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Let M be a compact, orientable, irreducible, ∂-irreducible, anannular 3-manifold with one component T of ∂M a torus. Suppose that r_1 and r_2 are two slopes on T. In this paper, we shall show that if $M(r_1)$ is reducible while $M(r_2)$ contains an essential annulus, then $\triangle(r_1, r_2) \leq 2$.

1. Introduction.

Let M be a compact, orientable, irreducible, ∂-irreducible, anannular 3-manifold with one component T of ∂M a torus. A slope r on T is a T-isotopy class of essential, unoriented, simple closed curves on T, and the distance between two slopes r_1 and r_2, denoted by $\triangle(r_1, r_2)$, is the minimal geometric intersection number among all the curves representing the slopes. For a slope r on T, we denote by $M(r)$ the surgered manifold obtained by attaching a solid torus J to M along T so that r bounds a disk in J. Now consider two distinct slopes r_1, r_2 on T. There are many results showing how constraints on the topology of $M(r_1)$ and $M(r_2)$ put constraints on $\triangle(r_1, r_2)$. For example, C. Gordon and J. Luecke [5] have shown that if $M(r_1)$ and $M(r_2)$ are reducible, then $\triangle(r_1, r_2) \leq 1$. C. Gordon [3] has shown that if M contains no essential torus, and $M(r_i)$ contains an essential torus, $i = 1, 2$, then $\triangle(r_1, r_2) \leq 8$. Y-Q Wu [8] has shown that if $M(r_1)$ and $M(r_2)$ are ∂-reducible, then $\triangle(r_1, r_2) \leq 1$. In this paper, we shall estimate $\triangle(r_1, r_2)$ when $M(r_1)$ is reducible, and $M(r_2)$ contains an essential annulus. The main result is the following theorem:

Theorem 1. *Let M be a compact, orientable, irreducible, ∂-irreducible, anannular 3-manifold with one component T of ∂M a torus. If r_1 and r_2 are two slopes on T, such that $M(r_1)$ is reducible while $M(r_2)$ contains an essential annulus, then $\triangle(r_1, r_2) \leq 2$.*

An example has been given by Hayashi and Motegi [6] showing that the bound 2 in Theorem 1 is the best possible in general.

Another proof of Theorem 1 has been obtained independently by Y-Q Wu.

2. Scharlemann cycle and parallel edges.

In what follows, we shall assume that $M(r_1)$ is reducible, and $M(r_2)$ contains an essential annulus. We may assume further that $M(r_2)$ is irreducible, ∂-irreducible (see [5], [7]).

Let V_i be the solid torus attached to M in forming $M(r_i)$, $i = 1, 2$. Consider the family of essential 2-spheres in $M(r_1)$ which intersect V_1 in a family of meridional discs, and let $S \subset M(r_1)$ be such a 2-sphere chosen so that $S \cap V_1$ has the minimal number, say n_1, of components. Similarly, let $A \subset M(r_2)$ be an essential annulus which intersects V_2 in a collection of meridian discs, the number of which, say n_2, is minimal among all such annuli. By assumptions, $n_1 > 2$ and $n_2 > 0$.

Now suppose that $F_1 = M \cap S$ and $F_2 = M \cap A$. Then F_1 is an essential planar surface in M with boundary slope r_1 while F_2 is an essential punctured annulus in M with boundary slope r_2. We may assume that the number of components of $F_1 \cap F_2$ is minimal subject to these conditions. Then no circle component of $F_1 \cap F_2$ bounds a disk in F_1 or F_2, and no arc component of $F_1 \cap F_2$ is boundary parallel in F_1 or F_2. Each component of ∂F_i lying in T is called an inner component of ∂F_i, $i = 1, 2$.

Let $\Gamma_1(\Gamma_2)$ be the graph in $S(A)$ obtained by taking the arc components of $F_1 \cap F_2$ as edges and taking the inner components of $\partial F_1(\partial F_2)$ as fat vertices.

We shall use the indices α and β to denote 1 or 2, with the convention that, when they are used together, $\{\alpha, \beta\}=\{1, 2\}$.

