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K-TYPES OF $SU(1, n)$ REPRESENTATIONS AND RESTRICTION OF COHOMOLOGY

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This paper shows that the highest weights of the K-types of any irreducible admissible representation of $SU(1, n)$ are determined by certain restriction maps from $\mathfrak{u}$ to $\mathfrak{u} \cap \mathfrak{k}$ cohomology. In particular, the image of these maps determines a set of points in a Cartan subalgebra. It is proved that the highest weights of the K-types are given by intersecting a translate of the root lattice with the closed convex hull of the points determined by the restriction maps.

1. Introduction.

A basic idea often employed in the study of representations of real reductive Lie groups is the notion of a K-type. In particular, if G is a real reductive Lie group, K its maximally compact subgroup, and X an admissible representation of G, then the representations of K appearing in X are called the K-types of X. The point is that compact groups are well understood and provide a powerful tool in the analysis of noncompact groups. The classical application of these ideas is Bargmann's description in [**1**] of the representations of $SL(2, \mathbb{R})$ based on the study on its $SO(2)$-types (see also [**4**], [**2**], [**3**], and especially [**6**] for an extensive list).

In many cases formulas for the K-types are known. For instance Blattner's formula ([**5**]) provides a wonderfully explicit description of the K-types of the discrete series (see [**8**] for a generalization). Unfortunately, even in these cases the formulas are combinatorially complex and it is often hard to determine whether a particular representation is a K-type.

A different approach, suggested by D. Vogan, is followed in [**13**]. There the object of study is the closed convex hull of the set of highest weights of the K-types. In the case of *finite dimensional* representations when G is $SU(1, n)$ or $SO(1, n)$, an algebraic method is developed for finding the "edges" of this closed convex hull. The point is that knowledge of just the edges is enough to reconstruct the whole closed convex hull. This already provides fairly sharp control of which representations can be K-types. Moreover, it can be seen in [**13**] that intersecting a translate of the root lattice with the closed convex hull recovers all the K-types.

Let X be an irreducible admissible representation of G, write $\mathfrak{g}$ for its complexified Lie algebra, and choose a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. The main algebraic tool used to construct the edges in [**13**] is a restriction map on cohomology

$$\tau : H^b(\mathfrak{u}, X) \to H^b(\mathfrak{u} \cap \mathfrak{k}, X).$$

In a sense that is made precise, it is shown there that the image of τ (as $\mathfrak{q}$ varies) determines all the edges (not lying in a Weyl chamber wall of K) and therefore determines all the K-types.

This paper generalizes [**13**]. The main result is Theorem 9. It says that the edges of the closed convex hull of the set of K-types of *any* irreducible admissible representation of $G = SU(1, n)$ are completely determined by the image of τ (in fact, only two parabolics are needed). Corollary 3 completes the circle by showing that all K-types may be recovered from this closed convex hull by intersecting it with a translate of the root lattice of K. All notation necessary to understand the precise result is contained in Section 6.

The layout of the paper is as follows: Section 2 sets up the notation, Section 3 lists the K-types of the induced representations of G, Section 4 lists the infinitesimal characters and a reducibility criterion, Section 5 gives the K-types of all irreducible representations of G, and Section 6 constructs the K-types in terms of the image of τ.

2. Notation.

Let $G = SU(1, n)$, $n > 1$, and write $K \cong U(n)$ for its maximally compact subgroup embedded into G as

$$K = \left\{ \begin{pmatrix} x & \\ & X \end{pmatrix} \mid x \in U(1),\, X \in U(n),\, x \det(X) = 1 \right\}.$$

Let $\mathfrak{g}_0 = \mathfrak{su}(1, n)$ be the Lie algebra of G and write $\mathfrak{g}$ for its complexification. This convention will be followed throughout the paper. For example, $\mathfrak{k}_0$ is the Lie algebra of K, isomorphic to $\mathfrak{u}(n)$, and $\mathfrak{k}$ is its complexification. Also write θ for the standard Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition.

Let T be the Cartan subgroup of K (and G) consisting of all diagonal matrices in G. If $x = (x_1, \dots x_m) \in \mathbb{R}^m$, define its *trace* by $\operatorname{tr}(x) = x_1 + \dots x_m$. With this notation and the identification of $i\mathfrak{t}_0^*$ with $i\mathfrak{t}_0$ via the standard dot product, the set of analytically integral weights on $\mathfrak{t}$ is

$$\hat{T} = \left\{ (\mu_0, \mu_1, \dots \mu_n) \in \left(\frac{1}{n+1} \mathbb{Z} \right)^{n+1} \mid \operatorname{tr}(\mu) = 0, \right.$$
$$\left. \mu_i - \mu_j \in \mathbb{Z} \text{ for } 0 \le i < j \le n \right\}.$$

Viewing T as a Cartan subgroup of K, we say $\mu \in \hat{T}$ is *positive* and write $\mu \in \hat{T}^+$ if $\mu_1 \geq \mu_2 \geq \ldots \mu_n$. By taking highest weights, $\hat{T}^+$ parameterizes the irreducible representations of K. We also write W_K for the Weyl group of K and W_G for the Weyl group of G with respect to $i\mathfrak{t}_0$. W_K acts on $i\mathfrak{t}_0$ as the set of all permutations of the last n coordinates and W_G acts on $i\mathfrak{t}_0$ as the set of all permutations.

