langle\phi,x_{\underline{(2)}}\rangle+r^{(2)}\otimes r^{(1)}{\triangleright}x_{\underline{(1)}}\langle\phi,x_{\underline{(2)}}\rangle-r^{(1)}{\triangleright}x_{\underline{(1)}}\otimes r^{(2)}\langle\phi,x_{\underline{(2)}}\rangle\\
&\quad+\phi_{\underline{(1)}\underline{(1)}}\otimes\phi_{\underline{(1)}\underline{(2)}}\langle\phi_{\underline{(2)}},x\rangle+r^{(2)}{\triangleright}\phi_{\underline{(1)}}\otimes r^{(1)}\langle\phi_{\underline{(2)}},x\rangle-r^{(1)}\otimes r^{(2)}{\triangleright}\phi_{\underline{(1)}}\langle\phi_{\underline{(2)}},x\rangle\\
&\quad+2\delta{r_+}^{(1)}\langle\phi,{r_+}^{(2)}{\triangleright}x\rangle
\end{aligned}$$

$$\begin{aligned}
&{\rm ad}_x\delta\phi\\
&=[x,\phi_{\underline{(1)}}]\otimes\phi_{\underline{(2)}}+\phi_{\underline{(1)}}\otimes[x,\phi_{\underline{(2)}}]+[x,r^{(2)}{\triangleright}\phi]\otimes r^{(1)}
\end{aligned}$$

$$
\begin{aligned}
&+ r^{(2)}\triangleright\phi\otimes[x,r^{(1)}]-[x,r^{(1)}]\otimes r^{(2)}\triangleright\phi-r^{(1)}\otimes[x,r^{(2)}\triangleright\phi]\\
=\,& x_{\underline{(1)}}\langle\phi_{\underline{(1)}},x_{\underline{(2)}}\rangle\otimes\phi_{\underline{(2)}}+\phi_{\underline{(1)(1)}}\langle\phi_{\underline{(1)(2)}},x\rangle\otimes\phi_{\underline{(2)}}+2{r_+}^{(1)}\langle\phi_{\underline{(1)}},{r_+}^{(2)}\triangleright x\rangle\otimes\phi_{\underline{(2)}}\\
&+\phi_{\underline{(1)}}\otimes x_{\underline{(1)}}\langle\phi_{\underline{(2)}},x_{\underline{(2)}}\rangle+\phi_{\underline{(1)}}\otimes\phi_{\underline{(2)(1)}}\langle\phi_{\underline{(2)(2)}},x\rangle\\
&+\phi_{\underline{(1)}}\otimes 2{r_+}^{(1)}\langle\phi_{\underline{(2)}},{r_+}^{(2)}\triangleright x\rangle-r^{(2)}\triangleright\phi\otimes r^{(1)}\triangleright x+r^{(1)}\triangleright x\otimes r^{(2)}\triangleright\phi\\
&+[x,r^{(2)}\triangleright\phi]\otimes r^{(1)}-r^{(1)}\otimes[x,r^{(2)}\triangleright\phi].
\end{aligned}
$$

In a similar way, one has

$$
\begin{aligned}
&-\mathrm{ad}_\phi\delta x\\
=\,& x_{\underline{(1)(1)}}\langle\phi,x_{\underline{(1)(2)}}\rangle\otimes x_{\underline{(2)}}+\phi_{\underline{(1)}}\langle\phi_{\underline{(2)}},x_{\underline{(1)}}\rangle\otimes x_{\underline{(2)}}+2{r_+}^{(1)}\langle\phi,{r_+}^{(2)}\triangleright x_{\underline{(1)}}\rangle\otimes x_{\underline{(2)}}\\
&+x_{\underline{(1)}}\otimes x_{\underline{(2)(1)}}\langle\phi,x_{\underline{(2)(2)}}\rangle+x_{\underline{(1)}}\langle\phi_{\underline{(2)}},x_{\underline{(2)}}\rangle\otimes\phi_{\underline{(1)}}\\
&+x_{\underline{(1)}}\otimes 2{r_+}^{(1)}\langle\phi,{r_+}^{(2)}\triangleright x_{\underline{(2)}}\rangle\\
&+r^{(2)}\triangleright\phi\otimes r^{(1)}\triangleright x-r^{(1)}\triangleright x\otimes r^{(2)}\triangleright\phi\\
&+r^{(2)}\otimes[r^{(1)}\triangleright x,\phi]-[r^{(1)}\triangleright x,\phi]\otimes r^{(2)}.
\end{aligned}
$$

Adding the latter two expressions and comparing with $\delta([x,\phi])$ we see that the terms of the form $r^{(2)}\triangleright\phi\otimes r^{(1)}\triangleright x$ etc. immediately cancel, the terms of the form $\phi_{\underline{(1)}}\langle\phi_{\underline{(2)}},x_{\underline{(2)}}\rangle\otimes x_{\underline{(2)}}$ etc. (involving Lie cobrackets of both x and ϕ) cancel by antisymmetry of the Lie cobrackets, and the terms of the form $x_{\underline{(1)(1)}}\langle\phi,x_{\underline{(1)(2)}}\rangle\otimes x_{\underline{(2)}}$ etc. (involving iterated Lie cobrackets of either x or ϕ) cancel using antisymmetry of the Lie cobrackets and the coJacobi identity $(\mathrm{id}\otimes\underline{\delta})\underline{\delta}+\text{cyclic}=0$ for $\mathfrak{b}$ and $\mathfrak{c}$. Hence the 1-cocycle identity for this case reduces to the more manageable

$$
\begin{aligned}
&r^{(2)}\otimes r^{(1)}\triangleright x_{\underline{(1)}}\langle\phi,x_{\underline{(2)}}\rangle+r^{(2)}\triangleright\phi_{\underline{(1)}}\otimes r^{(1)}\langle\phi_{\underline{(2)}},x\rangle\\
&\quad+[2{r_+}^{(1)},r'^{(1)}]\otimes r'^{(2)}\langle\phi,{r_+}^{(2)}\rangle-\text{flip}\\
&=2{r_+}^{(1)}\langle\phi_{\underline{(1)}},{r_+}^{(2)}\triangleright x\rangle\otimes\phi_{\underline{(2)}}+2{r_+}^{(1)}\langle\phi,{r_+}^{(2)}\triangleright x_{\underline{(1)}}\rangle\otimes x_{\underline{(2)}}\\
&\quad+[x,r^{(2)}\triangleright\phi]\otimes r^{(1)}-[r^{(1)}\triangleright x,\phi]\otimes r^{(2)}-\text{flip}
\end{aligned}
$$

where '-flip' means to subtract all the same expressions with the opposite tensor product. We used antisymmetry of the Lie cobrackets and the quasitriangular of $\mathfrak{g}$ for $\delta {r_+}^{(1)}$. One then has to put in the stated definitions of the Lie brackets $[x,r^{(2)}\triangleright\phi]$ and $[r^{(1)}\triangleright x,\phi]$ and use $\mathfrak{g}$-covariance of the pairing, and of the braided-Lie brackets and cobrackets to obtain equality.