Number the inner components of ∂F_α, $\partial_1 F_\alpha, \dots, \partial_{n_\alpha} F_\alpha$, so that they appear consecutively on T. By construction, each inner component $\partial_i F_\alpha$ of ∂F_α intersects each inner component $\partial_j F_\beta$ of ∂F_β in exactly $\Delta(r_1, r_2)$ points. The ends of the edges in Γ_α may be labeled by an integer $k \in \{1, 2, \dots, n_\beta\}$ as follows. Let x be the intersection of an edge e of Γ_α with one of its vertices $\partial_i F_\alpha$, then x is labeled k, where $\partial_k F_\beta$ is the unique vertex of Γ_β, such that $x \in e \cap \partial_i F_\alpha \cap \partial_k F_\beta$. Thus when we travel around $\partial_i F_\alpha$, the labels appear in the order $1, \dots, n_\beta, \dots, 1, \dots, n_\beta$ (repeated $\triangle(r_1, r_2)$ times).

Now fix an orientation on F_α, and let each inner component $\partial_i F_\alpha$ of ∂F_α have the induced orientation. Two inner components of ∂F_α are said to be parallel if they, when given the induced orientation by F_α, are homologous on T; otherwise they are antiparallel. Two vertices of Γ_α are said to be parallel if the corresponding inner components of ∂F_α are parallel; otherwise they are antiparallel.

Parity rule [2]. An edge connects parallel vertices of Γ_α if and only if it connects antiparallel vertices of Γ_β.

Two edges of Γ_α are said to be parallel if they, together with some arcs in ∂F_α, bound a disk in F_α. A cycle σ in Γ_α is a subgraph of Γ_α homeomorphic to a circle. The length of a cycle is the number of edges contained in it. A cycle σ in Γ_α is called a Scharlemann cycle if it bounds a disk face of Γ_α and the edges of σ connect parallel vertices of Γ_α, and have the same two labels at their ends. A length two Scharlemann cycle is called an S-cycle. A length two cycle $\sigma' = \{e_1', e_2'\}$ in Γ_α is called an extended S-cycle if there is an S-cycle $\sigma = \{e_1, e_2\}$ in Γ_α such that e_i' and e_i are adjacent parallel edges in Γ_α, $i = 1, 2$.

Let x be a vertex of Γ_α. An edge of Γ_β is called an x-edge if it has label x at one of its two ends. We denote by Γ_β^x the subgraph of Γ_β consisting of all the vertices of Γ_β and the x-edges connecting parallel vertices of Γ_β. A disk face D of Γ_β^x is called an x-face.

Lemma 2.1. *Let W be a compact, irreducible, ∂-irreducible 3-manifold, and let A_0 be a non-separating annulus properly embedded in W. Then A_0 is essential in W.*

Proof. Let D be a compressing disk of A_0. If one component of ∂A_0 is essential on ∂W, then W is ∂-reducible, a contradiction. If the two components of ∂A_0 are trivial on ∂W, then W contains a non-separating 2-sphere. Thus W is reducible, a contradiction.

Now let D be a ∂-compressing disk of A_0, such that $\partial D = a \cup b$, where a is an arc on A_0, and b is an arc on ∂W. If the two components of ∂A_0 bounds an annulus A_1 on ∂W, and $b \subset A_1$, then $A_0 \cup A_1$ is non-separating in W, and the surface obtained by doing a 2-surgery on $A_0 \cup A_1$ along D is a non-separating 2-sphere in W. Thus W is reducible, a contradiction. If not, then the band connected sum of the two components of ∂A_0 along b on ∂W, say C, bounds a disk in W, and C is essential on ∂W. Thus W is ∂-reducible, a contradiction. □

Lemma 2.2. *If Γ_α contains a Scharlemann cycle, then F_β is separating.*

Proof. By Lemma 2.1 of [5], F_1 is separating when Γ_2 contains a Scharlemann cycle.

Now let σ be a Scharlemann cycle of Γ_1 with label pair $\{1, 2\}$, D be the disk face bounded by σ in Γ_1, and let A_1 be the annulus bounded by $\partial_1 F_2$ and $\partial_2 F_2$ on T, such that the interior of A_1 is disjoint from A. Let D_i be the disks in A bounded by $\partial_i F_2$, and let $T' = (A - D_1 \cup D_2) \cup A_1$. Then T' is a punctured torus. Let A' be the surface obtained by doing a 2-surgery on T' along D, then A' is an annulus in $M(r_2)$, such that $|A' \cap V_2| < n_2$. If F_2 is non-separating, then A' is also non-separating. By Lemma 2.1, A' is essential, contradicting the minimality of n_2. □

Lemma 2.3. *Let σ be a Scharlemann cycle of Γ_1, then the edges in σ can not lie in a disk of A.*

Proof. Suppose, otherwise, that the edges in σ lie in a disk of A. Then $M(r_2)$ contains a lens space as a factor (by the proof of Lemma 2.8 of [1]).