Let A be the subalgebra of G defined by $\exp(\mathfrak{a}_0)$ where $\mathfrak{a}_0 \subseteq \mathfrak{p}$ is the subalgebra given by

$$\mathfrak{a}_0 = \left\{ a_\nu \equiv \begin{pmatrix} 0 & & \nu \\ & \ddots & \\ \nu & & 0 \end{pmatrix} \mid \nu \in \mathbb{R} \right\}.$$

By conjugation, we may pull back the standard trace form on the diagonal matrices to $\mathfrak{a}$ so that $a_{\nu_1} \cdot a_{\nu_2} = 2\nu_1\nu_2$. We use this to identify $\mathfrak{a}$ and $\mathfrak{a}^*$. We further identify $\mathbb{C}$ with $\mathfrak{a}$ by mapping $z \in \mathbb{C}$ to $a_z \in \mathfrak{a}$. By the identification of $\mathfrak{a}$ and $\mathfrak{a}^*$, z acts on $\mathfrak{a}$ by $z \cdot a_\nu = 2z\nu$.

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system so $\Sigma = \{\pm\frac{1}{2}, \pm 1\}$ with multiplicities $2(n-1)$ and 1, respectively. Set $\Sigma^+ = \{\frac{1}{2}, 1\}$ and let P be the corresponding parabolic subgroup with $P = MAN$ its Langlands decomposition. In particular,

$$M = \left\{ \begin{pmatrix} x & & \\ & X & \\ & & x \end{pmatrix} \mid x \in U(1),\, X \in U(n-1),\, x^2 \det(X) = 1 \right\}$$

and is a double cover of $U(n-1)$.

Let S be the Cartan subgroup of M consisting of all diagonal matrices in M and write $H = SA$ as a Cartan subgroup of G. The set of analytically integral weights on S is

$$\begin{aligned}\hat{S} = \Big\{ &(x_0, x_1, \ldots x_n) \in \mathbb{R}^{n+1} \mid x_0 = x_n,\, \operatorname{tr}(x) = 0,\ x_0 - x_1 \in \frac{1}{2}\mathbb{Z}, \\ &\qquad\qquad x_i - x_j \in \mathbb{Z} \text{ for } 1 \leq i < j \leq n-1 \Big\} \\ = \Big\{ &(x_0, x_1, \ldots x_n) \mid x_1 \in \frac{1}{n+1}\mathbb{Z},\, x_i - x_j \in \mathbb{Z} \text{ for } 1 \leq i < j \leq n-1, \\ &\qquad\qquad x_0 = x_n = -\frac{1}{2}\sum_{j=1}^{n-1} x_j \Big\}.\end{aligned}$$

We say $x \in \hat{S}$ is *positive* and write $x \in \hat{S}^+$ if $x_1 \geq x_2 \geq \ldots x_{n-1}$. By taking highest weights, $\hat{S}^+$ parameterizes the irreducible representations of M.

3. K-types of induced modules.

For $\mu \in \hat{T}$, let V_μ^K be the irreducible representation of K with extremal weight μ. Use similar notation for irreducible representations of M. We also write $V_\mu^K|_M$ to signify restriction to M. Since the branching law for restriction from $U(n)$ to $U(n-1)$ is well known ([**14**]), it is easy to determine how restriction works from K to M:

Theorem 1. *Let $\mu = (\mu_0, \mu_1, \dots \mu_n) \in \hat{T}^+$. Then*

$$V_\mu^K|_M = \bigoplus_{x \in \Phi_\mu} V_x^M$$

where

$$\Phi_\mu = \{(x_0, x_1, \dots x_n) \mid x_0 = x_n,\ \mathrm{tr}(x) = 0,\ x_i - \mu_0 \in \mathbb{Z} \textit{ for } 1 \le i \le n-1,\\ \mu_1 \ge x_1 \ge \mu_2 \ge x_2 \ge \dots \mu_{n-1} \ge x_{n-1} \ge \mu_n\}.$$

By Frobenius reciprocity, we then have

Corollary 1. *Let $x = (x_0, x_1, \dots x_n) \in \hat{S}^+$. Then*

$$\mathrm{Ind}_M^K(V_x^M) = \bigoplus_{\mu \in \Phi_x} V_\mu^K,$$

where

$$\Phi_x = \{(\mu_0, \mu_1, \dots \mu_n) \mid \mathrm{tr}(\mu) = 0,\ \mu_i - x_1 \in \mathbb{Z} \textit{ for } 0 \le i \le n,\\ \mu_1 \ge x_1 \ge \mu_2 \ge x_2 \ge \dots \mu_{n-1} \ge x_{n-1} \ge \mu_n\}.$$

For Langlands parameters $x \in \hat{S}$ and $\nu \in \mathfrak{a}^*$, write $I(x,\nu)$ for the normalized induced module $\mathrm{Ind}_P^G(V_x^M \otimes e^\nu)$. Since $I(x,\nu)|_K \cong \mathrm{Ind}_M^K(V_x^M)$, Corollary 1 describes the K-types of $I(x,\nu)$. If $\mathrm{Re}(\nu) > 0$, write $J(x,\nu)$ for the unique irreducible Langlands quotient of $I(x,\nu)$.