Note that by comparing the Lie bosonisation formulae with the braided group case, we can read off the Lie double-bosonisation formulae from the braided group case given in the required left-module form in the appendix of [**18**]. The only subtlety is that in the Lie case we can eliminate the categorical pairing ev (corresponding to the categorical dual $\mathfrak{b}^\star$ in the finite-dimensional case): $\mathfrak{c},\mathfrak{b}$ are categorically paired by $\mathrm{ev}:\mathfrak{c}\otimes\mathfrak{b}\to k$ *iff* $\langle\ ,\ \rangle=-\mathrm{ev}$ is a ($\mathfrak{g}$-equivariant) ordinary duality pairing. Then one obtains the $[x,\phi]$

relations as stated. Finally, in [**15**] it is explicitly shown that the double-bosonisation is built on the tensor product vector space. The analogous arguments now prove that the Lie double bosonisation is built on the direct sum vector space. □

Proposition 3.11. *Let $\mathfrak{b}\in {}_{\mathfrak{g}}\mathcal{M}$ be a finite-dimensional braided-Lie bialgebra with dual $\mathfrak{b}^*$. Then the double-bosonisation $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}{\triangleright\!\!\!<}\mathfrak{b}^{*\mathrm{op}}$ is quasitriangular, with*

$$r^{\mathrm{new}}=r+\sum_a f^a\otimes e_a,$$

where $\{e_a\}$ is a basis of $\mathfrak{b}$ and $\{f^a\}$ is a dual basis, and r is the quasitriangular structure of $\mathfrak{g}$. If $\mathfrak{g}$ is factorisable then so is the double-bosonisation.

Proof. We show first that the Lie cobracket of the double-bosonisation has the form $\delta=\mathrm{d}r_{\mathrm{new}}$. With summation over a understood, we have

$$\begin{aligned}
&[\phi,{r^{(1)}}_{\mathrm{new}}]\otimes {r^{(2)}}_{\mathrm{new}}+{r^{(1)}}_{\mathrm{new}}\otimes[\phi,{r^{(2)}}_{\mathrm{new}}]\\
&=[\phi,r^{(1)}]\otimes r^{(2)}+r^{(1)}\otimes[\phi,r^{(2)}]-[\phi,f^a]_{\mathfrak{b}^*}\otimes e_a+f^a\otimes[\phi,e_a]\\
&=-[\phi,f^a]_{\mathfrak{b}^*}\otimes e_a-r^{(1)}{\triangleright}\phi\otimes r^{(2)}-r^{(1)}\otimes r^{(2)}{\triangleright}\phi\\
&\quad -f^a\otimes e_{a\underline{(1)}}\langle\phi,e_{a\underline{(2)}}\rangle-f^a\otimes\phi_{\underline{(1)}}\langle\phi_{\underline{(2)}},e_a\rangle-f^a\otimes 2{r_+}^{(1)}\langle\phi,{r_+}^{(2)}{\triangleright}e_a\rangle\\
&=\underline{\delta}\phi-r^{(1)}{\triangleright}\phi\otimes r^{(2)}-r^{(1)}\otimes r^{(2)}{\triangleright}\phi+2{r_+}^{(2)}{\triangleright}\phi\otimes {r_+}^{(1)}=\delta\phi
\end{aligned}$$

as required. Here $[f^a,\phi]_{\mathfrak{b}^*}\otimes e_a=f^a\otimes e_{a\underline{(1)}}\langle\phi,e_{a\underline{(2)}}\rangle$ since both evaluate against $x\in\mathfrak{b}$ to $x_{\underline{(1)}}\langle\phi,x_{\underline{(2)}}\rangle$. The suffix $\mathfrak{b}^*$ is to avoid confusion with the Lie bracket inside the double-bosonisation, which is that of $\mathfrak{b}^{*\mathrm{op}}$ on these elements. Similarly,

$$\begin{aligned}
&[x,{r^{(1)}}_{\mathrm{new}}]\otimes {r^{(1)}}_{\mathrm{new}}+{r^{(1)}}_{\mathrm{new}}\otimes[x,{r^{(2)}}_{\mathrm{new}}]\\
&=-r^{(1)}{\triangleright}x\otimes r^{(2)}-r^{(1)}\otimes r^{(2)}{\triangleright}x+[x,f^a]\otimes e_a+f^a\otimes[x,e_a]\\
&=-r^{(1)}{\triangleright}x\otimes r^{(2)}-r^{(1)}\otimes r^{(2)}{\triangleright}x+f^a\otimes[x,e_a]+x_{\underline{(1)}}\langle f^a,x_{\underline{(2)}}\rangle\otimes e_a\\
&\quad +{f^a}_{\underline{(1)}}\langle {f^a}_{\underline{(2)}},x\rangle\otimes e_a+2{r_+}^{(1)}\otimes {r_+}^{(2)}{\triangleright}x\\
&=\underline{\delta}x-r^{(1)}{\triangleright}x\otimes r^{(2)}+r^{(2)}\otimes r^{(1)}{\triangleright}x=\delta x.
\end{aligned}$$

Here ${f^a}_{\underline{(1)}}\langle {f^a}_{\underline{(2)}},x\rangle\otimes e_a=-f^a\otimes[x,e_a]$ as both evaluate against $\phi\in\mathfrak{b}^*$ to $\phi_{\underline{(1)}}\langle\phi_{\underline{(2)}},x\rangle$. Since the Lie cobracket of the double-bosonisation is antisymmetric, we conclude also that $2r_{+\mathrm{new}}$ is ad-invariant.