Since $\partial M(r_2) \neq \phi$, $M(r_2)$ is reducible, contradicting our assumptions on $M(r_2)$. $\square$

Proposition 2.4. *Γ_α can not contain two Scharlemann cycles with distinct label pairs.*

Proof. By Theorem 2.4 of [**5**], Γ_2 can not contain two Scharlemann cycles with distinct label pairs.

Now suppose, otherwise, that Γ_1 contains two Scharlemann cycles σ_1 and σ_2, with label pairs $\{x, y\}$ and $\{x', y'\}$ respectively, such that $\{x, y\} \neq \{x', y'\}$. By Lemma 2.2, n_2 is even.

Now consider the edges of σ_1 and σ_2 as they lie in Γ_2, joining the vertices x, y and x', y'. By Lemma 2.3, there exists an annulus $E \subset A$, such that

1) one component of ∂E is one component of ∂A, say $\partial_1 A$, and another component of ∂E is contained in int$A$;

2) the number of vertices of Γ_2 in E is at most $n_2/2$;

3) int$E$ contains the edges of one of the two Scharlemann cycles, say σ_1, and the corresponding vertices x, y.

Let E' be an annulus containing x', y' and the edges of σ_2, such that one component of $\partial E'$ is the remaining component of ∂A, say $\partial_2 A$, and another component of $\partial E'$ is contained in int$A$. If $\{x, y\} \cap \{x', y'\} = \phi$, then we may assume $E \cap E' = \phi$.

Now let D be the face of Γ_1 bounded by σ_1, let H be the 3-cell in V_2 between the consecutive meridional disks of V_2 corresponding to x and y, and let N be a regular neighborhood of $E \cup H \cup D$ in $M(r_2)$. Then the frontier of N is an annulus A' properly embedded in $M(r_2)$, whose two boundary components are $\partial_1 A \times \{-1\}$ and $\partial_1 A \times \{1\}$, and the union of N and $D_0 \times [-1, 1]$ along $\partial_1 A \times [-1, 1]$ is a punctured lens space whose fundamental group has order the length of σ_1, where D_0 is a disk. Similarly, let D' be the face of Γ_1 bounded by σ_2, let H' be the 3-cell in V_2 between the consecutive meridional disks of V_2 corresponding to x', y', and let N' be a regular neighborhood of $E' \cup H' \cup D'$. Then the frontier of N' is an annulus A'' properly embedded in $M(r_2)$, whose two boundary components are $\partial_2 A \times \{-1\}$ and $\partial_2 A \times \{1\}$, and the union of N' and $D_0 \times [-1, 1]$ along $\partial_2 A \times [-1, 1]$, say M_1, is a punctured lens space whose fundamental group has order the length of σ_2, where D_0 is a disk. We may assume that $N \cap N' = \phi$ (moving ∂N slightly off A if $\{x, y\} \cap \{x', y'\} \neq \phi$). We claim that A' is essential in $M(r_2)$.

Suppose, otherwise, that A' is not essential in $M(r_2)$. Since $M(r_2)$ is ∂-irreducible, A' is incompressible in $M(r_2)$. Now let D_1 be a ∂-compressing disk of A', such that $\partial D_1 = a \cup b$, $a \subset \partial M(r_2)$, and $b \subset A'$.

Case 1. $a \subset \partial_1 A \times [-1, 1]$.

Now either $M(r_2)$ is reducible, or the union of N and $D_0 \times [-1,1]$ along $\partial_1 A \times [-1,1]$ is a 3-cell, a contradiction.

Case 2. $a \subset \partial M(r_2) - \partial_1 A \times (-1,1)$.

If $\partial M - \partial_1 A \times (-1,1)$ is not an annulus, then either $M(r_2)$ is reducible, or $M(r_2)$ is ∂-reducible, a contradiction. If $\partial M - \partial_1 A \times (-1,1)$ is an annulus, then $\partial M(r_2)$ is a torus, and the union of $M(r_2)-$ int$N$ and $D_0 \times [-1,1]$ along $\partial M - \partial_1 A \times (-1,1)$ is a 3-cell, but it contains M_1 as a factor, a contradiction.