4. Character Equalities.

For $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$, write λ for the infinitesimal character $x + \rho_M + \nu$ of $I(x,\nu)$. After conjugating $\mathfrak{h}$ to $\mathfrak{t}$, we may take $\lambda \in \mathfrak{t}$ as

$$\lambda = \left(x_0 + \nu, x_1 + \frac{n}{2} - 1, x_2 + \frac{n}{2} - 2, \dots x_{n-1} - \frac{n}{2} + 1, x_n - \nu\right). \tag{4.1}$$

We say λ is *nonsingular* if no two coordinates are the same.

Using the action of the Weyl group, it is straightforward to write down all induced modules with the same infinitesimal character. The following notation simplifies the results.

Definition 1. Let $s_1 : \mathbb{R}^{n+1} \times \mathbb{C} \to \mathbb{R}^{n+1}$ by $s_1(x, v) = \widehat{x}$ where

$$\hat{x}_i = \begin{cases} x_0 + \nu - \frac{n}{2} & \text{if } i = 0 \\ x_i & \text{if } 1 \le i \le n-1 \\ x_n - \nu + \frac{n}{2} & \text{if } i = n. \end{cases}$$

For each $a, b \in \mathbb{Z}$ with $0 \le a < b \le n$, define $s_{2,a,b} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{C}$ by $s_{2,a,b}(\widehat{x}) = (x', \nu')$ where

$$x'_i = \begin{cases} \frac{1}{2}(\hat{x}_a + \hat{x}_b - a + n - b) & \text{if } i = 0, n \\ \hat{x}_{i-1} + 1 & \text{if } 1 \le i \le a \\ \hat{x}_i & \text{if } a+1 \le i \le b-1 \\ \hat{x}_{i+1} - 1 & \text{if } b \le i \le n-1, \end{cases}$$

$$\nu' = \frac{1}{2}(\hat{x}_a - \hat{x}_b - a + b)$$

and define $s_{a,b} : \mathbb{R}^{n+1} \times \mathbb{C} \to \mathbb{R}^{n+1} \times \mathbb{C}$ by the composition $s_{2,a,b} \circ s_1$. It is easy to verify that the inverse, $s_{a,b}^{-1}$, is determined by defining $s_{2,a,b}^{-1}(y', w') = \widehat{y}$ where

$$\widehat{y}_i = \begin{cases} y'_{i+1} - 1 & \text{if } 0 \le i \le a-1 \\ y'_0 + w' - \frac{n}{2} + a & \text{if } i = a \\ y'_i & \text{if } a+1 \le i \le b-1 \\ y'_n - w' + \frac{n}{2} + b - n & \text{if } i = b \\ y'_{i-1} + 1 & \text{if } b+1 \le i \le n \end{cases}$$

and defining $s_1^{-1}(\widehat{y}) = (y, w)$ where

$$w = \frac{1}{2}(\widehat{y}_0 - \widehat{y}_n + n),$$

$$y_i = \begin{cases} \widehat{y}_0 - w + \frac{n}{2} & \text{if } i = 0 \\ \widehat{y}_i & \text{if } 1 \le i \le n-1 \\ \widehat{y}_n + w - \frac{n}{2} & \text{if } i = n. \end{cases}$$

Definition 2. For $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$, define $\lambda = \lambda(x, \nu)$ to be

$$\lambda = \left(x_0 + \nu, x_1 + \frac{n}{2} - 1, x_2 + \frac{n}{2} - 2, \dots x_{n-1} - \frac{n}{2} + 1, x_n - \nu\right).$$

Observe that

$$\lambda_1 > \lambda_2 > \dots \lambda_{n-1}.$$

If $\nu \in \mathbb{R}^{\ge 0}$, define $c, d \in \mathbb{Z}$, $0 \le c < d \le n$, so that $\lambda_c > \lambda_0 \ge \lambda_{c+1}$ and $\lambda_{d-1} \ge \lambda_n > \lambda_d$. Define

$$(\widetilde{x}, \widetilde{\nu}) = s_{c,d}^{-1}(x, \nu).$$

Observe that if $\widetilde{\lambda} = \lambda(\widetilde{x}, \widetilde{\nu})$, then

$$\widetilde{\lambda}_0 \ge \widetilde{\lambda}_1 > \widetilde{\lambda}_2 > \dots \widetilde{\lambda}_{n-1} \ge \widetilde{\lambda}_n.$$

We say that $\nu \in \mathbb{C}$ is *non-negative* if either $\mathrm{Re}(\nu) > 0$ or both $\mathrm{Re}(\nu) = 0$ and $\mathrm{Im}(\nu) \geq 0$. Since it is easy to see that $I(x, \nu)$ and $I(x, -\nu)$ have the same infinitesimal character for $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$, we may assume in the theorem below that ν, ν' are non-negative without loss of generality. We will use the notation from Definitions 1 and 2 without further comment. It is then easy to check that:

Theorem 2. *Fix $x, x' \in \hat{S}^+$ and $\nu, \nu' \in \mathbb{C}$ both non-negative. If $\lambda_0 - \lambda_1 \notin \mathbb{Z}$, then $I(x, \nu)$ and $I(x', \nu')$ have the same infinitesimal character if and only if $x = x'$ and $\nu = \nu'$. If $\lambda_0 - \lambda_1 \in \mathbb{Z}$, then $I(x, \nu)$ and $I(x', \nu')$ have the same infinitesimal character if and only if $(x', \nu') = s_{a,b}(\widetilde{x}, \widetilde{\nu})$ for some a, b with $0 \leq a < b \leq n$.*

Using Theorem 2, the Subrepresentation theorem, and the Langland's classification, it is easy to identify almost all the induced modules $I(x, \nu)$ that are irreducible. The calculations later in this paper or a few R-group calculations suffice to clear up the remaining ambiguities. But since this is known (Kraljevic, [**11**], Proposition 1, §3 from [**12**], Theorems 7.5 and 8.7) and not so important for the purpose of this paper, we simply state the result.