Finally, we verify the CYBE for r_{new}. Actually, once $\delta=\mathrm{d}r_{\mathrm{new}}$ has been established, the CYBE is equivalent to

$$(\delta\otimes\mathrm{id})r_{\mathrm{new}}={r^{(1)}}_{\mathrm{new}}\otimes {r'^{(1)}}_{\mathrm{new}}\otimes[{r^{(2)}}_{\mathrm{new}},{r'^{(2)}}_{\mathrm{new}}]$$

(see [**10**]). Note that

$$\underline{\delta}f^a\otimes e_a=f^a\otimes f^b\otimes[e_a,e_b]$$

(sum over a, b) since evaluation against $x, y \in \mathfrak{b}$ gives $[x, y]$ in both cases. Then

$$\begin{aligned}(\delta\otimes \mathrm{id})r_{\mathrm{new}} &= (\delta\otimes \mathrm{id})r + \delta f^a\otimes e_a\\ &= r^{(1)}\otimes r'^{(1)}\otimes[r^{(2)}, r'^{(2)}] + f^a\otimes f^b\otimes[e_a, e_b]\\ &\quad + r^{(2)}{\triangleright} f^a\otimes r^{(1)}\otimes e_a - r^{(1)}\otimes r^{(2)}{\triangleright} f^a\otimes e_a\\ &= r^{(1)}\otimes r'^{(1)}\otimes[r^{(2)}, r'^{(2)}] + f^a\otimes f^b\otimes[e_a, e_b]\\ &\quad - f^a\otimes r^{(1)}\otimes[r^{(2)}, e_a] + r^{(1)}\otimes f^a\otimes[r^{(2)}, e_a]\end{aligned}$$

as required. We used $\mathfrak{g}$-covariance of the pairing, so that $\xi{\triangleright} f^a\otimes e_a = -f^a\otimes \xi{\triangleright} e_a = -f^a\otimes[\xi, e_a]$ for all $\xi\in\mathfrak{g}$.

If $\mathfrak{g}$ is factorisable then $2r_{+\mathrm{new}}$ as a map $(\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}{\triangleright\!\!\!<}\mathfrak{b}^{*\mathrm{op}})^*\to\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}{\triangleright\!\!\!<}\mathfrak{b}^{*\mathrm{op}}$ has $\mathfrak{g}$ in its image, by restricting to $\mathfrak{g}$. It has $\mathfrak{b}$ in its image by restricting to $\mathfrak{b}$, and $\mathfrak{b}^*$ in its image by restricting to $\mathfrak{b}^*$. So the double-bosonisation is again factorisable. Explicitly, if we denote by K the bilinear form on $\mathfrak{g}$ corresponding to the inverse of $2r_+$ as a map, we have

$$K_{\mathrm{new}}(x\oplus\xi\oplus\phi, y\oplus\eta\oplus\psi) = \langle\psi, x\rangle + K(\xi,\eta) + \langle\phi, y\rangle.$$

□

There is also a more general double-bisum construction $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{f}{\triangleright\!\!\!<}\mathfrak{c}^{\mathrm{op}}$ containing biproducts $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{f}$ and $\mathfrak{f}{\triangleright\!\!\!<}\mathfrak{c}^{\mathrm{op}}$ (with $\mathfrak{c},\mathfrak{b}\in{}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ suitably paired braided-Lie bialgebras) and reducing to the double-bosonisation in the case when $\mathfrak{c},\mathfrak{b}$ are in the image of the functor in Lemma 3.8.

Double bosonisation reduces to Drinfeld's double $D(\mathfrak{b})$ when $\mathfrak{g}=0$ (then a braided-Lie bialgebra reduces to an ordinary Lie bialgebra). And because it preserves factorisability, it provides an inductive construction for new factorisable quasitriangular Lie bialgebras from old ones. We will see in the next section that it can be used as a coordinate free version of the idea of adjoining a node to a Dynkin diagram (adjoining a simple root vector in the Cartan-Weyl basis). Moreover, building up $\mathfrak{g}$ iteratively like this also builds up the quasitriangular structure r. Finally, the triangular decomposition implies, in particular, examples of Lie algebra splittings and hence of matched pairs of Lie algebras as in [**9**]. Thus, $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}{\triangleright\!\!\!<}\mathfrak{b}^{*\mathrm{op}} = (\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}){\bowtie}\mathfrak{b}^{*\mathrm{op}}$ as Lie algebras, where $\mathfrak{b}{>\!\!\!\triangleleft}\mathfrak{g}$ (the semidirect sum by the given action of $\mathfrak{g}$ on $\mathfrak{b}$) and $\mathfrak{b}^{*\mathrm{op}}$ act on each other by

$$\phi{\triangleright} x = \langle\phi, x_{\underline{(1)}}\rangle x_{\underline{(2)}} - 2{r_+}^{(1)}\langle\phi, {r_+}^{(2)}{\triangleright} x\rangle,$$
$$\phi{\triangleright}\xi = 0,\quad \phi{\triangleleft} x = \langle\phi_{\underline{(1)}}, x\rangle\phi_{\underline{(2)}},\quad \phi{\triangleleft}\xi = -\xi{\triangleright}\phi$$

for $x\in\mathfrak{b}$, $\phi\in\mathfrak{b}^*$, $\xi\in\mathfrak{g}$. This is immediate from the Lie bracket stated in Theorem 3.10.

4. Parabolic Lie bialgebras and Lie induction.

In this section we give some concrete examples and applications of the above theory. We work over $\mathbb{C}$. We begin with the simplest example of a braided-Lie bialgebra, with zero Lie bracket and zero Lie cobracket. According to Definition 2.2 this means precisely modules of our background quasitriangular Lie bialgebras for which the infinitesimal braiding cocycle ψ vanishes.