By construction, $|A' \cap V_2| = |A \cap V_2| - 2$, contradicting the minimality of n_2. □

Lemma 2.5. (1) Γ_2 *contains no extended S-cycle.*

(2) Γ_2 *contains at most* $n_1/2+1$ *mutually parallel edges connecting parallel vertices.*

(3) Γ_2 *contains at most* $n_1 - 1$ *mutually parallel edges.*

(4) *If* Γ_1 *contains a great cycle, then* Γ_1 *contains a Scharlemann cycle.*

(5) *If* $n_\alpha \geq 3$*, and* Γ_β *contains two distinct Scharlemann cycles* σ_1 *and* σ_2*, then the edges of* σ_1 *are disjoint from the edges of* σ_2*.*

Proof. (1) is Lemma 2.3 of [**9**]. (2) is Lemma 2.4 of [**9**]. (3) is Lemma 2.6 of [**1**]. See also [**4**, Proposition 1.3]. (4) is Lemma 2.6.2 of [**2**]. (5) Suppose, otherwise, that one edge of σ_1 is contained in σ_2. Then $n_\alpha = 2$, a contradiction. □

Lemma 2.6. *Let* y *be a vertex of* Γ_β*.*

(1) *If* Γ_α *contains a* n*-sided* y*-face, such that* $2 \leq n \leq 3$*, then* Γ_α *contains a Scharlemann cycle.*

(2) *If* F_β *is separating, and* Γ_α *contains a* y*-face* f*, then* Γ_α *contains a Scharlemann cycle in* f*.*

Proof. (1) Suppose that Γ_α contains a n-sided y-face, such that $2 \leq n \leq 3$. Then Γ_α contains a great cycle. By Lemma 2.5(4), Γ_α contains a Scharlemann cycle. (2) is Lemma 2.2 of [**5**]. □

3. Reduced graph.

Let G be a graph in a surface S. The reduced graph of G is the graph obtained from G by amalgamating each complete set of mutually parallel edges of G to a single edge.

Lemma 3.1. *One of* Γ_1 *and* Γ_2 *satisfies*

$(*)$. *Each vertex is incident to an edge connecting it to an antiparallel vertex.*

This follows immediately from the proof of [**9**, Lemma 2.6].

Let G_α be the subgraph of Γ_α consisting of all the vertices of Γ_α and the edges connecting parallel vertices. We first suppose that Γ_2 has property $(*)$. A component $F^{'}$ of G_2 is called an extremal component if there is a disc D in $\widehat{A}$ such that $D \cap G_2 = F^{'}$, where $\widehat{A}$ is the 2-sphere obtained by capping off the two components of ∂A with disks. In this case $F = D \cap \Gamma_2$ is a graph in D. If e is an edge in Γ_2 connecting a vertex of $F^{'}$ to an antiparallel vertex, then $e \cap D$ is an edge of F connecting that vertex to ∂D. Such an edge is called a boundary edge of F. Property $(*)$ means that each vertex of F belongs to a boundary edge.

Lemma 3.2. *Let Γ be a graph in a disk with no 1-sided disk face or two sided disk face, such that every vertex of Γ belongs to a boundary edge, then either Γ contains only one vertex, or there are at least two vertices of valency at most 3, each of which belongs to a single boundary edge.*

This follows immediately from the proof of [**2**, Lemma 2.6.5].

Lemma 3.3. *If Γ_2 has property $(*)$, then there exists at least one vertex of Γ_2, such that among the families of ends around it, there are at most two families which are ends of edges connecting it to parallel vertices. Furthermore, if there are two such families, they are successive.*

Proof. Since G_2 contains at least two extremal components, there is an extremal component $F^{'}$ of G_2, such that the correspond disc D of $F^{'}$ in $\widehat{A}$ contains at most one component of ∂A, say $\partial_1 A$, and the remaining component of ∂A is disjoint from D. Thus F contains at most one 1-sided disk face. Furthermore, if F contains a 1-sided disk face f_1, then $\partial_1 A \subset \mathrm{int} f_1$. Let $\bar{F}$ be the reduced graph of F in D. Let S be the graph obtained from $\bar{F}$ by removing the edge bounding f_1. (Let $S = \bar{F}$ when $\bar{F}$ contains no 1-sided disk face.) Then S contains no 1-sided disk face, and it contains at most one 2-sided disk face. Furthermore, if S contains a 2-sided disk face f_2, then $\partial_1 A (\subset \mathrm{int} f_1) \subset \mathrm{int} f_2$. Let $\bar{S}$ be the reduced graph of S in D. Let D_0 be the disk bounded by $\partial_1 A$ in D, and let $\bar{F}_0$ be the reduced graph of F in $D - \mathrm{int} D_0$.