Theorem 3. *For $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$, let λ be the character $\lambda(x, \nu)$. Then $I(x, \nu)$ is reducible if and only if $\lambda_0 - \lambda_1 \in \mathbb{Z}$ and either $\lambda_0 - \lambda_{c+1} \neq 0$ or $\lambda_0 - \lambda_{d-1} \neq 0$.*

Note that reducibility always implies $\nu \in \frac{1}{2}\mathbb{Z}$.

5. K-types of Langlands quotients.

In this section we record the K-types of each irreducible representation of G. The Langland's classification says that every irreducible representation is a discrete series representation, limit of discrete series representation, an irreducible tempered representation of the form $I(x, i\nu)$ with $x \in \hat{S}^+$ and $\nu \in \mathbb{R}^{\geq 0}$, or one of the $J(x, \nu)$ with $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > 0$. Hence Corollary 1 and Theorem 3 yield the K-types of most irreducible representations. The only ones yet to be determined are the discrete series, limit of discrete series, and the irreducible quotients $J(x, \nu)$ in the cases where $I(x, \nu)$ is reducible.

We begin by studying the reducible $I(x, \nu)$ with nonsingular character. They may be parameterized as follows. Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. Choose the unique $x \in \hat{S}^+$ and $\nu \in \frac{1}{2}\mathbb{Z}^{>0}$ so that $\lambda = \lambda(x, \nu)$. Write

$$I_{a,b}(\lambda) = I(s_{a,b}(x, \nu))$$

and

$$J_{a,b}(\lambda) = J(s_{a,b}(x, \nu))$$

for $a, b \in \mathbb{Z}$ with $0 \le a < b \le n$. In particular, $I_{0,n}(\lambda) = I(x,\nu)$. By Section 4, the set of all $I_{a,b}(\lambda)$ encompass the set of all reducible principal series representations ($\mathrm{Re}(\nu) > 0$) with nonsingular character.

The set of discrete series representations of G may be parameterized as follows. Let $\omega_a \in W_G$, $0 \le a \le n$, be defined by

$$\omega_a(t_0, t_1, \dots t_n) = (t_a, t_0, t_1 \dots t_{a-1}, t_{a+1}, \dots t_n).$$

For each λ from the previous paragraph, write

$$J_{a,a}(\lambda)$$

for the discrete series representation with infinitesimal character $\omega_a\lambda$ associated to the G chamber determined by $\omega_a\lambda$. The set of all $J_{a,a}(\lambda)$ encompass all discrete series representations.. Define Λ_a to be the highest weight of its lowest K-type. It is easy to verify that

$$(\Lambda_a)_i = \begin{cases} \widehat{x}_a + n - 2a & \text{if } i = 0 \\ \widehat{x}_{i-1} + 1 & \text{if } 1 \le i \le a \\ \widehat{x}_i - 1 & \text{if } a+1 \le i \le n \end{cases} \tag{5.1}$$

and that the K-types of $J_{a,a}(\lambda)$ are contained in the cone $\Lambda_a + C_a$ where

$$C_a = \{(t_0, t_1, \dots t_n) \mid \mathrm{tr}(t) = 0,\ t_1, \dots t_a \in \mathbb{R}^{\ge 0},\ t_{a+1}, \dots t_n \in \mathbb{R}^{\le 0}\}. \tag{5.2}$$

Using Corollary 1, it easy to explicitly write the K-types for each $I_{a,b}(\lambda)$ and to check the following.

Corollary 2. *Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \le i < j \le n$. Let $a, b \in \mathbb{Z}$ with $0 \le a < b \le n$. The K-types of $I_{a,b}(\lambda)$ occur with multiplicity one. $I_{a,b}(\lambda)$ and $I_{a',b'}(\lambda)$ have K-types in common if and only if $(a', b') \in \{(a + \varepsilon_1, b + \varepsilon_2) \mid \varepsilon_1, \varepsilon_2 \in \{0, \pm 1\},\ 0 \le a' < b' \le n\}$.*

In the case $n = 2$, using only the Langlands classification, the information about K-types already determined, and a basic embedding result on discrete series ([**9**]), it is possible to give the semisimplification of each $I_{a,b}(\lambda)$ and to deduce the K-types of each $J_{a,b}(\lambda)$. However, things become too complicated for this line of reasoning to be sufficient for larger n. Thus we use the following well known description of the composition series of $I_{a,b}(\lambda)$ (see [**11**] Proposition 3, §7, [**12**] Theorem 7.5, [**15**]).