Proposition 4.1. *Let $\mathfrak{g}$ be a semisimple factorisable (s.s.f) Lie bialgebra and $\mathfrak{b}$ an isotypical representation such that $\Lambda^2\mathfrak{b}$ is isotypical. Then $\mathfrak{b}$ with zero bracket and zero cobracket is a braided-Lie bialgebra in ${}_{\tilde{\mathfrak{g}}}\mathcal{M}$, where $\tilde{\mathfrak{g}}$ is a central extension.*

Proof. Let $c = r_+{}^{(1)} r_+{}^{(2)}$ in $U(\mathfrak{g})$. Since r_+ is ad-invariant, c is central. Moreover, $2r_+ = \Delta c - (c \otimes 1 + 1 \otimes c)$ where Δ is the coproduct of $U(\mathfrak{g})$ as a Hopf algebra. Since $\mathfrak{b}$ is assumed isotypical, the action of c on it is by multiplication by a scalar, say λ_1. Since $\Lambda^2\mathfrak{b}$ is assumed isotypical, the action of c on it, which is the action of Δc in each factor, is also multiplication by a scalar, say λ_2. Then $\psi(x \otimes y) = (\Delta c - (c \otimes 1 + 1 \otimes c)){\triangleright}(x \otimes y - y \otimes x) = (\lambda_2 - 2\lambda_1)(x \otimes y - y \otimes x) = \lambda(x \otimes y - y \otimes x)$ say, where λ is a constant.

Now, $\mathfrak{b}$ with the zero bracket and cobracket is not a braided group in ${}_{\mathfrak{g}}\mathcal{M}$ unless our cocycle ψ vanishes. However, in the present case we can neutralise the cocycle with a central extension. Thus, let $\tilde{\mathfrak{g}} = \mathbb{C} \oplus \mathfrak{g}$ with $\mathbb{C}$ spanned by ς, say. We take the Lie bracket, quasitriangular structure and Lie cobracket

$$[\xi, \varsigma] = 0, \quad \tilde{r} = r - \frac{\lambda}{2}\varsigma \otimes \varsigma, \quad \delta\varsigma = 0$$

for all $\xi \in \mathfrak{g}$. In this way, $\tilde{\mathfrak{g}}$ becomes a quasitriangular Lie bialgebra. We consider $\mathfrak{b} \in {}_{\tilde{\mathfrak{g}}}\mathcal{M}$ by $\varsigma{\triangleright}x = x$ for all $x \in \mathfrak{b}$. The infinitesimal braiding on $\mathfrak{b}$ in this category is $\tilde{\psi}(x \otimes y) = 2\tilde{r}_+{\triangleright}(x \otimes y - y \otimes x) = \psi(x \otimes y) - \lambda(x \otimes y - y \otimes x) = 0$. So $\mathfrak{b}$ is a braided-Lie bialgebra in this category. $\square$

The constant λ is the infinitesimal analogue of the so-called quantum group normalisation constant. The central extension is the analogue of the central extension by a 'dilaton' needed for the quantum planes to be viewed as braided groups [7]. We see now the infinitesimal analogue of this phenomenon.

Next, we can apply Theorem 3.5 and obtain a Lie bialgebra $\mathfrak{b}{>\!\!\!\triangleleft}\tilde{\mathfrak{g}}$ as the bosonisation of $\mathfrak{b}$. Moreover, double-bosonisation provides a still bigger and factorisable Lie algebra containing $\mathfrak{b}{>\!\!\!\triangleleft}\tilde{\mathfrak{g}}$.

Corollary 4.2. *Let $\mathfrak{g}$ be simple and strictly quasitriangular, and $\mathfrak{b}$ a finite-dimensional irreducible representation with $\Lambda^2\mathfrak{b}$ isotypical. Then the double bosonisation $\mathfrak{b}{>\!\!\!\triangleleft}\tilde{\mathfrak{g}}{\triangleright\!\!\!<}\mathfrak{b}^*$ from Theorem 3.10 is again simple, strictly quasitriangular and of strictly greater rank.*

Proof. The Lie bracket in the double-bosonisation in Theorem 3.10, and the form of $\tilde{r}$ are

$$[\xi, x] = \xi{\triangleright}x, \quad [\varsigma, x] = x, \quad [\xi, \phi] = \xi{\triangleright}\phi, \quad [\varsigma, \phi] = -\phi, \quad [\xi, \varsigma] = 0$$

$$[x, \phi] = 2{r_+}^{(1)}\langle\phi, {r_+}^{(2)}{\triangleright}x\rangle - \lambda\varsigma\langle\phi, x\rangle$$

for all $\xi \in \mathfrak{g}$, $x \in \mathfrak{b}$ and $\phi \in \mathfrak{g}^*$. Consider $I \subseteq \mathfrak{b} \oplus \mathfrak{g} \oplus \mathbb{C} \oplus \mathfrak{b}^*$ an ideal of the double-bosonisation. Let $I_\mathfrak{b}, I_{\mathfrak{b}^*}, I_\mathfrak{g}, I_\mathbb{C}$ be the components of I in the direct sum. By the relation $[\xi, x] = \xi{\triangleright}x$, $I_\mathfrak{b} \subseteq \mathfrak{b}$ is a subrepresentation under $\mathfrak{g}$. Since $\mathfrak{b}$ is irreducible, $I_\mathfrak{b}$ is either zero or $\mathfrak{b}$. Similarly for $I_{\mathfrak{b}^*}$. Likewise $I_\mathfrak{g}$ is zero or $\mathfrak{g}$ as $\mathfrak{g}$ is simple. Finally, $I_\mathbb{C}$ is zero or $\mathbb{C}$ by linearity. We therefore have 16 possibilities to consider for whether $\mathbb{C}, \mathfrak{g}, \mathfrak{b}, \mathfrak{b}^*$ are contained or not in I. (i) If $\mathfrak{g}$ is contained, then since $\mathfrak{b}$ is irreducible, the relation $[\xi, x] = \xi{\triangleright}x$ spans $\mathfrak{b}$ for any fixed x, and hence is certainly not always zero. So $\mathfrak{b}$ is contained, and likewise $\mathfrak{b}^*$ is contained if $\mathfrak{g}$ is. In this case, the $[x, \phi]$ relation means that $\mathbb{C}$ is contained and I is the whole space. (ii) If $\mathfrak{b}$ is contained then the $[x, \phi]$ relation and $2r_+$ nondegenerate means that $\mathfrak{g}$ and $\mathbb{C}$ are contained and hence I is the whole space. (iii) Similarly if $\mathfrak{b}^*$ is contained. (iv) Finally, if $\mathbb{C}$ is contained then the relation $[\varsigma, x] = x$ implies that $\mathfrak{b}$ is contained and hence I is the whole space. Hence I is zero or the whole space, as required. The new quasitriangular structure is non-zero since its component in $\mathfrak{g} \otimes \mathfrak{g}$ is non-zero. The rank is clearly increased by at least 1 due to the addition of ς. ☐