To prove the lemma, we need only to prove that $\bar{F}_0$ contains a vertex of valency at most 3, which belongs to a boundary edge. If $F^{'}$ contains only one vertex v, then v has valency 3 in $\bar{F}_0$, and it belongs to a boundary edge. If $F^{'}$ contains two vertices, then the one which does not belong to the edge bounding f_1, has valency at most 3 in $\bar{F}_0$. Assume now that $F^{'}$ contains at least three vertices.

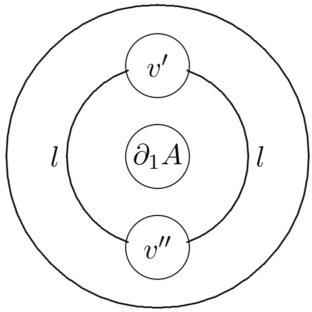

Figure 1.

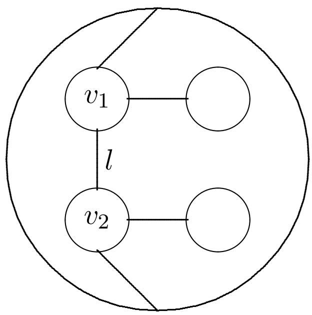

Figure 2.

Case 1. S contains no 2-sided disk face.

By Lemma 3.2, S contains at least two vertices of valency at most 3, say v_1 and v_2, each of which belongs to a single boundary edge.

Suppose that $\partial_1 A$ is contained in a fat edge l of S (as in Fig. 1). If $v_i \notin \{v^{'}, v^{''}\}$, i =1, or 2, then v_i has valency at most 3 in $\bar{F}_0$. If $\{v_1, v_2\} = \{v^{'}, v^{''}\}$, then the edges incident to one of v_1 and v_2 are as in one of Figures 2-4.

(1) the edges incident to one of v_1 and v_2 are as in Fig. 2.

Let $S^{'}$ be the graph obtained from S by taking the union of v_1, l and v_2 as a fat vertex, say v. Then $S^{'}$ contains no 1-sided disk face or 2-sided disk face. By Lemma 3.2, $S^{'}$ contains at least two vertices of valency at most 3, each of which belongs to a single boundary edge, and the one which is not equal to v, has valency at most 3 in $\bar{F}_0$.

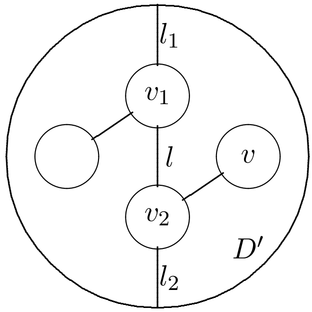

Figure 3.

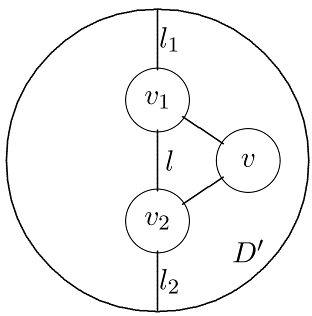

Figure 4.

(2) the edges incident to one of v_1 and v_2 are as in one of Figures 3-4.

Now $l_1 \cup v_1 \cup l \cup v_2 \cup l_2$ separates D into two discs $D^{'}$ and $D^{''}$. Since $\partial_1 A$ is contained in l, the graph $S \cap D^{'}$ contains no 1-sided disk face or 2-sided disk face. If $S \cap D^{'}$ contains only one vertex v, then v has valency at most 3 in $\bar{F}_0$. If $S \cap D^{'}$ contains at least two vertices, then $S \cap D^{'}$ contains at least two vertices of valency at most 3, each of which belongs to a single boundary edge, and the one which is not equal to v, has valency at most 3 in $\bar{F}_0$.