Theorem 4. *Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \le i < j \le n$. Let $a, b \in \mathbb{Z}$ with $0 \le a < b \le n$. The socle filtration of $I_{a,b}(\lambda)$ is*

$$\begin{tikzcd} & J_{a,b}(\lambda) \arrow[dl, dash] \arrow[dr, dash] & \\ J_{a,b-1}(\lambda) \arrow[dr, dash] & & J_{a+1,b}(\lambda) \arrow[dl, dash] \\ & J_{a+1,b-1}(\lambda) & \end{tikzcd}$$

where the bottom row does not occur if $a + 1 = b$.

Combining this theorem with our knowledge of the K-types allows us to prove:

Lemma 1. *Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \le i < j \le n$. Let $0 \le a \le b \le n$. Then $J_{a,b}(\lambda)$ and $J_{a',b'}(\lambda)$ have a K-type in common if and only if $a = a'$ and $b = b'$.*

Proof. Corollary 1 and Equations 5.1 to 5.2 imply that if either a and a' or b and b' differ by more than one, then they have no K-types in common. On the other hand, if either differs by one, then Theorem 4 allows us to embed $J_{a,b}$ and $J_{a',b'}$ into some $I_{a'',b''}$ and $I_{a''',b'''}$ where either a'' and a''' or b'' and b''' differ by more than one so that Corollary 1 again says that they have no K-types in common. $\square$

This allows us to determine the K-types for each $J_{a,b}(\lambda)$ (which includes the discrete series).

Theorem 5. *Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \le i < j \le n$. Let $a, b \in \mathbb{Z}$ with $0 \le a \le b \le n$. The K-types of $J_{a,b}(\lambda)$ appear with multiplicity one. Choose the unique $x \in \hat{S}^+$ and $\nu \in \frac{1}{2}\mathbb{Z}^{>0}$ such that $\lambda = \lambda(x,\nu)$. The highest weights of the K-types of $J_{a,b}(\lambda)$ are*

$$\begin{aligned}&\{(\mu_0, \mu_1, \dots \mu_n) \mid \operatorname{tr}(\mu) = 0,\ \mu_i - x_1 \in \mathbb{Z} \text{ for } 0 \le i \le n,\\ &\mu_1 \ge \hat{x}_0 + 1 \ge \dots \mu_a \ge \hat{x}_{a-1} + 1 > \hat{x}_a \ge \mu_{a+1} \ge \hat{x}_{a+1} \ge \dots\\ &\ \ge \mu_b \ge \hat{x}_b > \hat{x}_{b+1} - 1 \ge \mu_{b+1} \ge \hat{x}_{b+2} - 1 \ge \dots \mu_{n-1} \ge \hat{x}_n - 1 \ge \mu_n\}.\end{aligned}$$

This notation includes the natural collapsing of certain μ. For instance, if $a = b$ the above inequalities reduce to

$$\begin{aligned}&\mu_1 \ge \hat{x}_0 + 1 \ge \dots \mu_a \ge \hat{x}_{a-1} + 1 > \hat{x}_a\\ &\qquad\qquad \hat{x}_a > \hat{x}_{a+1} - 1 \ge \mu_{a+1} \ge \hat{x}_{a+2} - 1 \ge \dots \mu_{n-1} \ge \hat{x}_n - 1 \ge \mu.\end{aligned}$$

Proof. This follows using Lemma 1, Theorem 4, and Corollary 1. For instance, the K-types of $J_{a,b}$ for $0 < a \le b < n$ are the K-types that occur in both $I_{a-1,b}$ and $I_{a,b+1}$. The other cases are handled similarly. $\square$

We turn our attention to the reducible $I(x,\nu)$ with singular character. They may be parameterized as follows. Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_c = \lambda_{c+1} > \dots \lambda_n$, $0 \le c \le n-1$, and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \le i < j \le n$. Choose the unique $x \in \mathbb{R}^{n+1}$ and $\nu \in \frac{1}{2}\mathbb{Z}^{>0}$ so that $\lambda = \lambda(x,\nu)$. Write

$$\begin{aligned}I^-_{a,c+1}(\lambda) &= I(s_{a,c+1}(x,\nu))\\ J^-_{a,c+1}(\lambda) &= J(s_{a,c+1}(x,\nu))\end{aligned}$$

for each $0 \le a < c$ and

$$\begin{aligned}I^+_{c,b}(\lambda) &= I(s_{c,b}(x,\nu))\\ J^+_{c,b}(\lambda) &= J(s_{c,b}(x,\nu))\end{aligned}$$

for each $c+1 < b \leq n$.

The set of discrete series representations of G may be parameterized as follows. Continue with the same λ from the previous paragraph and recall the elements $\omega_a \in W_G$ from the discussion of the discrete series. Write

$$J^-_{c,c+1}(\lambda)$$

for the limit of discrete series representation with infinitesimal character λ corresponding to the chamber determined by ω_c. It is immediate that Equation 5.1 (with $a = c$) gives the lowest K-type and that Equation 5.2 describes a cone containing all of its K-types. Similarly write

$$J^+_{c,c+1}(\lambda)$$

for the limit of discrete series representation with infinitesimal character λ corresponding to the chamber determined by ω_{c+1}. It is immediate that Equations 5.1 and 5.2 (with $a = c+1$) describe its K-types. The set of all $J^{\pm}_{c,c+1}(\lambda)$ encompass all the limits of discrete series. Moreover, it is possible to check that $I(s_{c,c+1}(x,\nu))$ (notice $s_{c,c+1}(\nu) = 0$) splits as a direct sum of $J^-_{c,c+1}(\lambda) \bigoplus J^+_{c,c+1}(\lambda)$.