Thus the double-bosonisation in Theorem 3.10 provides an inductive construction for simple strictly quasitriangular Lie bialgebras. It is possible to see that the fundamental representations of su_n or so_n take us up to su_{n+1} and so_{n+1}, i.e., precisely take us up the ABD series in the usual classification of Lie algebras. Moreover, we see the role of the single bosonisation in Theorem 3.5:

Example 4.3. Consider $\mathfrak{g} = su_2$ with the Drinfeld-Sklyanin quasitriangular structure. The 2-dimensional irreducible representation $\mathfrak{b}$ is a braided-Lie bialgebra via Proposition 4.1. Its bosonisation $\mathbb{C}^2{>\!\!\!\triangleleft}\widetilde{su_2}$ is the maximal parabolic of the double bosonisation $\mathbb{C}^2{>\!\!\!\triangleleft}\tilde{\mathfrak{g}}{\triangleright\!\!\!<}\mathbb{C}^2 = su_3$. Explicitly, it is the Lie algebra of su_2 and

$$[x, y] = 0, \quad [X_+, x] = 0, \quad [X_+, y] = x, \quad [X_-, x] = y, \quad [X_-, y] = 0$$
$$[H, x] = x, \quad [H, y] = -y, \quad [\varsigma, H] = 0, \quad [\varsigma, X_\pm] = 0,$$
$$[\varsigma, x] = x, \quad [\varsigma, y] = y$$

where $\{x, y\}$ are a basis of $\mathbb{C}^2$ and $H, X_\pm$ are the standard su_2 Chevalley generators. The Lie cobracket on the generators is

$$\delta\varsigma = 0, \quad \delta X_\pm = \frac{1}{2}X_\pm \wedge H, \quad \delta x = \frac{1}{2}x \wedge h$$

where $h = -\frac{1}{2}H - \frac{3}{2}\varsigma$ and $\wedge = (\mathrm{id} - \tau) \circ \otimes$.

Proof. Note that we work over $\mathbb{C}$, but there is are natural real forms justifying the notation. Here $\Lambda^2\mathfrak{b}$ is the 1-dimensional (i.e., spin 0) representation of su_2. The standard quasitriangular structure of su_2 is

$$r = \frac{1}{4}H \otimes H + X_+ \otimes X_-.$$

Then $c = r_+{}^{(1)}r_+{}^{(2)}$ is twice the quadratic Casimir in its usual normalisation. Hence its value in the $(2j+1)$ dimensional (i.e., spin j) irreducible representation is $j(j+1)$. In the present case, we have $\lambda = 0.(0+1)-2.\frac{1}{2}(\frac{1}{2}+1) = -\frac{3}{2}$ in Proposition 4.1. We therefore make the central extension to $\tilde{\mathfrak{g}}$ and apply Theorem 3.5. The Lie algebra of the bosonisation is given by the action of $\widetilde{\mathfrak{g}}$. Its explicit form in the representation $\rho(X_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\rho(X_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is $X_+ \triangleright x = 0$, $X_+ \triangleright y = x$, $X_- \triangleright x = y$, $X_- \triangleright y = 0$, $H \triangleright x = x$ and $H \triangleright y = -y$, giving the Lie bracket stated. The Lie cobracket is $\delta x = 0 + \tilde{r}^{(2)} \wedge \tilde{r}^{(1)} \triangleright x = \frac{1}{4}H \wedge H \triangleright x + \frac{3}{4}\varsigma \wedge \varsigma \triangleright x = \frac{1}{2}x \wedge h$ as stated. We identify $X_\pm = X_{\pm 1}, H = H_1$ as a sub-Lie algebra of su_3 and $x = X_{-2}, h = H_2$ as the remaining Chevalley generators of its standard maximal parabolic. Finally, let $\mathfrak{b}^*$ have dual basis $\{\phi, \psi\}$. By a similar computation to the above, we obtain $s\tilde{u}_2 \triangleright\!\!\!< \mathbb{C}^2$ with Lie bracket

$$[\phi, \psi] = 0, \quad [X_+, \phi] = -\psi, \quad [X_+, \psi] = 0, \quad [X_-, \phi] = 0, \quad [X_-, \psi] = -\phi$$
$$[H, \phi] = -\phi, \quad [H, \psi] = \psi, \quad [\varsigma, \phi] = -\phi, \quad [\varsigma, \psi] = -\psi.$$

Among the further $\mathfrak{b}, \mathfrak{b}^*$ brackets in the double bosonisation in Theorem 3.10, we have $[x, \phi] = 2r_+{}^{(1)}\langle \phi, r_+{}^{(2)} \triangleright x\rangle + \frac{3}{2}\varsigma\langle \phi, x\rangle = \frac{1}{2}H\langle \phi, H \triangleright x\rangle + 0 + \frac{3}{2}\varsigma\langle \phi, x\rangle = -h$. From these relations we find that $\phi = X_{+2}$ and $\psi = X_{+12}$ explicitly identifies the double bosonisation as su_3. The Lie cobracket on ϕ is $\delta\phi = \tilde{r}^{(2)} \triangleright \phi \wedge \tilde{r}^{(1)} = \frac{1}{4}H \triangleright \phi \wedge H + \frac{3}{4}\varsigma \triangleright \phi \wedge \varsigma = \frac{1}{2}\phi \wedge h$. This conforms with the standard Lie cobracket for su_3. Indeed, the quasitriangular structure of the double bosonisation in Theorem 3.10 reproduces the Drinfeld-Sklyanin quasitriangular structure of su_3. □

This is far from the only braided-Lie bialgebra in the category of $\widetilde{su_2}$-modules, however.