Now we suppose that $\partial_1 A$ is not contained in a fat edge. Since S contains no 2-sided disk face, the one of v_1 and v_2 which does not belong to the edge bounding f_1, has valency at most 3 in $\bar{F}_0$.

Case 2. S contains a 2-sided disk face f_2.

Now $\partial_1 A \subset \mathrm{int} f_2$. That means that $\partial_1 A$ is contained in a fat edge of $\bar{S}$. By Lemma 3.2, $\bar{S}$ contains at least two vertices of valency at most 3, each of which belongs to a single edge. Using $\bar{S}$ to take place of S in Case 1, we can proof that $\bar{F}_0$ contains a vertex of valency at most 3, which belongs to a boundary edge. □

Proposition 3.4. *If Γ_2 has property $(*)$, then Γ_2 contains at least one vertex, such that around it, all the endpoints of edges connecting it to parallel vertices are successive, and there are at most $n_1 + 2$ of them.*

Proof. This follows immediately from Lemma 2.5(2) and Lemma 3.3. □

Proposition 3.5. *If Γ_2 does not have property $(*)$, then Γ_1 contains at least one vertex, such that around it, all the endpoints of edges connecting to parallel vertices are successive, and there are at most $n_2 - 1$ of them.*

Proof. By Lemma 3.1, Γ_1 has property $(*)$. Now let $F^{'}$ be an extremal component of G_1, and let D be the corresponding disc. In this case $F = D \cap \Gamma_1$ is a graph in D. By Lemma 3.2, $\bar{F}$ contains at least one vertex, say x, of valency at most 3, which belongs to a single boundary edge. That implies that in Γ_1, there are at most two families of parallel edges connecting x to parallel vertices, and if there are two such families, then they are successive. Since Γ_2 does not have property $(*)$, Γ_2 contains one vertex, such that each of the edges incident to it connects it to a parallel vertex. That implies that all edges in Γ_1 with this vertex as a label connect nonparallel vertices, hence the above two families of parallel edges contains at most $n_2 - 1$ edges. □

4. The proof of Theorem 1.

Let G be a graph on a surface S. In this section, we shall denote by V the number of vertices of G, E the number of edges of G and F the number of disk faces of G. By the Euler characteristic formula, $V - E + F \geq \chi(S)$.

Proposition 4.1. *If $n_2 = 1$, then $\triangle(r_1, r_2) \leq 1$.*

Proof. Suppose, otherwise, that $\triangle(r_1, r_2) \geq 2$. Since $n_2 = 1$, Γ_2 contains only one vertex, say x. It is easy to see that x has valency 2 in $\bar{\Gamma}_2$. Hence Γ_2 contains n_1 mutually parallel edges, contradicting Lemma 2.5(3). □

Proposition 4.2. *If $n_2 = 2$, then $\triangle(r_1, r_2) \leq 2$.*

Proof. Suppose, otherwise, that $\triangle(r_1, r_2) \geq 3$. Since $n_2 = 2$, each vertex of $\bar{\Gamma}_2$ has valency 4 as in Fig. 5 (otherwise Γ_2 contains n_1 mutually parallel edges). By Lemma 2.5(2), l_1 and l_2 contains at most $n_1 + 2$ edges. If l_1 and l_2 occupy at most $n_1 + 1$ edges, then one of l_3 and l_4 occupies at least n_1 mutually parallel edges, contradicting Lemma 2.5(3).

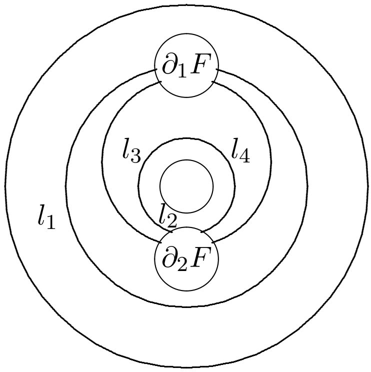

Figure 5.

Now we suppose that that l_1 and l_2 occupy n_1+2 edges. Then Γ_2 contains an S-cycle. By Lemma 2.2, F_1 is separating, and $n_1 \geq 4$. By Lemma 2.5(3), each of l_3 and l_4 occupies $n_1 - 1$ edges.

Case 1. $\partial_1 F_2$ and $\partial_2 F_2$ are parallel.