The composition series of $I(x,\nu)$ is also well know in the singular setting (see [**11**] Proposition 3, §7, [**12**] Theorem 7.5) and closely parallels Theorem 4.

Theorem 6. *Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_c = \lambda_{c+1} > \dots \lambda_n$, $0 \leq c \leq n-1$, and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. For $0 \leq a < c$, the socle filtration of $I^-_{a,c+1}(\lambda)$ is*

$$\begin{array}{ccc} J^-_{a,c+1}(\lambda) & & \\ & \diagdown & \\ & & J^-_{a+1,c+1}(\lambda). \end{array}$$

For $c+1 < b \leq n$, the socle filtration of $I^+_{c,b}(\lambda)$ is

$$\begin{array}{ccc} & & J^+_{c,b}(\lambda) \\ & \diagup & \\ J^+_{c,b-1}(\lambda). & & \end{array}$$

As in Theorem 5, Theorem 6 allows us to immediately determine the K-types for each $J^-_{a,c+1}(\lambda)$ and $J^+_{c,b}(\lambda)$ (which includes the limits of discrete series).

Theorem 7. *Fix λ with $\lambda_0 > \lambda_1 > \dots \lambda_c = \lambda_{c+1} > \dots \lambda_n$, $0 \leq c \leq n-1$, and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. Choose the unique $x \in \mathbb{R}^{n+1}$ and $\nu \in \frac{1}{2}\mathbb{Z}^{>0}$*

so that $\lambda = \lambda(x, \nu)$. *For* $0 \leq a \leq c$, *the* K*-types of* $J^{-}_{a,c+1}(\lambda)$ *are*

$$\begin{aligned}\{(\mu_0, \mu_1, \dots \mu_n) \mid \operatorname{tr}(\mu) = 0,\ \mu_i - x_1 \in \mathbb{Z} \textit{ for } 0 \leq i \leq n,\\ \mu_1 \geq \hat{x}_0 + 1 \geq \dots \mu_a \geq \hat{x}_{a-1} + 1 > \hat{x}_a \geq \mu_{a+1} \geq \hat{x}_{a+1} \geq\\ \dots \mu_c \geq \hat{x}_c \geq \mu_{c+1} \geq \hat{x}_{c+2} - 1 \geq \dots \mu_{n-1} \geq \hat{x}_n - 1 \geq \mu_n\}.\end{aligned}$$

For $c + 1 \leq b \leq n$, *the* K*-types of* $J^{+}_{c,b}(\lambda)$ *are*

$$\begin{aligned}\{(\mu_0, \mu_1, \dots \mu_n) \mid \operatorname{tr}(\mu) = 0,\ \mu_i - x_1 \in \mathbb{Z} \textit{ for } 0 \leq i \leq n,\\ \mu_1 \geq \hat{x}_0 + 1 \geq \dots \mu_c \geq \hat{x}_{c-1} + 1 \geq \mu_{c+1} \geq \hat{x}_{c+1} \geq \dots\\ \mu_b \geq \hat{x}_b > \hat{x}_{b+1} - 1 \geq \mu_{b+1} \geq \hat{x}_{b+2} - 1 \geq \dots \mu_{n-1} \geq \hat{x}_n - 1 \geq \mu_n\}.\end{aligned}$$

Corollary 1 and Theorems 5 and 7 give the K-types for all irreducible representations of G.

6. Restriction of Cohomology.

We begin this section by recalling some notation from [**13**].

Definition 3. Fix a $(\mathfrak{g}, K)$ module X and a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ where $\mathfrak{l}$ is the Levi component and $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$. Let τ be the map on cohomology

$$\tau : H^b(\mathfrak{u}, X) \to H^b(\mathfrak{u} \cap \mathfrak{k}, X)$$

induced by restricting $\operatorname{Hom}(\bigwedge^b \mathfrak{u}, X) \to \operatorname{Hom}(\bigwedge^b \mathfrak{u} \cap \mathfrak{k}, X)$.

Write $\Delta^+(\mathfrak{k}, \mathfrak{t})$ for the positive roots of $\mathfrak{k}$ corresponding to the choice of $\hat{T}^+$ and write ρ_K for the half sum these roots.

Definition 4. Fix a $(\mathfrak{g}, K)$ module X and a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. Let ν be the highest weight of an $L \cap K$ representation appearing in $H^b(\mathfrak{u} \cap \mathfrak{k}, X)$ and choose $w \in W_K$ so that $w(\nu + \rho_K)$ is positive. Define

$$\nu_K = w(\nu + \rho_K) - \rho_K.$$

By Kostant's Borel-Weil theorem ([**10**]), $V^K_{\nu_K}$ appears in $X|_K$. We say that ν_K is the *associated* K-type to the $L \cap K$-type ν.

Definition 5. Fix a $(\mathfrak{g}, K)$ module X. Let $\mathcal{C}$ be the closed convex hull in $i\mathfrak{t}_0^*$ of the set of highest weights of the K-types appearing in X. Given $\mu \in \hat{T}^+$ a K-type of X, we say that μ lies on the *geometric edge* of the set of K-types of X if it lies on the boundary of $\mathcal{C}$ as a subset of $i\mathfrak{t}_0^*$.