Example 4.4. Consider $\mathfrak{g} = su_2$ with the Drinfeld-Sklyanin quasitriangular structure. The 3-dimensional irreducible representation $\mathfrak{b}$ is a braided-Lie bialgebra via Proposition 4.1. Its bosonisation $\mathbb{R}^3 \rtimes \widetilde{so_3}$ is the maximal parabolic of the double bosonisation so_5. Explicitly, it is the Lie algebra of so_3 and

$$[x_i, x_j] = 0, \quad [e_i, x_j] = \textstyle\sum_k \epsilon_{ijk} x_k, \quad [\varsigma, x_j] = x_j$$

where $i, j, k = 1, 2, 3$ and ϵ is the totally antisymmetric tensor with $\epsilon_{123} = 1$. Here $\{e_i\}$ are the vector basis of so_3. The Lie cobracket is

$$\delta e_1 = \imath e_1 \wedge e_3, \quad \delta e_2 = \imath e_2 \wedge e_3, \quad \delta e_3 = 0, \quad \delta \varsigma = 0,$$
$$\delta x_1 = (\imath e_1 + e_2) \wedge x_3 + x_2 \wedge e_3 + \varsigma \wedge x_1$$
$$\delta x_2 = x_3 \wedge (e_1 - \imath e_2) + e_3 \wedge x_1 + \varsigma \wedge x_2,$$
$$\delta x_3 = (e_1 - \imath e_2) \wedge x_2 + x_1 \wedge (\imath e_1 + e_2) + \varsigma \wedge x_3.$$

Proof. Here $\Lambda^2\mathfrak{b}$ is also the 3-dimensional (i.e., spin 1) representation. Hence, from the first part of the proof of Proposition 4.1, we have $\lambda = 1.(1+1) - 2.1.(1+1) = -2$. The Lie algebra so_3 in the vector basis is $[e_1, e_2] = e_3$ and cyclic rotations of this, and the Drinfeld-Sklyanin quasitriangular structure in this basis is [**10**, Ex. 8.1.13]

$$r = -\sum_i e_i \otimes e_i + \imath(e_1 \otimes e_2 - e_2 \otimes e_1).$$

We add $\varsigma \otimes \varsigma$ to give the quasitriangular structure $\tilde{r}$. The action on $\mathbb{C}^3$ with basis x_i is $[e_1, x_2] = x_3$ and cyclic rotations of this. This immediately provides the Lie algebra of the bosonisation. The Lie cobracket from Theorem 3.5 is

$$\delta x_i = \imath e_2 \wedge [e_1, x_i] - \imath e_1 \wedge [e_2, x_i] + \sum_{j,k} e_j \wedge \epsilon_{ijk} x_k + \varsigma \wedge x_i$$

with computes as stated. □

This example is manifestly the Lie algebra of motions plus dilation of $\mathbb{R}^3$, as a sub-Lie algebra of the conformal Lie algebra $so(1, 4)$, equipped now with a Lie bialgebra structure. At the level of complex Lie algebra, it is the maximal parabolic of so_5. The generator ς is called the 'dilaton' in the corresponding quantum groups literature. We likewise obtain natural maximal parabolics for the whole ABD series by bosonisation of the fundamental representation $\mathfrak{b}$.

On the other hand, these steps for other Lie algebras can involve less trivial braided-Lie bialgebras $\mathfrak{b}$ (with non-zero bracket and cobracket). The general case is as follows. We consider simple Lie algebras $\mathfrak{g}$ associated to root systems in the usual conventions. Positive roots are denoted α, with length d_α. The Cartan-Weyl basis has root vectors $X_{\pm\alpha}$ and Cartan generators H_i corresponding to the simple roots α_i. We define $d_\alpha H_\alpha = \sum_i n_i d_i H_i$ if $\alpha = \sum_i n_i \alpha_i$. We take the Drinfeld-Sklyanin quasitriangular structure in its general form

$$r = \sum_\alpha d_\alpha X_\alpha \otimes X_{-\alpha} + \frac{1}{2} \sum_{ij} A_{ij} H_i \otimes H_j, \tag{12}$$

where $A_{ij}=d_i(a^{-1})_{ij}$. Here a is the Cartan matrix. The corresponding Lie cobracket is $\delta X_{\pm i}=\frac{d_i}{2}X_{\pm i}\wedge H_i$ and $\delta H_i=0$ on the generators.

Proposition 4.5. *Let i_0 be a choice of simple root such that its deletion again generates the root system of a simple Lie algebra, $\mathfrak{g}_0$. Let $\mathfrak{b}_-\subset\mathfrak{g}$ be the standard (negative) Borel and let $\mathfrak{f}\subset\mathfrak{b}_-$ denote the sub-Lie algebra excluding all vectors generated by X_{-i_0}. Both $\mathfrak{b}_-$ and $\mathfrak{f}$ are sub-Lie bialgebras of $\mathfrak{g}$ and*

$$\mathfrak{b}_-\underset{\hookleftarrow}{\overset{\pi}{\to}}\mathfrak{f},\quad \pi(H_i)=H_i,\quad \pi(X_{-\alpha})=\begin{cases}0 & \text{if } \alpha \text{ contains } \alpha_{i_0}\\ X_{-\alpha} & \text{else}\end{cases}$$

is a split Lie bialgebra projection. Then $\mathfrak{b}=\ker\pi$ is the Lie ideal generated by X_{-i_0} in $\mathfrak{b}_-$ and is a braided-Lie bialgebra in ${}^{\mathfrak{f}}_{\mathfrak{f}}\mathcal{M}$ by Theorem 3.7.

Proof. Here $\mathfrak{f}$ is generated by all the H_i and only those X_{-j} where $j\neq i_0$, i.e. spanned by the H_i and $\{X_{-\alpha}\}$ such that α does not contain α_{i_0}. It is clearly a sub-Lie algebra of $\mathfrak{b}_-$. We show first that it is a sub-Lie bialgebra. First of all, note that the Lie coproduct in $\mathfrak{g}$ has the general form

$$\delta X_{\pm\alpha}=\frac{d_\alpha}{2}X_{\pm\alpha}\wedge H_\alpha+\sum_{\beta+\gamma=\alpha}c_{\pm\beta,\pm\gamma}X_{\pm\beta}\wedge X_{\pm\gamma}$$

where the sum is over positive root β,γ adding up to α and the c are constants. The proof is by induction (being careful about signs). From the Lie bialgebra cocycle axiom and the induction hypothesis,