Let x be a vertex of Γ_1. By the parity rule, Γ_2^x contains at least six $(3n_2)$ edges. Since $\bar{\Gamma}_2$ has four edges, Γ_2^x contains a 2-sided disk face. By Lemma 2.5(1) and Lemma 2.5(3), Γ_2 contains an S-cycle, one label of which is x, for any given vertex x in Γ_1. Since $n_1 \geq 3$, Γ_2 contains two Scharlemann cycles with distinct label pairs, contradicting Proposition 2.4.

Case 2. $\partial_1 F_2$ and $\partial_2 F_2$ are antiparallel.

Let Γ be the subgraph of Γ_1 consisting of all the vertices of Γ_1 and the edges in l_3. Since F_1 is separating, Γ is not connected. By the Euler characteristic formula, Γ contains a disk face. By the proof of Proposition 1.3 of [**4**], M contains an essential annulus, a contradiction. □

In the following arguments, we shall assume that $n_\alpha \geq 3$.

Proposition 4.3. *If* Γ_2 *has property* $(*)$, *then* $\triangle(r_1, r_2) \leq 2$.

Proof. Suppose, otherwise, that $\triangle(r_1, r_2) \geq 3$. By Proposition 3.4, there exists a vertex of Γ_2, say y, such that Γ_1^y contains at least $2n_1 + l$ edges, where $l \geq -2$. By the Euler characteristic formula, Γ_1^y contains at least $n_1 + l + 2 \geq n_1$ disk faces. Since there are n_1 adjacent edges at y connecting it to antiparallel vertices, there is a great y-cycle in Γ_1. By Lemma 2.5(4), Γ_1 contains a Scharlemann cycle. By Lemma 2.2, F_2 is separating. By Lemma 2.6(2) and Proposition 2.4, Γ_1 contains at least n_1 Scharlemann cycles with the same label pair, say $\{1, 2\}$. Now suppose that Γ_1 contains m Scharlemann cycles with label pair $\{1, 2\}$. Then $m \geq n_1$. By Lemma 2.5(5), Γ_1^1 contains at least $2m$ edges. By the Euler characteristic formula, Γ_1^1 contains at least $2m - n_1 + 2 \geq m + 2$ disk faces. By Lemma 2.6(2), Γ_1 contains at least $m + 2$ Scharlemann cycles. Thus Γ_1 contains two Scharlemann cycles with distinct label pairs, contradicting Proposition 2.4. □

Proposition 4.4. *If* Γ_2 *does not have property* $(*)$*, then* $\triangle(r_1, r_2) \leq 2$.

Proof. Suppose, otherwise, that $\triangle(r_1, r_2) \geq 3$. By Proposition 3.5, there exists a vertex of Γ_1, say x, such that Γ_2^x contains $2n_2 + l$ edges, where $l \geq 1$. By the Euler characteristic formula, Γ_2^x contains at least $n_2 + l$ disk faces. By the parity rule and Proposition 3.5, there are $n_2 - l$ vertices of Γ_2, each of which is incident to an edge connecting it to an antiparallel vertex. That means that there are at least $n_2 - l$ edges of Γ_2^x, each of which is on the boundary of only one disk face of Γ_2^x. We claim that Γ_2^x contains either a 2-sided disk face, or a 3-sided disk face.

If Γ_2^x contains no 2-sided disk face or 3-sided disk face. Then $4F \leq 2(2n_2 + l) - (n_2 - l)$. Thus $F \leq 3/4(n_2 + l) < n_2 + l$, a contradiction.

Now by Lemma 2.6(1), Γ_2 contains a Scharlemann cycle. By Lemma 2.2, F_1 is separating. By Lemma 2.6(2) and Proposition 2.4, Γ_2 contains at least $n_2 + l$ Scharlemann cycles with the same label pair, say $\{1, 2\}$. Now suppose that Γ_2 contains m Scharlemann cycles with label pair $\{1, 2\}$. Then $m \geq n_2 + l$, where $l \geq 1$. By Lemma 2.5(5), Γ_2^1 contains at least $2m$ edges. By the Euler characteristic formula, Γ_2^1 contains at least $m + l$ disk faces. By Lemma 2.6(2), Γ_2 contains at least $m + l$ Scharlemann cycles. Thus Γ_2 contains two Scharlemann cycles with distinct label pairs, contradicting Proposition 2.4. □

Theorem 1 follows immediately from Propositions 4.1-4.4.

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