It is hoped that the associated K-types to the $L \cap K$-types appearing in the image of τ describe the K-types lying on geometrical edges (as long the edge is not completely contained in a Weyl chamber wall of K). Thus it is hoped that knowledge of the image of τ completely determines $\mathcal{C}$ and therefore goes a long way towards describing all the K-types.

Definition 6. Fix a $(\mathfrak{g}, K)$ module, X. If $\mu \in \hat{T}^+$, the *multiplicity* of μ in X, $m(\mu)$, is the multiplicity of V_μ^K in $X|_K$. Extend this definition as follows. For $\mu \in \hat{T}$ with $\mu + \rho_K$ singular, define $m_e(\mu) = 0$. For $\mu \in \hat{T}$ with $\mu + \rho_K$ nonsingular, there exists a unique $w \in W_K$ so that $w(\mu + \rho_K) - \rho_K \in \hat{T}^+$. Define

$$m_e(\mu) = (-1)^{l(w)} m(w(\mu + \rho_K) - \rho_K)$$

where $l(w)$ is the length of w in W_K.

If $\mathfrak{b}$ is any Lie algebra and $\mathfrak{b}'$ is a toral subalgebra, write $\Delta(\mathfrak{b}, \mathfrak{b}')$ for the set of roots of $\mathfrak{b}$ with respect to $\mathfrak{b}'$.

Definition 7. Fix a $(\mathfrak{g}, K)$ module X and a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. Choose any $w \in W_K$ so that

$$w\Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t}) \subseteq \Delta^+(\mathfrak{k}, \mathfrak{t})$$

and write

$$m = \min\{\dim(\mathfrak{u} \cap \mathfrak{p}),\ l(w) + 1\}.$$

Given a K-type $\mu \in \hat{T}$ of X, we say that μ lies on an *algebraic* $\mathfrak{q}$-*edge* of the set of K-types if

$$m_e(\mu + 2\rho_A) = 0$$

for every nonempty collection A consisting of elements in $\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t})$ of order at most m (here $2\rho_A$ is the sum of the roots in A).

Theorem 8. *Fix a $(\mathfrak{g}, K)$ module X and a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. If $\mu \in \hat{T}$ is a K-type of X lying on an algebraic $\mathfrak{q}$-edge, then there is another θ-stable parabolic and an $L \cap K$-type ν in the image of τ whose associated K-type is μ.*

Proof. By Theorem 3.4 in [**13**] (using the notation in Definition 7 above),

$$\tau : H^{l(w)}(w\mathfrak{u}, X) \to H^{l(w)}(w\mathfrak{u} \cap \mathfrak{k}, X)$$

is surjective on the $L \cap K$-types $\nu = w(\mu + \rho_K) - \rho_K$. Choosing w of minimal length, we may assume that ν appears in $H^{l(w)}(w\mathfrak{u} \cap \mathfrak{k}, X)$ (by Lemma 2.2 in [**13**]) and therefore that ν appears in the image of τ. But since $\nu_K = \mu$, we are done. ☐

We now apply Theorem 8 to the irreducible representations of G to show that the geometric edge can be constructed from the image of τ.

Theorem 9. *Let X be an irreducible representation of $G = SU(1,n)$. Then any K-type of X lying on a geometric edge is the associated K-type to an $L \cap K$-type lying in the image of $\tau : H^*(\mathfrak{u}, X) \to H^*(\mathfrak{u} \cap \mathfrak{k}, X)$ for some θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$.*

Proof. The idea of the proof is to use the explicit description of K-types (Corollary 1 and Theorems 5, 7, and 3) and to show that every K-type of X lying on a geometric edge lies on an algebraic $\mathfrak{q}$-edge for some θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. Theorem 8 then finishes the proof. Since this is merely a matter of coming up with $\mathfrak{q}$ and checking the appropriate definitions, we only give the details in the case of $X = J_{a,b}(\lambda)$ with $\lambda_0 > \lambda_1 > \dots \lambda_n$, $\lambda_i - \lambda_j \in \mathbb{Z}$ for $0 \le i < j \le n$, and $a, b \in \mathbb{Z}$ with $0 \le a \le b \le n$. The argument in the other cases is identical. Let $\mu \in \hat{T}^+$. Then μ is a K-type of $J_{a,b}(\lambda)$ if and only if it satisfies the integrality condition in Theorem 5 and $\xi_i^- \ge \mu_i \ge \xi_i^+$, $1 \le i \le n$, where $\xi_i^\pm$ (possibly equal to $\pm\infty$) are identified explicitly in Theorem 5. It is easy to verify that $\xi_{i-1}^+ \ge \xi_i^-$. Thus μ lies on a geometric edge if and only if either $\xi_i^- = \mu_i$ or $\mu_i = \xi_i^+$, for some i, $1 \le i \le n$. Write $E_i^\pm$, respectively, for the set of K-types lying on a geometric edge satisfying either $\xi_i^\pm = \mu_i$, respectively. Let $\varepsilon_i = (0, \dots, 0, 1, 0, \dots 0)$ with i zeros before the one and let $\varepsilon_{i,j} = \varepsilon_i - \varepsilon_j$ be the usual root vector in $\Delta(\mathfrak{g}, \mathfrak{t})$. For each edge define $\mathfrak{q}_i^\pm = \mathfrak{l}_i^\pm + \mathfrak{u}_i^\pm$ to be the θ-stable parabolic subalgebra of $\mathfrak{g}$ generated by $\pm\varepsilon_i$. In particular,