$$\begin{aligned}&\delta([X_i,X_\alpha])\\ &=\frac{d_\alpha}{2}[X_i,X_\alpha]\wedge H_\alpha+\frac{d_\alpha}{2}X_\alpha\wedge[X_i,H_\alpha]+\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}[X_i,X_\beta]\wedge X_\gamma\\ &\quad+\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}X_\beta\wedge[X_i,X_\gamma]-\frac{d_i}{2}[X_\alpha,X_i]\wedge H_i-\frac{d_i}{2}X_i\wedge[X_\alpha,H_i]\\ &=\frac{d_{\alpha+\alpha_i}}{2}[X_i,X_\alpha]\wedge H_{\alpha+\alpha_i}+\alpha(d_iH_i)X_i\wedge X_\alpha\\ &\quad+\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}[X_i,X_\beta]\wedge X_\gamma+\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}X_\beta\wedge[X_i,X_\gamma]\end{aligned}$$

if $\alpha+\alpha_i$ is a positive root. We used the identities $[d_\alpha H_\alpha,X_i]=\alpha(d_iH_i)X_i$ and $[d_iH_i,X_\alpha]=\alpha(d_iH_i)X_\alpha$. Since all positive root vectors are obtained by iterated Lie brackets of the X_i, we conclude the result (the argument for negative roots is similar).

From this form, it is clear first of all that δ restricts to $\mathfrak{b}_-\to\mathfrak{b}_-\otimes\mathfrak{b}_-$, so this becomes a sub-Lie bialgebra of $\mathfrak{g}$ (this is well-known). Moreover, if α does not involve α_{i_0} then neither can positive β,γ such that $\beta+\gamma=\alpha$. Hence $\mathfrak{f}$ is a Lie sub-bialgebra of $\mathfrak{b}_-$.

Finally, π is clearly a Lie algebra map by considering the cases separately. For elements of $\mathfrak{f}\otimes\mathfrak{f}$ we know that $\pi\circ[\ ,\ ]=[\ ,\ ]=[\pi(\ ),\pi(\ )]$ since $\mathfrak{f}$ is closed, while if α involves α_{i_0} then so does $\alpha+\beta$ and $\pi([X_{-\alpha},X_{-\beta}])=0=[\pi(X_{-\alpha}),\pi(X_{-\beta})]$. Moreover, π is a Lie coalgebra map on $\mathfrak{f}$ since $\delta\mathfrak{f}\subset\mathfrak{f}\otimes\mathfrak{f}$ as shown above. Finally, $(\pi\otimes\pi)\delta X_{-i_0}=0=\delta\pi(X_{-i_0})$ from the simple form of δ on the generators.

Therefore we may apply Theorem 3.7 and obtain a braided-Lie bialgebra $\mathfrak{b}=\ker\pi$. Here $\mathfrak{b}\subset\mathfrak{b}_-$ is the Lie ideal generated by X_{-i_0}, i.e., spanned by $\{X_{-\alpha}\}$ where α contains α_{i_0}.

The braided-Lie cobracket of $\mathfrak{b}$ from Theorem 3.7 is

$$\underline{\delta}X_{-\alpha}=\sum c_{-\beta,-\gamma}X_{-\beta}\wedge X_{-\gamma},$$

the part of the Lie cobracket $\delta X_{-\alpha}$ in which both β,γ contain α_{i_0}. The action of $\mathfrak{f}$ is by Lie bracket in $\mathfrak{b}_-$ and the Lie coaction of $\mathfrak{f}$ is $\beta(X_{-\alpha})=-\frac{d_\alpha}{2}H_\alpha\otimes X_{-\alpha}+\sum c_{-\beta,-\gamma}X_{-\beta}\otimes X_{-\gamma}$ where the sum is the part of $\delta X_{-\alpha}$ where β does not contain α_{i_0}. $\square$

This constructs the required braided-Lie bialgebra for the general case. Although obtained here in the category of $D(\mathfrak{f})$-modules, this action is compatible with an action of the central extension $\tilde{\mathfrak{g}_0}\subset\mathfrak{g}$. It is easy to see that there is a unique element $\varsigma\in\mathfrak{g}$, $\varsigma\notin\mathfrak{g}_0$ which commutes with the image of $\mathfrak{g}_0$. It is determined by the Cartan matrices of $\mathfrak{g},\mathfrak{g}_0$. Viewed in $\mathfrak{g}$, this $\tilde{\mathfrak{g}_0}$ acts on $\mathfrak{g}$ by the adjoint action and this action restricts to $\mathfrak{b}$. In this way, $\mathfrak{b}$ becomes a braided-Lie bialgebra in ${}_{\tilde{\mathfrak{g}_0}}\mathcal{M}$. One may then recover $\mathfrak{g}=\mathfrak{b}{>\!\!\!\triangleleft}\tilde{\mathfrak{g}_0}{\triangleright\!\!\!<}\mathfrak{b}^{*\mathrm{op}}$ from Theorem 3.10.

Example 4.6. When $\mathfrak{g}=g_2$ and $\mathfrak{g}_0=su_2$, we obtain the 5-dimensional braided-Lie bialgebra where su_2 acts as the $4\oplus 1$ dimensional (i.e., the spin $\frac{3}{2}$ and spin 0 representations). Both the Lie bracket and the Lie cobracket are not identically zero.

Proof. We take the Cartan matrix for $\mathfrak{g}$ as $\begin{pmatrix}2&-1\\-3&2\end{pmatrix}$. We take $i_0=1$ so that the required su_2 is spanned by $H_2,X_{\pm 2}$. The negative roots vectors $X_{-1},X_{-21},X_{-221},X_{-2221}$ span the 4-dimensional representation of su_2, the eigenvalues of the adjoint action of $-\frac{1}{2}H_2$ being $-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}$ respectively. These and the remaining negative root vector X_{-12221} (which forms a 1-dimensional trivial representation of su_2) are a basis of $\mathfrak{b}$. We then restrict the Lie bracket to $\mathfrak{b}$, the only non-zero entries being

$$[X_{-1},X_{-2221}]=X_{-12221}=[X_{-221},X_{-21}].$$

This is a central extension (by a cocycle) of the zero bracket on the 4-dimensional representation. The Lie cobracket can then be computed by projection of the Lie cobracket in g_2. Since (as one may easily verify) the infinitesimal braiding is nontrivial, both the braided-Lie bracket and braided-Lie cobracket on $\mathfrak{b}$ are not identically zero. The element $\varsigma = -2H_1 - H_2$ commutes with su_2 and acts as the identity in the 4-dimensional part of $\mathfrak{b}$. □

Example 4.7. When $\mathfrak{g} = sp_6$ and $\mathfrak{g}_0 = sp_4$, we obtain the 5-dimensional braided-Lie bialgebra where sp_4 acts in the $4 \oplus 1$ dimensional representation. Here the 4-dimensional representation is the fundamental one of sp_4. Both the Lie bracket and the Lie cobracket are not identically zero.