$$\Delta(\mathfrak{q}^\pm, \mathfrak{t}) = \{\varepsilon_{r,s} \mid \pm\, \varepsilon_i \cdot \varepsilon_{r,s} \ge 0 \text{ where } 0 \le r, s \le n,\, r \ne s\}$$

and $\Delta(\mathfrak{u}^\pm \cap \mathfrak{p}, \mathfrak{t}) = \{\mp\varepsilon_{0,i}\}$ so that $\dim(\mathfrak{u}^\pm \cap \mathfrak{p}) = 1$. There are two motivating factors behind this choice of $\mathfrak{q}^\pm$. The first condition is that $\mathfrak{l}^\pm$ is supposed to describe the direction of $E^\pm$; *i.e.*, if $\mu, \mu' \in E^\pm$ then their difference should be in the span of the roots of $\mathfrak{l}^\pm$. The second condition is that $\mathfrak{u}^\pm$ should point towards the outside of the K-types; *i.e.*, if $\mu \in E^\pm$ then the sum of μ and any non-zero root in $\mathfrak{u}^\pm$ should not be a K-type. It is easy to check that $\mathfrak{q}^\pm$ is the largest parabolic satisfying these both conditions.

Let $w^\pm \in W_K$ be defined by the cyclic permutations $w^+ = (1, 2, \dots i)$ and $w^- = (n, n-1, \dots i)$. Then $l(w^+) = i - 1$ and $l(w^-) = n - i$. These elements are chosen so that $w^\pm \Delta(\mathfrak{u}^\pm \cap \mathfrak{k}, \mathfrak{t}) \subseteq \Delta^+(\mathfrak{k}, \mathfrak{t})$. In particular, $\omega^+\mathfrak{q}^+$ is generated by ε_1 and $\omega^-\mathfrak{q}^-$ is generated by ε_n. Let μ be a K-type in $E^\pm$. Then μ lies on the algebraic $\mathfrak{q}^\pm$-edge if and only if $m_e(\mu \mp \varepsilon_{0,i}) = 0$. In fact, we prove a much stronger statement that is special to $SU(1, n)$: that $m_e(\mu \mp r\varepsilon_{0,i}) = 0$ for any $r \in \mathbb{R}^{>0}$. For this it suffices to set $y = \omega(\mu \mp r\varepsilon_{0,i} + \rho_K) - \rho_K$ for any $\omega \in W_G$ and show that y cannot be a K-type of $J_{a,b}$.

It is convenient to shift everything by ρ_K. Therefore write $\widetilde{\mu}$ for $\mu + \rho_K$ and employ similar notation for ξ and y. Thus assume we have $\tilde{\xi}^\pm$ satisfying $\tilde{\xi}_{k-1}^+ > \tilde{\xi}_k^-$, $1 \le k \le n$, $\widetilde{\mu}$ satisfying $\tilde{\xi}_k^- \ge \tilde{\mu}_k \ge \tilde{\xi}_k^+$ and either $\tilde{\xi}_i^\pm = \tilde{\mu}_i$, and $\tilde{y} = \omega(\tilde{\mu} \mp r\varepsilon_{0,i})$ with $r > 0$. We show there is always some k, $1 \le k \le n$, so that $\tilde{y}_k$ fails to lie between $\tilde{\xi}_k^\pm$. View ω as a permutation and consider the case where $\omega(i) = i$ first. In this case $\tilde{y}_i = \tilde{\mu}_i \mp r$ and therefore fails to lie between $\tilde{\xi}_i^-$ and $\tilde{\xi}_i^+$. On the other hand, say $\omega(i) = j \ne i$. Then

$\tilde{y}_j = \tilde{\mu}_i$ and therefore lies between $\tilde{\xi}_i^{\pm}$. However, this makes it impossible for $\tilde{y}_j$ to lie between $\tilde{\xi}_j^{\pm}$ since $\tilde{\xi}_i^{\pm}$ and $\tilde{\xi}_j^{\pm}$ are disjoint intervals. This finishes the proof. □

Corollary 3. *Let X be any irreducible representation of $SU(1,n)$. Let $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i$, $i = 1, n$, be the two maximal proper θ-stable parabolic subalgebras of $\mathfrak{g}$ generated by ε_1 and $-\varepsilon_n$, respectively. Let E be the set of associated K-types to the $L \cap K$-types in the images of*

$$\tau : H^*(\mathfrak{u}_i, X) \to H^*(\mathfrak{u}_i \cap \mathfrak{k}, X).$$

Then the closed convex hull of the set of highest weights of the K-types of X is equal to the closed convex hull of E. Moreover, $\mu \in \mathfrak{t}^*$ is a K-type of X if and only if μ lies in the closed convex hull of E and differs from some element of E by an element of the root lattice of $\mathfrak{g}$.

Proof. This follows immediately from the Theorem 9 (noting that all the parabolics used in the proof were K conjugate to either $\mathfrak{q}_1$ or $\mathfrak{q}_n$) and the explicit description of K-types. □

While this is a strong statement, the generalization to other groups cannot always be as nice. In particular, "gaps" may appear in the set of K-types. But in any case, it is still conjectured that the image of τ is enough to describe the closed convex hull of the set of K-types.

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