Proof. We take the Cartan matrices for $\mathfrak{g}$ and $\mathfrak{g}_0$ as

$$a = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad a_0 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

where $i_0 = 1$. We identify $sp_4 = C_2$ inside sp_6 as the root vectors $X_{\pm 2}$, $X_{\pm 3}$, $X_{\pm 23}$ and Cartan vectors H_2, H_3. The negative root vectors X_{-1}, X_{-12}, X_{-123}, X_{-1223} form the 4-dimensional representation of sp_4. These and the remaining negative root vector X_{-11223} (which forms a 1-dimensional trivial representation of sp_4) are a basis of $\mathfrak{b}$. We then restrict the Lie bracket to $\mathfrak{b}$ and find that this is again a cocycle central extension of the zero Lie bracket on the 4-dimensional representation, the only non-zero entries being

$$[X_{-12}, X_{-123}] = \frac{1}{2} X_{-11223}, \quad [X_{-1}, X_{-1223}] = X_{-11223}.$$

The infinitesimal braiding and the Lie cobracket are also nontrivial, as one may verify by further computation. The element $\varsigma = -(H_1 + H_2 + H_3)$ commutes with sp_4 and acts as the identity in the 4-dimensional part of $\mathfrak{b}$. □

These examples show that the general case need not depart too far from the setting of Proposition 4.1 and Corollary 4.2; we need to make a central extension of the underlying irreducible representation to define $\mathfrak{b}$. By construction, $\mathfrak{b} {>\!\!\!\triangleleft} \tilde{\mathfrak{g}}_0$ is once again the maximal parabolic of $\mathfrak{g}$ associated to α_{i_0}. A similar construction works for more roots missing, giving non-maximal parabolics of the double-bosonisation. We simply define π setting to zero all the root vectors containing the roots to be deleted in defining $\mathfrak{g}_0$. Clearly, the extreme example of this is $\mathfrak{f} = \mathfrak{t}$ (the Cartan subalgebra) so that $\pi(H_i) = H_i$ and $\pi(X_{-\alpha}) = 0$. Then $\mathfrak{b} = \mathfrak{n}_-$ (the Lie algebra generated by the X_{-i}) is a braided-Lie bialgebra in ${}^{\mathfrak{t}}_{\mathfrak{t}}\mathcal{M}$ with

$$\delta X_{-i} = 0, \quad \psi(X_{-\alpha} \otimes X_{-\beta}) = (\alpha, \beta) X_{-\alpha} \wedge X_{-\beta},$$

$$h {\triangleright} X_{-\alpha} = -\alpha(h), \quad \beta(X_{-\alpha}) = -\frac{d_\alpha}{2} H_\alpha \otimes X_{-\alpha},$$

for all $h \in \mathfrak{t}$. The coaction here is induced from the action as in Lemma 3.8, where $\mathfrak{t}$ is a quasitriangular Lie algebra with zero bracket, zero cobracket and $r = \frac{1}{2}\sum_{ij} A_{ij} H_i \otimes H_j$. In this way we can also view $\mathfrak{n}_- \in {}_{\mathfrak{t}}\mathcal{M}$ and $\mathfrak{g} = \mathfrak{n}_- \bowtie\!\!\!\triangleleft \mathfrak{t} \triangleright\!\!\!\bowtie \mathfrak{n}_+$ via Theorem 3.10, where we identify $(\mathfrak{n}_-)^{*\mathrm{op}} = \mathfrak{n}_+$ via the Killing form.

5. Concluding remarks.

We have given here the basic theory of braided-Lie algebras, obtained by infinitesimalising the existing theory of braided groups. We also outlined in Section 4 its application to the inductive construction of simple Lie algebras with their standard quasitriangular structures. Further variations of these constructions are certainly possible, and by making them one should be able to also obtain the other strictly quasitriangular Lie bialgebras structures in the Belavin-Drinfeld classification [**1**]. For example, there is a twisting theory of quantum groups [**4**] and braided groups [**19**]. An infinitesimal version of the latter would allow one to introduce additional twists at each stage of the inductive construction of the simple Lie algebra.

Also, although we have (following common practice) named our Lie algebras by their natural real forms, our Lie algebras in Section 4 were complex ones. There is a theory of $*$-braided groups (real forms of braided groups) as well as their corresponding bosonisations and double-bosonisations [**19**], [**18**]. The infinitesimal version of these should yield, for example, $so(1,4)$ as a real form arising from the double-bosonisation of the 3-dimensional braided-Lie bialgebra in Example 4.4. The construction of natural compact real forms and the classification of real forms would be a further goal. These are some directions for further work.

Finally, just as Lie bialgebras extend to Poisson-Lie groups, so braided-Lie bialgebra structures typically extend to the associated Lie group B of $\mathfrak{b}$, at least locally. First, one needs to exponentiate $\psi \in Z^2_{\mathrm{ad}}(\mathfrak{b}, \mathfrak{b}\otimes\mathfrak{b})$ to a group cocycle $\Psi \in Z^2_{\mathrm{Ad}}(B, \mathfrak{b}\otimes\mathfrak{b})$. Since $\mathrm{d}\underline{\delta} = \psi$, we should likewise exponentiate $\underline{\delta}$ to the group as a map $D : B \to \mathfrak{b}\otimes\mathfrak{b}$ with coboundary Ψ, and define from this a 'braided-Poisson bracket'. The latter will not, however, respect the group product in the usual way but rather up to a 'braiding' obtained from ψ. Details of these braided-Poisson-Lie groups and the example of the Kirillov-Kostant braided-Poisson bracket from Example 3.3 extended to the group manifold (e.g., to SU_2) will be developed elsewhere.

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Received May 26, 1998. Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge. Most of the paper was written when I was on Leave 1995-1996 at the Dept. of Mathematics, Harvard University, Cambridge, MA 02138.

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